MA20220 - Ordinary Differential Equations and Control
Semester 1
Lecture Notes on ODEs and
Laplace Transform (Parts I and II)

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0.1 Revision

In science and engineering, mathematical models often lead to an equation that contains an unknown function together with some of its derivatives. Such an equation is called a differential equation (DE).

We will use following notation for derivatives of a function x of a scalar argument t:

$$\dot{x} := \frac{dx}{dt} = x'(t), \ \ \ddot{x} := \frac{d^2x}{dt^2} = x''(t), \ \ etc.$$

Examples.

1. *Free fall: consider free fall for a body of mass $m > 0$ – the equation of motion is*

$$m\frac{d^2h}{dt^2} = -mg,$$

where $h(t)$ is the height at time t, and $-mg$ is the force due to gravity. To find the solutions of this problem, rewrite the equation as $\ddot{h}(t) = -g$ and integrate twice to obtain $\dot{h}(t) = -gt + c_1$ and

$$h(t) = -\frac{1}{2}gt^2 + c_1t + c_2,$$

for two constants $c_1, c_2 \in \mathbb{R}$. An interpretation of the constants of integration c_1 and c_2 is as follows: at $t = 0$, the height is $h(0) = c_2$, so c_2 is the initial height, and similarly $\dot{h}(0) = c_1$ is the initial velocity.

2. *Radioactive decay: the rate of decay is proportional to the amount $x(t)$ of radioactive material present at time t: $\dot{x}(t) = -k\,x(t)$, for some constant $k > 0$. The solution of this equation is $x(t) = Ce^{-kt}$. The initial amount of radioactive substance at time $t = 0$ is $x(0) = C$.*

Remark. *Note that the solutions to differential equations are not unique; for each choice of c_1 and c_2 in the first example there is a solution of the form $h(t) = (-1/2)gt^2 + c_1t + c_2$. Likewise, for each choice of constant C in $x(t) = Ce^{-kt}$ there exists a solution to the problem $\dot{x}(t) = -kx(t)$. Those integration constants are often found from initial conditions (IC). We hence often deal with initial value problems (IVPs), which consist of a differential equation together with some initial conditions.*

Example. *Solve the initial value problem*

$$\dot{y}(t) + ay(t) = 0, \quad y(0) = 2.$$

Solution. Rewriting the differential equation gives

$$\frac{\dot{y}(t)}{y(t)} = -a.$$

Since $\frac{d}{dt}(\ln y(t)) = \dot{y}(t)/y(t)$, it follows that $\ln y(t) = \int \dot{y}(t)/y(t)dt$; hence

$$y(t) = \exp\left(\int \frac{\dot{y}(t)}{y(t)}dt\right) \quad \Longrightarrow \quad y(t) = e^{-\int a dt}.$$

Hence, $y(t) = e^{-at+c} = Ce^{-at}$ ($C := \exp(c)$). The initial condition yields

$$y(0) = 2 \iff C = 2.$$

Therefore, the solution of the initial value problem is the function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y(t) = 2e^{-at}, \quad \forall t \in \mathbb{R}.$$

□

Chapter 1

Systems of Linear Autonomous Ordinary Differential Equations

1.1 Writing linear ODEs as First-order systems

Any linear ordinary differential equation, of any order, can be written in terms of a first order system. This is best illustrated by an example.

Example. *Consider the equation*

$$\ddot{x}(t) + 2\dot{x}(t) + 3x(t) = t. \tag{1.1}$$

In this case, set $X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$, *i.e.* $X(t)$ *is a two-dimensional vector-valued function of* t. *Then*

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -3x - 2\dot{x} + t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix} \tag{1.2}$$

So

$$\dot{X} = AX + g,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}, \ g(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

□

The system (1.2) and the equation (1.1) are equivalent in the sense that $x(t)$ solves the equation if and only if X solves the system.

(For more examples for the above reduction see Problem Sheet 1, QQ 1–3.)

This motivates studying first order ODE systems. The most general first order system is of the form

$$B(t)\dot{x}(t) + C(t)x(t) = f(t),$$

where $B, C : \mathbb{R} \to \mathbb{C}^{n\times n}$, $f, x : \mathbb{R} \to \mathbb{C}^n$. Henceforth, we do not employ any special notation for vectors and vector-value functions (like underlining or using bold cases), to simplify the notation, and on the assumption that what is a vector will be clear from the context. If $B^{-1}(t)$ exists for all $t \in \mathbb{R}$, the equation may be rewritten to obtain:

$$\dot{x}(t) = -B^{-1}(t)C(t)x(t) + B^{-1}(t)f(t)$$

or

$$\boxed{\dot{x}(t) = A(t)x(t) + g(t),} \tag{1.3}$$

where $A(t) = -B^{-1}(t)C(t)$ and $g(t) = B^{-1}(t)f(t)$.

Definition. *1. Equation* (1.3) *(in fact, a system of equations) is called the standard form of a first order system of ordinary differential equations.*

2. If $A(t)$ and $g(t)$ do not depend on t, the system is called autonomous.

3. If $g \equiv 0$, the system (1.3) *is called homogeneous, otherwise ($g \not\equiv 0$) it is called inhomogeneous.*

1.2 Autonomous Homogeneous Systems

We consider initial-value problems for autonomous homogeneous systems, i.e we assume A is a constant, generally complex-valued, $n \times n$ matrix.

Theorem (Existence and Uniqueness of IVPs)**.** *Let $A \in \mathbb{C}^{n\times n}$ and $x_0 \in \mathbb{C}^n$. Then the initial value problem*

$$\dot{x}(t) = Ax(t), \quad x(t_0) = x_0 \tag{1.4}$$

has a unique solution.

Proof. (Sketch)

We first argue that a solution to (1.4) exists and is given by $x(t) = \exp\left((t-t_0)A\right)x_0$, where $\exp\left((t-t_0)A\right)$ is an appropriately understood matrix exponential.

Definition (Matrix Exponential)**.** *For a matrix $Y \in \mathbb{C}^{n\times n}$, the function $\exp(\cdot Y) : \mathbb{R} \to \mathbb{C}^{n\times n}$ is defined by*

$$\exp(tY) = \sum_{k=0}^{\infty} \frac{1}{k!}(tY)^k = I + tY + \frac{1}{2}t^2Y^2 + \ldots, \quad \forall t \in \mathbb{R}. \tag{1.5}$$

(We adopt the conventions $0! = 1$, and $Y^0 = I$ where I is the unit matrix.)

Remarks. *The following hold true*

Fact 1: *The series* (1.5) *converges for all $t \in \mathbb{R}$ (meaning, the series of matrices converges for each component of the matrix);*

Fact 2: *The derivative satisfies*

$$\frac{d}{dt}\left(\exp(tY)\right) = Y\exp(tY) = \exp(tY)Y.$$

We define $x(t) = \exp((t-t_0)A)x_0$. Then

$$x(t_0) = \exp((t_0 - t_0)A)x_0 = \exp(0A)x_0 = Ix_0 = x_0.$$

Hence $x(t_0) = x_0$. Also, $\dot{x}(t) = A\exp((t-t_0)A)x_0 = Ax(t)$, so $x(t)$ is a solution.

