Random walks on percolation clusters

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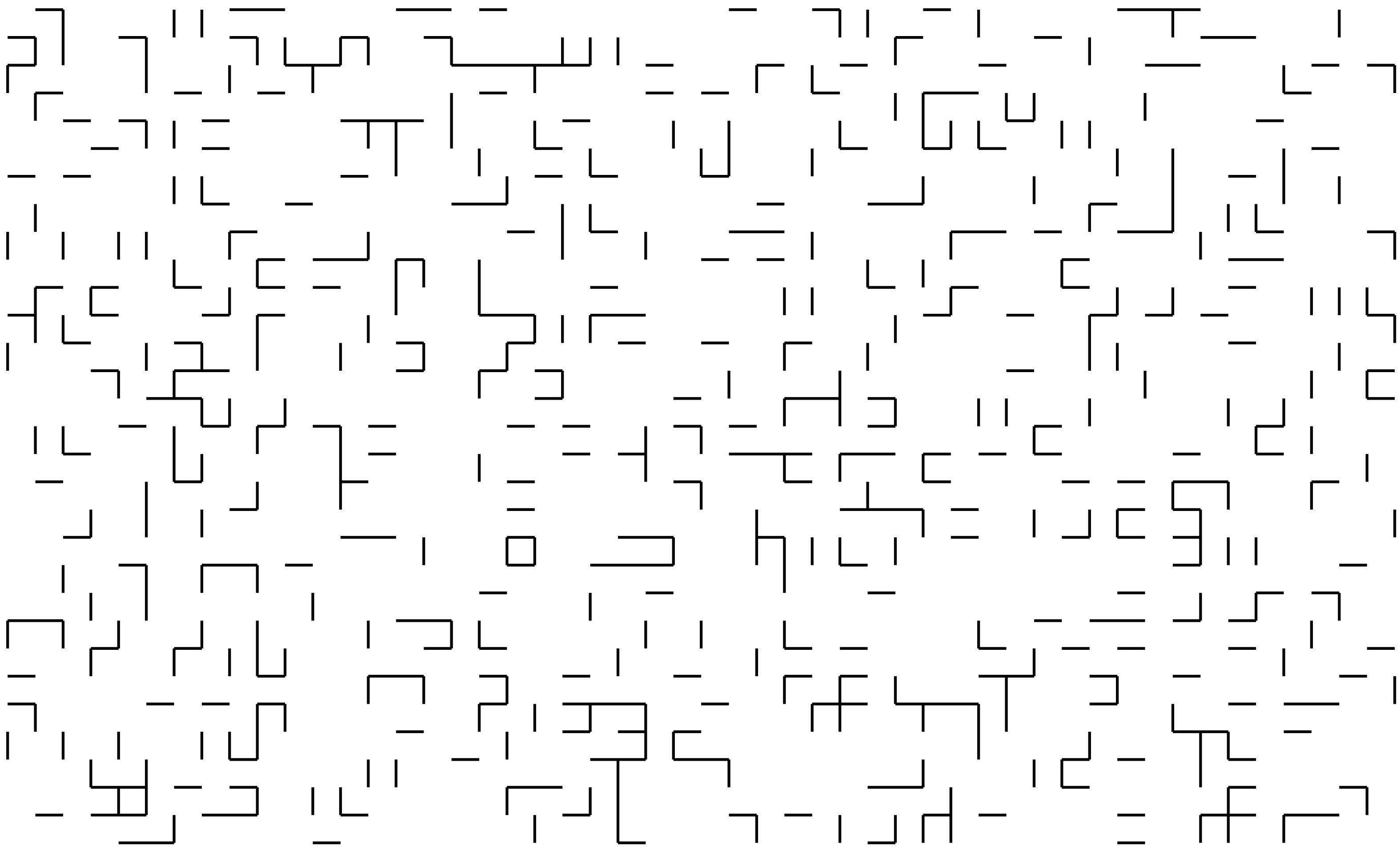
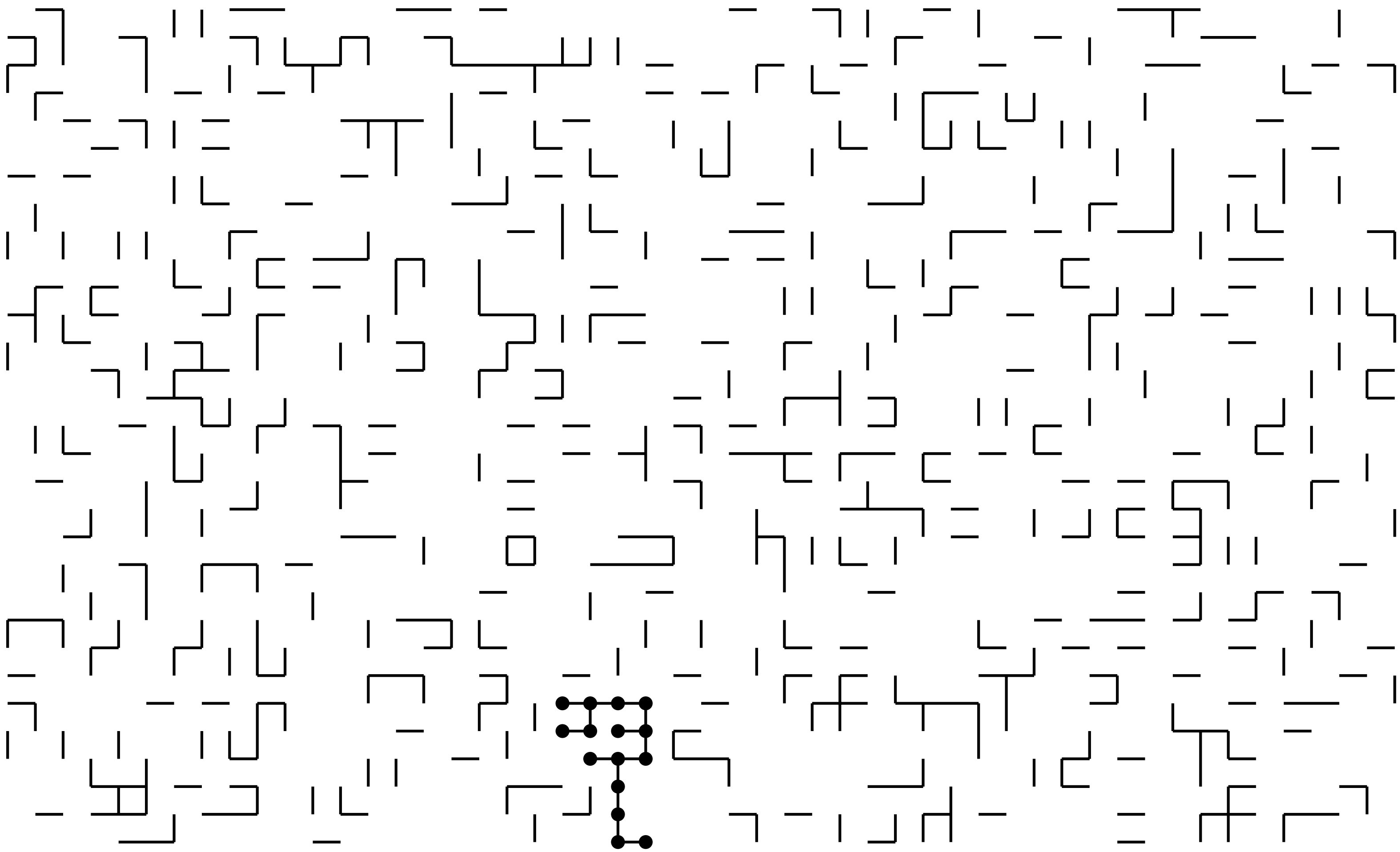
- Mathematicians are just developing the tools to start thinking about such questions!

