MODERN ASPECTS OF BRANCHING PROCESSES: Lecture 1 (Edinburgh): Galton-Watson processes and random walk

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0.1 Basic Notions

Definition. A Galton-Watson process is a Markov chain $\{Z(n), n = 0, 1, 2, ..\}$ on nonnegative integers. Its transition function is specified by a probability law $\{p_k, k = 0, 1, ...\}$, $p_k \geq 0, \sum p_k = 1$ with

$$P_{ij} = \mathbf{P}\left\{Z(n+1) = j | Z(n) = i\right\} = \begin{cases} p_j^{*i} & \text{if} \quad i \geq 1, j \geq 0 \\ \delta_{0j} & \text{if} \quad i = 0, j \geq 0. \end{cases}$$

where

$$p_j^{*i} = \sum_{j_1+...+j_i=j} p_{j_1} p_{j_2} ... p_{j_i}.$$

Generating functions.

It is usually denoted by F and is viewed as a function of a real variable $s \in [0, 1]$:

$$F(s) = \mathbb{E}[s^{\xi}] = \sum_{k=0}^{\infty} \mathbb{P}(\xi = k) s^k = \sum_{k=0}^{\infty} p_k s^k, 0 \leq s \leq 1, \quad (1)$$

in terms of a random variable ξ giving the offspring of an individual, or in terms of its distribution $p_0, p_1, p_2, \ldots$. For geometric offspring size distribution we have

$$F(s) = \sum_{k=0}^{\infty} q p^k s^k = \frac{q}{1 - ps}.$$

It is not difficult to understand that

$$Z(n+1) = \xi_{n1} + ... + \xi_{nZ(n)},$$

where $\xi_{ni} \stackrel{d}{=} \xi$ are iid. Iterations

$$F_0(s) = s, F_{n+1}(s) = F_n(F(s)).$$

In particular, given $Z(0) = 1$

$$\begin{aligned} F(n+1,s) \quad : &= \mathbf{E}s^{Z(n+1)} = \mathbf{E}\left[\mathbf{E}\left[s^{Z(n+1)}|Z(n)\right]\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[s^{\xi_{n1}+...+\xi_{nZ(n)}}\right]Z(n)\right] \\ &= \mathbf{E}\left(\mathbf{E}s^{\xi}\right)^{Z(n)} = F(n,F(s)) = ... = F_{n+1}(s). \end{aligned}$$

0.2 Classification

$$A = \mathbf{E}\xi = \mathbf{E}Z(1) = F^{'}(1).$$

The process is called subcritical if $A < 1$, critical, if $A = 1$ and supercritical, if $A > 1$.

The expacted number of individuals and the second factorial moment for the number of particles at the n-th generation can be calculated by

$$\mathbf{E}Z(n) = \left(\mathbf{E}s^{Z(n)}\right)'|_{s=1} = (F_n(s))'|_{s=1} = \left(F^{'}(1)\right)^n = A^n$$

and

$$\mathbf{E}[Z(n)(Z(n)-1)] = A\mathbf{E}[Z(n-1)(Z(n-1)-1)] + F''(1)A^{2(n-1)}.$$

Hence, given $Z(0) = 1$ we get

$$\mathbf{E}[Z(n)(Z(n)-1)] = F''(1)\frac{A^{n-1}(A^n-1)}{A-1},$$

if $A \neq 1$ and $\mathbf{E}[Z(n)(Z(n)-1)] = F''(1)n$ in the critical case. Consequently with $\sigma^2 = Var[\xi] = F''(1) - A(A-1)$ and $Z(0) = 1$ it follows that

$$Var[Z(n)] = \begin{cases} \sigma^2\frac{A^{n-1}(A^n-1)}{A-1} & if \ \ A \neq 1, \\ \sigma^2 n & if \ A = 1. \end{cases} \tag{2}$$

