Optimal Stopping and Applications
Example 2: American options

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1 Introduction

1.1 The market model

We consider a financial market consisting of two primary assets, a *risk-free* bond B and a stock S whose dynamics under the unique *risk-neutral* measure $\mathbb{P}$ are given by

$$\begin{aligned} \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t \\ \mathrm{d}S(t) &= rS(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t) \\ B(0) &= 1 \\ S(0) &= x \end{aligned} \tag{1}$$

where r, σ are deterministic constants with $\sigma > 0$ and $W(t)$ is a Brownian motion under $\mathbb{P}$. We refer to r as the *interest rate* and to σ as the *volatility* of S. We denote by $\{\mathscr{F}_t\}_{t\geq 0}$ the natural augmented filtration of W. It is easy to verify that (1) under $\mathbb{P}$ has the unique strong solution

$$B(t) = \mathrm{e}^{rt} \tag{2}$$

$$S(t) = x\mathrm{e}^{\sigma W(t)+(r-\sigma^2/2)t} \tag{3}$$

It is not difficult to check that $\mathrm{e}^{-rt}S(t)$ is a $\{\mathscr{F}_t\}$-martingale under $\mathbb{P}$.

1.2 Pricing formula

Theorem 1 (Fundamental theorem of asset pricing)**.** *Let $T > 0$ and D be a $\mathbb{P}$-integrable and $\mathscr{F}_T$-measurable random variable, which we interpret as the value of some* derivative security *at time T. The* arbitrage-free *price of D at time $t \in [0; T]$ is given by*

$$D(t) = \mathbb{E}[\mathrm{e}^{-r(T-t)}D|\mathscr{F}_t] \tag{4}$$

Moreover $\mathrm{e}^{-rt}D(t)$ is a $\{\mathscr{F}_t\}$-martingale under $\mathbb{P}$.

Proof. See [8] Chapter 5.

1.3 European and American options

Definition 1. *A **European [American] call option** C^{Eur} [C^{Am}] with **strike price** $K > 0$ and **time of maturity** $T > 0$ on the **underlying asset** S is a contract defined as follows*

- *The **holder** of the option has, exactly at time T [at any time $t \in [0;T]$], the right but not the obligation to **buy** one share of the underlying asset S at price K from the **underwriter** of the option.*

Definition 2. *A **European [American] put option** P^{Eur} [P^{Am}] with **strike price** $K > 0$ and **time of maturity** $T > 0$ on the **underlying asset** S is a contract defined as follows*

- *The **holder** of the option has, exactly at time T [at any time $t \in [0;T]$], the right but not the obligation to **sell** one share of the underlying asset S at price K to the **underwriter** of the option.*

We fix a strike $K > 0$ and a time of maturity $T > 0$. By theorem 1, the *arbitrage-free* prices of a European call [put] at time 0 is given by

$$C^{Eur} = \mathbb{E}[e^{-rT}(S(T) - K)^+] \tag{5}$$
$$P^{Eur} = \mathbb{E}[e^{-rT}(K - S(T))^+] \tag{6}$$

which can be expressed in a closed formula, the *Black-Scholes formula*[1].

Now suppose we are the owner of an American call [put] option. Since we can exercise the option at any time $t \in [0;T]$, we choose an $\{\mathscr{F}_t\}$-stopping time $\tau \in [0,T]$ taking values in $[0,T]$. At time T we own $e^{r(T-\tau)}(K - S(\tau))$[2]. Since we may choose any $\{\mathscr{F}_t\}$-stopping time $\tau \in [0,T]$, theorem 1 implies[3] that the arbitrage-free price of an American call [put] option at time 0 is given by

$$C^{Am} = \sup_{\tau \in [0,T]} \mathbb{E}[e^{-r\tau}(S(\tau) - K)^+] \tag{7}$$
$$P^{Am} = \sup_{\tau \in [0,T]} \mathbb{E}[e^{-r\tau}(K - S(\tau))^+] \tag{8}$$

It is obvious that $C^{Am} \geq C^{Eur}$ and $P^{Am} \geq P^{Eur}$, since we can choose the exercise strategy $\tau = T$. The following theorem states in which cases the latter strategy is indeed the best that we can do.

Theorem 2.

1. *Suppose $r \geq 0$. Then $C^{Am} = C^{Eur}$.*
2. *Suppose $r = 0$. Then $P^{Am} = P^{Eur}$.*

Proof.

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[1] See for instance [1] p 100 et seq.

[2] If we exercise before time T, we invest our money for the rest of the time up to T in the risk-less bond

[3] More precisely this follows once we have shown the existence of an optimal stopping time.

1. Fix $\tau \in [0,T]$. Define $g_1 : \mathbb{R}^+ \to \mathbb{R}^+$ by $g_1(x) = (x - \mathrm{e}^{-rT}K)^+$. Clearly g_1 is convex. Jensen's inequality for conditional expectations, the fact that $e^{-rt}S(t)$ is a martingale and the optional sampling theorem yield

$$\begin{aligned} C^{Eur} &= \mathbb{E}[e^{-rT}(S(T)-K)^+] = \mathbb{E}[g_1(e^{-rT}S(T))] \\ &= \mathbb{E}[\mathbb{E}[g_1(e^{-rT}S(T))|\mathscr{F}_\tau]] \geq \mathbb{E}[g_1(\mathbb{E}[e^{-rT}S(T)|\mathscr{F}_\tau])] \\ &= \mathbb{E}[g_1(e^{-r\tau}S(\tau))] = \mathbb{E}[(e^{-r\tau}S(\tau) - \mathrm{e}^{-rT}K)^+] \\ &\geq \mathbb{E}[e^{-r\tau}(S(\tau)-K)^+] \end{aligned} \tag{9}$$

Since $\tau \in [0,T]$ was arbitrary, we have $C^{Eur} \geq C^{Am}$, which together with $C^{Eur} \leq C^{Am}$ yields the claim.

