# Lecture 3 (Edinburgh): Changes of measures and probability of survival of branching processes in random environment

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## 1 Change of measure 1

Besides the measure $\mathbf{P}$ we consider another probability measure $\hat{\mathbf{P}}^+$. In order to define this measure let $\mathcal{F}_n$, $n \geq 0$ be the $\sigma$–field of events generated by the random variables $f_0, \dots, f_{n-1}$ and $Z(0), \dots, Z(n)$. These $\sigma$–fields form a filtration $\mathcal{F}$.

**Lemma 1** *The random variables $V(S_n)I_{\{L_n \geq 0\}}$, $n = 0, 1, \dots$ form a martingale with respect to $\mathcal{F}$ under $\mathbf{P}$.*

*Proof.* Let $B$ and $D$ be Borel sets in $\mathbb{N}_0^n$ and $\mathcal{P}^n$, respectively. Recall identities of the first lecture $\mathbf{E}V(x+X) = V(x)$, $x \geq 0$ and the fact that $V(x) = 0$ for $x < 0$. Conditioning first on the environment $f_0, .f_1, ...$ and then on $\mathcal{F}_n$ and using the independence of $f_0, f_1, \dots$ we obtain

$$\begin{aligned} & \mathbf{E}\left[V(S_{n+1});\ L_{n+1} \geq 0, Z(0) = z, (f_0, \dots, f_{n-1}) \in D, (Z(1), \dots, Z(n)) \in B\right] \\ = \ & \mathbf{E}[V(X_{n+1} + S_n); L_n \geq 0, L_{n+1} \geq 0, Z(0) = z, (f_0, \dots, f_{n-1}) \in D, (Z(1), \dots, Z(n)) \in B] \\ = \ & \mathbf{E}[V(S_n)\ ;\ L_n \geq 0, Z(0) = z, (f_0, \dots, f_{n-1}) \in D, (Z(1), \dots, Z(n)) \in B]. \end{aligned} \tag{1}$$

By definition of conditional expectation, (1) implies

$$\mathbf{E}[V(S_{n+1})I_{\{L_{n+1} \geq 0\}} \,|\, \mathcal{F}_n] = V(S_n)I_{\{L_n \geq 0\}} \quad \mathbf{P} - a.s.,$$

which is the desired martingale property.

The proof of the lemma is complete.

Taking into account $V(0) = 1$ we may introduce probability measures $\hat{\mathbf{P}}_n^+$ on the $\sigma$-fields $\mathcal{F}_n$ by means of the densities

$$d\hat{\mathbf{P}}_n^+ \ := \ V(S_n)I_{\{L_n \geq 0\}}\, d\mathbf{P}\ .$$

Because of the martingale property the measures are consistent, i.e., $\mathbf{P}_{n+1}^+|\mathcal{F}_n = \mathbf{P}_n^+$. Therefore (choosing a suitable underlying probability space), there exists a probability measure $\hat{\mathbf{P}}^+$ on the $\sigma$-field $\mathcal{F}_\infty := \bigvee_n \mathcal{F}_n$ such that

$$\hat{\mathbf{P}}^+|\mathcal{F}_n \ = \ \hat{\mathbf{P}}_n^+\,, \quad n \geq 0\,. \tag{2}$$

We note that (2) can be rewritten as

$$\hat{\mathbf{E}}^+ Y_n \;=\; \mathbf{E}[Y_n V(S_n); L_n \geq 0] \tag{3}$$

for every $\mathcal{F}_n$–measurable non-negative random variable $Y_n$. This change of measure is the well-known Doob $h$-transform from the theory of Markov processes. In particular, under $\hat{\mathbf{P}}^+$ the process $S$ becomes a Markov chain with state space $\mathbb{R}_0^+$ and transition kernel

$$\hat{\mathbf{P}}^+(x; dy) \;:=\; \frac{1}{V(x)} \mathbf{P}\{x + X \in dy\} V(y) \;, \quad x \geq 0 \;.$$

In our context $\hat{\mathbf{P}}^+$ arises from conditioning:

**Lemma 2** *Assume A1. For $k \in \mathbb{N}$ let $Y_k$ be a bounded real-valued $\mathcal{F}_k$–measurable random variable. Then, as $n \to \infty$,*

$$\mathbf{E}[Y_k \mid L_n \geq 0] \;\to\; \hat{\mathbf{E}}^+ Y_k \;.$$

*More generally, let $Y_1, Y_2, \ldots$ be a uniformly bounded sequence of real-valued random variables adapted to the filtration $\mathcal{F}$, which converges $\hat{\mathbf{P}}^+$–a.s. to some random variable $Y_\infty$. Then, as $n \to \infty$,*

$$\mathbf{E}[Y_n \mid L_n \geq 0] \;\to\; \hat{\mathbf{E}}^+ Y_\infty \;.$$

**Proof**. For $x \geq 0$ write, as before, $m_n(x) := \mathbf{P}\{L_n \geq -x\}$. Then for $k \leq n$ conditioning on $\mathcal{F}_k$ gives

$$\mathbf{E}[Y_k \mid L_n \geq 0] \;=\; \mathbf{E}\left[ Y_k \frac{m_{n-k}(S_k)}{m_n(0)} \;;\; L_k \geq 0 \right] \;.$$

We know from Lecture 1 and properties of slowly varying functions that for any $k$ and $x > 0$

$$\lim_{n \to \infty} \frac{m_{n-k}(x)}{\mathbf{P}(\gamma > n)} = \lim_{n \to \infty} \frac{m_{n-k}(x)}{m_n(0)} = V(x) \,.$$

Besides, according to Lecture 1

$$\frac{m_{n-k}(x)}{m_n(0)} \leq CV(x)$$

that allows us to apply the dominated convergence theorem giving

$$\lim_{n \to \infty} \mathbf{E}[Y_k \mid L_n \geq 0] \;=\; \mathbf{E}\left[ Y_k \lim_{n \to \infty} \frac{m_{n-k}(S_k)}{m_n(0)} \;;\; L_k \geq 0 \right] = \mathbf{E}\left[ Y_k V(S_k) \;;\; L_k \geq 0 \right]$$

and proving the first claim of the lemma.

For the second claim let $\sigma > 1$. Using again the same arguments as earlier and (3), we obtain for $k \le n$

$$\begin{aligned}\left|\mathbf{E}[Y_n - Y_k \mid L_{\lfloor \sigma n \rfloor} \ge 0]\right| &\le \mathbf{E}\left[|Y_n - Y_k| \frac{m_{\lfloor (\sigma-1)n \rfloor}(S_n)}{m_{\lfloor \sigma n \rfloor}(0)} \; ; \; L_n \ge 0\right] \\ &\le c\left(\frac{\sigma-1}{\sigma}\right)^{-(1-\rho)} \mathbf{E}[|Y_n - Y_k| V(S_n) \; ; \; L_n \ge 0] \\ &= c\left(\frac{\sigma-1}{\sigma}\right)^{-(1-\rho)} \hat{\mathbf{E}}^+ |Y_n - Y_k| \; ,\end{aligned}$$

where $c$ is some positive constant. Letting first $n \to \infty$ and then $k \to \infty$ the right-hand side vanishes by the dominated convergence theorem. Thus, using the first part of the lemma, we conclude

$$\mathbf{E}[Y_n \; ; \; L_{\lfloor \sigma n \rfloor} \ge 0] \; = \; (\, \hat{\mathbf{E}}^+ Y_\infty + o(1))\, \mathbf{P}\{L_{\lfloor \sigma n \rfloor} \ge 0\} \; .$$

Consequently, for some $c > 0$

$$\begin{aligned}|\mathbf{E}[Y_n \; ; \; L_n \; &\ge \; 0] - \; \hat{\mathbf{E}}^+ Y_\infty \, \mathbf{P}\{L_n \ge 0\}| \\ &\le \left|\mathbf{E}[Y_n \; ; \; L_{\lfloor \sigma n \rfloor} \ge 0] - \hat{\mathbf{E}}^+ Y_\infty \, \mathbf{P}\{L_{\lfloor \sigma n \rfloor} \ge 0\}\right| \; + \; c\, \mathbf{P}\{L_n \ge 0, \, L_{\lfloor \sigma n \rfloor} < 0\} \\ &\le \left(o(1) + c\left(1 - \sigma^{-(1-\rho)}\right)\right) \mathbf{P}\{L_n \ge 0\},\end{aligned}$$

where for the last inequality we also used results of Lecture 1 again. Since $\sigma$ may be chosen arbitrarily close to 1, we have

$$\mathbf{E}[Y_n \; ; \; L_n \ge 0] - \hat{\mathbf{E}}^+ Y_\infty \, \mathbf{P}\{L_n \ge 0\} \; = \; o(\mathbf{P}\{L_n \ge 0\}) \, ,$$

which is the second claim of the lemma.

The change of measure has a natural interpretation: Under $\hat{\mathbf{P}}^+$ the chain $S$ can be viewed as a random walk conditioned to never hit the strictly negative half line.

The next statement is an easy consequence of the previous result.