0.3 Calculation of iterations for the pure geometric reproduction law

$$F(s) = \sum_{k=0}^{\infty} qp^k s^k = \frac{q}{1-ps}.$$

Clearly, $F^{'}(1) = A = p/q$. Further we have

$$1 - F(s) = \frac{p(1-s)}{1-ps}$$

and

$$\begin{aligned} & \frac{1}{1-F(s)} - \frac{1}{A(1-s)} \\ = & \frac{1-ps}{p(1-s)} - \frac{q}{p(1-s)} = 1. \end{aligned}$$

Thus,

$$\frac{1}{1-F_n(s)} - \frac{1}{A(1-F_{n-1}(s))} = \frac{1}{1-F(F_{n-1}(s))} - \frac{1}{A(1-F_{n-1}(s))} = 1$$

or

$$\frac{1}{1-F_n(s)} = 1 + \frac{1}{A(1-F_{n-1}(s))} = 1 + \frac{1}{A} + \frac{1}{A^2(1-F_{n-2}(s))} = \ldots.$$

The end of this is a simple closed form,

$$\begin{aligned} \frac{1}{1-F_n(s)} &= 1 + (1/A) + (1/A)^2 + \ldots + (1/A)^{n-1} + 1/A^n(1-s) \\ &= \begin{cases} \frac{A^n-1}{A^{n-1}(A-1)} + \frac{1}{A^n(1-s)} & \text{if} \quad A \neq 1 \\ n + \frac{1}{1-s} & \text{if} \quad A = 1. \end{cases} \end{aligned}$$

Therefore,if $A \neq 1$ then

$$1 - F_n(s) = \frac{A^n(A-1)(1-s)}{A(A^n-1)(1-s) + A - 1}. \tag{3}$$

and if $A = 1$ then

$$1 - F_n(s) = \frac{1}{n + (1-s)^{-1}}.$$

Survival probability: if $A = p/q \neq 1$ then

$$\begin{aligned} \mathbf{P}(Z(n) > 0) &= 1 - F_n(0) \\ &= \frac{A^n(A-1)}{A(A^n-1) + A - 1} = \frac{A^{n+1}(1-1/A)}{A^{n+1}-1} \\ &= \frac{\left(\frac{p}{q}\right)^n (1 - \frac{p}{q})}{1 - \left(\frac{p}{q}\right)^{n+1}}, \end{aligned}$$

if $A = 1$ then

$$\mathbf{P}\left(Z(n) > 0\right) = \frac{1}{n+1}.$$

In particular, if $A > 1$ then

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}\left(Z(n) > 0\right) &= \lim_{n\to\infty} \frac{A^{n+1}(1-1/A)}{A^{n+1}-1} \\ &= 1 - \frac{1}{A}.\end{aligned}$$

0.4 Extinction probability

$$F_n(s) = \mathbf{E}s^{Z(n)} = \sum_{k=0}^{\infty} P\left(Z(n) = k\right) s^k,$$

$$F_n(0) = \mathbf{P}(Z(n) = 0) \le \mathbf{P}(Z(n+1) = 0) = F_{n+1}(0).$$

It follows that the sequence

$$\mathbf{P}(n) = \mathbf{P}(\text{ extinction by generation } n) = \mathbf{P}(Z(n) = 0) = F_n(0), n = 1, 2 \ldots$$

must increase to the extinction probability, which we denote by P,

$$\lim_{n\to\infty} \mathbf{P}(n) = P.$$

Since $F(0) < r = F(r)$

$$\mathbf{P}(n) = F_n(0) = F(F_{n-1}(0)) = F(P(n-1)) < F(r) = r$$

and the function F is continuous, it follows that $P = F(P)$. Hence $P = r$.

Thus, the subcritical and critical processes die with probability 1 while supercritical with probability $P < 1$ being the smallest root of $F(s) = s$, $s \in [0, 1)$.