To establish uniqueness, let $x(t)$ and $y(t)$ be two solutions of (1.4) and consider $h(t) := x(t) - y(t)$. Then $\dot{h} = \dot{x} - \dot{y} = Ax - Ay = A(x-y) = Ah$ and hence $\dot{h} = Ah$ and $h(t_0) = x(t_0) - y(t_0) = 0$. We will show that $h(t) \equiv 0$. It suffices to show that $\|h(t)\| \equiv 0$, where $\|\cdot\|$ denotes the length of a vector. Then

$$\|h(t)\| = \left\|\int_{t_0}^{t} Ah(s)ds\right\| \leq \left|\int_{t_0}^{t} \|Ah(s)\|ds\right| \leq \left|\int_{t_0}^{t} |A|\,\|h(s)\|\,ds\right|, \tag{1.6}$$

where $|A| := \max_{x\neq 0}\left(\|Ax\|\|x\|^{-1}\right)$, and we have used the fact that the length of an integral of a vector-function is less or equal the integral of a length. The function

$$f(t) := \exp(-|t-t_0||A|)\left|\int_{t_0}^{t} |A|\|h(s)\|\,ds\right|$$

satisfies $f(t_0) = 0$, and $f(t) \geq 0\ \forall t$. On the other hand, evaluating f' and using (1.6) we conclude: $f'(t) \leq 0$ for $t \geq t_0$ and $f'(t) \geq 0$ for $t \leq t_0$. [For example for $t \geq t_0$,

$$f'(t) = \exp(-(t-t_0)|A|)|A|\left[-\int_{t_0}^{t} |A|\|h(s)\|\,ds + \|h(t)\|\right] \leq 0$$

by (1.6).] Taken together this implies that $f(t) \equiv 0$, hence $\|h(t)\| \equiv 0$, implying $h(t) \equiv 0$ as required. □

To compute $\exp(tA)$ explicitly is generally not so easy. (This will be discussed in more detail later.)

We will aim at constructing appropriate number of "linearly independent" solutions of the system $\dot{x} = Ax$. Hence first

Definition (Linear Independence of Functions)**.** *Let $I \subset \mathbb{R}$ be an interval. The vector functions $y_i : I \to \mathbb{C}^n$, $i = 1, \ldots, m$ are said to be linearly independent on I if*

$$\sum_{i=1}^{m} c_i\, y_i(t) = 0, \quad \forall t \in I \quad \textit{implies} \quad c_1 = c_2 = \cdots = c_m = 0.$$

The functions y_i are said to be linearly dependent if they are not linearly independent, i.e. if $\exists\, c_1, c_2, \ldots, c_m$ not all zero such that

$$\sum_{i=1}^{m} c_i y_i(t) = 0, \quad \forall t \in I.$$

Example. *Let $I = [0, 1]$ and $y_1(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}$, $y_2(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$.*

Claim *y_1 and y_2 are linearly independent.*

Proof *Assume they are not; then $\exists c_1, c_2$ not both zero such that*

$$c_1 y_1(t) + c_2 y_2(t) = 0, \quad \forall t \in I \iff \begin{cases} c_1 e^t + c_2 = 0 \\ c_1 t e^t + c_2 t = 0, \end{cases} \qquad \forall t \in I.$$

If c_2 is non-zero then clearly c_1 is non-zero too (and the other way round), and hence in particular $e^t = -c_2/c_1$ is constant which yields a contradiction. Thus y_1 and y_2 are linearly independent.

□

For more examples see Sheet 1 QQ 4–5.

1.3 Linearly Independent Solutions

Consider the homogeneous autonomous system

$$\dot{x} = Ax, \tag{1.7}$$

where $A \in \mathbb{C}^{n \times n}$ and $x = x(t) : \mathbb{R} \to \mathbb{C}^n$. We prove that there exists n and only n linearly independent solutions of (1.7).

Theorem (Existence of n and no more than n Linearly Independent Solutions)**.** *There exist n linearly independent solutions of system* (1.7)*. Any other solution can be written as*

$$x(t) = \sum_{i=1}^{n} c_i x_i(t), \quad c_i \in \mathbb{C}. \tag{1.8}$$

Proof. Let $v_1, \ldots, v_n \in \mathbb{C}^n$ be n linearly independent vectors. Let $x_i(t)$ be the unique solutions of the initial value problems

$$\dot{x}_i = Ax_i, \quad x(t_0) = v_i, \quad i = 1, \ldots, n.$$

Now the functions $x_i(t)$ are linearly independent (see Problem 4 (a) Sheet 1), so the existence of n linearly independent solutions is assured. Now let $x(t)$ be an arbitrary solution of (1.7), with $x(t_0) = x_0 \in \mathbb{C}^n$. Since the v_i, $i = 1, ..., n$, are linearly independent, they form a basis for $\mathbb{C}^n$, so in particular $\exists c_1, \ldots, c_n \in \mathbb{C}$ such that

$$x_0 = \sum_{i=1}^{n} c_i v_i.$$

Now define

$$y(t) = \sum_{i=1}^{n} c_i x_i(t), \quad y : \mathbb{R} \to \mathbb{C}^n.$$

Then,

$$\dot{y}(t) = \sum_{i=1}^{n} c_i \dot{x}_i(t) = \sum_{i=1}^{n} c_i A x_i(t) = A \sum_{i=1}^{n} c_i x_i(t) = Ay(t).$$

Further,

$$y(t_0) = \sum_{i=1}^{n} c_i x_i(t_0) = \sum_{i=1}^{n} c_i v_i = x_0.$$

Hence, by uniqueness of solution to the initial value problem $\dot{y} = Ay$ with $y(t_0) = x_0$, it follows that $y(t) = x(t)$. □

A Method for Determining the Solutions

Some linearly independent solutions are found by seeking solutions of the form

$$x(t) = e^{\lambda t} v, \tag{1.9}$$

where $v \in \mathbb{C}^n$ is a non-zero constant vector. In this case,

$$\dot{x}(t) = \lambda e^{\lambda t} v,$$

so to satisfy $\dot{x} = Ax$, it is required that $\lambda e^{\lambda t} v = A e^{\lambda t} v \iff Av = \lambda v$, $v \neq 0$. Hence $x(t) = e^{\lambda t} v \not\equiv 0$ is a solution of $\dot{x} = Ax$ if and only if λ is an eigenvalue of A with $v \in \mathbb{C}^n$ being corresponding eigenvector.

Example. *Consider*

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x(t), \ \textit{hence} \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}. \tag{1.10}$$

The eigenvalues of A are found by solving:

$$\begin{aligned} \det(A - \lambda I) = 0 &\iff \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0 \\ &\iff (1-\lambda)^2 - 4 = 0 \iff \lambda^2 - 2\lambda - 3 = 0 \\ &\iff (\lambda - 3)(\lambda + 1) = 0 \\ &\iff \lambda_1 = 3, \quad \lambda_2 = -1. \end{aligned}$$

The corresponding eigenvectors (denoting v^1 and v^2 the components of the vector v)

$\lambda = 3$:

$$\begin{aligned} Av = 3v \iff (A - 3I)v = 0 &\iff \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = 0 \\ &\iff \begin{cases} -2v^1 & +v^2 = 0 \\ 4v^1 & -2v^2 = 0 \end{cases} \\ &\Rightarrow \text{can take } v = v_1 := \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \end{aligned}$$

as an eigenvector corresponding to $\lambda = 3$.

$\lambda = -1$:

$$\begin{aligned} (A + I)v = 0 &\iff \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = 0 \\ &\iff \begin{cases} 2v^1 + v^2 = 0 \\ 4v^1 + 2v^2 = 0 \end{cases} \\ &\Rightarrow \text{take} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \end{aligned}$$

So, in summary,

$$x_1(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_2(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

are solutions to $\dot{x} = Ax$. Further, since v_1 and v_2 are linearly independent vectors, the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Thus the general solution to (1.10) *is*

$$x(t) = c_1x_1(t) + c_2x_2(t) \iff x(t) = c_1e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants. □

Corollary. *Let $A \in \mathbb{C}^{n\times n}$. If A has n linearly independent eigenvectors, then the general solution of $\dot{x} = Ax$ is given by*

$$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i,$$

where v_i are the linearly independent eigenvectors with corresponding eigenvalues λ_i, and $c_i \in \mathbb{C}$ are arbitrary constants.