2. Fix $\tau \in [0,T]$. Define $g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ by $g_2(x) = (K-x)^+$. Clearly g_2 is convex. Jensen's inequality for conditional expectations, the fact that $S(t)$ is a martingale and the optional sampling theorem yield

$$\begin{aligned} C^{Eur} &= \mathbb{E}[(K-S(T))^+] = \mathbb{E}[g_2(S(T))] \\ &= \mathbb{E}[\mathbb{E}[g_2(S(T))|\mathscr{F}_\tau]] \geq \mathbb{E}[g_2(\mathbb{E}[S(T)|\mathscr{F}_\tau])] \\ &= \mathbb{E}[g_2(S(\tau))] = \mathbb{E}[(K-S(\tau))^+] \end{aligned} \tag{10}$$

Since $\tau \in [0,T]$ was arbitrary, we have $P^{Eur} \geq P^{Am}$, which together with $P^{Eur} \leq P^{Am}$ yields the claim.

Remark. If $r > 0$ the above argument breaks down for the American put. We will show below that in this case we have $P^{Am} > P^{Eur}$ and we will derive an explicit formula for difference $P^{Am} - P^{Eur}$.

2 Analytical Characterization of the Put Price

2.1 Formal definition of the problem

Let $\{\tilde{W}(s)\}_{s\geq 0}$ be a Brownian motion on some probability space $\{\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_s, \tilde{\mathbb{P}}\}$, where $\{\tilde{\mathscr{F}}_s\}_{s\geq 0}$ is the natural augmented filtration of $\tilde{W}$. Let $E := [0,\infty) \times (0,\infty)$ (*Perpetual American Put*) or $E = [0,T] \times (0,\infty)$; $0 < T < \infty$ (*Finite American Put*). Set $\Omega = E \times \tilde{\Omega}$, $\mathscr{F} = \mathscr{B}(E) \otimes \tilde{\mathscr{F}}$ and $\mathscr{G} = \{\emptyset, E\} \otimes \tilde{\mathscr{F}}$. For $s \geq 0$ and $\omega = (t, x, \tilde{\omega}) \in \Omega$ define

$$\begin{aligned} W(s)(\omega) &= \tilde{W}(s)(\tilde{\omega}) \\ S(s)(\omega) &= x\mathrm{e}^{\sigma W(s)(\tilde{\omega}) + (r - \sigma^2/2)s} \\ X(s)(\omega) &= (t+s, S(s)(\omega)) \end{aligned} \tag{11}$$

where r, σ are deterministic constants with $\sigma, r > 0$. Moreover, for $s \geq 0$, let $\mathscr{F}_s = \mathscr{B}(E) \otimes \tilde{\mathscr{F}}_s$ and $\mathscr{G}_s = \{\emptyset, E\} \otimes \tilde{\mathscr{F}}_s$. Finally define probability measures $\{\mathbb{P}_{(t,x)}\}_{(t,x)\in E}$ on $\{\Omega, \mathscr{F}\}$ and $\mathbb{P}$ on $\{\Omega, \mathscr{G}\}$ by $\mathbb{P}_{(t,x)} := \delta_t \otimes \delta_x \otimes \tilde{\mathbb{P}}$ and $\mathbb{P} := \mu \otimes \tilde{\mathbb{P}}$, where δ_t and δ_x denote Dirac measures and $\mu : \{\emptyset, E\} \mapsto [0,1]$ is defined by $\mu(\emptyset) = 0$; $\mu(E) = 1$. For convenience we set $\mathbb{P}_x := \mathbb{P}_{(0,x)}$. It is not difficult to check that $\{W(s)\}_{s\geq 0}$ is a Brownian motion on $\{\Omega, \mathscr{G}, \mathscr{G}_s, \mathbb{P}\}$ and $\{S(s)\}_{s\geq 0}$ and $\{X(s)\}_{s\geq 0}$ are strong Markov families on $\{\Omega, \mathscr{F}, \mathscr{F}_s, \{\mathbb{P}_x\}_{x>0}\}$ and $\{\Omega, \mathscr{F}, \mathscr{F}_s, \{\mathbb{P}_{(t,x)}\}_{(t,x)\in E}\}$.

Remark. Under $\mathbb{P}_{(t,x)}$ we interpret $S(s)$ as the value of a stock $\tilde{S}$ with volatility σ in a financial market with interest rate r at time $\mathbf{t}+\mathbf{s}$ given that $\tilde{S}(t)=x$.

We fix a strike price $K>0$. Define the *gain function* $G:E\mapsto[0,K]$ by $G(t,x):=\mathrm{e}^{-rt}(K-x)^{+}$. For $(t,x)\in E$ define the optimal stopping problem

$$\begin{aligned} V(t,x) &= \sup_{\tau\in[0,T-t]} \mathbb{E}_{(t,x)}[G(X(s))] \\ &= \sup_{\tau\in[0,T-t]} \mathbb{E}_{(t,x)}[\mathrm{e}^{-r(t+\tau)}(K-S(\tau))] \end{aligned} \tag{12}$$

where T is the upper boundary of the time coordinate of E and $\tau\in[0,T-t]$ is a stopping time[4] taking values in $[0,T-t]$[5]. Since G is bounded, $V(t,x)$ is defined for all $(t,x)\in E$. We call V the *value function.*

Remark. We interpret $V(t,x)$ as the arbitrage free price of an American put option with strike K and maturity T on $\tilde{S}$ *at time* 0 given that $\tilde{S}(t)=x$. Since we have a positive interest rate r, we cannot compare prices at different times directly, but need to discount appropriately. The price of an American put option *at time* t given $\tilde{S}(t)=x$ is given by

$$v(t,x)=\mathrm{e}^{rt}V(t,x)=\sup_{\tau\in[0,T-t]}\mathbb{E}_x[\mathrm{e}^{-r\tau}(K-S(\tau))^{+}] \tag{13}$$

We call v the *value* function.* Similarly we define

$$g(t,x)=\mathrm{e}^{rt}G(t,x)=(K-x)^{+} \tag{14}$$

which we call the *gain* function.* Even though V and G are the formal correct objects, which in addition carry the economic interpretation of time value of money, it turns out that v and g are the convenient mathematical objects to work with.

2.2 Elementary properties of the value* function

Lemma 1.

