Chapter 3

Control Theory

1 Introduction

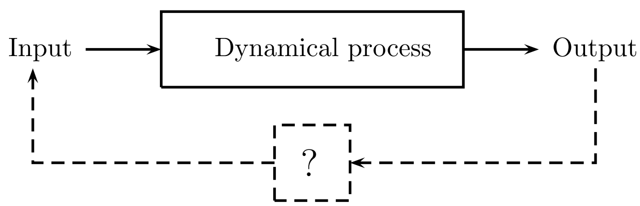
To control an object is to influence its behaviour so as to achieve some desired objective.

The object to be controlled is usually some dynamic process (a process evolving in time), for example, the UK economy, illness/disease in a patient, air temperature in this lecture theatre.

The controls are inputs to the process which can influence its evolution (for example, UK interest rates, medication in patient treatment, radiators in a heating system).

The evolution of the process may be monitored by observations or outputs (for example, UK inflation, medical biopsies, temperature sensors in a heating system).

A central concept in control theory is that of feedback: the use of information available from the observations or outputs to generate the controls or inputs so as to cause the process to evolve in some desirable manner.



Much of control theory is focussed on the question $\boxed{?}$.

We start with a mathematical description of the dynamic process linking the output to the input.

2 Transfer functions

We restrict attention to linear, autonomous, single-input, single-output systems of the form

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = \xi, \quad A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n \tag{1}$$

$$y(t) = c^T x(t) + du(t), \quad c \in \mathbb{R}^n,\ d \in \mathbb{R}\,, \tag{2}$$

where u is a scalar-valued input and y is a scalar-valued output. The variable $x(t)$ is referred to as the *state* of the system. By the variation of parameters formula, the solution of (1) is

$$t \mapsto x(t) = (\exp At)\xi + \int_0^t (\exp A(t-\tau))bu(\tau)\,\mathrm{d}\tau\,.$$

Therefore, the system output (2) is given by

$$t \mapsto y(t) = c^T(\exp At)\xi + \int_0^t c^T(\exp A(t-\tau))bu(\tau)\,\mathrm{d}\tau + du(t).$$

Taking Laplace transform,

$$\hat{y}(s) = c^T(sI-A)^{-1}\xi + \big(c^T(sI-A)^{-1}b + d\big)\hat{u}(s).$$

With an initial value $\xi = 0$, we have

$$\hat{y}(s) = G(s)\hat{u}(s), \quad G(s) := c^T(sI-A)^{-1}b + d. \tag{3}$$

Thus, with zero initial data, the function G relates the (transformed) input $\hat{u}$ to the (transformed) output $\hat{y}$. G is the *transfer function* of the system.

Recalling that the Dirac delta function δ has Laplace transform 1 and writing $f(t) = c^T(\exp At)b$, we see that

$$G(s) = \mathfrak{L}\{f\}(s) + d\mathfrak{L}\{\delta\}(s) = \mathfrak{L}\{f + d\delta\}(s)$$

and so G is the Laplace transform of the function

$$t \mapsto g(t) := c^T(\exp At)b + d\delta(t), \tag{4}$$

and, for $\xi = 0$, $y = g \star u$. The function g is referred to as the *impulse response* of system (1-2), so called because the output response of the system with $\xi = 0$ and impulsive input $u(t) = \delta(t)$ is given by

$$y(t) = (g \star \delta)(t) = g(t).$$

Recalling that an invertible matrix M has inverse given by

$$M^{-1} = \frac{1}{|M|}\mathrm{adj}\, M, \quad \text{where adj}\, M \text{ is the } \textit{adjugate} \text{ of } M,$$

we have

$$G(s) = \frac{1}{|sI - A|}\left(c^T \mathrm{adj}(sI - A)b + d|sI - A|\right).$$

Since $|sI - A|$ is a polynomial and the entries of $\mathrm{adj}(sI - A)$ are also polynomials, we see that the transfer function G is a *rational function*, that is a ratio of two polynomial functions:

$$G(s) = \frac{N(s)}{D(s)}, \quad N \text{ and } D \text{ polynomial.}$$

Moreover, since d and the entries of A, b and c are real numbers, the polynomials N and D have real coefficients.