Definition (Notation)**.** *For $A \in \mathbb{C}^{n\times n}$, the characteristic polynomial will be denoted by*

$$\pi_A(\lambda) := \det(A - \lambda I).$$

The set of eigenvalues of A (the "spectrum" of A) is denoted by

$$\mathrm{Spec}(A) = \{\lambda \in \mathbb{C} \mid \pi_A(\lambda) = 0\}. \quad \square$$

As is known from linear algebra, for a matrix A to have n linearly independent eigenvectors (equivalently, for A to be diagonalisable), it would suffice e.g. if A had n *distinct* eigenvalues or were symmetric. However, a matrix may have less than n linearly independent eigenvectors as the following example illustrates.

Example. *An example of a 2×2 matrix which does not have two linearly independent eigenvectors. Let*

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \textit{hence } \pi_A(\lambda) = (2-\lambda)^2 = 0 \iff \lambda = 2,$$

an eigenvalue with "algebraic multiplicity" 2. A corresponding eigenvector is found by solving

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = 0.$$

In the latter one can choose only one linearly independent eigenvector e.g. $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence

$$x_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

solves $\dot{x} = Ax$. But the second linearly independent solution still remains to be found.

Finding the Second Linearly Independent Solution

Try $x_2(t) = u(t)e^{\lambda t}$, for some function $u : \mathbb{R} \to \mathbb{C}^2$. Substituting gives

$$\begin{aligned} \dot{x}_2(t) = \dot{u}(t)e^{\lambda t} + \lambda u(t)e^{\lambda t} = Ax_2(t) &\iff \dot{u}(t)e^{\lambda t} + \lambda u(t)e^{\lambda t} = Au(t)e^{\lambda t} \\ &\iff \dot{u}(t) + \lambda u(t) = Au(t) \\ &\iff \dot{u}(t) = (A - \lambda I)u(t). \qquad (*) \end{aligned}$$

Now try $u(t) = a + t\,b$*, with* $a, b \in \mathbb{C}^2$ *constant vectors. Then* $\dot{u}(t) = b$*, so substituting into (*) gives*

$$b = (A - \lambda I)(a + t\,b) = (A - \lambda I)a + t(A - \lambda I)b$$

Comparing coefficients of 1 *and* t *yields*

$$(A - \lambda I)a = b, \quad (A - \lambda I)b = 0.$$

Thus one can choose b *as the eigenvector already found:* $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$*. This enables the equation to be solved for* a*:*

$$(A - \lambda I)a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\text{with } e.g. \;\; a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This gives

$$x_2(t) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{2t}.$$

Further, $x_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $x_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$*, so* $x_1(t)$ *and* $x_2(t)$ *are linearly independent solutions of*

$$\dot{x}(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x(t).$$

The general solution is therefore given by

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} t \\ 1 \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{C}$ *are arbitrary constants.* □

We shall now consider the case when an $n \times n$ matrix has less than n linearly independent eigenvectors in greater generality. Notice that in the above example $(A - \lambda I)b = 0$, and $(A - \lambda I)a = b \neq 0$ while $(A - \lambda I)^2 a = (A - \lambda I)b = 0$. Hence while b is an ordinary eigenvector, a may be viewed as a "generalised" eigenvector. This motivates the following generic

Definition (Generalised Eigenvectors)**.** *Let* $A \in \mathbb{C}^{n \times n}$*,* $\lambda \in \operatorname{Spec}(A)$*. Then a vector* $v \in \mathbb{C}^n$ *is called a generalised eigenvector of order* $m \in \mathbb{N}$*, with respect to* λ*, if the following two conditions hold:*

- $(A - \lambda I)^k v \neq 0, \quad \forall 0 \leq k \leq m - 1;$
- $(A - \lambda I)^m v = 0.$ □

In the above example, a is a generalised eigenvector of A with respect to $\lambda = 2$ of order $m = 2$. Notice that, in the light of the above definition, the ordinary eigenvectors are "generalised eigenvectors of order 1".

Following lemma gives a way of constructing a sequence of generalised eigenvectors as long as we have one, of order $m \geq 2$:

Lemma (Constructing Generalised Eigenvectors)**.** *Let* $\lambda \in \operatorname{Spec}(A)$ *and* v *be a generalised eigenvector of order* $m \geq 2$*. Then, for* $k = 1, \ldots, m - 1$*, the vector*

$$v_{m-k} := (A - \lambda I)^k v$$

is a generalised eigenvector of order $m - k$*.*

Proof. It is required to show that

- $(A-\lambda I)^l v_{m-k} \neq 0, \quad \forall 0 \leq l \leq m-k-1$;
- $(A-\lambda I)^{m-k} v_{m-k} = 0$.

Firstly,

$$(A-\lambda I)^l v_{m-k} = (A-\lambda I)^l (A-\lambda I)^k v = (A-\lambda I)^{l+k} v \neq 0, \quad 0 \leq l+k \leq m-1$$

whence

$$(A-\lambda I)^l v_{m-k} \neq 0, \quad 0 \leq l \leq m-k-1.$$

Moreover,

$$(A-\lambda I)^{m-k} v_{m-k} = (A-\lambda I)^{m-k} (A-\lambda I)^k v = (A-\lambda I)^m v = 0.$$

□

The need for incorporating the generalised eigenvectors arises as long as "there are not enough" ordinary eigenvectors, more precisely when the geometric multiplicity is different (i.e. strictly less) than the algebraic multiplicity, defined as follows.

Definition (Algebraic and Geometric Multiplicity)**.** *Let $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathrm{Spec}(A)$. Then λ has geometric multiplicity $m \in \mathbb{N}$ if m is the largest number for which m linearly independent eigenvectors exist. If $m = 1$, λ is said to be a simple eigenvalue. The algebraic multiplicity of λ is the multiplicity of λ as a root of $\pi_A(\lambda)$.*

Examples.

(i) Consider

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}. \quad \textit{Hence } \mathrm{Spec}(A) = \{2\}.$$

Then, by an earlier example, $\lambda = 2$ has geometric multiplicity 1, but algebraic multiplicity is 2.

(ii) Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then

$$\pi_A(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)^2(2-\lambda).$$

Hence $\mathrm{Spec}(A) = \{1, 2\}$ and $\lambda = 1$ has algebraic multiplicity 2, $\lambda = 2$ has algebraic multiplicity 1. For the eigenvectors,

$$\underline{\lambda = 1}: \quad (A - I)v = 0 \iff \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are two linearly independent eigenvectors, so $\lambda = 1$ has geometric multiplicity 2. Further,

$\underline{\lambda = 2}$*:*

$$(A - 2I)v = 0 \iff \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = 0, \; v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is an eigenvector. Hence $\lambda = 2$ is a simple eigenvalue. □

The following theorem provides a recipe for constructing m linearly independent solutions as long as we have a generalised eigenvector of order $m \geq 2$.

Theorem. *Let $A \in \mathbb{C}^{n\times n}$ and $v \in \mathbb{C}^n$ be a generalised eigenvector of order m, with respect to $\lambda \in \operatorname{Spec}(A)$. Define the following m vectors:*

$$\begin{array}{lcl} v_1 & = & (A-\lambda I)^{m-1}v \\ v_2 & = & (A-\lambda I)^{m-2}v \\ & \vdots & \\ v_{m-1} & = & (A-\lambda I)v \\ v_m & = & v. \end{array}$$

Then:

- *The vectors $v_1, v_2, \dots, v_{m-1}, v_m$ are linearly independent;*
- *The functions*

$$x_k(t) = e^{\lambda t}\sum_{i=0}^{k-1}\frac{t^i}{i!}v_{k-i}, \quad k \in \{1,\dots,m\}$$

  form a set of linearly independent solutions of $\dot{x} = Ax$.