Percolation

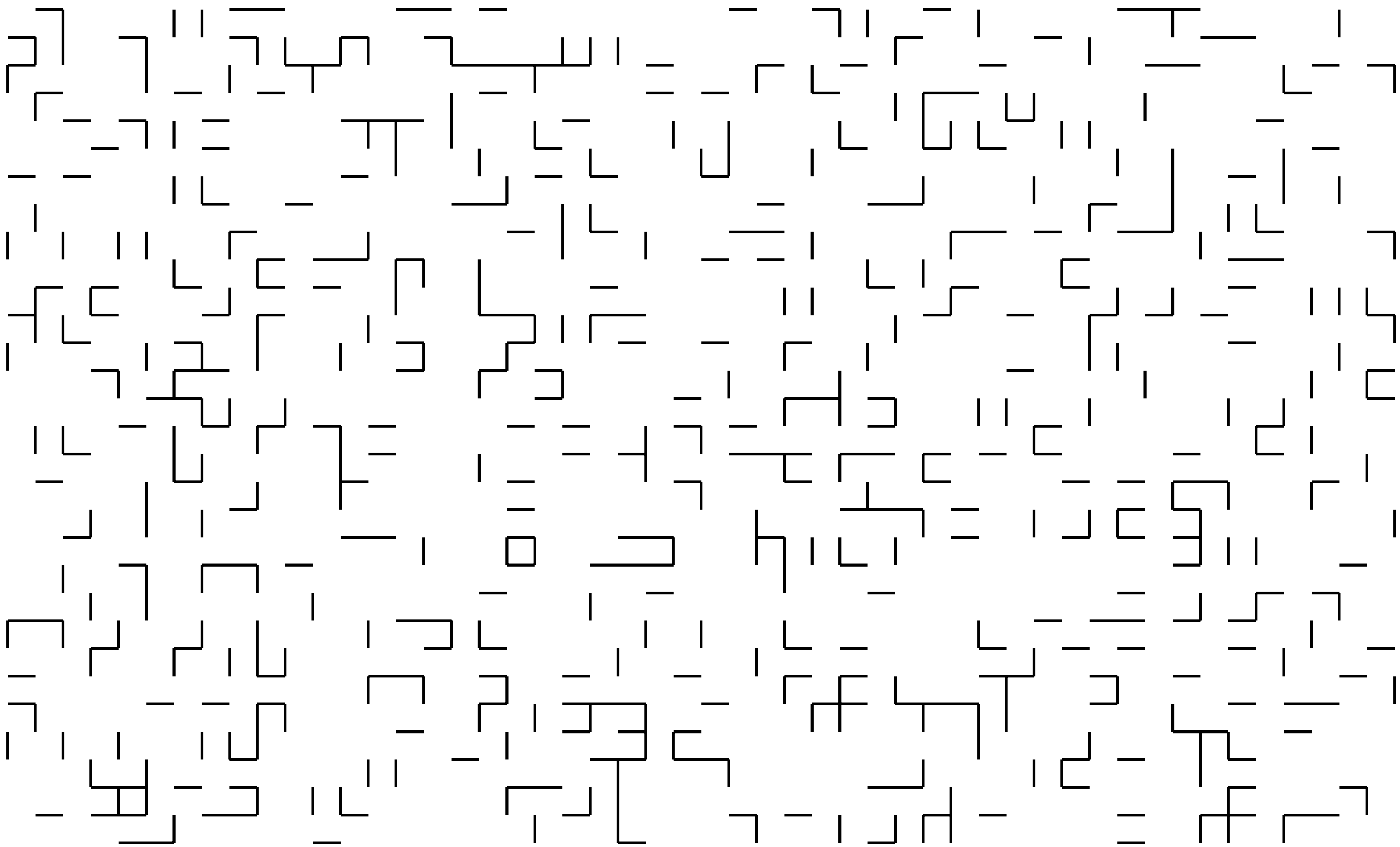
- Introduced by Broadbent and Hammersley (1957).
- Euclidean lattice $\mathbb{Z}^d$, edges (bonds) E_d.
- Fix $p \in [0,1]$. For $x \sim y$, let μ_{xy} be independent random variables with $\mathbb{P}(\mu_{xy} = 1) = p$, $\mathbb{P}(\mu_{xy} = 0) = 1 - p$.
  The bonds (edges) such that $\mu_{xy} = 1$ are called *open bonds*. Let $\mathcal{O}$ be the set of open bonds.
- The connected components of the graph $(\mathbb{Z}^d, \mathcal{O})$ are called *(open) clusters*.
- There exists $p_c \in (0,1)$ such that, a.s.,
  -if $p < p_c$, all clusters are finite,
  -if $p > p_c$, then there exists a unique infinite cluster, C_∞.
  -if $p = p_c$, no infinite cluster for $d = 2, d \geq 19$, believed $\forall d$.



$p = 0.2$



$p = 0.2$, largest cluster marked



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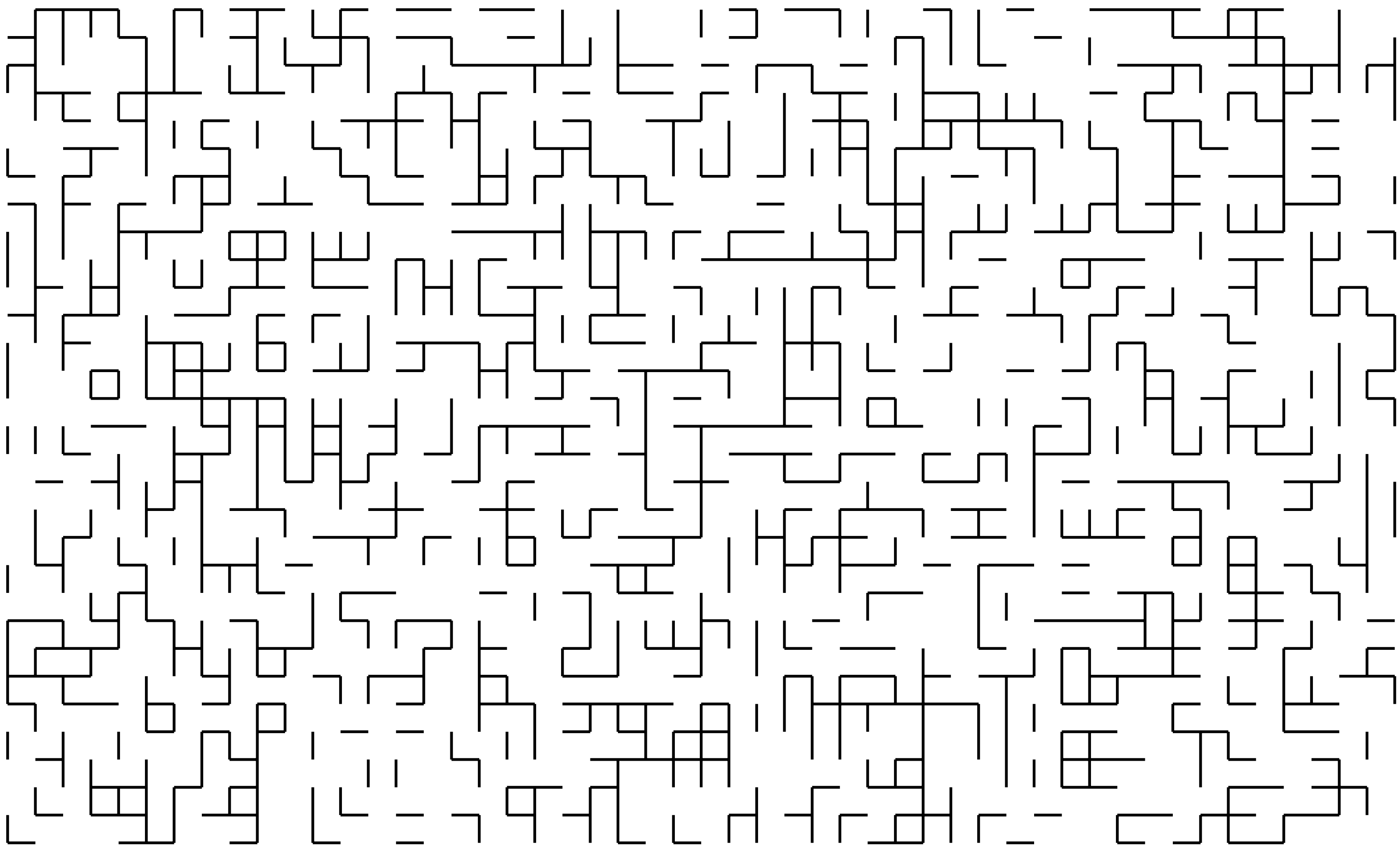

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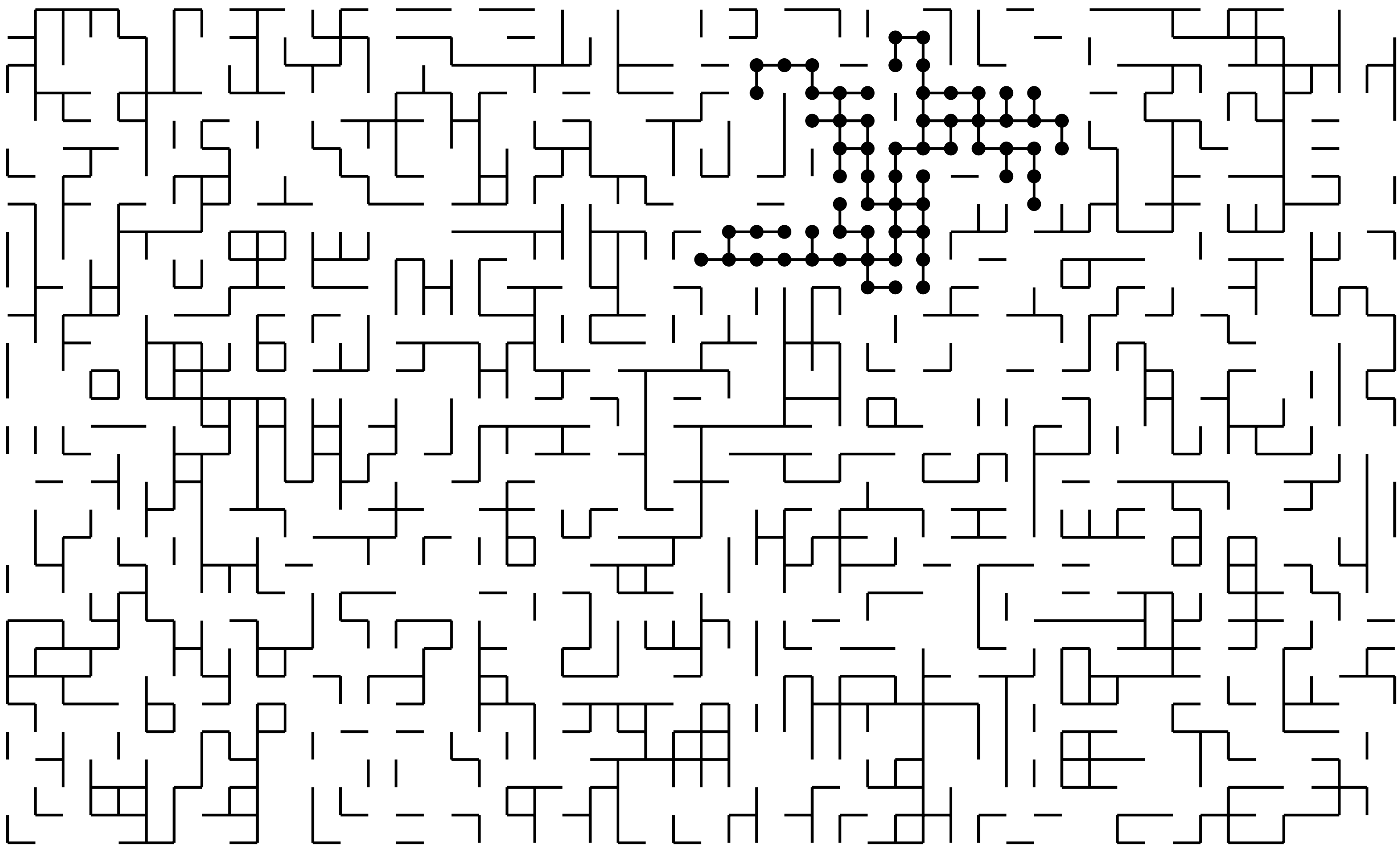