**Lemma 3** *Assume A1. For $k \in \mathbb{N}$ let $Y_k$ be a bounded real-valued $\mathcal{F}_k$–measurable random variable. Then, as $n \to \infty$,*

$$\mathbf{E}[e^{-zY_k} \mid L_n \ge 0] \; \to \; \hat{\mathbf{E}}^+ \left[e^{-zY_k}\right], \, z \in [0, \infty) \; .$$

*More generally, let $Y_1, Y_2, \ldots$ be a uniformly bounded sequence of real-valued random variables adapted to the filtration $\mathcal{F}$, which converges $\hat{\mathbf{P}}^+$–a.s. to some random variable $Y_\infty$. Then, as $n \to \infty$,*

$$\mathbf{E}[e^{-zY_n} \mid L_n \ge 0] \; \to \; \hat{\mathbf{E}}^+ \left[\, e^{-zY_\infty}\right], \; z \in [0, \infty) \; .$$

Thus, we have conditional limit theorems for convergence in distribution of the respective sequences of random variables.

## 2 Change of measure 2

Let $\{f_n^-\}_{n\geq 0}$ and $\{f_n^+\}_{n\geq 0}$ be two independent sequences (realizations) of the random environment and let $\{S_n^-\}_{n\geq 0}$ and $\{S_n^+\}_{n\geq 0}$ be the corresponding associate random walks. Later on any characteristics or random variables related with $\{f_n^-\}_{n\geq 0}$ and $\{f_n^+\}_{n\geq 0}$ , are supplied with the symbols $-$ or $+$, respectively. For instance, we write $L_n^+ = \min_{0\leq j\leq n} S_j^+$,

We need also the random variables

$$\Gamma^- = \min\{n \geq 1 : S_n^- \geq 0\}$$

and

$$\gamma^+ = \min\{n \geq 1 : S_n^+ < 0\}$$

and the event $\mathcal{A}_{k,p} := \{\Gamma^- > k, \gamma^+ > p\}$ .

We may now introduce probability measures $\hat{\mathbf{P}}_{k,p}$ on the $\sigma$-fields $\mathcal{F}_k^- \times \mathcal{F}_p^+$ by means of the densities

$$d\hat{\mathbf{P}}_{k,p} \ = d\left(\mathbf{P}_k^- \times \mathbf{P}_p^+\right) := \ e^D U(-S_k^-)V(S_p^+)I\left\{\Gamma^- > k, \gamma^+ > p\right\} \, d\left(\mathbf{P}^- \times \mathbf{P}^+\right) \ . \tag{4}$$

Because of the properties of the functions $U(x)$ and $V(x)$ the measures are consistent, i.e.,

$$\hat{\mathbf{P}}_{k+1,p}|\mathcal{F}_k^- \times \mathcal{F}_p^+ = \hat{\mathbf{P}}_{k,p}; \quad \hat{\mathbf{P}}_{k,p+1}|\mathcal{F}_k^- \times \mathcal{F}_p^+ = \hat{\mathbf{P}}_{k,p}$$

Therefore (choosing a suitable underlying probability space), there exists a probability measure $\hat{\mathbf{P}}$ on the $\sigma$-field $\mathcal{F}_\infty := \bigvee_{k,p}\left(\mathcal{F}_k^- \times \mathcal{F}_p^+\right)$ such that

$$\hat{\mathbf{P}}|\mathcal{F}_k^- \times \mathcal{F}_p^+ \ = \ \hat{\mathbf{P}}_{k,p}\,, \quad k,p \geq 0\,. \tag{5}$$

We note that (5) can be rewritten as

$$\hat{\mathbf{E}}\, Y_{k,p} \ = \ \mathbf{E}[Y_{k,p} e^D U(-S_k^-)V(S_p^+)I\left\{\Gamma^- > k, \gamma^+ > p\right\}] \tag{6}$$

for every $\mathcal{F}_k^- \times \mathcal{F}_p^+$–measurable non-negative random variable $Y_{k,p}$.

In particular,

$$\widehat{\mathbf{P}}_{k,p}(\mathcal{A}) = e^D \int_{\mathcal{A}} U\left(-S_k^-\right) V(S_p^+) I\{\mathcal{A}_{k,p}\} d(\mathbf{P}^- \times \mathbf{P}^+). \tag{7}$$

We use symbols $\widehat{\mathcal{L}}^{\pm}$ and $\widehat{\mathcal{L}}$ for the laws of distributions generated by the measures $\widehat{\mathbf{P}}^{\pm}$ and $\widehat{\mathbf{P}}$. Analogous agreement we keep for $\widehat{\mathbf{E}}^{\pm}$ and $\widehat{\mathbf{E}}$.