Proof.

- Seeking a contradiction, assume that the vectors are linearly dependent, so $\exists\, c_1,\dots,c_m$ not all zero such that

$$\sum_{i=1}^{m} c_i v_i = 0.$$

  Then there exists the "last" non-zero c_j: $c_j \neq 0$ and either $j = m$ or $c_i = 0$ for any $j < i \leq m$. Hence

$$\sum_{k=1}^{j} c_k v_k = 0 \iff \sum_{k=1}^{j} c_k (A-\lambda I)^{m-k}v = 0.$$

  Hence, pre-multiplying by $(A-\lambda I)^{j-1}$ gives

$$(A-\lambda I)^{j-1}\sum_{k=1}^{j} c_k (A-\lambda I)^{m-k}v = 0 \iff \sum_{k=1}^{j} c_k (A-\lambda I)^{m+j-k-1}v = 0.$$

  Now for $k = 1,\dots,j-1$ it follows that $m+j-k-1 \geq m$, so $(A-\lambda I)^{m+j-k-1}v = 0$ by definition of the generalised eigenvector. For $k = j$, it follows that $m+j-k-1 = m-1$, so $(A-\lambda I)^{m-1}v \neq 0$; this implies that $c_j = 0$ which is a contradiction. Hence the vectors are in fact linearly independent, as required.

- Since $v_1,\dots,v_m$ are linearly independent and $x_k(0) = v_k$, it follows that $x_1(t),\dots,x_m(t)$ are linearly independent functions (cf. Sheet 1 Q 4(a)). It remains to show that $\dot{x}_k(t) = Ax_k(t)$, $\forall 1 \leq k \leq m$. Indeed,

$$\begin{aligned} \dot{x}_k(t) &= \lambda e^{\lambda t}\sum_{i=0}^{k-1}\frac{t^i}{i!}v_{k-i} + e^{\lambda t}\sum_{i=0}^{k-2}\frac{t^i}{i!}v_{k-1-i} \\ &= e^{\lambda t}\left[\lambda\sum_{i=0}^{k-1}\frac{t^i}{i!}v_{k-i} + \sum_{i=0}^{k-2}\frac{t^i}{i!}v_{k-1-i}\right] \\ &= e^{\lambda t}\left[\frac{t^{k-1}}{(k-1)!}\lambda v_1 + \sum_{i=0}^{k-2}\frac{t^i}{i!}(\lambda v_{k-i} + v_{k-1-i})\right]. \end{aligned}$$

  Now for $2 \leq j \leq m$,

$$v_{j-1} = (A-\lambda I)^{m-j+1}v = (A-\lambda I)(A-\lambda I)^{m-j}v = (A-\lambda I)v_j = Av_j - \lambda v_j.$$

Thus $\lambda v_j + v_{j-1} = Av_j$, and hence (having also used $\lambda v_1 = Av_1$)

$$\dot{x}_k(t) = e^{\lambda t} \sum_{i=0}^{k-1} \frac{t^i}{i!} A v_{k-i} = A e^{\lambda t} \sum_{i=0}^{k-1} \frac{t^i}{i!} v_{k-i} = A x_k(t),$$

so $\dot{x}_k(t) = Ax_k(t)$ as required.

$\square$

Examples.

1. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \pi_A(\lambda) = (1-\lambda)^2(3-\lambda).$$

For the eigenvalue $\underline{\lambda = 1}$,

$$(A - I)v = 0 \iff \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = 0 \Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is an eigenvector with respect to $\lambda = 1$*. (There is only one linearly independent eigenvector, i.e the geometric multiplicity is* 1 *while the algebraic multiplicity is* 2*.) Hence seek* v_2 *such that* $(A-I)v_2 \neq 0$ *and* $(A-I)^2 v_2 = 0$*. Notice that*

$$(A - I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix},$$

so take $v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ *as a generalised eigenvector of order* 2*. Then set*

$$v_1 := (A - I)v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(notice v_1 *is an ordinary eigenvector, as expected).*

For the eigenvalue $\underline{\lambda = 3}$,

$$(A - 3I)v_3 = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} v_3 = 0,$$

so take $v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

Hence finally set

$$x_1(t) = e^t v_1 = \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}, \quad x_2(t) = e^t(v_2 + tv_1) = \begin{pmatrix} -2e^t \\ e^t \\ te^t \end{pmatrix},$$

and

$$x_3(t) = e^{3t} v_3 = \begin{pmatrix} 0 \\ 2e^{3t} \\ e^{3t} \end{pmatrix}.$$

The general solution to $\dot{x} = Ax$ *is hence:*

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t),$$

where $c_1, c_2, c_3 \in \mathbb{C}$ *are arbitrary constants.*

2. *Find three linearly independent solutions of*

$$\dot{x}(t) = Ax(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x(t).$$

In this case,

$$\pi_A(\lambda) = (1-\lambda)^3,$$

and since

$$(A - I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

there is only one (linearly independent) eigenvector, $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. *Hence* $\lambda = 1$ *has algebraic multiplicity* 3 *and geometric multiplicity one. Hence a generalised eigenvector* v_3 *of order* <u>*three*</u> *is required, i.e. such that:*

$$(A - I)v_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} v_3 \neq 0$$

$$(A - I)^2 v_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_3 \neq 0,$$

and $(A-I)^3 v_3 = [0]_{3\times 3} v_3 = 0$. *We can choose as such* $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. *Then set:*

$$v_2 := (A - I)v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_1 := (A - I)^2 v_3 = (A - I)v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, using the main theorem on the existence of solutions,

$$\begin{cases} x_1(t) = e^t v_1 \\ x_2(t) = e^t (v_2 + t\, v_1) \\ x_3(t) = e^t \left(v_3 + t\, v_2 + \dfrac{t^2}{2} v_1 \right) \end{cases}$$

are three linearly independent solutions. □

1.4 Fundamental Matrices

Assume that for $A \in \mathbb{C}^{n\times n}$ n linearly independent solutions $x_1(t), \ldots, x_n(t)$ of the system $\dot{x}(t) = Ax(t)$ have been found. Such n vector-functions constitute a <u>fundamental system</u>. Given such a fundamental system, there exist constants $c_i \in \mathbb{C}$ such that any other solution $x(t)$ can be written as $x(t) = \sum_{i=1}^{n} c_i x_i(t)$. Given a fundamental system we can define <u>fundamental matrix</u> as follows.

Definition (Fundamental Matrix)**.** *A fundamental matrix for the system $\dot{x}(t) = Ax(t)$ is defined by*

$$\Phi(t) = \Big(x_1(t) \mid x_2(t) \mid \cdots \mid x_n(t) \Big),$$

that is, $\Phi : \mathbb{R} \to \mathbb{C}^{n\times n}$, where the ith column is $x_i : \mathbb{R} \to \mathbb{C}^n$, and x_i, $i = 1, ..., n$, are n linearly independent solutions (i.e. a fundamental system). □

Then

$$\dot{\Phi}(t) = (\dot{x}_1(t) \mid \cdots \mid \dot{x}_n(t)) = (Ax_1(t) \mid \cdots \mid Ax_n(t)) = A\,(x_1(t) \mid \cdots \mid x_n(t)) = A\Phi(t).$$

Also note that $\Phi^{-1}(t)$ exists for all $t \in \mathbb{R}$, since the $x_i(t)$ are linearly independent.