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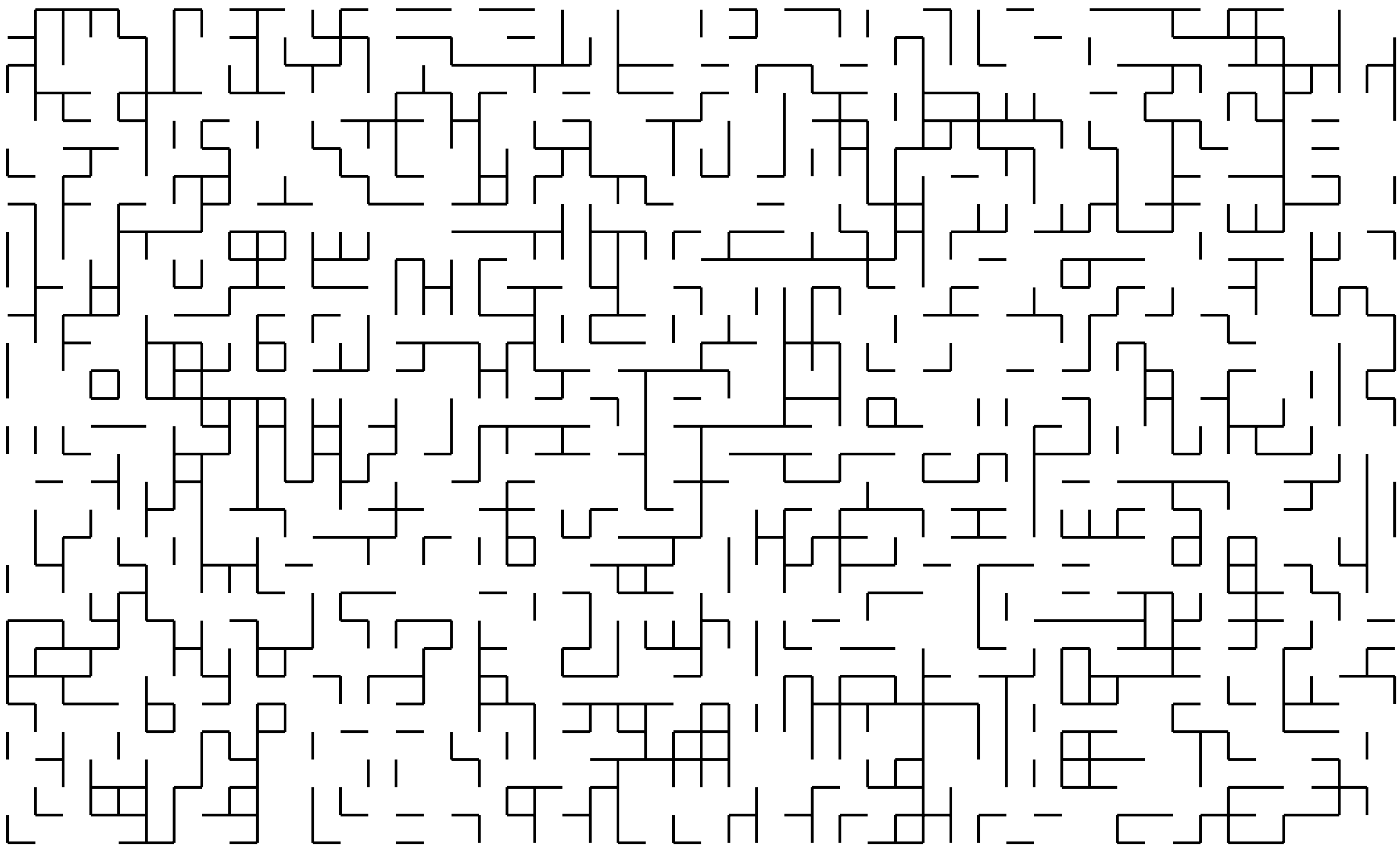

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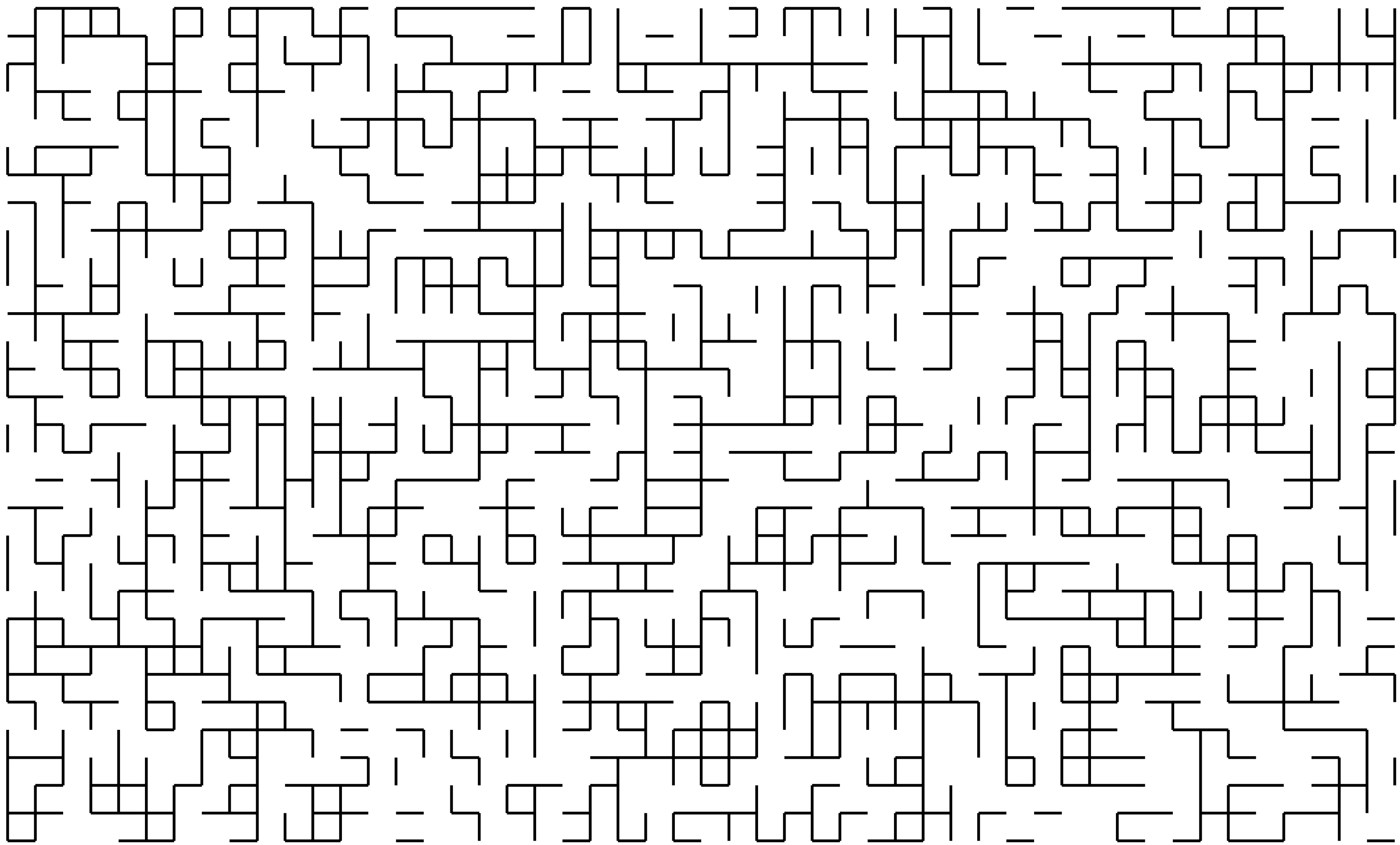

