Numerical solutions of SDEs with Markovian switching and jumps

Outline

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- Bally, Protter, & Talay (1995), The law of the Euler scheme for SDEs.
- Kloeden & Platen (1992), Numerical Solution of SDEs.
- Milshtein (1985), Weak approximation of solutions of systems of SDEs.
- Protter & Talay (1997), The Euler scheme for Lévy driven SDEs.
- Higham, Mao & Stuart (2002), Strong convergence of Euler-type methods for nonlinear SDEs.

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Outline

Linear scalar SDE

$$dx(t) = \mu x(t)dt + \sigma x(t)dB(t) \tag{1.1}$$

on $t \geq 0$ with initial value $x(0) = x_0 \in \mathbb{R}$, where μ and σ are real numbers.

The classical result: Theorem 1

Theorem

If $x_0 \neq 0$, then

$$\lim_{t\to\infty} \frac{1}{t} \log(|x(t)|) = \mu - \tfrac{1}{2}\sigma^2 \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) = 2\mu + \sigma^2.$$

The Euler-Maruyama (EM) method

Given a step size $\Delta > 0$, the EM method is to compute the discrete approximations $X_k \approx x(k\Delta)$ by setting $X_0 = x_0$ and forming

$$X_{k+1} = X_k(1 + \mu\Delta + \sigma\Delta B_k), \tag{1.2}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$.

Questions

Q1. If $\alpha - \frac{1}{2}\sigma^2 < 0$, is the EM method almost surely exponentially stable for sufficiently small Δ?

Q2. If $\alpha + \frac{1}{2}\sigma^2 < 0$, is the EM method exponentially stable in mean square for sufficiently small Δ?

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Q2. If $\alpha + \frac{1}{2}\sigma^2 < 0$, is the EM method exponentially stable in mean square for sufficiently small Δ?

Known results

- Saito and Mitsui (SIAM J. Numer. Anal. **33**, 1996) gave a positive answer to Q2.
- Higham (SIAM J. Numer. Anal. **38**, 2000) gave a positive answer to Q1 for the revised EM method

$$X_{k+1} = X_k(1 + \alpha\Delta + \sigma\xi_k),$$

where ξ_k's are i.i.d. with $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = 0.5$.

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An example

$$dx(t) = (x(t) - x^3(t))dt + 2x(t)dB(t). \tag{1.3}$$

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -1 \quad a.s. \tag{1.4}$$

Applying the EM to the SDE (1.3) gives

$$X_{k+1} = X_k(1 + \Delta - X_k^2\Delta + 2\Delta B_k).$$

Lemma

Given any initial value $X_0 \neq 0$ and any $\Delta > 0$,

$$P\left(\lim_{k\to\infty} |X_k| = \infty\right) > 0.$$

Theorem

If $\alpha - \frac{1}{2}\sigma^2 < 0$*, then for any* $\varepsilon \in (0,1)$ *there is a* $\Delta_1 \in (0,1)$ *such that for any* $\Delta < \Delta_1$*, the EM approximate solution has the property that*

$$\lim_{k\to\infty} \frac{1}{k\Delta} \log(|X_k|) \le (1-\varepsilon)(\alpha - \tfrac{1}{2}\sigma^2) < 0 \quad a.s. \tag{1.5}$$

Theorem

If $2\alpha + \sigma^2 < 0$, *then for any* $\varepsilon \in (0,1)$ *there is a* $\Delta_3 \in (0,1)$ *such that for any* $\Delta < \Delta_3$, *the EM approximate solution has the property that*

$$\lim_{k\to\infty} \frac{1}{k\Delta} \log(\mathbb{E}|X_k|^2) \leq (1-\varepsilon)(2\alpha+\sigma^2) < 0. \tag{1.6}$$

Open question

When $\alpha - \frac{1}{2}\sigma^2 > 0$, by Theorem 1, the true solution obeys

$$\lim_{t\to\infty} \frac{1}{t} \log(|x(t)|) > 0 \quad a.s.$$

namely $|x(t)|$ will tend to infinity almost surely. However, we still don't know if, for a sufficiently small Δ, the EM solution obeys

$$\lim_{k\to\infty} \frac{1}{k\Delta} \log(|X_k|) > 0 \quad a.s.$$

Outline

Outline

In control engineering, one frequently encounters dynamical systems whose state is described by two variables, a) one is continuous, b) one is discrete.

Example:

A thermostat (on/off). The temperature in a room is a continuous variable, and the state of the thermostat is discrete. The continuous and discrete parts cannot be described independently since they interact.