1. *If $T=\infty$, the function $t\mapsto v(t,x)$ is constant*
2. *If $T<\infty$, the function $t\mapsto v(t,x)$ is decreasing with $v(T,x)=(K-x)^{+}$.*

Proof. Let $0\le t_1\le t_2\le T$[6]. Then

$$\begin{aligned} v(t_1,x) &= \sup_{\tau\in[0,T-t_1]}\mathbb{E}_x[\mathrm{e}^{-r\tau}(K-S(\tau))^{+}] \\ &\ge \sup_{\tau\in[0,T-t_2]}\mathbb{E}_x[\mathrm{e}^{-r\tau}(K-S(\tau))^{+}] \\ &= v(t_2,x) \end{aligned} \tag{15}$$

Since $[0,\infty-t_1]=[0,\infty-t_2]$ and clearly $v(T,x)=(K-x)^{+}$ by definition for $T<\infty$, both assertions follow immediately.

[4] In general we allow τ to be an $\{\mathscr{F}_t\}$-stopping time. Note, however, that for fixed $(t,x)\in E$ for each $\{\mathscr{F}_t\}$-stopping time $\tau_{\mathscr{F}}$ there exists a $\{\mathscr{G}_t\}$-stopping time $\tau_{\mathscr{G}}$ with $\tau_{\mathscr{F}}=\tau_{\mathscr{G}}$ $\mathbb{P}_{(t,x)}$-a.s.. If necessary we will work with $\tau_{\mathscr{G}}$ rather than with $\tau_{\mathscr{F}}$, which will always be clear from the context.

[5] $\infty-t:=\infty$; moreover we allow $\tau=\infty$ if $T=\infty$

[6] We require $t<\infty$

Lemma 2. *The function $x \mapsto v(t,x)$ is convex and continuous*

Proof. Fix $t \in [0,T]$[7]. For $\tau \in [0, T-t]$, $x > 0$ define

$$u(x,\tau) := \mathrm{e}^{-r\tau}(K - x\mathrm{e}^{\sigma W(\tau)+(r-\sigma^2/2)\tau})^+ \tag{16}$$

It is straightforward to check that $x \mapsto u(x,\tau)$ is convex. By linearity of the integral it follows that $x \mapsto \mathbb{E}[u(x,\tau)]$ is convex. Moreover clearly

$$v(x,t) = \sup_{\tau\in[0,T-t]} \mathbb{E}[u(x,\tau)] \tag{17}$$

The assertion follows by the well-know facts that the supremum of convex functions is convex again, and that convex functions are continuous.

Lemma 3. *The function $(t,x) \mapsto v(t,x)$ is lsc.*

Proof. For $\tau \in [0,T]$ and $(t,x) \in E$ define

$$u(t,x,\tau) := \mathrm{e}^{-r(\tau\wedge(T-t))}(K - x\mathrm{e}^{\sigma W(\tau\wedge(T-t))+(r-\sigma^2/2)(\tau\wedge(T-t))})^+ \tag{18}$$

It is not difficult to check that $(t,x) \mapsto u(t,x,\tau)$ is continuous. By the dominated convergence theorem we get that $(t,x) \mapsto \mathbb{E}[u(t,x,\tau))]$ is continuous. Moreover clearly[8]

$$v(x,t) = \sup_{\tau\in[0,T]} \mathbb{E}[u(t,x,\tau)] \tag{19}$$

The assertion follows by the well-know fact that the supremum of lsc functions is lsc again.

2.3 Existence of an optimal stopping time

According to the Markovian approach to optimal stopping problems we define the *continuation set*

$$\begin{aligned} C &:= \{(t,x) \in [0,T) \times (0,\infty) : V(t,x) > G(t,x)\} \\ &= \{(t,x) \in [0,T) \times (0,\infty) : v(t,x) > g(x)\} \end{aligned} \tag{20}$$

and the *stopping set*

$$\begin{aligned} D &:= \{(t,x) \in [0,T] \times (0,\infty) : V(t,x) = G(t,x)\} \\ &= \{(t,x) \in [0,T] \times (0,\infty) : v(t,x) = g(x)\} \end{aligned} \tag{21}$$

Note that D is closed since v is lsc by Lemma 3 and g is continuous. Moreover we define the stopping time[9]

$$\tau_D := \inf\{s \geq 0 : X_s \in D\} \tag{22}$$

Lemma 4. *All points $(t,x) \in [0,T) \times [K,\infty)$ belong to the continuation set C.*

[7] Again we require $t < \infty$
[8] Note $\{\tau \wedge (T-t) : \tau \in [0,T]\} = \{\tau : \tau \in [0,T-t]\}$
[9] This is indeed a stopping time since D is closed and X is continuous.

Proof. Let $(t, x) \in [0, T) \times [K, \infty)$ and $0 < \epsilon < K$. Define the stopping time[10]

$$\tau_\epsilon := \inf\{s \geq 0 : S_s \leq K - \epsilon\} \wedge (T - t) \tag{23}$$

It is not difficult to show that $\mathbb{P}_{(t,x)}(0 < \tau_\epsilon < T - t) =: \alpha > 0$. Hence we have $V(t, x) \geq \alpha e^{-rT} \epsilon > 0 = G(t, x)$, which establishes the claim.

Now define $w(t, x) = v(x, t) + x$. Lemma 4 implies

$$C = \{(t, x) \in [0, T) \times (0, \infty) : w(t, x) > K\} \tag{24}$$

$$D = \{(t, x) \in [0, T) \times (0, \infty) : w(t, x) = K\} \cup \{T\} \times (0, \infty) \tag{25}$$

Lemma 5. *The function $x \mapsto w(t, x)$ is convex and increasing. Moreover* $\lim_{x \downarrow 0} w(t, x) = K$.

Proof. Convexity follows from convexity of $x \mapsto v(t, x)$ and $x \mapsto x$. The obvious inequality $(K - x)^+ + x \leq w(t, x) \leq K + x$, implies $K \leq w(t, x)\ \forall x \in (0, \infty)$ as well as $\lim_{x \downarrow 0} w(t, x) = K$. These two facts together with convexity of $x \mapsto w(t, x)$ imply immediately that $x \mapsto w(t, x)$ is increasing.