**Lemma 4** *Let condition A1 valid and let $Y_{l,p}$, $l = 1, 2, \ldots; p = 1, 2, \ldots$ be a tuple of uniformly bounded random variables such that $Y_{l,p}$ is measurable with respect to the $\sigma$-algebra $\mathcal{F}_l^- \times \mathcal{F}_p^+$ for any pair $l, p$. Then*

$$\lim_{\min(n,r)\to\infty} \mathbf{E}[Y_{l,p} \,|\, A_{n,r}] = \hat{\mathbf{E}} Y_{l,p}. \tag{8}$$

*More generally, if the tuples $Y_{n,r}$ converge as $\min(n,r)\to\infty$ to a random variable $Y_{\infty,\infty}$ $\hat{\mathbf{P}}$ a.s., then*

$$\lim_{\min(n,r)\to\infty}\mathbf{E}[Y_{n,r}\mid A_{n,r}]=\hat{\mathbf{E}}Y_{\infty,\infty}. \tag{9}$$

**Proof**. Relation (8) can be proved the same as Lemma 2. To demonstrate (9) observe that for any numbers $\sigma>1$ and $k\in\mathsf{N}$

$$\begin{aligned}
&|\mathbf{E}[Y_{n,r}-Y_{l,l}\mid A_{\sigma n,\sigma r}]|\\
\leq\ &\mathbf{E}\left[|Y_{n,r}-Y_{l,l}|\frac{\widetilde{m}^{-}_{(\sigma-1)n}(-S_n^-)}{\widetilde{m}^{-}_{\sigma n}(0)}\frac{m^{+}_{(\sigma-1)r}(S_r^+)}{m^{+}_{\sigma r}(0)}I\left\{A_{n,r}\right\}\right]\\
\leq\ &c\Big(\frac{\sigma-1}{\sigma}\Big)^{-\rho}\Big(\frac{\sigma-1}{\sigma}\Big)^{\rho-1}\mathbf{E}\left[|Y_{n,r}-Y_{k,k}|U^-(-S_n^-)V^+(S_r^+)I\left\{A_{n,r}\right\}\right]\\
=\ &c\Big(\frac{\sigma-1}{\sigma}\Big)^{-1}\hat{\mathbf{E}}[|Y_{n,r}-Y_{l,l}|].
\end{aligned}$$

Hence by the conditions of the lemma and the bounded convergence theorem we conclude that

$$\limsup_{l\to\infty}\limsup_{\min(n,r)\to\infty}|\mathbf{E}[Y_{n,r}-Y_{l,l}\mid A_{\sigma n,\sigma r}]|=0.$$

In particular,

$$\mathbf{E}[Y_{n,r}I\{A_{\sigma n,\sigma r}\}]=\Big(\hat{\mathbf{E}}Y_{\infty,\infty}+o(1)\Big)\,\mathbf{P}(A_{\sigma n,\sigma r}).$$

Consequently,

$$\begin{aligned}
&|\mathbf{E}[Y_{n,r}I\{A_{n,r}\}]-\hat{\mathbf{E}}Y_{\infty,\infty}\mathbf{P}(A_{n,r})|\\
&\quad\leq|\mathbf{E}[Y_{n,r}I\{A_{\sigma n,\sigma r}\}]-\hat{\mathbf{E}}Y_{\infty,\infty}\mathbf{P}(A_{\sigma n,\sigma r})|+c\,|\mathbf{P}(A_{\sigma n,\sigma r})-\mathbf{P}(A_{n,r})|\\
&\quad\leq(o(1)+c\left((1-\sigma^{-\rho})\sigma^{\rho-1}+(1-\sigma^{1-\rho})\right)))\mathbf{P}(A_{n,r}),
\end{aligned}$$

since from the results of Lecture 1 it follows that if $M_n^{(1)-}=\max_{1\leq k\leq n}S_k^-$ then

$$\begin{aligned}
&|\mathbf{P}(A_{\sigma n,\sigma r})-\mathbf{P}(A_{n,r})|\\
=\ &\left|\mathbf{P}(M_{\sigma n}^{(1)-}<0)\mathbf{P}(L_{\sigma r}\geq 0)-\mathbf{P}(M_n^{(1)-}<0)\mathbf{P}(L_r\geq 0)\right|\\
\leq\ &\left|\mathbf{P}(M_{\sigma n}^{(1)-}<0)-\mathbf{P}(M_n^{(1)-}<0)\right|\mathbf{P}(L_{\sigma r}\geq 0)\\
+|\mathbf{P}(L_r\ \geq\ &0)-\mathbf{P}(L_{\sigma r}\geq 0)|\mathbf{P}(M_n^{(1)-}<0)\\
\leq\ &c\left((1-\sigma^{-\rho})\sigma^{\rho-1}+(1-\sigma^{1-\rho})\right)\mathbf{P}(A_{n,r}).
\end{aligned}$$

Therefore,

$$\mathbf{E}[Y_{n,r}I\{A_{n,r}\}]-\hat{\mathbf{E}}Y_{\infty,\infty}\mathbf{P}(A_{n,r})=o(\mathbf{P}(A_{n,r})),$$

as $\min(n,r)\to\infty$ which is equivalent to (9).