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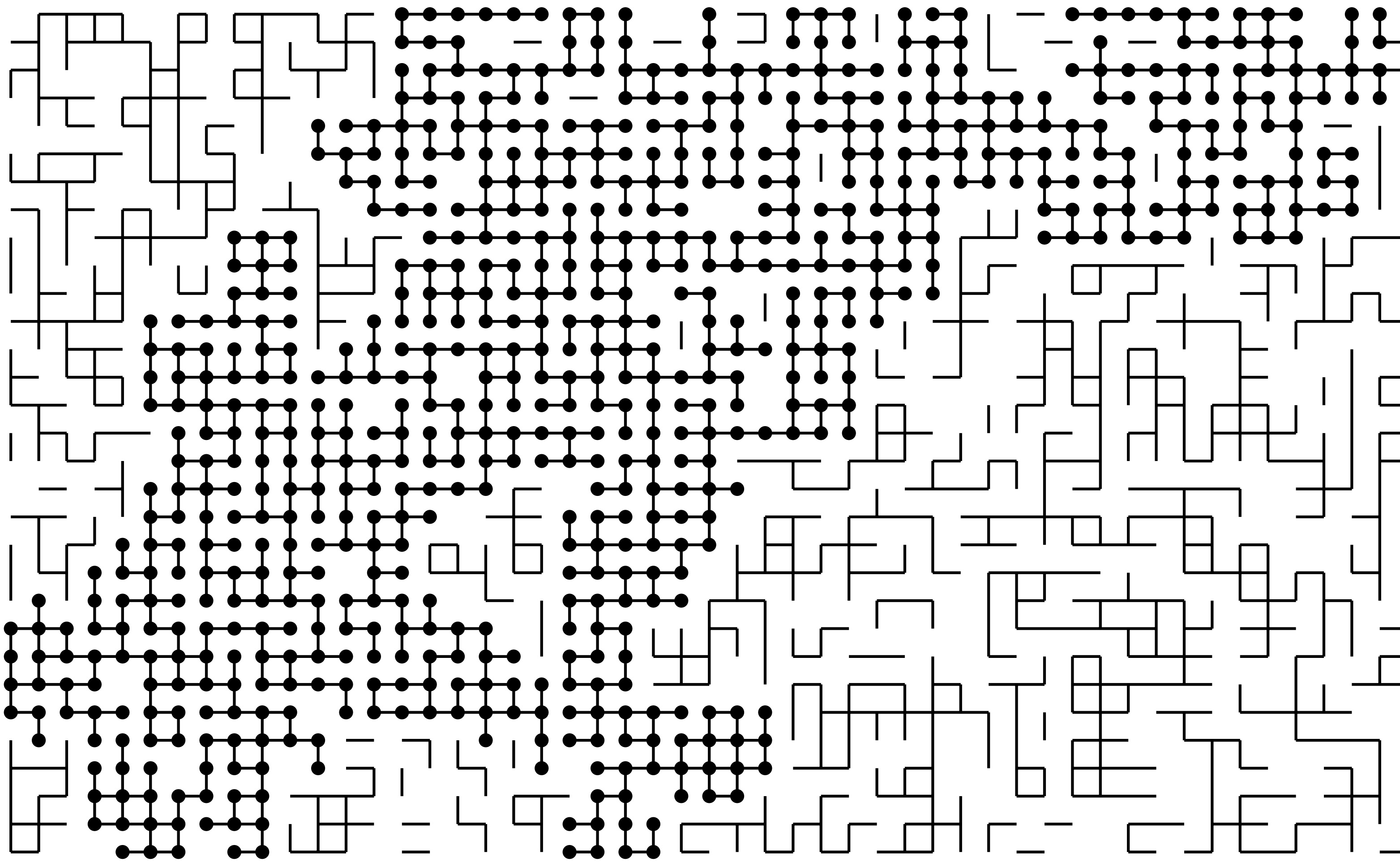

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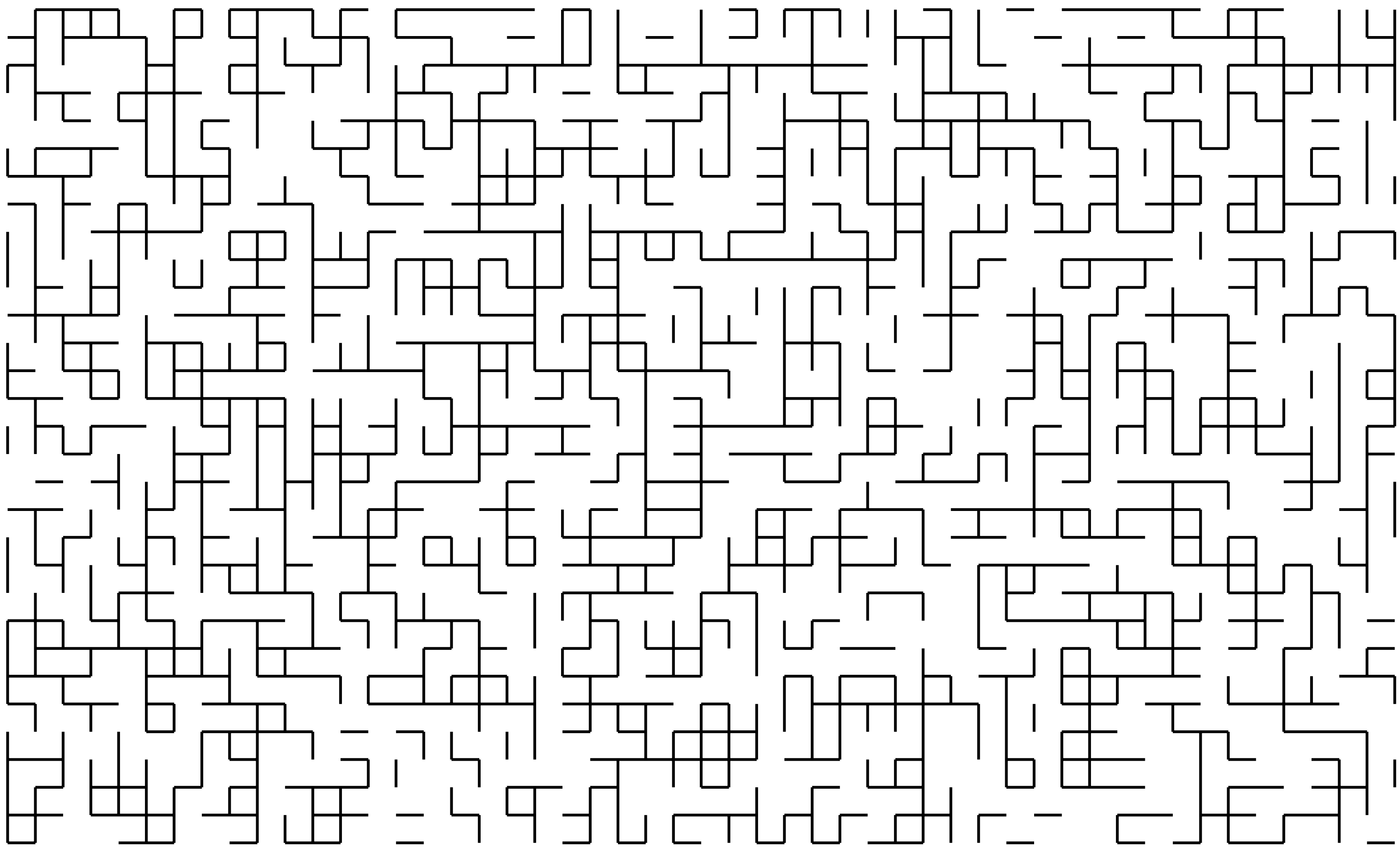

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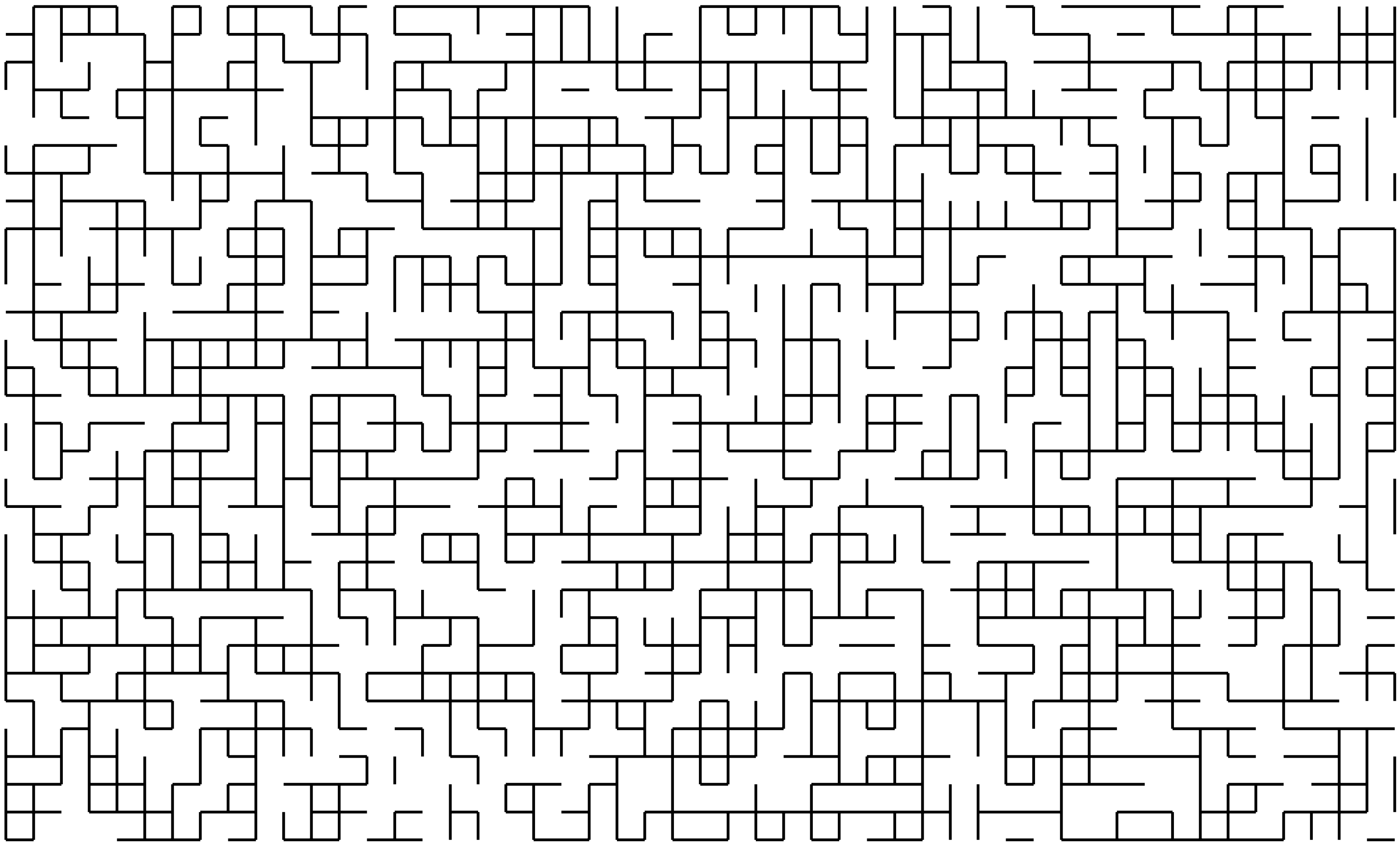