Consider n-dimensional hybrid Itô stochastic differential equations (SDEs) having the form

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t) \tag{2.1}$$

on $t \geq 0$ with initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$ and $r(0) = r_0 \in L_{\mathcal{F}_0}(\Omega; \mathbb{S})$. We assume that

$$f : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n \times m}$$

are sufficiently smooth for the existence and uniqueness of the solution. We also assume that

$$f(0, i) = 0 \quad \text{and} \quad g(0, i) = 0 \quad \forall i \in \mathbb{S}, \tag{2.2}$$

so equation (2.1) admits the zero solution, $x(t) \equiv 0$, whose stability is the issue under consideration.

$r(t)$, $t \geq 0$, is a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ and independent of the Brownian motion $B(\cdot)$. The corresponding generator is denoted $\Gamma = (\gamma_{ij})_{N \times N}$, so that

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & : \quad \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) & : \quad \text{if } i = j, \end{cases}$$

where $\delta > 0$. Here γ_{ij} is the transition rate from i to j and $\gamma_{ij} > 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$.

Let Δ be the stepsize, $t_k = k\Delta$ and for $k \geq 0$ and the discrete approximation $X_k \approx x(t_k)$ is formed by simulating from $X_0 = x_0,\ r_0^\Delta = r_0, r_k^\Delta = r(k\Delta)$ and, generally,

$$X_{k+1} = X_k + f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k, \qquad (2.3)$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$.

Outline

Lyapunov exponent

$$dx(t) = \mu(r(t))x(t)dt + \sigma(r(t))x(t)dB(t), \qquad x(0) \neq 0 \quad a.s., \tag{3.1}$$

on $t \geq 0$. Here we let $B(t)$ be a scalar Brownian motion.

Lyapunov exponent

It is known that the linear hybrid SDE (3.1) has the explicit solution

$$x(t) = x_0 \exp\left\{ \int_0^t [\mu(r(s)) - \tfrac{1}{2}\sigma^2(r(s))]ds + \int_0^t \sigma(r(s))dB(s) \right\}. \tag{3.2}$$

In the following, we assume that the Markov chain is irreducible.

Moment Lyapunov Exponent Theorem

Theorem

The second moment Lyapunov exponent of the SDE (3.1) is

$$\lim_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^2)=\sum_{j\in\mathbb{S}}\pi_j(2\mu_j+\sigma_j^2), \tag{3.3}$$

where we write $\mu(j)=\mu_j$ and $\sigma(j)=\sigma_j$. Hence the SDE (3.1) is exponentially stable in mean square if and only if

$$\sum_{j\in\mathbb{S}}\pi_j(2\mu_j+\sigma_j^2)<0. \tag{3.4}$$

Outline

Stability of the numerical solutions

Given a stepsize $\Delta > 0$, the EM method (2.3) applied to (3.1) gives $X(0) = x_0$ and

$$X_{k+1} = X_k \left[1 + \mu(r_k^\Delta)\Delta + \sigma(r_k^\Delta)\Delta B_k \right], \quad k \geq 1. \tag{3.5}$$

Stability of the numerical solutions

Theorem

The EM approximation (3.5) satisfies

$$\lim_{n\to\infty} \frac{1}{n\Delta}\mathbb{E}[X_n^2] = \sum_{j\in\mathbb{S}} \pi_j(2\mu_j + \sigma_j^2) + \Delta \sum_{j\in\mathbb{S}} \pi_j\left(\tfrac{1}{2}\sigma_j^2 - (\mu_j + \sigma_j^2)^2\right) + O(\Delta^2), \quad \text{as } \Delta \to 0. \tag{3.6}$$

Hence, the numerical method matches the exponential mean-square stability or instability of the SDE, for sufficiently small Δ.

Nonlinear systems

Extend the numerical method to continuous time. Thus, we let

$$\bar{X}(t) = X_k, \quad \bar{r}(t) = r_k^{\Delta}, \qquad \text{for } t \in [t_k, t_{k+1}), \tag{4.1}$$

and take our continuous-time EM approximation to be

$$X(t) = x_0 + \int_0^t f(\bar{X}(s), \bar{r}(s))ds + \int_0^t g(\bar{X}(s), \bar{r}(s))dB(s). \tag{4.2}$$

Note that $X(t_k) = \bar{X}(t_k) = X_k$, that is, $X(t)$ and $\bar{X}(t)$ interpolate the discrete numerical solution.