Lemma 4 and 5 imply that there exist a function $b : [0, T) \to [0, K)$ such that

$$C = \{(t, x) \in [0, T) \times (0, \infty) : x > b(t)\} \tag{26}$$

$$D = \{(t, x) \in [0, T) \times (0, \infty) : x \leq b(t)\} \cup \{T\} \times (0, \infty) \tag{27}$$

2.3.1 Infinite time horizon

For convenience set $v(x) := v(0, x)$. For $0 < b < K$ define the stopping time $\tau_b = \inf\{s \geq 0 : S(s) \leq b\}$ and let

$$v_b(x) := \mathbb{E}_x[e^{-r\tau_b}(K - S(\tau_b))^+] \tag{28}$$

The formula for the Laplace transform for the first passage time of a Brownian motion with drift[11] yields after some simple calculations

$$v_b(x) = \begin{cases} K - x & \text{if } 0 < x \leq b \\ (K - b)\left(\frac{x}{b}\right)^{-2r/\sigma^2} & \text{if } x \geq b \end{cases} \tag{29}$$

Define $v^*(x) = \sup_{b \in (0,K)} v_b(x)$. Elementary Calculus yields

$$v^*(x) = v_{b^*}(x) = \begin{cases} K - x & \text{if } 0 < x \leq b^* \\ (K - b^*)\left(\frac{x}{b^*}\right)^{-2r/\sigma^2} & \text{if } x \geq b^* \end{cases} \tag{30}$$

where $b^* = \frac{2r}{2r + \sigma^2} K$. It is straightforward to check that $v^* \in C^1((0, \infty))$ and $v^* \in C^2((0, b) \cup (b, \infty))$ with

$$v^*_x(x) = \begin{cases} -1 & \text{if } 0 < x \leq b^* \\ \frac{-2r}{\sigma^2 x} v^*(x) & \text{if } x \geq b^* \end{cases} \tag{31}$$

$$v^*_{xx}(x) = \begin{cases} 0 & \text{if } 0 < x < b^* \\ \frac{2r(2r + \sigma^2)}{\sigma^4 x^2} v^*(x) & \text{if } x > b^* \end{cases} \tag{32}$$

Define $V^*(t, x) = e^{-rt} v^*(x)$.

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[10] This is again a stopping time since $[0, K - \epsilon]$ is closed and S is continuous.

[11] See for instance [8] p 346 et seq (Theorem 8.3.2)

Theorem 3. $v^*(x) = v(x)$ *for* $x \in (0,\infty)$*. Moreover* τ_{b^*} *is the optimal stopping time for the Perpetual American Put.*

Proof. Since $V^* \in C^{1,1}(E) \cup C^{1,2}(E \backslash ([0,\infty) \times b))$ and $\mathbb{P}_x(S(t) = b^*) = 0$ for all $x \in (0,\infty)$ and all $t > 0$, we can apply a slightly generalized version of Itô's formula[12] to $V^*(t, S(t))$ and get

$$\begin{aligned} \mathrm{d}V^*(t,S(t)) &= -rV^*(t,S(t))\,\mathrm{d}t + V_x^*(t,S(t))\,\mathrm{d}S(t) \\ &\quad + \frac{1}{2} V_{xx}^*(t,S(t)) \mathbb{1}_{\{S(t)\neq b\}}\,\mathrm{d}\langle S(t), S(t)\rangle \\ &= -\mathrm{e}^{-rt} rK \mathbb{1}_{\{S(t)<b^*\}}\,\mathrm{d}t + \sigma S(t) V_x^*(t,S(t))\,\mathrm{d}W(t) \end{aligned} \tag{33}$$

Hence $V^*(t,S(t))$ is a $\{\mathscr{F}_t\}$-supermartingale[13] with $V^*(t,S(t)) \geq G(t,S(t))$[14]. Let $\tau \in [0,\infty]$ be a stopping time. Monotonicity of the integral and the optional sampling theorem yield

$$\mathbb{E}_x[G(\tau, S(\tau))] \leq \mathbb{E}_x[V^*(\tau, S(\tau))] \leq V^*(0,x) = v^*(x) \tag{34}$$

Taking the supremum in (34) over $\tau \in [0,\infty]$ yields $v^*(x) \geq v(x)$. On the other hand $v^*(x) \leq v(x)$ by definition. Hence

$$v^*(x) = v(x) = \mathbb{E}_x[v^*(\tau_b^*, S(\tau_b^*))] \tag{35}$$

q.e.d.

2.3.2 Finite time horizon

Since V is lsc by lemma 3 and G is continuous, τ_D is optimal in (12), since $\mathbb{P}_{t,x}(\tau_D < \infty) = 1$ by the main existence theorem of the Markovian approach (Theorem 3.7 of the lecture notes).

2.4 Elementary properties of b for finite time horizon

Lemma 6. *The function* b *is increasing with* $b^* \leq b(t) < K$*.*

Proof. Let $0 \leq t_1 < t_2 < T$. By Lemma 1 and the definitions of the functions v, g and b we have

$$g(b(t_1)) = v(t_1, b(t_1)) \geq v(t_2, b(t_1)) \geq g(b(t_1)) \tag{36}$$

Therefore $(t_2, b(t_1)) \in D$, which implies $b(t_2) \geq b(t_1)$. Moreover let $x \leq b^*$. Then by Theorem 3

$$v(0,x) \leq \sup_{\tau \in [0,\infty]} \mathbb{E}_x[\mathrm{e}^{-r\tau}(K - S(\tau))^+] = K - x = g(x) \tag{37}$$

whence $(0,x) \in D$, which implies $b(t) \geq b(0) \geq b^*$. Finally, Lemma 4 implies $b(t) < K$.

[12] confer [6] p 74 et seq

[13] $\int_0^t \sigma S(s) V_x^*(s,S(s))\,\mathrm{d}W(s)$ is a proper martingale since $|V_x^*(s,S(s))| \leq \mathrm{e}^{-rs} \leq 1$.

[14] Note that $v^*(x) \geq (K-x)^+$ for $x \in (0,\infty)$.

2.5 Further properties of the value* function

Lemma 7. *The function* $x \mapsto v(t,x)$ *is decreasing and strictly decreasing for* $x \in (0,K]$*. Moreover* $\lim_{x\downarrow 0} v(t,x) = K$ *and* $\lim_{x\to\infty} v(t,x) = 0$.