1 Asymptotic behavior of the survival probability for subcritical processes

Theorem 1 *If $A < 1$ then*

$$\mathbf{P}\left(Z(n) > 0\right) = Q(n) \sim KA^n(1 + o(1)),\ \ K > 0,$$

if and only if

$$\begin{aligned}\mathbf{E}\xi \log^+ \xi &= \mathbf{E}Z(1) \log^+ Z(1) \\ &= \sum_{k=1}^{\infty} p_k k \log k < \infty.\end{aligned}$$

Note that this theorem implies

$$\frac{A^n}{Q(n)} = \frac{\mathbf{E}Z(n)}{\mathbf{P}\left(Z(n)>0\right)} = \mathbf{E}\left[Z(n)|Z(n)>0\right] \approx K^{-1},\ n\to\infty.$$

Theorem 2 *If $A<1$ then*

$$\lim_{n\to\infty}\mathbf{P}(Z(n)=k|Z(n)>0) = P_k^*, \sum_{k=1}^{\infty}P_k^* = 1$$

and

$$F^*(s) = \sum_{k=1}^{\infty}P_k^* s^k$$

satisfies

$$1-F^*(F(s)) = A(1-F^*(s)).$$

2 Branching processes and simple random walk

Branching process: Consider a branching process with geometric probability generating function for the offspring number:

$$F(s) = \frac{q}{1-ps} = \mathbf{E}s^{\xi},\ p+q=1,\ pq>0. \tag{4}$$

It follows from the consideration above that the probability of extinction of this process, being a solution of $F(P)=P$, is

$$P = \min\left\{\frac{q}{p},1\right\}$$

and, besides the standard recurrence relation

$$Z(n+1) = \xi_{n1}+...+\xi_{nZ(n)} \tag{5}$$

is valid, where ξ_{ni} are iid, $\xi_{ni}\overset{d}{=}\xi$ with $\mathbf{P}\left(\xi=j\right)=qp^j,\ j=0,1,....$

Random walk: Consider a random walk

$$S_0=0, S_k = X_1+...+X_k$$

with

$$\mathbf{P}\left(X_i=1\right)=p, \qquad \mathbf{P}\left(X_i=-1\right)=1-p=q.$$

Let S_k^* be the random walk stopped at zero at moment $\tau=\min\left\{k:S_k=-1\right\}$. It is known that

$$\mathbf{P}\left(\tau<\infty\right)=\min\left\{\frac{q}{p},1\right\}.$$

Set

$$Y(n) = \text{the number of } k \text{ such that } S_k^*=n, S_{k+1}^*=n-1.$$

Then the random variable

$$Y(1) = \text{the number of } k \text{ such that } S_k^* = 1, S_{k+1}^* = 0$$

has the following probability law:

$$\mathbf{P}(Y(1) = 0) = q, \ \mathbf{P}(Y(1) = 1) = pq$$

and, in general, the Geometric distribution with

$$\mathbf{P}(Y(1) = j) = \mathbf{P}(\eta = j) = qp^j.$$

Besides,

$$Y(n+1) = \eta_1^{(n)} + ... + \eta_{Y(n)}^{(n)}$$

where $\eta_i^{(n)} \overset{d}{=} \eta$.

Thus, we get the same stochastic process as in (5).

If $p \leq 1/2$ then the branching process dies out and if T is the moment of extinction then

$$\sigma = Z(0) + Z(1) + ... + Z(T-1)$$

is the total number of particles in the process and

$$\sigma = 2\tau - 1.$$

2.1 Local time of the simple random walk

Consider again a simple random walk

$$S_0 = 0, \, S_k = X_1 + ... + X_k$$

with

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = -1) = 1 - p = q, \ p < q.$$

Let S_k^* be the random walk stopped at zero at moment $\tau = \min\{k : S_k = -1\}$. It is known that

$$\mathbf{P}(\tau < \infty) = \frac{p}{q}.$$

Set

$$Z(n) = \text{the number of } k \text{ such that } S_k^* = n, \ S_{k+1}^* = n-1.$$