Some notation and terminology:

$\mathbb{R}[s]$ denotes the set of polynomials in the complex variable s with real coefficients.

$\mathbb{R}(s) := \{P/Q |\ P, Q \in \mathbb{R}[s],\ Q \neq 0\}$, the set of rational functions with real coefficients.

A rational function $R = P/Q \in \mathbb{R}(s)$ is said to be (a) *proper* if $\deg(P) \leq \deg(Q)$ and (b) *strictly proper* if $\deg(P) < \deg(Q)$.

Remarks.
(i) $R \in \mathbb{R}(s)$ is proper if, and only if, $\lim_{|s|\to\infty} |R(s)| < \infty$;
(ii) $R \in \mathbb{R}(s)$ is strictly proper if, and only if, $\lim_{|s|\to\infty} |R(s)| = 0$.

Examples.

$$\frac{s^2 + 1}{2s^2 + 5} \text{ is proper,} \qquad \frac{s}{3s^2 + 4} \text{ is strictly proper,} \qquad \frac{s^3 + 7}{s^2 + 2} \text{ is } \textit{not} \text{ proper.}$$

The transfer function $G(s) = c^T(sI - A)^{-1}b + d$ is such that

$$\lim_{|s|\to\infty} G(s) = \lim_{|s|\to\infty} \left(\frac{1}{s}c^T\left(I - \frac{1}{s}A\right)^{-1}b + d\right) = d$$

and so is proper. Moreover, G is strictly proper if, and only if, $d = 0$.

2.1 Poles and zeros

A number $z \in \mathbb{C}$ is a *zero* of $R \in \mathbb{R}(s)$ if $\lim_{s \to z} R(s) = 0$. A number $p \in \mathbb{C}$ is a *pole* of $R \in \mathbb{R}(s)$ if $|R(s)| \to \infty$ as $s \to p$.

Examples.

(i) If $R(s) = \dfrac{s-1}{s-2}$, then 1 is a zero and 2 is a pole.

(ii) If $R(s) = \dfrac{s-1}{s^2 - 3s + 2}$, then

$$R(s) = \frac{s-1}{(s-1)(s-2)} = \frac{1}{s-2}$$

and so R has pole 2 and no zeros.

(iii) If $R(s) = \dfrac{(s-1)^2}{s^2 - 3s + 2}$, then

$$R(s) = \frac{(s-1)^2}{(s-1)(s-2)} = \frac{s-1}{s-2}$$

and so R has zero 1 and and pole 2.

Two polynomials are said to be *coprime* if they have no common linear factors.

Examples.
(i) $P(s) = s - 1$ and $Q(s) = s^2 - 5s + 6 = (s-2)(s-3)$ have no common linear factors and so are coprime.
(ii) $P(s) = s - 1$ and $Q(s) = s^2 - 3s + 2 = (s-1)(s-2)$ have $(s-1)$ as a common linear factor and so are not coprime.

Let $R = P/Q \in \mathbb{R}(s)$. By cancelling all common linear factors in P and Q, we may write $R = \tilde{P}/\tilde{Q}$ where $\tilde{P}$ and $\tilde{Q}$ are coprime. It then follows that (i) $z \in \mathbb{C}$ is a zero of R if, and only if, z is a root of $\tilde{P}$, and (ii) $p \in \mathbb{C}$ is a pole of R if, and only if, p is a root of $\tilde{Q}$.