Proof. The claim is trivial for $t = T$, so assume $t < T$. Lemma 5 implies $\lim_{x\downarrow 0} v(t,x) = \lim_{x\downarrow 0} w(t,x) = K$. Moreover by (32) for $x \geq K$

$$\begin{aligned} v(x,t) &= \sup_{\tau\in[0,T-t]} \mathbb{E}_x[\mathrm{e}^{-r\tau}(K - S(\tau))^+] \\ &\leq \sup_{\tau\in[0,\infty]} \mathbb{E}_x[\mathrm{e}^{-r\tau}(K - S(\tau))^+] \\ &= (K - b^*)\left(\frac{x}{b^*}\right)^{-2r/\sigma^2} \end{aligned} \tag{38}$$

which implies $\lim_{x\to\infty} v(t,x) = 0$. Since $0 \leq v(t,x) \leq K$ by definition and $x \mapsto v(t,x)$ is convex by Lemma 2, $x \mapsto v(t,x)$ is decreasing. Moreover clearly $v(t,x) > 0$ for $x < K$. Again convexity of $x \mapsto v(t,x)$ implies that $x \mapsto v(t,x)$ is strictly decreasing for $x \in (0,K]$.

Lemma 8. *The function* v *is continuous in* E*.*

Proof. For $t \geq 0$ define $M(t) := \sup_{0\leq s\leq t} |W(s)|$. Fix $x \in (0,\infty)$ and let $0 \leq t_1 < t_2 \leq T$. Denote by τ_1 the $\{\mathscr{G}_s\}$-optimal stopping time for $v(t_1,x)$ and define $\tau_2 := \tau_1 \wedge (T - t_2)$. Clearly $\tau_1 \geq \tau_2$ with $\tau_1 - \tau_2 \leq t_2 - t_1$. By stationary and independent increments $\{W(\tau_2 + t) - W(\tau_2)\}_{t\geq 0}$ is independent of $\mathscr{G}_{\tau_2}$ and equal in law to $\{W(t)\}_{t\geq 0}$. Recalling that $\mathrm{e}^{-rt}S(t)$ is a martingale and $v(t_1,x) \geq v(t_2,x)$ we get

$$\begin{aligned} 0 &\leq v(t_1,x) - v(t_2,x) \\ &\leq \mathbb{E}_x[\mathrm{e}^{-r\tau_1}(K - S(\tau_1))^+] - \mathbb{E}_x[\mathrm{e}^{-r\tau_2}(K - S(\tau_2))^+] \\ &\leq \mathbb{E}_x[\mathrm{e}^{-r\tau_2}[(K - S(\tau_1))^+ - (K - S(\tau_2))^+]] \\ &\leq \mathbb{E}_x[\mathrm{e}^{-r\tau_2}(S(\tau_2) - S(\tau_1))^+] \\ &\leq \mathbb{E}_x[\mathrm{e}^{-r\tau_2}S(\tau_2)(1 - \mathrm{e}^{\sigma(W(\tau_1)-W(\tau_2))+(r-\sigma^2/2)(\tau_1-\tau_2)})^+] \\ &\leq \mathbb{E}_x[\mathrm{e}^{-r\tau_2}S(\tau_2)\mathbb{E}[(1 - \mathrm{e}^{\sigma(W(\tau_1)-W(\tau_2))+(r-\sigma^2/2)(\tau_1-\tau_2)})^+|\mathscr{G}_{\tau_2}]] \\ &\leq \mathbb{E}[\mathrm{e}^{-r\tau_2}S(\tau_2)\mathbb{E}[(1 - \mathrm{e}^{\sigma W(\tau_1-\tau_2)+(r-\sigma^2/2)(\tau_1-\tau_2)})^+]] \\ &\leq x\,\mathbb{E}[(1 - \mathrm{e}^{-\sigma M(t_2-t_1)-|(r-\sigma^2/2)|(t_2-t_1)})^+] \end{aligned} \tag{39}$$

Define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(t) := \mathbb{E}[(1 - \mathrm{e}^{-\sigma M(|t|)-|(r-\sigma^2/2)||t|})^+]$. By dominated convergence h is continuous at 0 with $h(0) = 0$. Now fix $(t_0,x_0) \in E$ and let $\{(t_n,x_n)\}_{n\geq 1}$ be a sequence in E with $\lim_{n\to\infty}(t_n,x_n) = (t_0,x_0)$. Then by Lemma 2 and (39) we have

$$\begin{aligned} \limsup_{n\to\infty} |v(t_n,x_n) - v(t_0,x_0)| &\leq \limsup_{n\to\infty} |v(t_n,x_n) - v(t_0,x_n)| \\ &\quad + \limsup_{n\to\infty} |v(t_0,x_n) - v(t_0,x_0)| \\ &\leq \limsup_{n\to\infty} x_n h(t_n - t_0) + 0 = 0 \end{aligned} \tag{40}$$

Hence v is continuous in E.

Lemma 9. *The function v is $C^{1,2}$ in C and satisfies there $v_x \leq 0$ and $v_t \leq 0$ as well as $v_{xx}(t,x) \geq \frac{2r}{\sigma^2 x^2} v(x,t)$.*

Proof. Denote by L_X the infinitesimal generator of X. It is not difficult to establish that

$$L_X = \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2} \tag{41}$$

Now fix $(t_0, x_0) \in C$ and let $r_0 > 0$ such that $\overline{B} := \overline{B}_{r_0}(t_0, x_0) \subset C$. Now consider the following PDE

$$\begin{aligned} L_X U &= 0 \quad \text{in } B \\ U &= V \quad \text{on } \partial B \end{aligned} \tag{42}$$

Since V is continuous by Lemma 8, standard PDE results[15] state, that there exist a unique solution U of (42) in $C^{1,2}(B) \cap C^0(\overline{B})$. Let $(t,x) \in B$ and $\epsilon > 0$ be arbitrary. Since $\overline{B}$ is compact and $U, V \in C^0(\overline{B})$ with $U = V$ on ∂B there exists $0 < r_1 < r_0$ such that $(t,x) \in B_* := B_r(t_0, x_0)$ and $|U - V| \leq \epsilon$ on ∂B_*. Let $U^* : E \mapsto \mathbb{R}$ be a $C^{1,2}$-extension of $U|_{\overline{B}_*}$ [16]. Now applying Itô's formula to $U^*(X_s)$ yields

$$\mathrm{d}U^*(X_s) = L_X U^*(X_s)\,\mathrm{d}s + \sigma S(s) U_x^*(X_s)\,\mathrm{d}W(s) \tag{43}$$