This is a branching process with geometric offspring distribution. Then for the local time $\ell(t)$ of the stopped random walk at level t :

$$\begin{aligned} \ell(t) &= \text{the number of } k \text{ such that } S_k^* = t \\ &= (\text{the number of } k \text{ such that } S_{k-1}^* = t-1 \text{ and } S_k^* = t) \\ &\quad + \big(\text{the number of } k \text{ such that } S_{k-1}^* = t+1 \text{ and } S_k^* = t\big) \\ &= (\text{the number of } k \text{ such that } S_{k-1}^* = t \text{ and } S_k^* = t-1) \\ &\quad + \big(\text{the number of } k \text{ such that } S_{k-1}^* = t+1 \text{ and } S_k^* = t\big) \\ &= Z(t) + Z(t+1), \ t = 0, 1, 2, ..., \end{aligned}$$

where $Z(t)$ is the number of particles at moment t in a branching process with offspring generating function $F(s)=q(1-ps)^{-1}$. Hence, to find the distribution of $\ell(t)$ it is necessary to study the joint distribution of $(Z(t),Z(t+1))$ for the processes with geometric probability generating functions. In fact, we establish the desired result in the general situation.

Theorem 3 *If $A<1$ then for any fixed $m=0,1,...$*

$$\lim_{n\to\infty}\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}|Z(n)>0\right]=\frac{F^*\left(s_1F(s_2)\right)-F^*\left(s_1F(s_2F_m(0))\right)}{A^{m+1}}.$$

Proof. We have

$$\begin{aligned}
&\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)};Z(n)>0\right]\\
=\ &\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}\right]-\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)};Z(n)=0\right].
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}\right] &= \mathbf{E}\left[\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}\right]|Z(n-m-1)\right]\\
&= \mathbf{E}\left[s_1^{Z(n-m-1)}\mathbf{E}\left[s_2^{Z(n-m)}|Z(n-m-1)\right]\right]\\
&= \mathbf{E}\left[s_1^{Z(n-m-1)}F^{Z(n-m-1)}(s_2)\right]=F_{n-m-1}(s_1F(s_2))
\end{aligned}$$

while

$$\begin{aligned}
&\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)};Z(n)=0\right]\\
=\ &\mathbf{E}\left[\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)};Z(n)=0\right]|Z(n-m-1);Z(n-m)\right]\\
=\ &\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}\mathbf{E}\left[I\left\{Z(n)=0\right\}|Z(n-m)\right]\right]\\
=\ &\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}\mathbf{P}(Z(n)=0|Z(n-m))\right]\\
=\ &\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}F_m^{Z(n-m)}(0)\right]=F_{n-m-1}(s_1F(s_2F_m(0))).
\end{aligned}$$

As a result

$$\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)};Z(n)>0\right]=F_{n-m-1}(s_1F(s_2))-F_{n-m-1}(s_1F(s_2F_m(0))).$$

Therefore

$$\begin{aligned}
\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}|Z(n)>0\right] &= \frac{F_{n-m-1}(s_1F(s_2))-F_{n-m-1}(s_1F(s_2F_m(0)))}{1-F_n(0)}\\
&= \frac{1-F_{n-m-1}(0)}{1-F_n(0)}\,\frac{F_{n-m-1}(s_1F(s_2))-F_{n-m-1}(s_1F(s_2F_m(0)))}{1-F_{n-m-1}(0)}.
\end{aligned}$$

Now

$$\lim_{n\to\infty}\frac{1-F_{n-m-1}(0)}{1-F_n(0)}=\lim_{n\to\infty}\frac{1-F_{n-m-1}(0)}{1-F_{m+1}(F_{n-m-1}(0))}=\frac{1}{F'_{m+1}(1)}=\frac{1}{A^{m+1}}\quad(6)$$

while as $n \to \infty$

$$\begin{aligned}
& \frac{F_{n-m-1}(s_1F(s_2)) - F_{n-m-1}(s_1F(s_2F_m(0)))}{1 - F_{n-m-1}(0)} \\
= \; & \frac{(F_{n-m-1}(s_1F(s_2)) - F_{n-m-1}(0)) - (F_{n-m-1}(s_1F(s_2F_m(0))) - F_{n-m-1}(0))}{1 - F_{n-m-1}(0)} \\
\to \; & F^*\left(s_1F(s_2)\right) - F^*\left(s_1F(s_2F_m(0))\right). \qquad (7)
\end{aligned}$$

Combining (6) and (7) proves the theorem.