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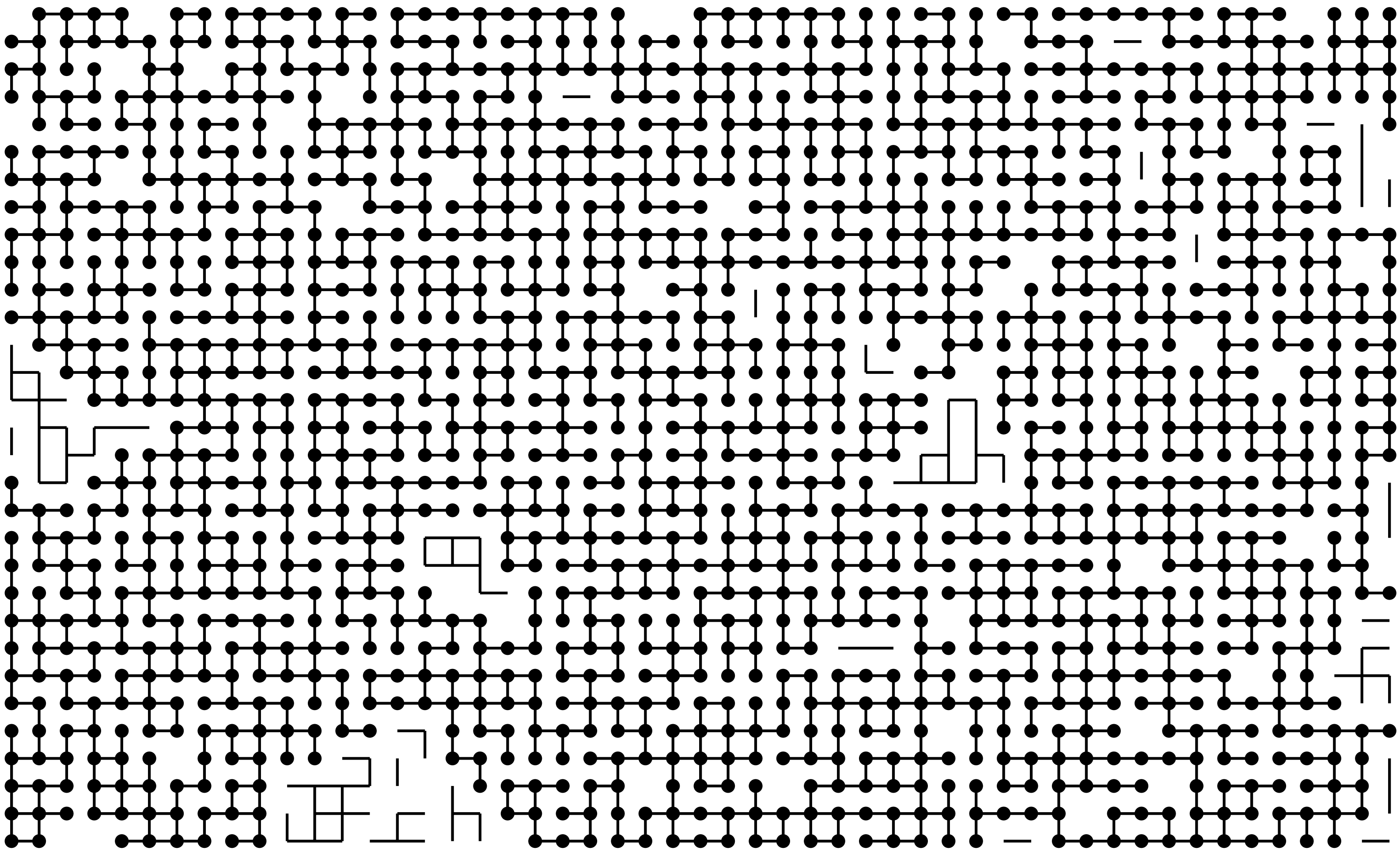

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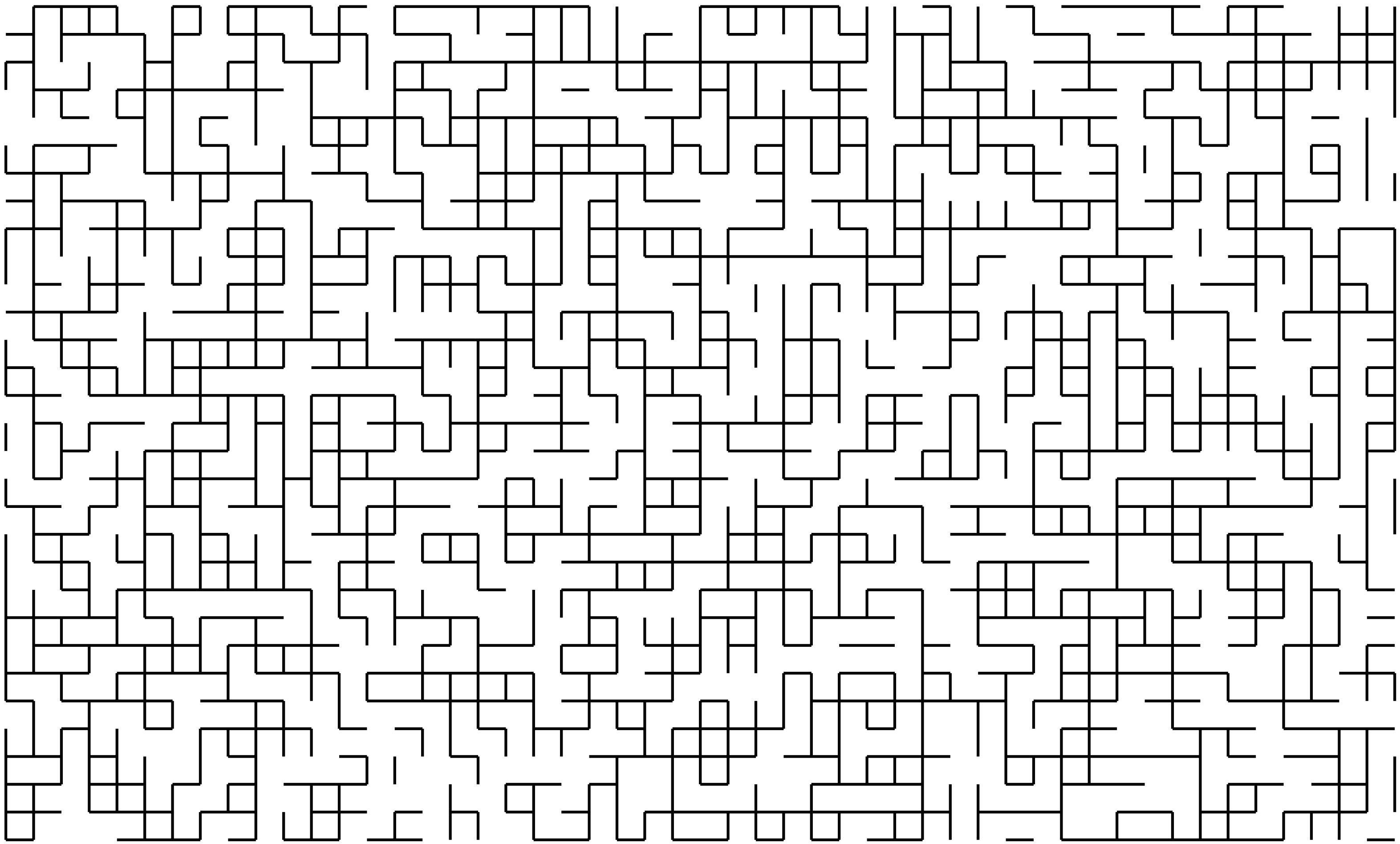