Proposition 2.1. *Let $G \in \mathbb{R}(s)$ be a transfer function given by*

$$G(s) = c^T(sI - A)^{-1}b + d, \quad A \in \mathbb{R}^{n \times n},\ b \in \mathbb{R}^n,\ c \in \mathbb{R}^n,\ d \in \mathbb{R}\,.$$

(i) If $p \in \mathbb{C}$ is a pole of G, then p is an eigenvalue of A.

(ii) The converse of statement (i) is not true in general.

Proof. (i) Assume $p \in \mathbb{C}$ is a pole of G. Seeking a contradiction, suppose that p is not an eigenvalue of A. Then $|pI - A| \neq 0$ and so

$$\frac{1}{|pI - A|}\big(c^T \mathrm{adj}(pI - A)b\big) + d =: q \in \mathbb{C}\,.$$

it follows that

$$\lim_{s \to p} |G(s)| = \lim_{s \to p} \left| \frac{1}{|sI - A|}\big(c^T \mathrm{adj}(sI - A)b\big) + d \right| = |q| < \infty$$

which contradicts the fact that p is a pole of G. Therefore, $p \in \mathrm{spec}(A)$.

(ii) Assume $p \in \mathrm{spec}(A)$. If $c = 0$ or $b = 0$, then $G(s) = d$ which has no poles (and so $p \in \mathbb{C}$ cannot be a pole). A less trivial counterexample is the following. Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad d = 0.$$

then

$$G(s) = \frac{1}{(s+1)(s-2)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s-2}{(s+1)(s-2)} = \frac{1}{s+1}$$

and so 2 is an eigenvalue of A but not a pole of G. ■

3 Stability

3.1 Asymptotic stability

Consider the homogeneous system

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n} \tag{5}$$

Clearly, $t \mapsto x(t) = 0$ is a solution of (5): this solution is referred to as the *equilibrium* solution. System (5) is said to be *asymptotically stable* if, for all ξ, the solution of the initial-value problem $\dot{x} = Ax$, $x(0) = \xi$, approaches the equilibrium as $t \to \infty$:

$$x(t) = (\exp At)\xi \to 0 \quad \text{as } t \to \infty\,.$$

Recall that every solution of (5) is a linear combination of functions of the form

$$t \mapsto e^{\lambda t}\left[v_k + tv_{k-1} + \cdots \frac{t^{k-1}}{(k-1)!}v_1\right], \quad k = 1, ..., m,$$

where $\lambda \in \text{spec}(A)$ with associated generalized eigenvector v (of order m) and

$$v_1 = (A-\lambda I)^{m-1}v, \;\; v_2 = (A-\lambda I)^{m-2}v, \;\; \cdots \;\; , v_{m-1} = (A-\lambda I)v, \;\; v_m = v.$$

It immediately follows that every solution of (5) approaches the equilibrium as $t \to \infty$ if, and only if, $\text{Re}\,(\lambda) < 0$ for all $\lambda \in \text{spec}(A)$.

Proposition 3.1. *System (5) is asymptotically stable if, and only if, Re*$\,(\lambda) < 0$ *for all* $\lambda \in$ *spec*(A).

3.2 Bounded-input bounded-output stability

Consider the control system with zero initial state $x(0) = 0$

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = 0, \quad A \in \mathbb{R}^{n\times n},\; b \in \mathbb{R}^n \\ &y(t) = c^T x(t) + du(t), \;\; c \in \mathbb{R}^n, \;\; d \in \mathbb{R} \end{aligned}\right\} \tag{6}$$

with impulse response $t \mapsto g(t) := c^T(\exp At)b(t) + d\delta(t)$. This system is said to be *bounded-input, bounded-output (BIBO) stable* if it has the property that the output function $y = g \star u$ is bounded whenever the input function u is bounded. Furthermore, it can be shown that BIBO stability admits the following characterization.

Definition. System (6) is BIBO stable if, and only if, there exists a constant $\gamma > 0$ such that, for each bounded input function u, the corresponding output function $y = g \star u$ satisfies

$$\sup_{t\ge 0} |y(t)| \le \gamma \sup_{t \ge 0} |u(t)|\,.$$

BIBO stability of (6) is intimately connected with the poles of its transfer function.