Example. *For*

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix},$$

two linearly independent solutions of the system $\dot{x}(t) = Ax(t)$ are (see Example in §1.3 above)

$$x_1(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_2(t) = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The functions $x_1(t), x_2(t)$ constitute a fundamental system, and the matrix

$$\Phi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

is a fundamental matrix for $\dot{x}(t) = Ax(t)$. □

Lemma. *If $\Phi(t)$ and $\Psi(t)$ are two fundamental matrices for $\dot{x}(t) = Ax(t)$, then $\exists C \in \mathbb{C}^{n\times n}$ (a constant matrix) such that $\Phi(t) = \Psi(t)C$, $\forall t \in \mathbb{R}$.*

Proof. Write $\Phi(t) = (x_1(t) \mid \cdots \mid x_n(t))$ and $\Psi(t) = (y_1(t) \mid \cdots \mid y_n(t))$. Since $y_1(t), \ldots, y_n(t)$ constitute a fundamental system, each $x_j(t)$ can be written in terms of $y_1(t), \ldots, y_n(t)$, so there exist constants $c_{ij} \in \mathbb{C}$ such that

$$x_j(t) = \sum_{i=1}^{n} c_{ij} y_i(t), \quad \forall 1 \le j \le n.$$

The above vector identity, for the k-th components, $k = 1, ..., n$, reads $\Phi_{kj}(t) = \sum_{i=1}^{n} c_{ij}\Psi_{ki}(t)$. This is equivalent to $\Phi(t) = \Psi(t)C$, with $C = [c_{ij}]_{n\times n}$, the matrix with (i, j)th entry c_{ij}. □

We show next that the matrix exponential is a fundamental matrix.

Theorem. *The matrix function $\Phi(t) = \exp(tA)$ is a fundamental matrix for the system $\dot{x}(t) = Ax(t)$.*

Proof.

$$\frac{d}{dt}(\exp(tA)) = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^{k+1} = A \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = A\exp(tA).$$

Hence $\dot{\Phi}(t) = A\Phi(t)$. Write $\Phi(t) = (x_1(t) \mid \cdots \mid x_n(t))$, where $x_i(t)$ is the ith column of Φ. For

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

it follows that $x_i(t) = \Phi(t)e_i$ and thus

$$\dot{x}_i(t) = \dot{\Phi}(t)e_i = A\Phi(t)e_i = Ax_i(t).$$

Hence $\dot{x}_i(t) = Ax_i(t)$ and $x_1(t), \ldots, x_n(t)$ are linearly independent functions because $x_1(0) = e_1, \ldots, x_n(0) = e_n$ are linearly independent vectors. Therefore, $\Phi(t) = \exp(tA)$ is a fundamental matrix, by definition. □

1.4.1 Initial Value Problems Revisited

Let $A \in \mathbb{C}^{n\times n}$, and seek $\Phi : \mathbb{R} \to \mathbb{C}^{n\times n}$ such that $\dot{\Phi}(t) = A\Phi(t)$ and $\Phi(t_0) = \Phi_0$, for some initial condition $\Phi_0 \in \mathbb{C}^{n\times n}$. If $\Psi(t)$ is a fundamental matrix and $\Psi(t_0) = \Psi_0$, set $\Phi(t) = \Psi(t)\Psi_0^{-1}\Phi_0$.

Claim $\Phi(t)$ solves the above initial value problem.

Proof

- Firstly,
$$\dot{\Phi}(t) = \dot{\Psi}(t)\Psi_0^{-1}\Phi_0 = A\Psi(t)\Psi_0^{-1}\Phi_0 = A\Phi(t).$$
- Moreover,
$$\Phi(t_0) = \Psi(t_0)\Psi_0^{-1}\Phi_0 = \Psi_0\Psi_0^{-1}\Phi_0 = \Phi_0,$$
so the initial condition is satisfied. □

Corollary. *The unique solution to the above initial value problem is given by*

$$\Phi(t) = \exp((t - t_0)A)\Phi_0.$$

Proof: Exercise. □

Example. *Solve the initial value problem*

$$\dot{\Phi}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \Phi(t), \quad \Phi(0) = I.$$

Solution. Firstly, (see the example above)

$$\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

is a fundamental matrix. Then, using the result from above,

$$\Phi(t) = \Psi(t)\Psi_0^{-1} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}.$$

□

Corollary. *Let $\Phi(t)$ be a fundamental matrix of $\dot{x} = Ax$. Then* $\exp(tA) = \Phi(t)\Phi^{-1}(0)$.

Proof. Since $\exp(tA)$ is a fundamental matrix, $\exp(tA) = \Phi(t)C$ for some constant matrix $C \in \mathbb{C}^{n\times n}$. At $t = 0$, $I = \Phi(0)C$, so $C = \Phi^{-1}(0)$ (recall that $\Phi(t)$ is invertible for all $t \in \mathbb{R}$). □

Hence determining a fundamental matrix essentially determines $\exp(tA)$.

Example. *From above, the system*

$$\dot{\Phi}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \Phi(t)$$

has a fundamental matrix given by

$$\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}.$$

Moreover $\Psi(t)\Psi^{-1}(0)$ is given by

$$\begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix} = \exp\left(t\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\right).$$

An alternative view (computing the matrix exponential via diagonalisation):
With $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$*, it follows that* $\lambda_1 = 3$ *and* $\lambda_2 = -1$ *are eigenvalues with corresponding eigenvectors*

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Write $Av_1 = \lambda_1 v_1$ *and* $Av_2 = \lambda_2 v_2$ *into one matrix equation:*

$$A\,(v_1 \mid v_2) = (v_1 \mid v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \iff AT = TD,$$

for matrices T *and* D *as defined above. Notice that* D *is diagonal and* T *is invertible since its columns are linearly independent, so* $A = TDT^{-1}$ *and* $T^{-1}AT = D$.
Generally, given $A \in \mathbb{C}^{n\times n}$ *and* $T \in \mathbb{C}^{n\times n}$ *invertible such that*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix},$$

then

$$\begin{aligned} \exp(tA) = \exp(tTDT^{-1}) &= \sum_{k=0}^{\infty} \frac{1}{k!} t^k \left(TDT^{-1}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} t^k T D^k T^{-1} \\ &= T \left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k D^k \right) T^{-1} \\ &= T \exp(tD) T^{-1} \\ &= T \begin{pmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{pmatrix} T^{-1}. \end{aligned}$$

Returning to the example and taking the above into account,

$$\exp\left(t \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \right) = -\frac{1}{4} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}.$$

□

The above shows how $\exp(tA)$ can be computed when A is diagonalisable (equivalently, has n linearly independent eigenvectors, taking which as a basis in fact diagonalises the matrix). More generally, when A is not necessarily diagonalisable, we can still present it in some "standard form" (called Jordan's normal form) by incorporating the generalised eigenvectors. By Corollary, the columns of a fundamental matrix are of the form $e^{At}v$, for some $v \in \mathbb{C}^n$. Now

$$e^{At}v = e^{\lambda t I} e^{(A-\lambda I)t} v = e^{\lambda t} \left[\sum_{k=0}^{\infty} \frac{1}{k!} (A - \lambda I)^k t^k \right] v.$$

Thus, to simplify any calculation, seek $\lambda \in C$ and $v \in C^n$ such that $(A - \lambda I)^k v = 0$ for all $k \geq m$, for some $k \in \mathbb{N}$; in other words, seek generalised eigenvectors.