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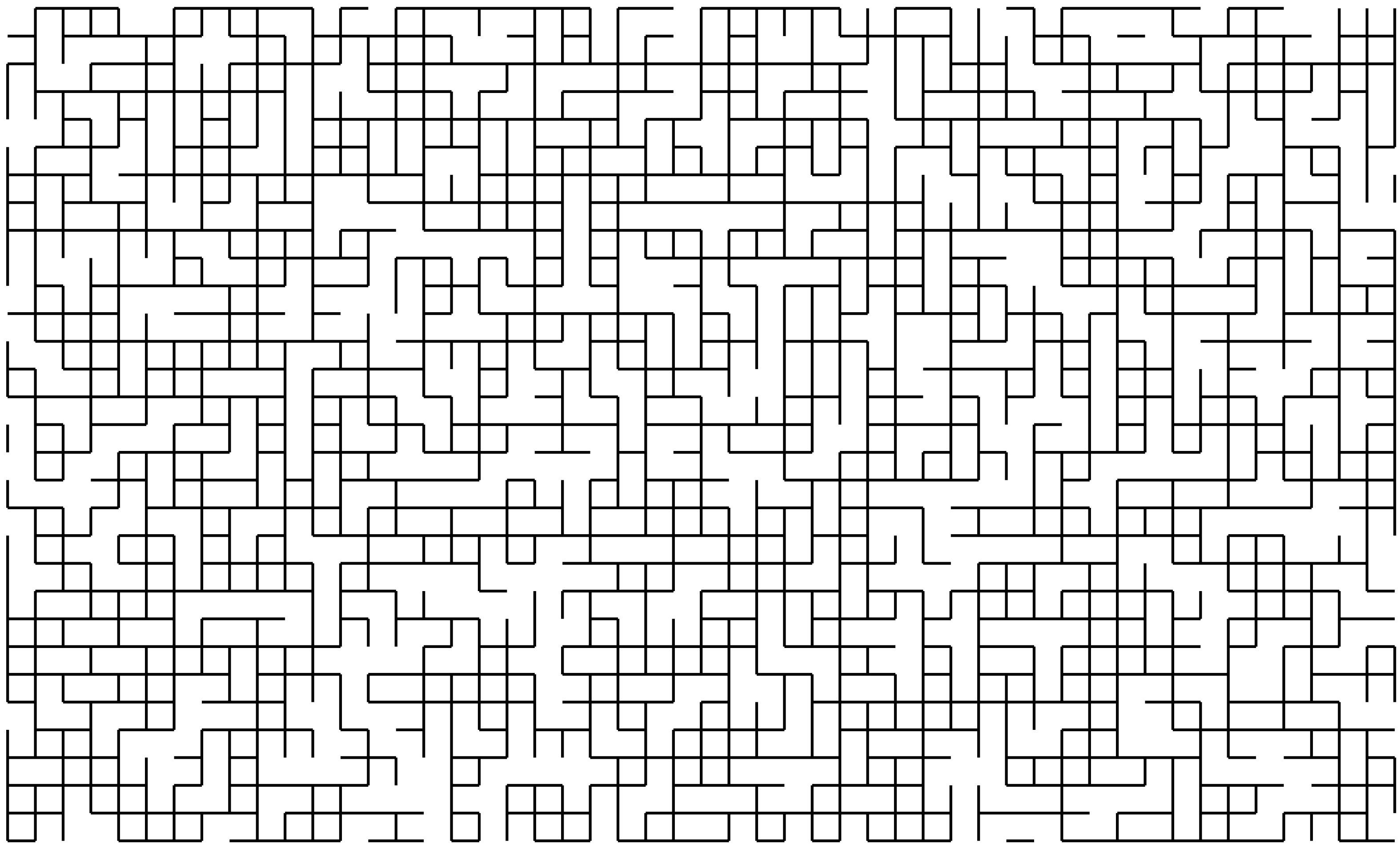

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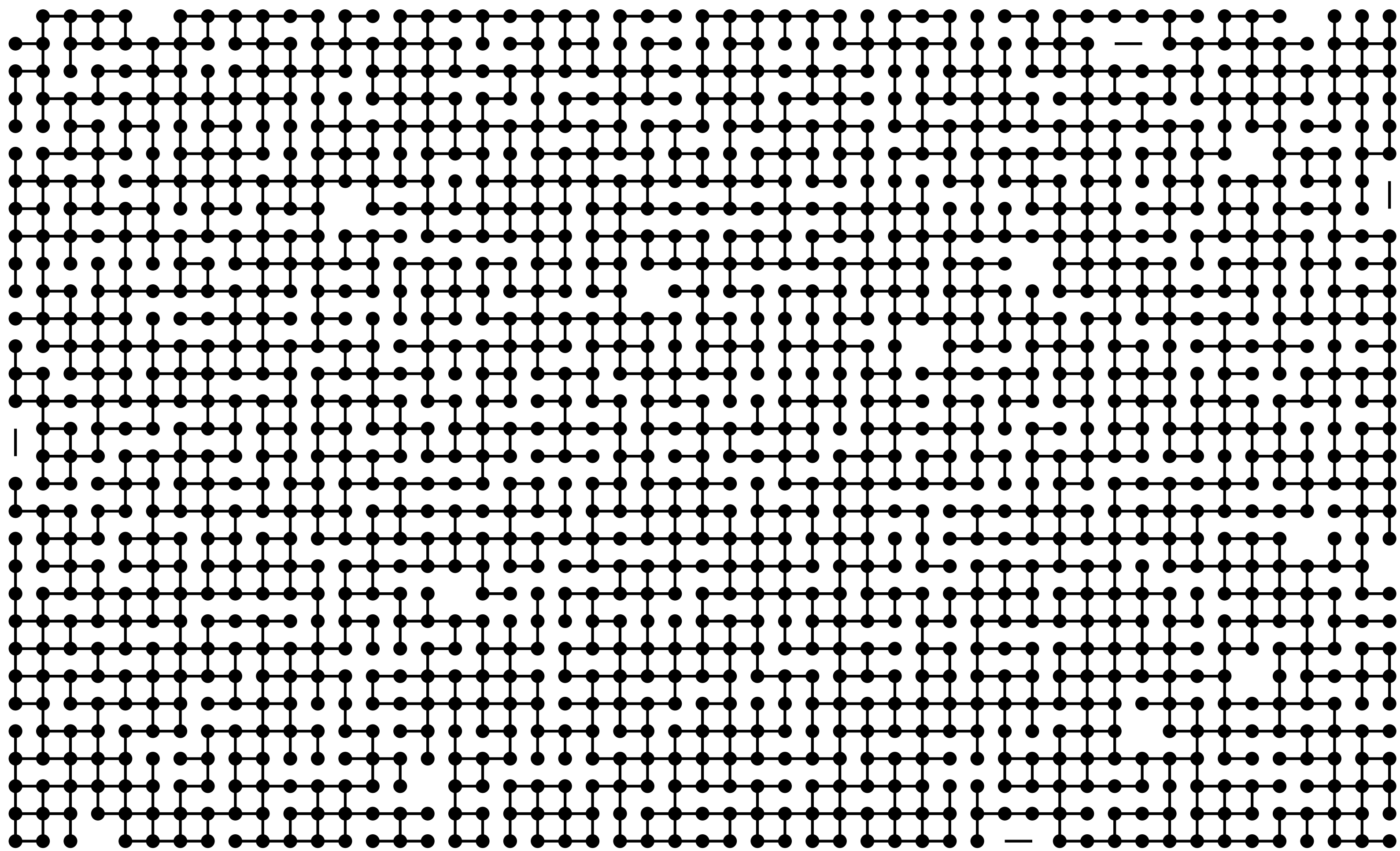

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Graphs

- Let $\Gamma = (G, E_\Gamma)$ be an infinite connected locally finite graph. Define weights or conductances $\mu_{xy} = \mu_{yx}$ for $x \sim y$. We allow $\mu_{xx} > 0$. Let $\mu_{xy} = 0$ if $x \not\sim y$. Set

$$\mu_x = \sum_y \mu_{xy},$$

and extend μ to a measure on G. The volume of $B(x,r)$, a ball in the graph of radius r at x, is $V(x,r) = \sum_{y \in B(x,r)} \mu_y$.

- Discrete Laplacian:

$$\Delta f(x) = \frac{1}{\mu_x} \sum_y \mu_{xy}(f(y) - f(x)). \qquad (1)$$

Random walks

- Continuous time random walk $Y = (Y_t, t \in [0, \infty))$ on (Γ, μ). If $Y_t = x$, then the probability of a jump to $y \sim x$ in $(t, t+\delta]$ is $\approx \delta\mu_{xy}/\mu_x$.
- Let $q_t(x, y)$ be the transition density of Y (w.r.t. μ), i.e.

$$\mathbb{P}^x(Y_t = y) = q_t(x, y)\mu_x.$$

Then $q_t(x, y) = q_t(y, x)$ satisfies the discrete heat equation (time continuous, space discrete)

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t).$$

- The discrete random walk $X = (X_n, n = 0, 1, \dots)$ on (Γ, μ). If $X_n = x$, then the probability of a jump to $y \sim x$ is μ_{xy}/μ_x.
- Let $p_n(x, y)$ be the transition density of Y (w.r.t. μ), i.e.

$$\mathbb{P}^x(X_n = y) = p_n(x, y)\mu_x.$$

Then $p_n(x, y) = p_n(y, x)$ satisfies the discrete heat equation (discrete time and space):

$$u(x, n+1) - u(x, n) = \Delta u(x, n).$$

Note that to deal with bipartite graphs we use $\hat{p}_n(x, y) = p_{n+1}(x, y) + p_n(x, y)$.

Random walk on $\mathcal{C}_\infty$

- We work in the supercritical case $p > p_c$. Fix a percolation configuration ω. Let $G = \mathcal{C}_\infty(\omega)$, E be the open bonds in $\mathcal{C}_\infty(\omega)$. This defines an (infinite, connected) weighted graph. Let Y_t be the continuous time random walk on $(\mathcal{C}_\infty(\omega), \mu(\omega))$. Its transition density is
$$q_t^\omega(x,y)\mu_y(\omega) = P_\omega^x(Y_t = y).$$
- The discrete version was called the 'ant in the labyrinth' by De Gennes 1976.
- We can consider myopic ants - for which $\mu_{xx} = 0$ and blind ants for which $\mu_{xx} = 2d - \sum_{y \neq x} \mu_{xy}$.
- Grimmett, Kesten, Zhang, 1993: Y is transient iff $d \geq 3$.

- Problems for the random walk Y on $\mathcal{C}_\infty$:
(1) Gaussian bounds (GB) on $q_t^\omega(x,y)$.
(2) Central limit theorem/ Invariance principle for Y.
(3) A local limit theorem for Y.
- CLT for $\mathbb{Z}^d$. Let $Y_t^{(n)} = n^{-1} Y_{n^2 t}$. Then

$$\mathbb{P}^0(Y_t^{(n)} \in U) \to \int_U (2\pi C_d t)^{-d/2} \exp(-\frac{|x|^2}{2C_d t}) dx$$

- Invariance principle for $\mathbb{Z}^d$ (Donsker 1951):

$$(Y_t^{(n)}, t \geq 0) \Rightarrow (C_d^{1/2} W_t, t \geq 0)$$

where W is Brownian motion.