**Lemma 5** *Let condition A1 be valid and let $Y$ and $Y_{l,p}$, $l,p\in\mathsf{N}$ - be a tuple of random variables meeting the conditions of Lemma 4. If $Y_n^*$, $n\in\mathsf{N}$ is a sequence of uniformly bounded random variables such that $Y_n^*$ is measurable with respect to the $\sigma$-algebra $\mathcal{F}_n$ for any $n$ and*

$$\mathbf{E}[Y_n^*\,|\,\tau(n)=l]=\mathbf{E}[Y_{l,n-l}\,|\,\mathcal{A}_{l,n-l}],\quad 0\le l\le n,$$

*then*

$$\lim_{n\to\infty}\mathbf{E}[Y_n^*]=\hat{\mathbf{E}}[Y_{\infty,\infty}]. \tag{10}$$

**Proof**. In view of the generalized arcsine law for any $\varepsilon>0$ there exists $\delta\in\left(0,2^{-1}\right)$ such that $\mathbf{P}\left(\tau\left(n\right)\notin\left[n\delta,n(1-\delta)\right]\right)<\varepsilon$ for all sufficiently large $n$. Now to prove the lemma it suffices to use the total probability formula with respect to $\{\tau(n)=k\},0\le k\le n$, and to apply Lemma 4.

Clearly, under the conditions above $Y_n^*\to Y_{\infty,\infty}$ in distribution.

We need the following statement.

**Theorem 6** *We have*

$$\widehat{\mathbf{E}}^{+}[\sum_{k=0}^{\infty}e^{-S_k}]<\infty,\ \widehat{\mathbf{E}}^{-}[\sum_{k=0}^{\infty}e^{S_k}]<\infty.$$

**Proof**. Let

$$\Gamma_0^*=0,\Gamma_{k+1}^*=\min\left\{n>\Gamma_k^*:S_n\ge S_{\Gamma_k^*}\right\}$$

be weak ascending ladder epochs. By the duality principle for random walks (see Feller, Volume II) we have

$$\begin{aligned}
&\sum_{p=0}^{\infty}\mathbf{P}\left(S_p\le x,\min_{0\le j\le p}S_j\ge 0\right)=\sum_{p=0}^{\infty}\mathbf{P}\left(S_p\le x,S_p\ge S_j,0\le j\le p\right)\\
=\ &\sum_{p=0}^{\infty}\sum_{k=0}^{p}\mathbf{P}\left(S_p\le x,\Gamma_k^*=p\right)=\sum_{k=0}^{\infty}\sum_{p=k}^{\infty}\mathbf{P}\left(S_p\le x,\Gamma_k^*=p\right)\\
=\ &\sum_{k=0}^{\infty}\mathbf{P}\left(S_{\Gamma_k^*}\le x\right)=U^*(x)
\end{aligned}$$

and, therefore, $U^*(x)$ is the renewal function for the sequence

$$X_0^*=0,X_{k+1}^*=S_{\Gamma_{k+1}^*}-S_{\Gamma_k^*}$$

Recall the following Key Renewal theorem (see Feller, Volume II):

If the distribution of $X_{k+1}^*$ is nonarithmetic then for any fixed $h$ as $x\to\infty$

$$U^*(x+h)-U^*(x)\to\frac{h}{EX_1^*}$$

(even if $EX_1^* = \infty$). The same is true for arithmetic distributions for $h = k\lambda$, where $\lambda$ is the span of the distribution of $X_1^*$. In particular, we have that for ANY renewal function there exists a constant $c$ such that

$$U^*(x) \leq c(x+1), \ x \geq 0.$$

By definition,

$$\begin{aligned}
\widehat{\mathbf{E}}^{+}[\sum_{p=0}^{\infty} e^{-S_p}] &= \sum_{p=0}^{\infty} \mathbf{E}\left[e^{-S_p} V(S_p) I\{\gamma > p\}\right] \\
&= \sum_{p=0}^{\infty} \int_0^{\infty} e^{-x} V(x) d\mathbf{P}\left(S_p \leq x, \gamma > p\right) \\
&= \int_0^{\infty} e^{-x} V(x) d\left(\sum_{p=0}^{\infty} \mathbf{P}\left(S_p \leq x, \min_{0 \leq j \leq p} S_j \geq 0\right)\right) \\
&= \int_0^{\infty} e^{-x} V(x) dU^*(x).
\end{aligned}$$