Nonlinear systems

Definition

The hybrid SDE (2.1) is said to be exponentially stable in mean square if there is a pair of positive constants λ *and* M *such that, for all initial data* $x_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$ *and* $r(0) = r_0 \in L_{\mathcal{F}_0}(\Omega; \mathbb{S})$*,*

$$\mathbb{E}|x(t)|^2 \le M\mathbb{E}|x_0|^2 e^{-\lambda t}, \quad \forall t \ge 0. \tag{4.3}$$

We refer to λ *as a rate constant and* M *as a growth constant.*

Nonlinear systems

Definition

For a given stepsize $\Delta > 0$, the EM method (4.2) is said to be exponentially stable in mean square on the hybrid SDE (2.1) if there is a pair of positive constants γ and H such that for all initial data $x_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$ and $r_0 \in L_{\mathcal{F}_0}(\Omega; \mathbb{S})$

$$\mathbb{E}|X(t)|^2 \le H\mathbb{E}|x_0|^2 e^{-\gamma t}, \quad \forall t \ge 0. \tag{4.4}$$

We refer to γ as a rate constant and H as a growth constant.

Nonlinear systems

Assumption (Global Lipschitz)

There is a positive constant K such that

$$|f(x,i) - f(y,i)|^2 \vee |g(x,i) - g(y,i)|^2 \le K|x-y|^2 \tag{4.5}$$

for all $(x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$.

Recalling (2.2) we observe from this assumption that the linear growth condition

$$|f(x,i)|^2 \vee |g(x,i)|^2 \le K|x|^2 \tag{4.6}$$

holds for all $(x,i) \in \mathbb{R}^n \times \mathbb{S}$.

Nonlinear systems

Lemma

If (4.6) holds, then for all sufficiently small Δ the continuous EM approximate solution (4.2) satisfies, for any $T > 0$,

$$\sup_{0 \le t \le T} \mathbb{E}|X(t)|^2 \le B_{x_0,T}, \tag{4.7}$$

where $B_{x_0,T} = 3\mathbb{E}|x_0|^2 e^{3(T+1)KT}$. Moreover, the true solution of (2.1) also obeys

$$\sup_{0 \le t \le T} \mathbb{E}|x(t)|^2 \le B_{x_0,T}. \tag{4.8}$$

Nonlinear systems

Lemma

If (4.6) holds, then for all sufficiently small Δ, $\bar{X}(t)$ *in (4.1) obeys*

$$\mathbb{E}\int_0^T |f(\bar{X}(s), r(s)) - f(\bar{X}(s), \bar{r}(s))|^2 ds \le \beta_T \Delta \sup_{0\le t\le T} \mathbb{E}|\bar{X}(t)|^2 \tag{4.9}$$

and

$$\mathbb{E}\int_0^T |g(\bar{X}(s), r(s)) - g(\bar{X}(s), \bar{r}(s))|^2 ds \le \beta_T \Delta \sup_{0\le t\le T} \mathbb{E}|\bar{X}(t)|^2 \tag{4.10}$$

for any $T > 0$, *where* $\beta_T = 4KTN[1 + \max_{1\le i\le N}(-\gamma_{ii})]$.

Nonlinear systems

Lemma

Under (2.2) and Assumption GL, for all sufficiently small Δ the continuous EM approximation $X(t)$ and true solution $x(t)$ obey

$$\sup_{0\le t\le T} \mathbb{E}|X(t)-x(t)|^2 \le \left(\sup_{0\le t\le T} \mathbb{E}|X(t)|^2\right) C_T \Delta \tag{4.11}$$

for any $T>0$, where

$$C_T = 4(T+1)[\beta_T + K2T(1+2K)]e^{8K(T+1)T}.$$

Nonlinear systems

Lemma

Let (2.2) and Global Lipschitz condition hold. Assume that the hybrid SDE (2.1) is exponentially stable in mean square, satisfying (4.3). Then there exists a $\Delta^\star > 0$ *such that for every* $0 < \Delta \leq \Delta^\star$ *the EM method is exponentially stable in mean square on the SDE (2.1) with rate constant* $\gamma = \frac{1}{2}$ *and growth constant* $H = 2Me^{\frac{1}{2}[1+(4\log M)/]}$.

Nonlinear systems

Lemma

Let (2.2) and Assumption (9) hold. Assume that for a stepsize $\Delta > 0$, the numerical method is exponentially stable in mean square with rate constant γ and growth constant H. If Δ satisfies

$$C_{2T} e^{\gamma T} (\Delta + \sqrt{\Delta}) + 1 + \sqrt{\Delta} \le e^{\frac{1}{4}\gamma T} \quad \text{and} \quad C_T \Delta \le 1, \quad (4.12)$$

where $T := 1 + (4 \log H)/\gamma$, then the hybrid SDE (2.1) is exponentially stable in mean square with rate constant $= \frac{1}{2}\gamma$ and growth constant $M = 2He^{\frac{1}{2}\gamma T}$.