Corollary 4 *If $A < 1$ and $\mathbf{E}\xi\log^+\xi < \infty$ then*

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbf{E}\left[s_1^{Z(n-m-1)}s_2^{Z(n-m)}|Z(n) > 0\right] = \frac{s_1s_2F^{*\prime}\left(s_1F(s_2)\right)F'(s_2)}{A}K$$

where K is the same as in the theorem describing the asymptotic behavior of the survival probability of a subcritical process.

Proof. As $m \to \infty$

$$\begin{aligned}
& \frac{F^*\left(s_1F(s_2)\right) - F^*\left(s_1F(s_2F_m(0))\right)}{A^{m+1}} \\
\approx \; & s_1s_2F^{*\prime}\left(s_1F(s_2)\right)F'(s_2)\frac{1 - F_m(0)}{A^{m+1}} \to \frac{s_1s_2F^{*\prime}\left(s_1F(s_2)\right)F'(s_2)}{A}K.
\end{aligned}$$

We know that for the geometric case

$$F^*\left(s\right) = \frac{(1-A)s}{1-As} = \frac{(q-p)s}{q-ps}.$$

From here by direct calculations we get

Corollary 5 *If the offspring generating function is geometric then for $A = p/q < 1$*

$$\begin{aligned}
& \lim_{n\to\infty}\mathbf{E}\left[s^{\ell(n-m-1)}|\max_k S_k^* > n\right] \\
= \; & \lim_{n\to\infty}\mathbf{E}\left[s^{Z(n-m-1)+Z(n-m)}|Z(n) > 0\right] = \frac{F^*\left(sF(s)\right) - F^*\left(sF(sF_m(0))\right)}{A^{m+1}} \\
= \; & \frac{(1-A)^2pqs^2}{A\left(1-2ps\right)\left(1 - A^{m+1} - p(2 - A^{m+1} - A^m)s\right)}.
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
& \frac{F^*(sF(s)) - F^*(sF(sF_m(0)))}{A^{m+1}} \\
= & \frac{1}{A^{m+1}} \left(\frac{(1-A)qs}{1-2ps} - \frac{(1-A)qs}{1-p(1+F_m(0))s} \right) \\
= & \frac{(1-A)qs}{A^{m+1}} \left(\frac{1}{1-2ps} - \frac{1}{1-p(1+F_m(0))s} \right) \\
= & \frac{(1-A)pqs^2(1-F_m(0))}{A^{m+1}(1-2ps)(1-p(1+F_m(0))s)} \\
= & \frac{(1-A)^2 pqs^2}{A(1-2ps)(1-A^{m+1})(1-p(1+F_m(0))s)}
\end{aligned}
$$

and since

$$
1 + F_m(0) = 2 - \frac{A^m(1-A)}{1-A^{m+1}} = \frac{2 - A^{m+1} - A^m}{1-A^{m+1}}
$$

this changes to

$$
\frac{(1-A)^2 pqs^2}{A(1-2ps)(1-A^{m+1} - p(2-A^{m+1}-A^m)s)}.
$$

Letting $m \to \infty$ we get the following statement.

Corollary 6 *If the offspring generating function is geometric then for $A = p/q < 1$*

$$
\begin{aligned}
& \lim_{m\to\infty} \lim_{n\to\infty} \mathbf{E}\left[s^{\ell(n-m)} | \max_k S_k^* > n \right] \\
= & \lim_{m\to\infty} \lim_{n\to\infty} \mathbf{E}\left[s^{Z(n-m-1)+Z(n-m)} | Z(n) > 0 \right] = \frac{s^2 F^{*\prime}(sF(s))\, F'(s)}{A} K \\
= & \frac{(1-A)^2 pqs^2}{A(1-2ps)^2} = \frac{(q-p)^2 s^2}{(1-2ps)^2}.
\end{aligned}
$$