The critical case

In the critical case there is no infinite cluster with probability 1 (at least for $d = 2, d \geq 19$). In this case we must define an 'Incipient infinite cluster' (IIC). This critical cluster should have fractal structure. For $d = 2$ it can be described via an SLE.

- Kesten (1986): random walk on the IIC in $d = 2$ is subdiffusive.
- Barlow & Kumagai (2006): random walk on the IIC on a tree ('$d = \infty$') has sub-Gaussian heat kernel estimates. Croydon (2006), the scaling limit is Brownian motion on the continuum random tree.
- Barlow, Jarai, Kumagai and Slade (2007), random walk on high dimensional spreadout oriented percolation.

Two types of invariance principle

- Y is a RW in a random environment $\mathcal{C}_\infty(\omega)$.
  Let $\mathbb{P}$ be the probability measure for the percolation configuration $\mathcal{C}_\infty$.
  Let P^x_ω be the probability measure for Y on $\mathcal{C}_\infty(\omega)$ starting at $x \in \mathcal{C}_\infty$.
- *Quenched or almost sure.* The Invariance principle for Y holds (w.r.t. P^x_ω) for a set of environments ω with $\mathbb{P}$ probability 1.
- *Averaged, or 'annealed'.* The Invariance principle for Y holds w.r.t. $\mathbb{P} \times P^x_\omega$.
- De Masi, Ferrari, Goldstein, Wick 1989: The averaged invariance principle holds for processes in stationary ergodic random environments. In particular, this holds for Y on $\mathcal{C}_\infty$.

Delmotte's theorem

Theorem *(T. Delmotte, 1999). Let* (Γ, μ) *be a weighted graph. (Assume* Γ *locally finite,* $\mu_{xy} \in [C^{-1}, C]$ *whenever* $x \sim y$*.) The following are equivalent:*

(a) Solutions of the heat equation on G *satisfy a Parabolic Harnack inequality (PHI).*

(b) (Γ, μ) *satisfies volume doubling (VD) and a Poincare inequality (PI).*

(c) $q_t(x, y)$ *satisfies Gaussian bounds :*

$$\frac{c_1 e^{-c_2 d(x,y)^2/t}}{V(x, t^{1/2})} \leq q_t(x, y) \leq \frac{c_3 e^{-c_4 d(x,y)^2/t}}{V(x, t^{1/2})},$$

if $t \geq \max(1, |x - y|)$*.*

Poincare inequality for graphs

- Let $B = B(x,r)$, $f : B \to \mathbb{R}$. Then

$$\sum_{x \in B} (f(x) - \overline{f})^2 \mu_x \le C_P r^2 \sum_{x,y \in B} (f(y) - f(x))^2 \mu_{xy}$$
$$= C_P r^2 \mathcal{E}_B(f,f).$$

As usual $\overline{f}$ is the real number which minimises the LHS.

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- An example of a graph for which the PI fails is two copies of $\mathbb{Z}^d$ ($d \ge 3$) connected at their origins.
  If $f = 1$ on one copy, $f = -1$ on the other and $B = B(x,r)$ then LHS $\approx r^d$ while the RHS $\approx r^2$.

Bounds on q_t

- The natural idea is to try to apply Delmotte's theorem.
- However, neither VD nor PI hold for $\mathcal{C}_\infty$. The reason is that if we look far enough we can find arbitrarily large 'bad regions'.

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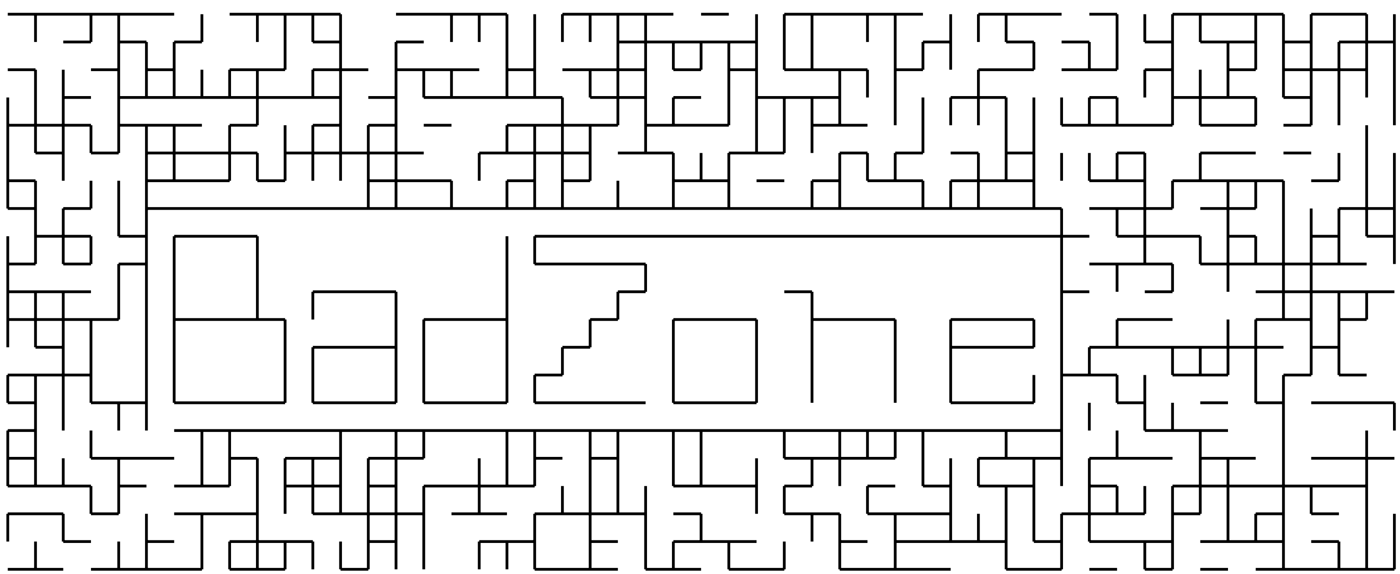

Obtaining Gaussian bounds for $\mathcal{C}_\infty$

- For the on-diagonal bound isoperimetric or Nash inequality ideas lead to (Mathieu and Remy (2004))

$$\sup_y q_t^\omega(x,y) \le c t^{-d/2},$$

for $t \ge S_x(\omega)$.

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$$\sup_y q_t^\omega(x,y) \le c t^{-d/2},$$

for $t \ge S_x(\omega)$.

- The next, and hardest, step in controlling $q_t(x,y)$ is to obtain 'off-diagonal' bounds, i.e.

$$q_t^\omega(x,y) \le \theta(t, |x-y|),$$

where $\theta(t,r) \to 0$ as $r \to \infty$.

Gaussian bounds

Theorem 1. *(Barlow, 2004) Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. S_x with $\mathbb{P}_p(S_x \geq n) \leq c\exp(-n^{\varepsilon_d})$ and (non-random) constants $c_i = c_i(d, p)$ such that the transition density of Y satisfies,*

$$\frac{c_1}{t^{d/2}} e^{-c_2|x-y|_1^2/t} \leq q_t^\omega(x, y) \leq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|_1^2/t}, \qquad (GB)$$

for $x, y \in \mathcal{C}_\infty(\omega)$, $t \geq \max(S_x(\omega), 1)$.

Note. The randomness of the environment is taken care of by the $S_x(\omega)$, which will be small for most points, and large for the rare 'bad points'.
The same bounds hold for the discrete transition density.

Quenched invariance principles

- **Theorem 2.** *(Sidoravicius and Sznitman, 2004 ($d \geq 4$), Berger and Biskup, 2005, Mathieu and Piatnitski, 2005). A quenched or a.s. invariance principle holds for Y.*
- The BB, MP papers used the *corrector*. This is a (random) function $\chi(\omega, x) : \mathcal{C}_\infty(\omega) \to \mathbb{R}^d$ such that $h(x) = x + \chi(x)$ is harmonic.
- This implies that if $q_t^{(n,\omega)}(x, y) = n^d q^\omega_{n^2 t}(\lfloor nx \rfloor, \lfloor ny \rfloor)$, then for $f \in C_K(\mathbb{R}^d)$, with $\mathbb{P}$-probability 1,

$$\int q_t^{(n,\omega)}(x, y) f(y) dy \to \int k_t(x, y) f(y) dy$$

where, $k_t(x, y) = (2\pi D)^{-d/2} \exp(-|x - y|^2 / 2Dt)$, $D > 0$.

PHI and Local Limit Theorem

The Gaussian bounds for $q_t^\omega(x,y)$ lead to a Parabolic Harnack inequality. This gives Hölder continuity of $q_t^\omega(x,y)$, and will allow us to replace the integrals by pointwise expressions.

We say a Ball $B(x,R)$ in the graph is good if it has a PI and a C such that $\mu(B(x,R)) \geq CR^d$. It is very good if all balls of a reasonable size in $B(x,R)$ are good.
We prove our PHI for very good balls.