Theorem 3.2. *System (6) is BIBO stable if, and only if, each pole of the transfer function* $G(s) = c^T(sI-A)^{-1}b + d$ *has negative real part.*

Proof. For notational convenience, write $g_0(t) = c^T(\exp At)b$, in which case the impulse response is given by $g(t) = g_0(t) + d\delta(t)$ (with Laplace Transform G) and $y = g \star u = g_0 \star u + du$.

Necessity: Assume that (6) is BIBO stable. For each $m \in \mathbb{N}$, define a function u_m by

$$u_m(t) = \begin{cases} \left.\begin{array}{ll} +1, & g_0(m-t) \geq 0 \\ -1, & g_0(m-t) < 0 \end{array}\right\}, & 0 \leq t \leq m \\ \quad 0, & t > m \end{cases}$$

and so

$$g_0(m-t)u_m(t) = \begin{cases} |g_0(m-t)|, & 0 \leq t \leq m \\ 0, & t > m\,. \end{cases}$$

Clearly, for all $m \in \mathbb{N}$, $\sup_{t\geq 0} |u_m(t)| = 1$ and so by BIBO stability, $|(g \star u_m)(m)| \leq \gamma$. Therefore,

$$\begin{aligned} \int_0^m |g_0(t)|\,\mathrm{d}t = \int_0^m |g_0(m-t)|\,\mathrm{d}t = \int_0^m g_0(m-t)u_m(t)\,\mathrm{d}t = (g_0 \star u_m)(m) \\ = (g \star u_m)(m) - du_m(m) \leq |(g \star u_m)(m)| + |d| \leq \gamma + |d| \quad \forall\, m \in \mathbb{N}. \end{aligned}$$

Therefore,

$$\int_0^\infty |g_0(t)|\,\mathrm{d}t \leq \gamma + |d|.$$

Write $G_0 = \mathfrak{L}\{g_0\}$. Note that $G = G_0 + d$ and so G and G_0 have the same set of poles. Let $s \in \mathbb{C}$ be such that $\mathrm{Re}\,(s) \geq 0$. Then

$$\infty > \gamma + |d| \geq \int_0^\infty |g_0(t)|\,\mathrm{d}t \geq \int_0^\infty |g_0(t)|e^{-t\mathrm{Re}\,(s)}\,\mathrm{d}t \geq \left|\int_0^\infty g_0(t)e^{-st}\,\mathrm{d}t\right|$$

and so $|G_0(s)| \leq \gamma + |d|$ for all $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) \geq 0$. Therefore, G_0 has no poles p with $\mathrm{Re}\,(p) \geq 0$. We have now shown that BIBO stability of (6) implies that every pole of its transfer function has negative real part. It remains to prove the reverse implication.

Sufficiency. If G has no poles, then $G = d$ and the BIBO property holds with $\gamma = |d|$. Now, assume that G has at least one pole. Let p_i, $i = 1, ..., l$ denote the distinct poles of G and let m_i denote their multiplicities. Assume that $\mathrm{Re}\,(p_i) < 0$, $i = 1, ..., l$. Expanding G_0 in partial fractions, we have

$$G_0(s) = \sum_{i=1}^{l}\sum_{j=1}^{m_i} \frac{c_{ij}}{(s-p_i)^j}$$

for some constants $c_{ij} \in \mathbb{C}$. Note that $c_{ij}/(s-p^i)^j$ is the Laplace Transform of the function g_{ij} given by

$$g_{ij}(t) = \frac{c_{ij}t^{j-1}e^{p_i t}}{(j-1)!}, \quad i = 1, ..., l, \quad j = 1, ..., m_i$$