Define the stopping time

$$\tau_{B_*^c} := \inf\{s \geq 0 : X_s \in B_*^c\} \tag{44}$$

Note that $L_X = 0$ in $\overline{B}_*$ by (42). Hence (43) and the Optional Sampling theorem yield[17]

$$U(t,x) = U^*(t,x) = \mathbb{E}_{(t,x)}[U^*(X_{\tau_{B_*^c}})] = \mathbb{E}_{(t,x)}[U(X_{\tau_{B_*^c}})] \tag{45}$$

On the other hand, since $\tau_{B_*^c} \leq \tau_D$ we get by the Strong Markov property

$$\begin{aligned} \mathbb{E}_{(t,x)}[V(X_{\tau_{B_*^c}})] &= \mathbb{E}_{(t,x)}[\mathbb{E}_{X_{\tau_{B_*^c}}}[G(X_{\tau_D})]] \\ &= \mathbb{E}_{(t,x)}[\mathbb{E}_{(t,x)}[G(X_{\tau_{B_*^c}+\tau_D})|\mathscr{F}_{\tau_{B_*^c}}]] \\ &= \mathbb{E}_{(t,x)}[G(X_{\tau_{B_*^c}+\tau_D})] = \mathbb{E}_{(t,x)}[G(X_{\tau_D})] \\ &= V(t,x) \end{aligned} \tag{46}$$

Putting (45) and (46) together yields

$$\begin{aligned} |V(t,x) - U(t,x)| &= |\mathbb{E}_{(t,x)}[V(X_{\tau_{B_*^c}}) - U(X_{\tau_{B_*^c}})]| \\ &\leq \mathbb{E}_{(t,x)}[|V(X_{\tau_{B_*^c}}) - U(X_{\tau_{B_*^c}})|] \\ &\leq \mathbb{E}_{(t,x)}[\,\epsilon\,] = \epsilon \end{aligned} \tag{47}$$

Since $\epsilon > 0$ was arbitrary, we get $V(t,x) = U(t,x)$. Thus $V = U$ in $\overline{B}$, which in particular implies that V is $C^{1,2}$ in (t_0, x_0). Since $(t_0, x_0) \in C$ was chosen

[15] See for instance [3] for a proof

[16] More precisely $U^* \in C^{1,2}(E)$ and $U^* = U$ on $\overline{B}_*$.

[17] Note that a priory we only know that $\int_0^t \sigma S(s) U_x^*(X_s)\,\mathrm{d}W(s)$ is local martingale. Therefore we use a localizing sequence $\{\tau_n\}_{n\geq 0}$ of stopping times and observe that U_x is bounded on $\overline{B}_*$. The dominated convergence theorem then yields the desired result.

arbitrary, it follows that V and therefore also v are in $C^{1,2}$ in C. By Lemma 1 and Lemma 7 we get that $v_x \leq 0$ and $v_t \leq 0$. Moreover, an easy calculation using $L_X V = 0$ in C yields

$$\begin{aligned} v_{xx}(t,x) &= \frac{2}{\sigma^2 x^2}(rv(t,x) - v_t(t,x) - v_x(t,x)) \\ &\geq \frac{2r}{\sigma^2 x^2} v(t,x) \end{aligned} \tag{48}$$

Lemma 10. *The function $x \mapsto v(t,x)$ is differentiable at $b(t)$ with $v_x = g_x$.*

Proof. Fix $t^* \in [0,T)$. Since $x \mapsto v(t^*,x)$ is convex by Lemma 2, the right-hand derivative $\frac{\partial^+ v}{\partial x}(t^*,x)$ exist for all $x \in (0,\infty)$. Denote $x^* = b(t^*) < K$. Then

$$\begin{aligned} \frac{\partial^+ v}{\partial x}(t^*,x^*) &= \lim_{\epsilon \downarrow 0} \frac{v(t^*,x^*+\epsilon) - v(t^*,x^*)}{\epsilon} \\ &\geq \lim_{\epsilon \downarrow 0} \frac{g(x^*+\epsilon) - g(x^*)}{\epsilon} = -1 \end{aligned} \tag{49}$$

Define $\tau_{x^*} := \inf\{s \geq 0 : S(s) \leq x^*\}$ and denote by τ^ξ the $\{\mathscr{G}_s\}$-optimal stopping time for $v(t^*, x^*+\xi)$ for $\xi \geq 0$. Since b is increasing by lemma 6, clearly $\tau^\xi \leq \tau_{x^*}$ under $\mathbb{P}_{(t^*,x^*+\xi)}$ for all $\xi \geq 0$. Moreover by (29) we have

$$\begin{aligned} \liminf_{\xi \downarrow 0} \mathbb{E}[e^{-r\tau^\xi}] &\geq \liminf_{\xi \downarrow 0} \mathbb{E}_{(t^*,x^*+\xi)}[e^{-r\tau_{x^*}}] \\ &= \lim_{\xi \downarrow 0} \left(\frac{x^*+\xi}{x^*} \right)^{-2r/\sigma^2} = 1 \end{aligned} \tag{50}$$

This implies that for any sequence $\{\xi_n\}_{n \geq 1}$ in $\mathbb{R}^+$ with $\lim_{n\to\infty} \xi_n = 0$ we have $\lim_{n\to\infty} \tau^{\xi_n} = 0$ in probability. For $t \geq 0$ define $M(t) := \sup_{0 \leq s \leq t} |W(s)|$ and for convenience set $\Sigma(t) := e^{\sigma W(t) + (r-\sigma^2/2)t}$ and $\Theta^\pm(t) := e^{\pm \sigma M(t) \pm |(r-\sigma^2/2)|t}$. Let $\epsilon > 0$ be arbitrary and $\{\xi_n\}_{n\geq 1}$ a sequence in $\mathbb{R}^+$ with $\lim_{n\to\infty} \xi_n = 0$. After possibly discarding a subsequence we may assume that $\lim_{n\to\infty} \tau^{\xi_n} = 0$ a.s. Then it holds