The following fundamental Theorem from linear algebra ensures that there always exist n linearly independent generalised eigenvectors. (Hence, we are always able to construct n linearly independent solutions in terms of those, via the earlier developed recipes.)

Theorem (Primary Decomposition Theorem)**.** *Let* $A \in \mathbb{C}^{n\times n}$*, and let*

$$\pi_A(\lambda) = \prod_{j=1}^{l} (\lambda_j - \lambda)^{m_j}$$

with the λ_j *distinct eigenvalues of* A *and* $\sum_{j=1}^{l} m_j = n$*. Then, for each* $j = 1, \ldots, l$*,* $\exists m_j$ *linearly independent (generalised) eigenvectors with respect to* λ_j*, and the combined set of* n *generalised eigenvectors is linearly independent.*

A proof will be given in Algebra-II (Spring semester), and is not required at present. □

Example. *Consider the system*

$$\dot{x}(t) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} x(t) = Ax(t).$$

Now $\mathrm{Spec}(A) = \{0, 0, 1\} = \{0, 1\}$*, and the corresponding eigenvectors are found by solving*

$$\underline{\lambda = 1}: \quad (A - I)v_1 = 0 \iff \begin{pmatrix} 0 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\underline{\lambda = 0}: \quad Av = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = 0 \Rightarrow v = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

(i.e. geometric multiplicity is 1 *vs algebraic multiplicity* 2*). Now examine*

$$A^2 = \begin{pmatrix} 1 & 2 & 11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and find $v_3 \in \mathbb{C}^3$ *such that* $A^2 v_3 = 0$ *and* $Av_3 \neq 0$*, that is, find a generalised eigenvector of order* 2*. Take e.g.* $v_3 = \begin{pmatrix} 11 \\ 0 \\ -1 \end{pmatrix}$*. Hence set*

$$v_2 := (A - \lambda I)v_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 11 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix}$$

(notice that $v_2 = 4v$ *is an ordinary eigenvector).*

Then v_1, v_2, v_3 *are linearly independent. Hence the general solution is*

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + c_3 \left(\begin{pmatrix} 11 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} \right),$$

for constants $c_1, c_2, c_3 \in \mathbb{C}$*.* □

1.5 Non-Homogeneous Systems

Let $A \in \mathbb{C}^{n\times n}$ and consider the system

$$\dot{x}(t) = Ax(t) + g(t) \tag{1.11}$$

for $x : \mathbb{R} \to \mathbb{C}^n$ and $g : \mathbb{R} \to \mathbb{C}^n$. Let $\Phi(t)$ be a fundamental matrix of the corresponding homogeneous system $\dot{x}(t) = Ax(t)$.

Seek a solution of (1.11) in the "variation of parameters" form, i.e. $x(t) = \Phi(t)u(t)$ for some function $u : \mathbb{R} \to \mathbb{C}^n$. For $x(t) = \Phi(t)u(t)$,

$$\dot{x}(t) = \dot{\Phi}(t)u(t) + \Phi(t)\dot{u}(t) = A\Phi(t)u(t) + \Phi(t)\dot{u}(t) = Ax(t) + \Phi(t)\dot{u}(t) = Ax(t) + g(t).$$

So the condition on u is that

$$\Phi(t)\dot{u}(t) = g(t).$$

Hence

$$\dot{u}(t) = \Phi^{-1}(t)g(t) \quad \Longrightarrow \quad u(t) = \int_{t_0}^{t} \Phi^{-1}(s)g(s)ds + C,$$

where $C \in \mathbb{C}^n$. Hence

$$x(t) = \Phi(t)u(t) = \Phi(t)\left[\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds + C\right].$$

Thus the general solution of (1.11) is given by:

$$x(t) = \Phi(t)C + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds, \tag{1.12}$$

which is known as the variation of parameters formula.

Given the initial value problem

$$\dot{x}(t) = Ax(t) + g(t), \quad x(t_0) = x_0 \in \mathbb{C}^n, \tag{1.13}$$

C can be determined:

$$x(t) = \Phi(t)C + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds \Rightarrow x(t_0) = \Phi(t_0)C \Rightarrow C = \Phi^{-1}(t_0)x_0.$$

Thus the solution of the initial value problem (1.13) is given by

$$x(t) \; = \; \Phi(t)\Phi^{-1}(t_0)x_0 \; + \; \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds. \tag{1.14}$$

The solutions in these two cases are said to have been obtained by variation of parameters.

Example. *Solve the initial value problem*

$$\dot{x}_1 \; = \; -x_1 - x_2 + e^t, \;\; x_1(0) = 0$$

$$\dot{x}_2 \; = \; -3x_1 + x_2, \quad x_2(0) = 2.$$

Solution.

$$\Longleftrightarrow \quad \dot{x}(t) = Ax(t) + g(t) = \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} x(t) + \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Firstly, $\mathrm{Spec}(A) = \{-2, 2\}$ and

$$\lambda = 2 \rightsquigarrow v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \lambda = -2 \rightsquigarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}$$

is a fundamental matrix for the corresponding homogeneous system $\dot{x}(t) = Ax(t)$. Finding the inverse gives

$$\Phi^{-1}(t) = \frac{1}{4}\begin{pmatrix} e^{-2t} & -e^{-2t} \\ 3e^{2t} & e^{2t} \end{pmatrix},$$

and

$$\Phi^{-1}(0)x(0) = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}.$$

Thus, by variation of parameters,

$$x(t) = \begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} e^{2t} & e^{-2t} \\ -3e^{2t} & e^{-2t} \end{pmatrix}\int_0^t \frac{1}{4}\begin{pmatrix} e^{-2s} & -e^{-2s} \\ 3e^{2s} & e^{2s} \end{pmatrix}\begin{pmatrix} e^s \\ 0 \end{pmatrix} ds.$$

Integrating then gives via routine calculation

$$\begin{aligned} x(t) &= \frac{1}{2}\begin{pmatrix} -e^{2t} + e^{-2t} \\ 3e^{2t} + e^{-2t} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} e^{2t} - e^{-2t} \\ \\ -3e^{2t} + 4e^t - e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} \\ \\ \frac{3}{4}e^{2t} + e^t + \frac{1}{4}e^{-2t} \end{pmatrix}. \end{aligned}$$

□

Chapter 2

Laplace Transform

2.1 Definition and Basic Properties

Definition (Laplace Transform). *The Laplace transform of a function* $f : [0, \infty) \to \mathbb{R}$ *is the complex-valued function of the complex variable* s *defined by*

$$\mathcal{L}\{f(t)\}(s) = \hat{f}(s) := \int_0^\infty f(t)e^{-st}dt, \tag{2.1}$$

for such $s \in \mathbb{C}$ *that the integral on the right-hand side exists (in the improper sense).*

Remarks.

1. *If* $g : \mathbb{R} \to \mathbb{C}$ *is a function, then*

$$\int g(t)dt := \int \mathrm{Re}(g(t))dt + i \int \mathrm{Im}(g(t))dt.$$

2. *For* $z \in \mathbb{C}$,

$$|e^z| = |e^{\mathrm{Re}\, z + i\,\mathrm{Im}\, z}| = |e^{\mathrm{Re}\, z}||e^{i\,\mathrm{Im}\, z}|,$$

so $|e^z| = e^{\mathrm{Re}\, z}$ *for all* $z \in \mathbb{C}$.