and so

$$g_0(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i} g_{ij}(t)\,.$$

Since $\mathrm{Re}\,(p_i) < 0$, $i = 1, ..., l$, it follows that

$$\int_0^\infty |g_0(t)|\,\mathrm{d}t =: L < \infty\,.$$

Let u be a bounded input and write $U := \sup_{t\geq 0}|u(t)|$. The corresponding output satisfies

$$\begin{aligned}|y(t)| &= |(g\star u)(t)| \leq |(g_0\star u)(t)| + |du(t)| = |(u\star g_0)(t)| + |d||u(t)| \\ &\leq \int_0^t |u(t-\tau)g_0(\tau)|\,\mathrm{d}\tau + |d||u(t)| \leq U\int_0^t |g_0(\tau)|\,\mathrm{d}\tau + U|d| \\ &\leq [L+|d|]U \quad \forall\, t\geq 0.\end{aligned}$$

Writing $\gamma := L + |d|$, it follows that

$$\sup_{t\geq 0}|y(t)| \leq \gamma U = \gamma \sup_{t\geq 0}|u(t)|$$

and so system (6) is BIBO stable. ■

An immediate consequence of Proposition 2.1 and Theorem 3.2 is the following.

Proposition 3.3. *If $Re(\lambda) < 0$ for all $\lambda \in spec(A)$, then (6) is BIBO stable.*

3.3 Hurwitz stability criterion

Given a matrix $A \in \mathbb{R}^{n\times n}$ a basic question is: how do we check in an efficient manner whether or not the homogeneous system (5) is asymptotically stable. We know that (5) is asymptotically stable if, and only if, every eigenvalue of A has negative real part. This, in turn, is equivalent to the requirement that every root of the characteristic equation

$$|sI - A| = P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

should have negative real part.

In the context of the control system (6), with (proper rational) transfer function

$$G(s) = \frac{N(s)}{D(s)} = c^T(sI-A)^{-1}b + d, \quad N \text{ and } D \text{ coprime polynomials,}$$

an analogous question is: how do we check that (6) is BIBO stable. We know that (6) is BIBO stable if, and only if, every pole of G has negative real part. This, in turn, is equivalent to the requirement that every root of the polynomial equation

$$D(s) = 0$$

should have negative real part. Therefore, testing for asymptotic stability and BIBO stability involves a study of the roots of polynomial equations.

Definition: A polynomial $P \in \mathbb{R}[s]$ is said to be *stable* if every root of the equation $P(s) = 0$ has negative real part.

A natural question: can we deduce stability or non-stability of a given polynomial $P \in \mathbb{R}[s]$ by means that do not require computation of the roots? For example, the following proposition gives a necessary condition for stability of $P \in \mathbb{R}[s]$ (and so, if a given P fails to satisfy the necessary condition, then it cannot be stable).

Proposition 3.4. *If the polynomial $P \in \mathbb{R}[s]$ given by*

$$P(s) = \sum_{m=0}^{n} a_m s^m = a_n s^n + \cdots + a_0, \quad a_n \neq 0\,,$$

is stable, then the coefficients a_m, $m = 0, ..., n$, are all non-zero and have the same sign.

The Hurwitz criterion provides a necessary and sufficient condition for stability of $P \in \mathbb{R}[s]$ in terms of its coefficients.

Theorem 3.5. *Let $P \in \mathbb{R}[s]$ be given by*

$$P(s) = \sum_{m=0}^{n} a_m s^m = a_n s^n + \cdots + a_0, \quad a_n \neq 0\,.$$

Without loss of generality, assume $a_n > 0$. Let H be the $n \times n$ determinant

$$H := |h_{ij}| = \begin{vmatrix} a_{n-1} & a_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & \cdots & 0 & 0 & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_0 \end{vmatrix}$$