Hence by monotonicity

$$\begin{aligned}
\widehat{\mathbf{E}}^{+}[\sum_{p=0}^{\infty} e^{-S_p}] &= \int_0^{\infty} e^{-x} V(x) dU^*(x) \\
&\leq \sum_{k=0}^{\infty} e^{-k} V(k+1) U^*(k+1) \\
&\leq c_1 c_2 \sum_{k=0}^{\infty} e^{-k} (k+1)^2.
\end{aligned}$$

The arguments needed to prove the second statement are similar.

**Corollary 7** *Under the conditions of Theorem 6 as $k \to \infty$*

$$S_k \to +\infty \quad \widehat{\mathbf{P}}^{+} - a.s., \ S_k \to -\infty \quad \widehat{\mathbf{P}}^{-} - a.s.$$

**Proof.** This is a simple consequence of the statements of Theorem 6.

# 3 Properties of generating functions

Set

$$\begin{aligned}
f_{k,n}(s) :=& \quad f_k(f_{k+1}(...(f_{n-1}(s))...)), \ 0 \leq k \leq n-1, \ f_{n+1,n}(s) := s, \\
f_{n,m}(s) :=& \quad f_{n-1}(f_{n-2}(...(f_m(s))...)), \ n \geq m+1,
\end{aligned}$$

and

$$\chi_k(s) := \frac{1}{1-f_k(s)} - \frac{1}{f_k'(1)(1-s)}, \ 0 \leq s \leq 1. \tag{11}$$

**Lemma 8** *Let $f_k \neq 1, 0 \leq k \leq n-1$. Then for any $0 \leq s < 1$ and $0 \leq m \leq n-1$*

$$\frac{1}{1-f_{m,n}(s)} = \frac{e^{-S_n+S_m}}{1-s} + \sum_{k=m}^{n-1} \eta_{k,n}(s)\, e^{-S_k+S_m}, \tag{12}$$

*and*

$$\frac{e^{S_n-S_m}}{1-f_{n,m}(s)} = \frac{1}{1-s} + \sum_{j=m+1}^{n} \eta_{j,m}(s) e^{S_j-S_m} \tag{13}$$

*where for $k \leq n$*

$$\eta_{k,n}(s) := \chi_k(f_{k+1,n}(s)) \leq \eta_{k+1} = \frac{f_k''(1)}{(f_k'(1))^2} \tag{14}$$

*and for $j > m$*

$$\eta_{j,m}(s) := \chi_{j-1}(f_{j-1,m}(s)) \leq \eta_j.$$

*If the generating functions are geometric:*

$$f_i(s) = \frac{q_i}{1-p_i s},\ \ p_i + q_i = 1,\ \ p_i q_i > 0, i = 0, 1, ...,$$

*then $\eta_{k,n}(s) = 1$ for all $k$ and $n$.*

**Proof**. We have

$$\begin{aligned}
\frac{1}{1-f_{0,n}(s)} &= \frac{1}{1-f_{0,n}(s)} - \frac{1}{f_0'(1)(1-f_{1,n}(s))} + \frac{1}{f_0'(1)(1-f_{1,n}(s))} \\
&= \chi_0(f_{1,n}(s))e^{S_0-S_0} + \frac{1}{f_0'(1)(1-f_{1,n}(s))} - \frac{1}{f_0'(1)f_1'(1)(1-f_{2,n}(s))} \\
&\quad + \frac{1}{f_0'(1)f_1'(1)(1-f_{2,n}(s))} \\
&= \chi_0(f_{1,n}(s))e^{S_0-S_0} + \chi_1(f_{2,n}(s))e^{S_0-S_1} + \frac{1}{f_0'(1)f_1'(1)(1-f_{2,n}(s))} \\
&= ... = \sum_{k=0}^{n-1} \chi_k(f_{k+1,n}(s))e^{S_0-S_k} + \frac{e^{S_0-S_n}}{1-s} \\
&= \sum_{k=0}^{n-1} \eta_{k,n}(s)e^{S_0-S_k} + \frac{e^{S_0-S_n}}{1-s}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{1-f_{m,n}(s)} &= \frac{1}{1-f_{m,n}(s)} - \frac{1}{f'_m(1)(1-f_{m+1,n}(s))} \\
&\quad + \frac{1}{f'_m(1)(1-f_{m+1,n}(s))} \\
&= \chi_m(f_{m+1,n}(s))e^{S_m-S_m} \\
&\quad + \frac{1}{f'_m(1)(1-f_{m+1,n}(s))} - \frac{1}{f'_m(1)f'_{m+1}(1)(1-f_{m+2,n}(s))} \\
&\quad + \frac{1}{f'_m(1)f'_{m+1}(1)(1-f_{m+2,n}(s))} \\
&= \chi_m(f_{m+1,n}(s))e^{S_m-S_m} + \chi_{m+1}(f_{m+2,n}(s))e^{S_{m+1}-S_m} \\
&\quad + \frac{1}{f'_m(1)f'_{m+1}(1)(1-f_{m+2,n}(s))} \\
&= \ldots = \frac{e^{S_m-S_n}}{1-s} + \sum_{k=m}^{n-1} \eta_{k,n}(s)\, e^{S_m-S_k}.
\end{aligned}$$