3. *Not all functions possess Laplace transforms. Further, the Laplace transform of a function may generally only be defined for certain values of the variable* $s \in \mathbb{C}$*, for the improper integral in* (2.1) *to make sense. The following theorem determines when the improper integral makes sense.*

For the improper integral in (2.1) to make sense, we need

Definition (Exponential Order). *A function* $f : [0, \infty) \to \mathbb{R}$ *is said to be of* <u>*exponential order*</u> *if there exist constants* $\alpha, M \in \mathbb{R}$ *with* $M > 0$ *such that*

$$|f(t)| \le Me^{\alpha t}, \quad \forall t \in [0, \infty).$$

We prove

Theorem (Existence of the Laplace Transform). *Suppose that a function* $f : [0, \infty) \to \mathbb{R}$ *is piecewise continuous and of exponential order with constants* $\alpha \in \mathbb{R}$ *and* $M > 0$*. Then the Laplace transform* $\hat{f}(s)$ *exists for all* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > \alpha$.

Proof. Fix $s \in \mathbb{C}$ with $\operatorname{Re} s > \alpha$. For all $T > 0$,

$$\begin{aligned}\int_0^T |f(t)e^{-st}|dt = \int_0^T |f(t)||e^{-st}|dt &\le \int_0^T Me^{\alpha t}|e^{-st}|dt \\ &= M\int_0^T e^{\alpha t}e^{(-\operatorname{Re} s)t}dt = M\int_0^T e^{(\alpha-\operatorname{Re} s)t}dt \\ &= \left[\frac{M}{\alpha - \operatorname{Re} s}e^{(\alpha-\operatorname{Re} s)t}\right]_{t=0}^{t=T} \\ &= \frac{M}{\operatorname{Re} s - \alpha}\left[1 - e^{(\alpha-\operatorname{Re} s)T}\right] \le \frac{M}{\operatorname{Re} s - \alpha}.\end{aligned}$$

(Since $\alpha - \operatorname{Re} s < 0$, sending $T \to \infty$ gives $e^{(\alpha-\operatorname{Re}(s))T} \to 0$.)
Hence

$$\int_0^T |f(t)e^{-st}|dt \le \frac{M}{\operatorname{Re} s - \alpha}.$$

Now define, for $t \in [0, \infty)$,

$$\alpha(t) = \operatorname{Re}(f(t)e^{-st});$$

$$\beta(t) = \operatorname{Im}(f(t)e^{-st});$$

$$\alpha^+(t) = \max\{\alpha(t), 0\} \ge 0;$$

$$\alpha^-(t) = \max\{-\alpha(t), 0\} \ge 0;$$

$$\beta^+(t) = \max\{\beta(t), 0\} \ge 0;$$

$$\beta^-(t) = \max\{-\beta(t), 0\} \ge 0.$$

Note that $\alpha(t) = \alpha^+(t) - \alpha^-(t)$ and $\beta(t) = \beta^+(t) - \beta^-(t)$. Further,

$$0 \le \int_0^T \alpha^+(t)dt \le \int_0^T |\alpha(t)|dt \le \int_0^T |f(t)e^{-st}|dt \le \frac{M}{\operatorname{Re} s - \alpha} < \infty.$$

Hence

$$\lim_{T\to\infty}\int_0^T \alpha^+(t)dt =: \alpha_*^+ \text{ exists.}$$

Similarly,

$$\lim_{T\to\infty}\int_0^T \alpha^-(t)dt =: \alpha_*^- \text{ exists,}$$

and

$$\lim_{T\to\infty}\int_0^T \beta^\pm(t)dt =: \beta_*^\pm \text{ exist.}$$

As a result,

$$(\alpha_*^+ - \alpha_*^-)(t) + i(\beta_*^+ - \beta_*^-)(t) = \lim_{T\to\infty}\int_0^T (\alpha(t) + i\beta(t))dt = \int_0^\infty f(t)e^{-st}dt = \hat{f}(s)$$

exists. ☐

Example. *Consider $f(t) = e^{ct}$, where $c \in \mathbb{C}$. Then $f(t)$ is of exponential order with $M = 1$, $\alpha = \operatorname{Re}(c)$:*

$$|f(t)| = |e^{ct}| = e^{\operatorname{Re}(c)t}.$$

Then

$$\hat{f}(s) = \int_0^\infty e^{ct}e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{(c-s)t}dt$$
$$= \lim_{T\to\infty}\left[\frac{1}{c-s}e^{(c-s)t}\right]_0^T$$
$$= \lim_{T\to\infty}\left(\frac{1}{c-s}e^{(c-s)T} - \frac{1}{c-s}\right).$$

If $\operatorname{Re} s > \operatorname{Re} c$, *then*

$$\lim_{T\to\infty} e^{(c-s)T} = 0.$$

So for $s \in \mathbb{C}$ *with* $\operatorname{Re}(s) > \operatorname{Re}(c)$,

$$\int_0^\infty e^{ct}e^{-st}dt = \frac{1}{s-c}.$$

Hence

$$\mathcal{L}\left\{e^{ct}\right\}(s) \;=\; \frac{1}{s-c}, \quad \operatorname{Re} s > \operatorname{Re} c. \tag{2.2}$$

In particular for $c = 0$, $f(t) = 1$ *with Laplace transform*

$$\mathcal{L}\{1\} \;=\; \frac{1}{s}.$$

The following theorem establishes some basic properties of the Laplace transform.

Theorem. *Suppose that* $f(t)$ *and* $g(t)$ *are of exponential order. Then for* $\operatorname{Re}(s)$ *sufficiently large, the following properties hold:*

1. *Linearity: For* $a, b \in \mathbb{C}$,

$$\mathcal{L}\{af(t) + bg(t)\}(s) = a\mathcal{L}\{f(t)\}(s) + b\mathcal{L}\{g(t)\}(s) = a\hat{f}(s) + b\hat{g}(t).$$

2. *Transform of a Derivative:*

$$\mathcal{L}\{f'(t)\}(s) = s\hat{f}(s) - f(0). \tag{2.3}$$

3. *Transform of Integral:*

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\hat{f}(s).$$

4. *Damping Formula:*

$$\mathcal{L}\left\{e^{-at}f(t)\right\}(s) = \hat{f}(s+a).$$

5. *Delay Formula: For* $T > 0$,

$$\mathcal{L}\{f(t-T)H(t-T)\}(s) = e^{-sT}\hat{f}(s),$$

where

$$H(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases}$$

is the Heaviside step function.

Remarks.

- *Replacing in (2.3) f by f' gives*

$$\mathcal{L}\left\{f''(t)\right\}(s) = s\mathcal{L}\left\{f'\right\}(s) - f'(0) = s^2\hat{f}(s) - sf(0) - f'(0),$$

  and more generally, for the n-th derivative, inductively

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n\hat{f}(s) - \left[s^{n-1}f(0) + s^{n-2}f'(0) + \cdots + f^{(n-1)}(0)\right],$$

  where

$$f^{(i)}(t) := \frac{d^i f}{dt^i}$$

  is the ith derivative of f (exercise).

- *In property 5, if $f(t)$ is undefined for $t < 0$, we set $f(t) = 0$ for $t < 0$. In any case, the "delayed" function $f(t-T)H(t-T)$ coincides with $f(t-T)$ for $t \geq T$ and vanishes for $0 \leq t < T$.*

Proof.