3 Unconditional limit theorem for the supercritical case

Theorem 7 *If $A > 1, \sigma^2 < \infty$ then there exists a random variable W such that, as $n \to \infty$*

$$
W_n = \frac{Z(n)}{A^n} \to W \quad a.s.
$$

and

1)

$$\lim_{n\to\infty} \mathbf{E}\left(W - W_n\right)^2 = 0,$$

2)

$$\mathbf{E}W = 1, \qquad VarW = \sigma^2/(A^2 - A)$$

3)

$$\mathbf{P}(W = 0) = P = \mathbf{P}(Z(n) = 0 \text{ for some } n).$$

Proof. Clearly, $\mathbf{E}W_n = 1$ and

$$\begin{aligned}
\mathbf{E}\left[W_n | W_{n-1}\right] &= \mathbf{E}\left[\frac{Z(n)}{A^n} | \frac{Z(n-1)}{A^{n-1}}\right] = \frac{1}{A^n}\mathbf{E}\left[Z(n)|Z(n-1)\right] \\
&= \frac{1}{A^n}\mathbf{E}\left[\sum_{k=1}^{Z(n-1)} \xi_k^{(n-1)} | Z(n-1)\right] = \frac{Z(n-1)}{A^n}\mathbf{E}\xi = W_{n-1}
\end{aligned}$$

and, therefore, $\{W_n\}_{n\geq 1}$ form a non-negative martingale. Hence, there exists a random variable W such that as $n \to \infty$

$$W_n = \frac{Z(n)}{A^n} \to W \quad \text{a.s.}$$

From the previous results

$$\mathbf{E}W_n^2 = \frac{\mathbf{E}Z^2(n)}{A^{2n}} = \frac{\sigma^2(1 - A^{-n})}{A^2 - A} + 1$$

and, therefore,

$$\sup_n \mathbf{E}W_n^2 = \lim_{n\to\infty} \mathbf{E}W_n^2 = \frac{\sigma^2}{A^2 - A} + 1 < \infty.$$

Now by properties of martingales we have according to the Doob theorem (Doob, 1953, p.319) that 1) and 2) are valid.

If $r = \mathbf{P}(W = 0)$ then $\mathbf{E}W = 1$ implies $r < 1$ and

$$\begin{aligned}
r &= \sum_{k=0}^{\infty} \mathbf{P}\left(W = 0 | Z(1) = k\right) \mathbf{P}(Z(1) = k) = \\
&= \sum_{k=0}^{\infty} \mathbf{P}^k\left(W = 0 | Z(1) = 1\right) \mathbf{P}(Z(1) = k) = F(r).
\end{aligned}$$

Hence, $r = P$.

We see also that

$$\mathbf{E}e^{-\lambda W_n} = F\left(\mathbf{E}\left[e^{-\lambda A^{-1} W_{n-1}}\right]\right)$$

or, passing to the limit as $n \to \infty$, we see that $\varphi(\lambda) = \mathbf{E}e^{-\lambda W}$ satisfies

$$\varphi(\lambda) = F\left(\varphi\left(\frac{\lambda}{A}\right)\right). \tag{8}$$

If

$$F(s) = \frac{q}{1-ps} = \frac{1}{1+A(1-s)}$$

with $A = p/q > 1$, then

$$F_n(s) = 1 - \frac{A^n(A-1)(1-s)}{A(A^n-1)(1-s)+A-1}$$

and

$$\begin{aligned}\mathbf{E}e^{-\lambda W_n} &= 1 - \frac{A^n(A-1)(1-e^{-\lambda A^{-n}})}{A(A^n-1)(1-e^{-\lambda A^{-n}})+A-1} \\ &\to 1 - \frac{\lambda(A-1)}{\lambda A + A - 1} = \frac{1}{A} + \left(1-\frac{1}{A}\right)\frac{(1-\frac{1}{A})}{\lambda+1-\frac{1}{A}}\end{aligned}$$

and the limiting distribution is

$$\mathbf{P}\left(W \le x\right) = \frac{1}{A} + (1-\frac{1}{A})(1-e^{-(1-\frac{1}{A})x}).$$