(For $1 \leq 2i - j \leq n$, the general element satisfies $h_{ij} = a_{n-(2i-j)}$, otherwise $h_{ij} = 0$.) Let H_m, $m = 1, ..., n$ be the principal subdeterminants of H:

$$H_1 = a_{n-1}, \quad H_2 = \begin{vmatrix} a_{n-1} & a_n \\ a_{n-3} & a_{n-2} \end{vmatrix}, \quad H_3 = \begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix}, \quad ... \quad , H_n = a_0 H_{n-1}.$$

The polynomial P is stable if, and only if,

$$H_m > 0 \quad \forall\; m = 1, ..., n.$$

Example. Determine conditions on the coefficients which are necessary and sufficient for stability of the cubic polynomial $P \in \mathbb{R}[s]$:

$$P(s) = as^3 + bs^2 + cs + d, \quad a > 0.$$

Solution. In this case,

$$H = \begin{vmatrix} b & a & 0 \\ d & c & b \\ 0 & 0 & d \end{vmatrix}$$

with principal subdeterminants

$$H_1 = b, \quad H_2 = \begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad, \quad H_3 = dH_2 = d(bc - ad)$$

Therefore, necessary and sufficient conditions for stability are: $b > 0$, $d > 0$ and $bc - ad > 0$.

Example. For what values of the real parameter k is the following polynomial stable ?

$$P(s) = s^4 + 6s^3 + 11s^2 + 6s + k\,.$$

Solution. In this case,

$$H = \begin{vmatrix} 6 & 1 & 0 & 0 \\ 6 & 11 & 6 & 1 \\ 0 & k & 6 & 11 \\ 0 & 0 & 0 & k \end{vmatrix}$$

with principal subdeterminants

$$H_1 = 6, \quad H_2 = 60, \quad H_3 = 36(10-k), \quad H_4 = 36k(10-k).$$

Therefore, by Theorem 3.5, the polynomial is stable if, and only if, $0 < k < 10$.

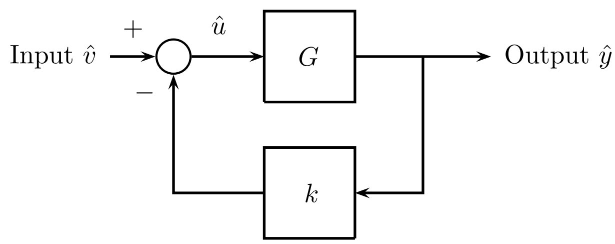
3.4 BIBO stabilization by output feedback

Consider system (6), with $d = 0$ and transfer function G given by

$$G(s) = c^T(sI - A)^{-1}b.$$

Suppose that this system is not BIBO stable. A question arises: can the system be rendered BIBO stable through the use of output feedback?

Write $u(t) = v(t) - ky(t)$, where v is a new input, $k \in \mathbb{R}$ and $-ky(t)$ is an output feedback component.



Taking Laplace transforms and with reference to the Figure, we have $\hat{y}(s) = G(s)\hat{u}(s) = G(s)\big[\hat{v}(s) - k\hat{y}(s)\big]$ and so the transfer function G_k of the feedback system from new input $\hat{v}$ to output $\hat{y}$ is given by

$$G_k(s) = \frac{G(s)}{1 + kG(s)}\,.$$

The question now is: does there exist a value k such that the feedback system is BIBO stable?

Theorem 3.6. *Assume (i) G has no zero with non-negative real part, and (ii) $\lim_{|s|\to\infty} sG(s) =: L > 0$. Then there exists $k^* > 0$ such that, for all $k > k^*$, G_k is the transfer function of a BIBO system.*

Proof. Write $G(s) = N(s)/D(s)$, where N and D are coprime polynomials. Without loss of generality, we may assume that D is a monic polynomial, that is, has leading coefficient 1. By hypothesis (ii), G is strictly proper with $\deg(D) = \deg(N) + 1$. Therefore, for some $n \in \mathbb{N}$ and real constants $a_0, \ldots, a_n$, $b_0, \ldots, b_n$, we have

$$N(s) = a_n s^n + \cdots + a_1 s + a_0,\ a_n \neq 0, \quad \text{and} \quad D(s) = s^{n+1} + b_n s^n + \cdots + b_1 s + b_0.$$