For pure geometric functions:

$$1-f_k(s) = 1 - \frac{q_k}{1-p_k s} = \frac{p_k(1-s)}{1-p_k s},\ f'_k(1) = p_k/q_k,$$

leading to

$$\chi_k(s) := \frac{1-p_k s}{p_k(1-s)} - \frac{q_k}{p_k(1-s)} = \frac{p_k(1-s)}{p_k(1-s)} = 1.$$

This gives

$$\frac{1}{1-f_{m,n}(s)} = \frac{e^{S_m-S_n}}{1-s} + \sum_{k=m}^{n-1} e^{S_m-S_k}. \tag{15}$$

In the general situation in view of $1-f(s) \geq f'(s)(1-s)$

$$\begin{aligned}
f'(1)\,\chi(s) &= \frac{f'(1)}{1-f(s)} - \frac{1}{1-s} \\
&= \frac{1}{1-f(s)} \frac{f'(1)(1-s) - (1-f(s))}{(1-s)} \\
&\leq \frac{f'(1) - f'(s)}{1-f(s)} = \sum_{k=1}^{\infty} k r_k(s)
\end{aligned} \tag{16}$$

where

$$r_k(s) = p_k \frac{1-s^{k-1}}{1-f(s)}.$$

Observe that

$$\frac{r_{k+1}(s)}{r_k(s)} = \frac{p_{k+1}\left(1-s^k\right)}{p_k\left(1-s^{k-1}\right)} = \frac{p_{k+1}}{p_k}\left(1 + \frac{1}{\sum_{j=1}^{k-1} s^{-j}}\right)$$

is increasing in $s$ for any $k \geq 1$ and that the $r_k(s)$ sums to 1 for any $0 \leq s < 1$. Hence the right-hand side of (16) is increasing in $s$ and, therefore

$$f'(1)\,\chi(s) \leq \sup_{s\in[0,1]} \frac{f'(1) - f'(s)}{1 - f(s)} = \frac{f''(1)}{f'(1)}.$$

The lemma is proved.

# 4 Probabilty of survival

**Theorem 9** *Assume that there exists a constant $C \in (0,\infty)$ such that*

$$\frac{f''(1)}{(f'(1))^2} \leq C \ \ a.s.$$

*and let the Spitzer condition be valid. Then the sequence of random variables*

$$\zeta_{0,n} := e^{-S_{\tau(n)}}\mathbf{P}_{\pi}(Z(n) > 0) = e^{-S_{\tau(n)}}(1 - f_{0,n}(0)),\ n = 0, 1, 2, ..., \quad (17)$$

*convereges in distribution as $n \to \infty$ to a random variable $\zeta \in [0,1]$ which is positive with probability 1.*

We prove this theorem into several steps

**Lemma 10** *Under the conditions of Theorem 9*

$$q_m := \lim_{n\to\infty} f_{m,n}(0) < 1 \qquad \hat{\mathbf{P}}^+ - \ a.s., \ m = 0, 1, ....$$

**Proof**. Existence of the limit is obvious. Let us show that $q_m < 1$ $\hat{\mathbf{P}}^+-$ a.s. Clearly,

$$\frac{1}{1 - f_{m,n}(0)} \uparrow$$

as $n \to \infty$. By Lemma 8 and the conditions of the lemma in question

$$\begin{aligned} \frac{1}{1 - f_{m,n}(0)} &= e^{-S_n + S_m} + \sum_{k=m}^{n-1} \eta_{k,n}(0)\, e^{-S_k + S_m} \\ &\leq (C+1)e^{S_m} \sum_{k=m}^{n} e^{-S_k} \leq (C+1)e^{S_m} \sum_{k=0}^{\infty} e^{-S_k} \end{aligned}$$

It follows from Theorem 6 that

$$\sum_{k=0}^{\infty} e^{-S_k} < \infty \qquad \hat{\mathbf{P}}^+ - \ \text{a.s.}$$

From this fact the statement of the lemma follows easily.