$$\begin{aligned} &\frac{\partial^+ v}{\partial x}(t^*,x^*) = \limsup_{n\to\infty} \frac{v(t^*,x^*+\xi_n) - v(t^*,x^*)}{\xi_n} \\ &\leq \limsup_{n\to\infty} \mathbb{E}[e^{-r\tau^{\xi_n}}((K-(x^*+\xi_n)\Sigma(\tau^{\xi_n}))^+ - (K - x^*\Sigma(\tau^{\xi_n}))^+)]/\xi_n \\ &\leq \limsup_{n\to\infty} \mathbb{E}[e^{-r\tau^{\xi_n}}(-\Sigma(\tau^{\xi_n}))\mathbb{1}_{\{(x^*+\xi_n)\Sigma(\tau^{\xi_n})<K\}}\mathbb{1}_{\{\tau^{\xi_n}<\epsilon\}}] \\ &\leq e^{-r\epsilon} \limsup_{n\to\infty} \mathbb{E}[-\Theta^-(\epsilon)\mathbb{1}_{\{(x^*+\xi_n)\Theta^+(\epsilon)<K\}}\mathbb{1}_{\{\tau^{\xi_n}<\epsilon\}}] \\ &= -e^{-r\epsilon}\mathbb{E}[\Theta^-(\epsilon))\mathbb{1}_{\{x^*\Theta^+(\epsilon)<K\}}] \end{aligned} \tag{51}$$

Letting $\epsilon \downarrow 0$ in (51) we get by dominated convergence[18]

$$\frac{\partial^+ v}{\partial x}(t^*,x^*) \leq -1 \tag{52}$$

[18] Clearly $\lim_{\epsilon\downarrow 0} \Theta^\pm(\epsilon) = 1$; recall moreover that $x^* < K$

which together with (49) implies $\frac{\partial^+ v}{\partial x}(t^*, x^*) = -1$. Finally we have

$$\begin{aligned}\frac{\partial^- v}{\partial x}(t^*, x^*) &= \lim_{\epsilon \downarrow 0} \frac{v(t^*, x^* - \epsilon) - v(t^*, x^*)}{-\epsilon} \\ &= \lim_{\epsilon \downarrow 0} \frac{g(x^* - \epsilon) - g(x^*)}{-\epsilon} = -1 \end{aligned} \tag{53}$$

Hence$x \mapsto v(t, x)$ is differentiable at $b(t)$ with $v_x = g_x$ (*smooth fit*).

Lemma 11. *The function $x \mapsto v(t, x)$ is C^1 with $-1 \leq v_x(t, x) \leq 0$.*

Proof. Fix $t^* \in [0, T)$. For $x > b(t^*)$ the assertion follows by lemma 7; for $x < b(t^*)$ this follows by the fact that $v(t, x) = g(x)$ in $(0, b(t^*)]$ and clearly $g(x) \in C^1((0, b(t^*)])$. Now let $x = b(t^*)$. Since $x \mapsto v(t^*, x)$ is differentiable at $b(t^*)$ by lemma 10 with $v_x(t^*, b(t^*)) = -1 = \lim_{x \uparrow b(t^*)} v_x(t^*, x)$, it remains to show that

$$\lim_{x \downarrow b(t^*)} v_x(t^*, x) = -1 \tag{54}$$

Since $x \mapsto v(t^*, x)$ is differentiable we clearly have

$$v_x(t^*, x) = \frac{\partial^+ v}{\partial x}(t^*, x) \tag{55}$$

Since $x \mapsto v(t^*, x)$ is convex, the function $x \mapsto \frac{\partial^+ v}{\partial x}(t^*, x)$ is right-continuous[19]. This fact together with (55) immediately establishes (54). Hence $x \mapsto v(t^*, x)$ is C^1. Finally, clearly $v_x(t^*, x) = -1$ for $x \in (0, b(t^*)]$ and hence by continuity of $x \mapsto v_x(t^*, x)$, convexity of $x \mapsto v(t^*, x)$ and lemma 9 we get $-1 \leq v_x(t^*, x) \leq 0$ for $x \in (0, b(t^*)]$.

2.6 Further properties of b for finite time horizon

Lemma 12. *The function b is continuous with $\lim_{t \uparrow T} b(t) = K$.*

Proof.

- Right-continuity: Let $t \in [0, T)$. Since b is increasing by Lemma 6, the right-hand limit $b(t+)$ exists with $b(t) \leq b(t+) < K$. Moreover by definition $(t, b(t)) \in D$ for $t \in [0, T)$. Since D is closed, it follows that $(t, b(t+)) \in D$. This together with Lemma 7 implies

$$\begin{aligned} 0 &\leq v(t, b(t+)) - v(t, b(t)) \\ &= (K - b(t+)) - (K - b(t)) \\ &= b(t) - b(t+) \leq 0 \end{aligned} \tag{56}$$

  Hence $b(t) = b(t+)$ and $t \mapsto b(t)$ is right-continuous.

- Left-continuity: Let $t \in (0, T]$. For convenience set $b(T) := K$. Since b is increasing, the left-hand limit $b(t-)$ exists with $b(t-) \leq b(t) \leq K$[20]. Moreover $(t, b(t)) \in D$ for $t \in [0, T)$. Since D is closed, it follows that

[19] For a proof see [5] p 142 et seq (Satz 7.7 iv).

[20] Recall that $b(t) < K$ for $t \in [0, T)$.

$(t, b(t-)) \in D$. Seeking a contradiction, suppose that $b(t-) < b(t)$. Set $x^* := (b(t-) + b(t))/2$ and let $t' < t$[21]. Then

$$b(t') \leq b(t-) < x^* < b(t) \leq K \tag{57}$$

which implies in particular that $(b(t'), x^*) \subset C$ and $(t, x^*) \in D$. By definition of C and Lemma 10 we have

$$\begin{aligned} v(t', b(t')) - g(b(t') &= 0 \\ v_x(t', b(t')) - g_x(b(t') &= 0 \end{aligned} \tag{58}$$

Finally, by Lemma 9 we have for $x \in (b(t'), x^*)$

$$\begin{aligned} v_{xx}(t', x) &\geq \frac{2r}{\sigma^2 x^2} v(t', x) \geq \frac{2r}{\sigma^2 x^2}(K - x) \\ &\geq \frac{2r}{\sigma^2 b(t')^2}(K - x^*) =: \gamma > 0 \end{aligned} \tag{59}$$