By hypothesis (ii), it follows that $a_n = L > 0$. Consider the rational function

$$\begin{aligned}
R(s) = \frac{1}{G(s)} - \frac{s}{a_n} &= \frac{D(s)}{N(s)} - \frac{s}{a_n} = \frac{a_n D(s) - sN(s)}{a_n N(s)} \\
&= \frac{a_n\big(s^{n+1} + b_n s^n + \cdots + b_0\big) - \big(a_n s^{n+1} + a_{n-1}s^n + \cdots + a_0 s\big)}{a_n\big(a_n s^n + \cdots + a_0\big)} \\
&= \frac{\big(a_n b_n - a_{n-1}\big)s^n + \text{terms of order} < n}{a_n^2 s^n + \text{terms of order} < n} \\
&\to \frac{a_n b_n - a_{n-1}}{a_n^2} < \infty \quad \text{as } |s| \to \infty\,.
\end{aligned}$$

Therefore, R is proper and its poles coincide with the zeros of G. Write

$$\mathbb{C}_0^+ := \{s \in \mathbb{C} |\ \ \mathrm{Re}\,(s) \geq 0\} \quad \text{(the closed right-half complex plane).}$$

By hypothesis (i), it follows that R has no pole in $\mathbb{C}_0^+$. We may now conclude that

$$\sup_{s \in \mathbb{C}_0^+} |R(s)| =: k^* < \infty\,.$$

Let $k > k^*$. The proof is complete if we can show that

$$G_k(s) = \frac{G(s)}{1 + kG(s)} = \frac{1}{(1/G(s)) + k} = \frac{1}{R(s) + k + (s/a_n)}$$

has no pole with non-negative real part. Seeking a contradiction, suppose that $p \in \mathbb{C}_0^+$ is a pole of G_k. Then

$$R(p) + k + \frac{p}{a_n} = 0 \quad \text{and so } p = -a_n\big(R(p) + k\big).$$

Now,

$$|\mathrm{Re}\,(R(p))| \leq \sup_{s \in \mathbb{C}_0^+} |\mathrm{Re}\,(R(s))| \leq \sup_{s \in \mathbb{C}_0^+} |R(s)| = k^* < k,$$

and so

$$0 < k - |\mathrm{Re}\,(R(p))| \leq k + \mathrm{Re}\,(R(p)).$$

Thus, we arrive at a contradiction:

$$0 \leq \mathrm{Re}\,(p) = -a_n\big(\mathrm{Re}\,(R(p)) + k\big) < 0.$$

Therefore, for all $k > k^*$, G_k has no pole with non-negative real part and so is the transfer function of a BIBO stable system. ■

Example. Consider the case

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c, \quad d = 0.$$

Because of the structure of b and c, in order to determine $G(s) = c^T(sI - A)^{-1}b$, we need only compute the top left element $E_{11}(s)$ of the matrix

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & 0 & -1 \\ 0 & s & -1 \\ -1 & 1 & s+1 \end{bmatrix}^{-1}.$$

In particular,

$$E_{11}(s) = \frac{s^2 + s + 1}{|sI - A|} = \frac{s^2 + s + 1}{s^3 - s - 1}, \quad \text{and so} \quad G(s) = E_{11}(s).$$

The hypotheses of Theorem 3.6 hold, and so there exists k^* such that, for all $k > k^*$, $G_k = G/(1 + kG)$ is the transfer function of a BIBO stable system. We can compute k^* by appealing to Theorem 3.2 and the Hurwitz criterion. In particular, if we can determine k^* such that the denominator polynomial of G_k is stable for all $k > k^*$, then, by Theorem 3.2, G_k is the transfer function of a BIBO stable system for all $k > k^*$. Now,

$$G_k(s) = \frac{G(s)}{1 + kG(s)} = \frac{s^2 + s + 1}{s^3 - s - 1 + k(s^2 + s + 1)} = \frac{s^2 + s + 1}{s^3 + ks^2 + (k-1)s + (k-1)}\,.$$

By the Hurwitz criterion, the denominator polynomial is stable if, and only if, $k > 1 =: k^*$.