Nonlinear systems

Theorem

Under (2.2) and global Lipschitz condition, the hybrid SDE (2.1) is exponentially stable in mean square if and only if there exists a $\Delta > 0$ such that the EM method is exponentially stable in mean square with rate constant γ, growth constant H, stepsize Δ and global error constant C_T for $T := 1 + (4 \log H)/\gamma$ satisfying (4.12).

STM

Given a step size $\Delta > 0$, with $X_0 = x_0$ and $r_0^\Delta = r_0$ the STM is defined for $k = 0, 1, 2, \cdots$ by

$$X_{k+1} = X_k + [(1-\theta)f(X_k, r_k^\Delta) + \theta f(X_{k+1}, r_k^\Delta)]\Delta + g(X_k, r_k^\Delta)\Delta B_k, \tag{5.1}$$

where $\theta \in [0, 1]$ is a fixed parameter. Note that with the choice $\theta = 0$, (5.1) reduces to the EM method.

STM

Define the continuous approximation.

$$X(t) = x_0 + \int_0^t [(1-\theta) f(z_1(s), \bar{r}(s)) + \theta f(z_2(s), \bar{r}(s))] ds + \int_0^t g(z_1(s), \bar{r}(s)) dB(s), \qquad (5.2)$$

where

$$z_1(t) = X_k,\ z_2(t) = X_{k+1} \text{ and } \bar{r}(t) = r_k^{\Delta} \text{ for } t \in [k\Delta, (k+1)\Delta).$$

STM

Lemma

Under GL condition, if Δ is sufficiently small that $\Delta\sqrt{K} < 1$, then equation (5.1) can be solved uniquely for X_{k+1} given X_k, with probability 1.

STM

Lemma

Under (4.6), for all sufficiently small Δ *(*$< 1/(2+2K)$ *at least), the continuous approximation* $X(t)$ *defined by (5.2) satisfies*

$$\sup_{0\le t\le T} \mathbb{E}|X(t)|^2 \le \alpha_T \mathbb{E}|x_0|^2, \quad \forall T \ge 0, \tag{5.3}$$

where $\alpha_T = 3 + 12K(T+1)e^{2(3+4K)(T+1)}$.

STM

Lemma

Under (4.6), for all sufficiently small Δ *(*$< 1/(4+6K)$ *at least),*

$$\mathbb{E}|X_{k+1}|^2 \le 2\mathbb{E}|X_k|^2, \quad \forall k \ge 0.$$

STM

Lemma

Under (4.6), for all sufficiently small Δ *(*$< 1/(4+6K)$ *at least), the continuous approximation* $X(t)$ *defined by (5.2) satisfies*

$$\sup_{0\le t\le T}\left\{\mathbb{E}|X(t)-z_1(t)|^2 \vee \mathbb{E}|X(t)-z_2(t)|^2\right\} \le 2(K+1)\Delta \sup_{0\le t\le T}\mathbb{E}|X(t)|^2, \quad (5.4)$$

for all $T>0$.

Nonlinear systems

Theorem

Under (2.2) and global Lipschitz condition, the hybrid SDE (2.1) is exponentially stable in mean square if and only if there exists a $\Delta > 0$ *such that the STM method is exponentially stable in mean square with rate constant* γ*, growth constant* H*, stepsize* Δ *and global error constant* C_T *for* $T := 1 + (4 \log H)/\gamma$ *satisfying (4.12).*

$$dx(t) = f(x$$

$$X_{k+1} = X_k + f(X_k, X_{[(k\Delta-\tau)/\Delta]})\Delta + g(X_k, X_{[(k\Delta-\tau)/\Delta]})\Delta B_k + \int_{\mathbb{R}^n} \gamma(X_k, X_{[(k\Delta-\tau)/\Delta]}, z))\Delta\tilde{N}_k(dz),$$

where $\Delta\tilde{N}_k(dz) = \tilde{N}((k+1)\Delta, dz) - \tilde{N}(k\Delta, dz)$.

Local Lipschitz condition

For each $R = 1, 2, \cdots$, there exists a constant L_R such that

$$|f(x,y) - f(\bar{x},\bar{y})|^2 + |g(x,y) - g(\bar{x},\bar{y})|^2 + \int_{\mathbb{R}^n} |\gamma(x,y,z) - \gamma(\bar{x},\bar{y},z)|^2 \nu(dz) \leq L_R(|x-\bar{x}|^2 + |y-\bar{y}|^2), \tag{6.2}$$

with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$.