A double application of the Fundamental Theorem of Calculus together with (58) yields

$$\begin{aligned} v(t', x^*) - g(x^*) &= \int_{b(t')}^{x^*} (v_x(t', y) - g_x(y))\,\mathrm{d}y \\ &= \int_{b(t')}^{x^*} \int_{b(t')}^{y} (v_{xx}(t', z) - g_{xx}(z))\,\mathrm{d}z\,\mathrm{d}y \\ &\geq \int_{b(t')}^{x^*} \int_{b(t')}^{y} \gamma\,\mathrm{d}z\,\mathrm{d}y = \gamma \frac{(x^* - b(t'))^2}{2} \end{aligned} \tag{60}$$

Taking the limit $t' \uparrow t$ and using that v is continuous yields

$$v(t, x^*) - g(x^*) \geq \gamma \frac{(x^* - b(t-))^2}{2} > 0 \tag{61}$$

Hence $(t, x^*) \notin D$ in contradiction to $(t, x^*) \in D$. Thus $b(t) = b(t-)$ and $t \mapsto b(t)$ is left-continuous with $\lim_{t \uparrow T} b(t) = K$.

Lemma 13. *The function $t \mapsto b(t)$ is convex and satisfies*

$$\lim_{t \uparrow T} \frac{\log(b(t)/K)}{\sigma\sqrt{(T - t)(-\log(8\pi r^2(T - t)/\sigma^2))}} = 1 \tag{62}$$

Proof. See [2].

Remark. This result will not be used in the following.

2.7 Early exercise premium representation

The above lemmata imply[22] that $V \in C^{0,1}(E \backslash \{T\} \times \{K\}) \cap C^{1,2}(E \backslash \Gamma(b(t)))$[23]. Moreover, since $\mathbb{P}_{(t,x)}(X(s) = b(t + s)) = 0$ for all $(t, x) \in (0, T) \times (0, \infty)$ and

[21] Note that $t' < T$.

[22] Note that clearly $V \in C^{1,2}(\mathrm{Int}\{D\})$.

[23] $\Gamma(b(t)) := \{(t, x) \in [0, T] \times (0, \infty) : x = b(t)\}$

all $0 < s < T-t$, we can apply a slightly generalized version of Itô's formula[24] to $V(X_s)$ and get

$$\begin{aligned} \mathrm{d}V(X_s) &= L_X V(X_s)\mathbb{1}_{\{X(s)\neq b(t+s)\}}\,\mathrm{d}s + V_x(X(s))\,\mathrm{d}S(t) \\ &= -\mathrm{e}^{-rs} rK\mathbb{1}_{\{S(s)<b(t+s)\}}\,\mathrm{d}t + \sigma S(t) V_x(X_s)\,\mathrm{d}W(t) \end{aligned} \tag{63}$$

From (63) we get immediately[25] for $(t,x)\in E$

$$E_{(t,x)}[V(X(T-t))] = V(t,x) - rK\int_0^{T-t} \mathrm{e}^{-r(t+s)}\,\mathbb{P}_{(t,x)}(S(s) < b(t+s))\,\mathrm{d}s \tag{64}$$

By the Markov Property using $(T-t)+\tau_D = T-t$ under $\mathbb{P}_{(t,x)}$ we have

$$\begin{aligned} E_{(t,x)}[V(X(T-t))] &= E_{(t,x)}[E_{X(T-t)}[G(X(\tau_D))]] \\ &= E_{(t,x)}[E_{(t,x)}[G(X((T-t)+\tau_D))|\mathscr{F}_{T-t}]] \\ &= E_{(t,x)}[G(X((T-t)+\tau_D))] \\ &= E_{(t,x)}[G(X((T-t)))] \end{aligned} \tag{65}$$

multiplying (64) with e^{-rt} yields using (65)

$$\begin{aligned} v(t,x) &= \mathrm{e}^{-r(T-t)} E_{(t,x)}[g(S(T-t))] \\ &\quad + rK\int_0^{T-t} \mathrm{e}^{-rs}\,\mathbb{P}_{(t,x)}(S(s) < b(t+s))\,\mathrm{d}s \end{aligned} \tag{66}$$

Plugging in $t=0$ in (66) yields after some algebra

$$\begin{aligned} V(0,x) &= E_x[\mathrm{e}^{-rT}(K-S(T))^+] \\ &\quad + rK\int_0^T \mathrm{e}^{-rs}\,\Phi\left(\frac{1}{\sigma\sqrt{s}}\left(\log\left(\frac{b(s)}{b(0)}\right) - \left(r-\frac{\sigma^2}{2}\right)s\right)\right)\mathrm{d}s \end{aligned} \tag{67}$$

where Φ denotes the cdf of a standard normal.

Remark. Formula (67) is called the *early exercise premium representation* of the value function. It shows that the value of an American put option with strike price K and maturity T is the sum of the value of an European put option with the same strike and maturity and the so-called *early exercise premium.*

2.8 Free boundary equation for b(t)

Theorem 4. *The function $t \mapsto b(t)$ is the unique solution in the class of continuous increasing functions $c : [0,T] \to \mathbb{R}$ satisfying $0 < c(t) < K$ for all $0 < t < \infty$ of the following* free-boundary integral equation

$$\begin{aligned} K - b(t) &= \mathrm{e}^{-r(T-t)}\int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-x}{b(t)}\right) - \left(r-\frac{\sigma^2}{2}\right)(T-t)\right)\right)\mathrm{d}x \\ &\quad + rK\int_0^{T-t} \mathrm{e}^{-rs}\,\Phi\left(\frac{1}{\sigma\sqrt{s}}\left(\log\left(\frac{b(t+s)}{b(t)}\right) - \left(r-\frac{\sigma^2}{2}\right)s\right)\right)\mathrm{d}s \end{aligned} \tag{68}$$

[24] confer [6] p 74 et seq

[25] Note that $\int_0^t \sigma S(s) V_x(s,S(s))\,\mathrm{d}W(s)$ is a proper zero-mean martingale since $|V_x(X_s)| \leq \mathrm{e}^{-rt} < 1$.

Proof.

- $t \mapsto b(t)$ is a solution of (68): Plugging $(t, b(t))$ in (66) and noting that $v(t, b(t)) = K - b(t)$ yields after some lengthy calculation (68).
- $t \mapsto b(t)$ is the unique solution of (68): See [6] p 386 - 392.

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