3.5 Integral control: essentially Question 2 of Problem Sheet 11

Consider system (6), with transfer function given by

$$G(s) = c^T(sI - A)^{-1}b + d$$

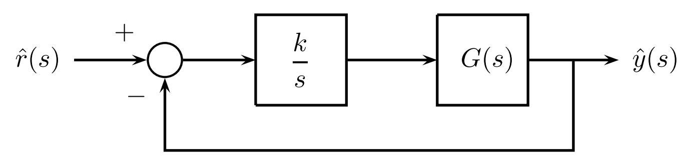
under integral control action

$$u(t) = k\int_0^t \big[r(\tau) - y(\tau)\big]\ \mathrm{d}\tau \tag{7}$$

where k is a real parameter and r is a reference input. The control objective is to cause the output y to approach the reference input r in the sense that $y(t) - r(t) \to 0$ as $t \to \infty$. Applying the Laplace transform to (7), we have

$$\hat{u}(s) = \frac{k}{s}\left[\hat{r}(s) - \hat{y}(s)\right] \tag{8}$$

and the overall controlled system (6-7) takes the form



Noting that

$$\hat{y}(s) = \frac{k}{s}G(s)\big[\hat{r}(s) - \hat{y}(s)\big],$$

the transfer function F_k from the reference input $\hat{r}$ to output $\hat{y}$ is given by

$$F_k(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = \frac{\frac{k}{s}G(s)}{1 + \frac{k}{s}G(s)} = \frac{kG(s)}{s + kG(s)}.$$

Now assume

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)},$$

in which case, we have

$$F_k(s) = \frac{k}{s^4 + 6s^3 + 11s^2 + 6s + k}.$$

Now apply the Hurwitz criterion to the denominator polynomial. The Hurwitz determinant is

$$H = \begin{vmatrix} 6 & 1 & 0 & 0 \\ 6 & 11 & 6 & 1 \\ 0 & k & 6 & 11 \\ 0 & 0 & 0 & k \end{vmatrix}$$

with principal subdeterminants

$$H_1 = 6, \quad H_2 = 60, \quad H_3 = 36(10 - k), \quad H_4 = 36k(10 - k).$$

Therefore, by the Hurwitz criterion, the polynomial is stable (and F_k is the transfer function of a BIBO system) if, and only if, $0 < k < k^* := 10$.

Finally, assume that the reference input is constant: $r(t) = r_0$ for all $t \geq 0$. Then $\hat{r}(s) = r_0/s$. Let $k \in (0, k^*)$ and let f_k be such that $F_k = \mathfrak{L}\{f_k\}$. Since F_k is the transfer function of a BIBO stable system, it follows that $f_k : [0, \infty) \to \mathbb{R}$ is exponentially decaying and so, by the final-value theorem, we may conclude

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (f_k \star r_0 H)(t) = r_0 \lim_{t\to\infty} (f_k \star H)(t) = r_0 F_k(0) = r_0,$$

and so the control objective is achieved.

The above is a particular example of the following general result.

Theorem 3.7. *Assume that (a) G is the transfer function of a BIBO stable system, (b) $G(0) > 0$, Then there exists $k^* > 0$ such that, for all $k \in (0, k^*)$, the feedback system is BIBO stable. If, in addition, the reference input is constant $r(t) = r_0$ for all $t \geq 0$, then, for all $k \in (0, k^*)$, $y(t) \to r_0 \quad$ as $t \to \infty$.*