Parabolic Harnack inequality

Let

$$Q(x,R,T) = [0,T] \times B(x,R),$$

and

$$Q_-(x,R,T) = [\tfrac{1}{4}T, \tfrac{1}{2}T] \times B(x, \tfrac{1}{2}R),$$

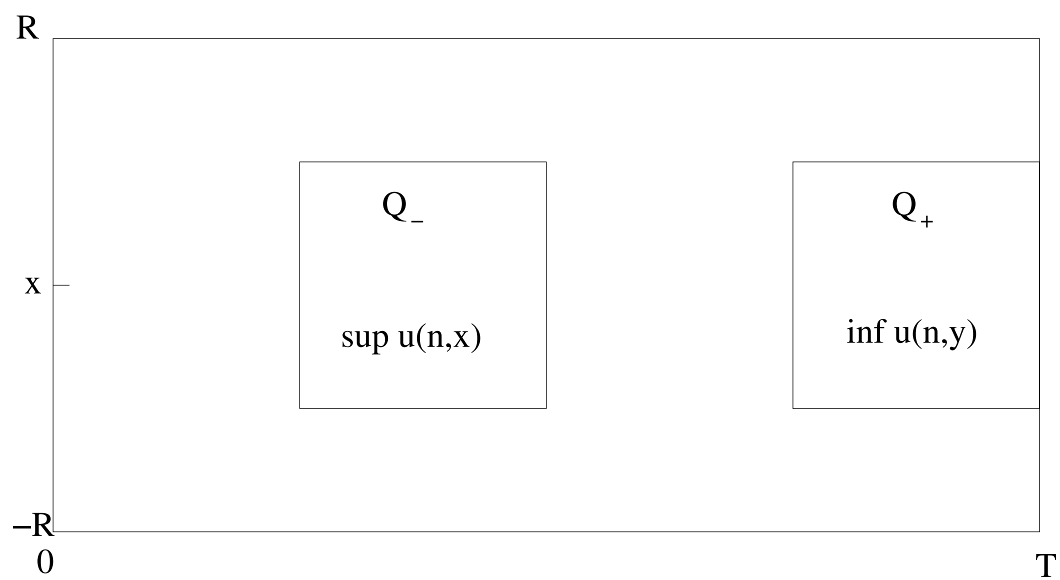
$$Q_+(x,R,T) = [\tfrac{3}{4}T, T) \times B(x, \tfrac{1}{2}R).$$

We say that a function $u(n,x)$ is *caloric* on Q if u is defined on $\overline{Q} = ([0,T] \cap \mathbb{Z}) \times \overline{B}(x,R)$, and

$u(n+1,y) - u(n,y) = \Delta u(n,y)$ for $0 \le n \le T-1$, $y \in B(x,R)$.

We say the parabolic Harnack inequality (PHI) holds with constant C_H for $Q = Q(x, R, T)$ if whenever $u = u(n, x)$ is non-negative and caloric on Q, then

$$\sup_{(n,x)\in Q_-} \hat{u}(n, x) \le C_H \inf_{(n,x)\in Q_+} \hat{u}(n, x). \quad (1)$$



The PHI

We assume that the conductivities μ_{xy} are bounded away from 0 and ∞ (μ_{xx} can be 0) and $\mu(B(x,r)) \leq cr^d$ for $r > 1$. Bounds on the heat kernel can be used to establish the PHI via a balayage argument.

Theorem 3.

Let $x_0 \in G$. Suppose that $R \geq 16$ and $B(x_0, R)$ is very good. Let $x_1 \in B(x_0, R/3)$, and $R_1 \log R_1 = R$. Then there exists a constant C_H such that the PHI (in both discrete and continuous time settings) holds with constant C_H for $Q(x_1, R_1, R_1^2)$.

By applying this PHI on a nested set of cubes we can control the oscillation in caloric functions.

Hölder Continuity

Let $x_0 \in G$. Suppose the PHI (with constant C_H) holds for $Q(x_0, R, R^2)$ for $R \geq s(x_0)$. Let $\theta = \log(2C_H/(2C_H - 1))/\log 2$, and

$$\rho(x_0, x, y) = s(x_0) \vee d(x_0, x) \vee d(x_0, y).$$

Let $r_0 \geq s(x_0)$, $t_0 = r_0^2$, and suppose that $u = u(n, x)$ is caloric in $Q = Q(x_0, r_0, r_0^2)$. Let $x_1, x_2 \in B(x_0, \frac{1}{2}r_0)$, and $t_0 - \rho(x_0, x_1, x_2)^2 \leq n_1, n_2 \leq t_0 - 1$. Then

$$|\hat{u}(n_1, x_1) - \hat{u}(n_2, x_2)| \leq c\Big(\frac{\rho(x_0, x_1, x_2)}{t_0^{1/2}}\Big)^\theta \sup_{Q_+} |\hat{u}|.$$

Let $k_t^{(D)}(x)$ be the Gaussian heat kernel in $\mathbb{R}^d$ with diffusion constant $D > 0$ and let $X_t^{(n)} = n^{-1/2} X_{\lfloor nt \rfloor}$. For $x \in \mathbb{R}^d$, set

$$H(x, r) = x + [-r, r]^d, \qquad \Lambda(x, r) = H(x, r) \cap \mathcal{G}. \qquad (2)$$

In general $\Lambda(x, r)$ will not be connected. Let

$$\Lambda_n(x, r) = \Lambda(xn^{1/2}, rn^{1/2}).$$

For $x \in \mathbb{R}^d$ let $g_n(x)$ be a closest point in $\mathcal{G}$ to $n^{1/2}x$, in the $|\cdot|_\infty$ norm.

Assumption 1 There exists a constant $\delta > 0$, and positive constants D, C_H, $C_i, a_{\mathcal{G}}$ such that the following hold.
(a) (CLT for X). For any $y \in \mathbb{R}^d$, $r > 0$,

$$P^0(X_t^{(n)} \in H(y,r)) \to \int_{H(y,r)} k_t^{(D)}(y)dy. \quad (3)$$

(b) There is an upper heat kernel bound

$$p_k(0,y) \le C_2 k^{-d/2}, \quad \forall y \in \mathcal{G}, k \ge C_3.$$

(c) For each $y \in \mathcal{G}$ there exists $s(y) < \infty$ such that the PHI holds with constant C_H for $Q(y, R, R^2)$ for $R \ge s(y)$.

(d) For any $r > 0$

$$\frac{\mu(\Lambda_n(x,r))}{(2n^{1/2}r)^d} \to a_{\mathcal{G}} \qquad \text{as } n \to \infty. \tag{4}$$

(e) For each $r > 0$ there exists n_0 such that, for $n \geq n_0$,

$$|x' - y'|_\infty \leq d(x', y') \leq (C_1|x' - y'|_\infty) \vee n^{1/2-\delta},$$
$$\text{for all } x', y' \in \Lambda_n(x, r).$$

(f) $n^{-1/2+\delta} s(g_n(x)) \to 0$ as $n \to \infty$.

Local version

Theorem 4
Let $x \in \mathbb{R}^d$ *and* $t > 0$*. Suppose Assumption 1 holds. Then*

$$\lim_{n\to\infty} n^{d/2}\hat{p}_{nt}(0, g_n(x)) = 2a_{\mathcal{G}}^{-1} k_t^{(D)}(x). \tag{5}$$

Proof idea: Let $\Lambda_n = \Lambda_n(x,\kappa) = \Lambda(n^{1/2}x, n^{1/2}\kappa)$ and recall $X_t^{(n)} = n^{-1/2}X_{\lfloor nt\rfloor}$. Let

$$\begin{aligned} J(n) &= P^0\Big(X_t^{(n)} \in \Lambda(x,\kappa)\Big) + P^0\Big(X_{t+1/n}^{(n)} \in \Lambda(x,\kappa)\Big) \\ &\quad -2\int_{\Lambda(x,\kappa)} k_t(y)dy. \end{aligned}$$

Then

$$
\begin{aligned}
J(n) \quad = \quad & \sum_{z\in\Lambda_n} \Big(\hat{p}_{nt}(0,z) - \hat{p}_{nt}(0,g_n(x))\Big)\mu_z \\
& +\mu(\Lambda_n)\hat{p}_{nt}(0,g_n(x)) - \mu(\Lambda_n)n^{-d/2}a_{\mathcal{G}}^{-1}2k_t(x) \\
& +2k_t(x)\Big(\mu(\Lambda_n)n^{-d/2}a_{\mathcal{G}}^{-1} - 2^d\kappa^d\Big) \\
& +2\int_{H(x,\kappa)}(k_t(x) - k_t(y))dy
\end{aligned}
$$

We want the second term and deal with the rest by our assumptions.

Uniform version

Assumption 2

(a) For any compact $I \subset (0, \infty)$, the CLT in Assumption 1 (a) holds uniformly for $t \in I$.

(b) There exist C_i such that

$$\hat{p}_k(0,x) \le C_2 k^{-d/2} \exp(-C_4 d(0,x)^2/k), \text{ for } k \ge C_3 \text{ and } x \in \mathcal{G}.$$

(c) We have a PHI as in Assumption 1 (c).

(d) Let $h(r)$ be the size of the biggest 'hole' in $\Lambda(0,r)$. More precisely, $h(r)$ is the supremum of the r' such that $\Lambda(y,r') = \emptyset$ for some $y \in H(0,r)$. Then $\lim_{r\to\infty} h(r)/r = 0$.

(e) There exist constants δ, C_1, C_H such that for each $x \in \mathbb{Q}^d$ Assumption 1 (d), (e) and (f) hold.

We now state a uniform version of our local limit result.

Theorem 5

Let $T_1 > 0$*. Suppose Assumption 2 holds. Then*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T_1} |n^{d/2}\hat{p}_{nt}(0, g_n(x)) - 2a_{\mathcal{G}}^{-1} k_t^{(D)}(x)| = 0. \qquad (6)$$

Application to Percolation

With some work we can show that for supercritical percolation clusters the assumptions of Theorem 5 hold.

Theorem 6

Let $T_1 > 0$. *Then there exist constants* a, D *such that* $\mathbb{P}_0$*-a.s.,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T_1} |n^{d/2}\hat{p}^{\omega}_{nt}(0, g^{\omega}_n(x)) - 2a^{-1}k^{(D)}_t(x)| = 0. \qquad (7)$$

This result holds for both blind and myopic ants as well as continuous time walks on $\mathcal{C}_\infty$.

The earlier theorems can be used to prove local limit theorems for random walks in a bounded random conductance model.