Theorem

Under a local Lipichitz condition and linear growth condition, if there exists a constant α such that $L_R^2 \leq \alpha(T)\log R$, then

$$E\left[\sup_{0\leq t\leq T}|x(t)-X(t)|^2\right]\leq C\Delta. \tag{6.3}$$

Sketch of the proof for Lyapunov exponent

There is a sequence of finite stopping times $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) I_{[\tau_k, \tau_{k+1})}(t), \quad t \geq 0.$$

For any integer $z > 0$, it then follows from (3.2) that

$$\begin{aligned} & |x(t \wedge \tau_z)|^2 \\ = \ & |x_0|^2 \exp\left\{ \int_0^{t \wedge \tau_z} [2\mu(r(s)) - \sigma^2(r(s))] ds + \int_0^{t \wedge \tau_z} 2\sigma(r(s)) dB(s) \right\} \\ = \ & \xi(t \wedge \tau_z) \prod_{k=0}^{z-1} \zeta_k, \end{aligned}$$

where

$$\begin{aligned}\xi(t\wedge\tau_z) =& |x_0|^2 \exp\Big\{\int_0^{t\wedge\tau_z}[2\mu(r(s))+\sigma^2(r(s))]ds\Big\},\\ \zeta_k =& \exp\Big\{-2\sigma^2(r(t\wedge\tau_k))(t\wedge\tau_{k+1}-t\wedge\tau_k)\\ &+2\sigma(r(t\wedge\tau_k))[B(t\wedge\tau_{k+1})-B(t\wedge\tau_k)]\Big\}.\end{aligned}$$

Let $\mathcal{G}_t = \sigma(\{r(u)\}_{u\geq 0}, \{B(s)\}_{0\leq s\leq t})$, i.e. the σ-algebra generated by $\{r(u)\}_{u\geq 0}$ and $\{B(s)\}_{0\leq s\leq t})$. Compute

$$\begin{aligned}\mathbb{E}|x(t\wedge\tau_z)|^2 &= \mathbb{E}\Big(\xi(t\wedge\tau_z)\prod_{k=0}^{z-1}\zeta_k\Big)\\ &= \mathbb{E}\Big\{\Big[\xi(t\wedge\tau_z)\prod_{k=0}^{z-2}\zeta_k\Big]\,\mathbb{E}\Big(\zeta_{z-1}\Big|\mathcal{G}_{t\vee\tau_{z-1}}\Big)\Big\} \quad (6.4)\end{aligned}$$

Define

$$\zeta_{z-1}(i) = \exp\Big\{-2\sigma_i^2(t\wedge\tau_z - t\wedge\tau_{z-1}) + 2\sigma_i[B(t\wedge\tau_z) - B(t\wedge\tau_{z-1})]\Big\}, i\in S.$$

By the well-known exponential martingale of a Brownian motion we have $\mathbb{E}\zeta_{z-1}(i) = 1$, for all $i \in S$. Then

$$\begin{aligned}\mathbb{E}\left(\zeta_{z-1}\middle|\mathcal{G}_{t\vee\tau_{z-1}}\right) &= \mathbb{E}\left(\sum_{i\in S} I_{\{r(t\wedge\tau_{z-1})=i\}}\zeta_{z-1}(i)\middle|\mathcal{G}_{t\vee\tau_{z-1}}\right)\\ &= \sum_{i\in S} I_{\{r(t\wedge\tau_{z-1})=i\}}\mathbb{E}\left(\zeta_{z-1}(i)\middle|\mathcal{G}_{t\vee\tau_{z-1}}\right) = 1.\end{aligned}$$

Substituting this into (6.4) yields

$$\mathbb{E}|x(t \wedge \tau_z)|^2 = \mathbb{E}\Big[\xi(t \wedge \tau_z) \prod_{k=0}^{z-2} \zeta_k\Big].$$

Repeating this procedure implies $\mathbb{E}|x(t \wedge \tau_z)|^2 = \mathbb{E}\xi(t \wedge \tau_z)$. Letting $z \to \infty$ we obtain

$$\mathbb{E}|x(t)|^2 = \mathbb{E}\xi(t) = \mathbb{E}\Big\{|x_0|^2 \exp\Big[\int_0^t [2\mu(r(s)) + \sigma^2(r(s))]ds\Big]\Big\} \tag{6.5}$$

Now, by the ergodic property of the Markov chain, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t [2\mu(r(s)) + \sigma^2(r(s))]ds = \sum_{j\in S} \pi_j(2\mu_j + \sigma_j^2) := \gamma \quad a.s \tag{6.6}$$