Introduce the notation

$$\zeta_{k,m}(s) := \frac{1 - f_{k,m}(s)}{e^{S_k - S_m}}$$

and

$$\zeta_k(s) := \frac{1 - f_{k,0}(s)}{e^{S_k}}.$$

**Lemma 11** *Under the conditions of Theorem 9 for any* $s \in [0,1)$ *and any* $m = 0, 1, ...$ *there exists* $\lim_{k\to\infty} \zeta_{k,m}(s) =: \zeta_{\infty,m}(s)$ *and* $\zeta_{\infty,m}(s) > 0$ $\hat{\mathbf{P}}^{-}-$ *a.s.*

**Proof**. Clearly, for $l + 1 \geq m$

$$\begin{aligned} \zeta_{k+1,m}(s) \quad &: \quad = \frac{1 - f_{k+1,m}(s)}{e^{S_{k+1} - S_m}} = \frac{1 - f_{k+1}(f_{k,m}(s))}{e^{S_{k+1} - S_m}} \\ &\leq \quad \frac{1 - f_{k,m}(s)}{e^{S_k - S_m}} = \zeta_{k,m}(s) \end{aligned}$$

proving existence of the limit. In particular,

$$\frac{e^{S_k - S_m}}{1 - f_{k,m}(s)} \uparrow$$

as $k \to \infty$. Further, by Lemma 8, the conditions of the lemma in question and the respective results of Lecture 2

$$\begin{aligned} \frac{e^{S_k - S_m}}{1 - f_{k,m}(s)} \quad &= \quad \frac{1}{1-s} + \sum_{j=m+1}^{k} \eta_{j,m}(s) e^{S_j - S_m} \\ &\leq \quad \frac{1}{1-s} + Ce^{-S_m} \sum_{j=0}^{\infty} e^{S_j} < \infty \qquad \hat{\mathbf{P}}^{-} - \text{ a.s.} \end{aligned}$$

The lemma is proved.

**Proof of Theorem 9**. We write

$$\zeta_{0,n} = \left(1 - f_{0,\tau(n)}(f_{\tau(n),n}(0))\right) e^{-S_{\tau(n)}},$$

set for $\lambda > 0$

$$Y_n^* = e^{-\lambda \zeta_{0,n}},$$

and consider the Laplace transform

$$\mathbf{E} e^{-\lambda \zeta_{0,n}} = \mathbf{E}\left[Y_n^*\right]$$

of the distribution of $\zeta_{0,n}$. Clearly, for the associated random walk

$$\mathbf{E}[Y_n^* \,|\, \tau(n) = k] = \mathbf{E}[Y_{k,n-k} \,|\, \mathcal{A}_{k,n-k}],$$

where

$$Y_{k,n-k} = \exp\left\{-\lambda\left(1 - f_{k,0}^{-}(f_{0,n-k}^{+}(0))\right) e^{-S_k^{-}}\right\}$$

and $\{f_n^-\}_{n\geq 0}$ and $\{f_n^+\}_{n\geq 0}$ are two independent sequences (realizations) of the random environment with $\{S_n^-\}_{n\geq 0}$ and $\{S_n^+\}_{n\geq 0}$ be the corresponding associate random walks.

From Lemmas 11 and 10 it follows that

$$\lim_{\min(n-k,k)\to\infty} Y_{k,n-k} =: Y_{\infty,\infty} = \exp\left\{-\lambda\left(1 - \zeta_{\infty,0}^{-}(q^{+})\right)\right\}$$

exists $\hat{\mathbf{P}}-$ a.s., where

$$\zeta_{\infty,0}^{-}(q^{+}) = \lim_{k,n-k\to\infty} \frac{1 - f_{k,0}^{-}(f_{0,n-k}^{+}(0))}{e^{S_k^{-}}}$$

and, moreover, $Y_{\infty,\infty} = Y(\lambda) < 1$ $\hat{\mathbf{P}}-$ a.s.

Hence, with $\zeta^{-}(s) := \zeta_{\infty,0}^{-}(s)$ and according to our previous results

$$\lim_{n\to\infty} \mathbf{E}\left[Y_n^*\right] = \lim_{n\to\infty} \mathbf{E}\left[e^{-\lambda\zeta_{0,n}}\right] = \hat{\mathbf{E}}\left[e^{-\lambda\zeta^{-}(q^{+})}\right].$$

The theorem is proved.