1. Follows from the linearity of the integration.
2. Integration by parts gives

$$\mathcal{L}\left\{f'(t)\right\}(s) = \int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt.$$

   Since $f(t)$ is of exponential order,

$$\lim_{t\to\infty} f(t)e^{-st} = 0,$$

   for all s with $\operatorname{Re} s$ sufficiently large. So

$$\mathcal{L}\left\{f'(t)\right\}(s) = -f(0) + s\mathcal{L}\left\{f(t)\right\}(s) = s\mathcal{L}\left\{f(t)\right\}(s) - f(0).$$

3. Define

$$F(t) = \int_0^t f(\tau)d\tau,$$

   so that

$$F(0) = 0, \quad F'(t) = f(t).$$

   Therefore, by property 2,

$$\mathcal{L}\left\{F'(t)\right\}(s) = s\mathcal{L}\left\{F(t)\right\}(s) - F(0).$$

   Hence

$$\mathcal{L}\left\{F(t)\right\}(s) = \frac{1}{s}\mathcal{L}\left\{F'(t)\right\}(s),$$

   so

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\}(s) = \frac{1}{s}\mathcal{L}\left\{f(t)\right\}(s).$$

4. For any $a \in \mathbb{C}$,

$$\mathcal{L}\left\{e^{-at}f(t)\right\}(s) = \int_0^\infty e^{-at}e^{-st}f(t)dt = \int_0^\infty e^{-(a+s)t}f(t)dt = \hat{f}(a+s).$$

5. In this case,

$$\mathcal{L}\left\{f(t-T)H(t-T)\right\}(s) = \int_0^\infty f(t-T)H(t-T)e^{-st}dt = \int_T^\infty f(t-T)e^{-st}dt.$$

   Let $\tau = t - T$, so that $\frac{d\tau}{dt} = 1$. Hence,

$$\mathcal{L}\left\{f(t-\tau)H(t-\tau)\right\}(s) = \int_0^\infty f(\tau)e^{-s(\tau+T)}d\tau = e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sT}\hat{f}(s).$$

□

Examples.

1. *From the exponential definition of* $\cos : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{L}\{\cos at\}(s) = \mathcal{L}\left\{\frac{1}{2}e^{iat} + \frac{1}{2}e^{-iat}\right\}(s) = \frac{1}{2}\mathcal{L}\left\{e^{iat}\right\}(s) + \frac{1}{2}\mathcal{L}\left\{e^{iat}\right\}(s),$$

by linearity of $\mathcal{L}$. *Therefore, using* (2.2), *for* $\operatorname{Re} s > 0$,

$$\mathcal{L}\{\cos at\}(s) = \frac{1}{2}\frac{1}{s-ia} + \frac{1}{2}\frac{1}{s+ia} = \frac{s}{s^2+a^2}.$$

2. *Using the transform of a derivative,*

$$\mathcal{L}\{\sin at\}(s) = s\mathcal{L}\left\{-\frac{1}{a}\cos(at)\right\}(s) + \frac{1}{a} = -\frac{s}{a}\frac{s}{s^2+a^2} + \frac{1}{a} = \frac{a}{s^2+a^2}.$$

3. *Let* $f(t) = t^n$, $n \in \mathbb{N}$. *Then,*

$$t^n = n\int_0^t \tau^{n-1} d\tau,$$

so

$$\mathcal{L}\{t^n\}(s) = \mathcal{L}\left\{n\int_0^\tau \tau^{n-1} d\tau\right\}(s) = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}(s) = \frac{n!}{s^{n+1}},$$

by induction (Exercise: Sheet 5 Q 4).

4. *From example 1,*

$$\mathcal{L}\{\cos at\}(s) = \frac{s}{s^2+a^2},$$

so by the damping formula,

$$\mathcal{L}\left\{e^{-\lambda t}\cos(at)\right\}(s) = \mathcal{L}\{\cos(at)\}(s+\lambda) = \frac{s+\lambda}{(s+\lambda)^2+a^2}.$$

5. *Consider the piecewise continuous function*

$$f(t) = \begin{cases} t, & 0 \le t \le T \\ T, & T < t \le 2T \\ 0, & t > 2T. \end{cases}$$

To find $\hat{f}(s)$, *first write* f *in terms of the Heaviside step function:*

$$\begin{aligned} f(t) &= t(1 - H(t-T)) + T(H(t-T) - H(t-2T)) \\ &= t - (t-T)H(t-T) - TH(t-2T), \end{aligned}$$

so

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{t\}(s) - \mathcal{L}\{(t-T)H(t-T)\}(s) - T\mathcal{L}\{H(t-2T)\}(s),$$

by linearity. Therefore, using the delay formula,

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{s^2} - e^{-sT}\frac{1}{s^2} - \frac{T}{s}e^{-2sT}.$$

2.2 Solving Differential Equations with the Laplace Transform

The Laplace transform turns a differential equation into an algebraic equation, the solutions of which can be transformed back into solutions of the differential equation. Again, this is best illustrated by example.

Examples.

1. *Consider the second order initial value problem*

$$x''(t) - 3x'(t) + 2x(t) = 1, \quad x(0) = x'(0) = 0.$$

Taking the Laplace transform of both sides gives

$$\mathcal{L}\left\{x'' - 3x' + 2x\right\}(s) = \mathcal{L}\left\{1\right\}(s) = \frac{1}{s}$$

whence

$$\mathcal{L}\left\{x''\right\}(s) - 3\mathcal{L}\left\{x'\right\}(s) + 2\mathcal{L}\left\{x\right\}(s) = \frac{1}{s},$$

by linearity. Therefore,

$$s^2\hat{x} - sx(0) - x'(0) - 3s\hat{x} + 3x(0) + 2\hat{x} = \frac{1}{s},$$

which gives

$$s^2\hat{x} - 3s\hat{x} + 2\hat{x} = \frac{1}{s} \iff \hat{x}(s^2 - 3s + 2) = \frac{1}{s},$$

so

$$\hat{x}(s) = \frac{1}{s(s-1)(s-2)}.$$

Using partial fractions and then inverting Laplace transform gives

$$\hat{x}(s) = \frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2} \quad \Longrightarrow \quad x(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t}.$$

2. *To solve the initial value problem*

$$x''(t) + 4x(t) = \cos 3t, \quad x(0) = 1, \quad x'(0) = -3,$$

taking Laplace transforms gives

$$s^2\hat{x} - sx(0) - x'(0) + 4\hat{x} = \frac{s}{s^2+9} \iff s^2\hat{x} - s + 3 + 4\hat{x} = \frac{s}{s^2+9}$$
$$\Rightarrow (s^2+4)\hat{x} = \frac{s}{s^2+9} + s - 3,$$

so

$$\begin{aligned}\hat{x}(s) &= \frac{1}{s^2+4}\left(\frac{s}{s^2+9} + s - 3\right)\\ &= \frac{s-3}{s^2+4} + \frac{s}{(s^2+4)(s^2+9)}\\ &= \frac{s-3}{s^2+4} + \frac{1}{5}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)\\ &= \frac{1}{5}\left(\frac{6s-15}{s^2+4} - \frac{s}{s^2+9}\right).\end{aligned}$$

Inverting the Laplace transform gives

$$x(t) = \frac{1}{5}\left(6\cos 2t - \frac{15}{2}\sin 2t - \cos 3t\right).$$

3. *The Laplace Transform is also useful for solving* <u>*systems*</u> *of linear ordinary differential equations. For example, consider*

$$\begin{cases} \dot{x}_1(t) - 2x_2(t) = 4t, & x_1(0) = 2; \\ \\ \dot{x}_2(t) + 2x_2(t) - 4x_1(t) = -4t - 2, & x_2(0) = -5. \end{cases}$$

Taking Laplace transforms gives

$$s\hat{x_1} - x_1(0) - 2\hat{x_2} = \frac{4}{s^2}, \quad s\hat{x_2} - x_2(0) + 2\hat{x_2} - 4\hat{x_1} = -\frac{4}{s^2} - \frac{2}{s}$$

and using the initial conditions gives

$$s\hat{x_1} - 2 - 2\hat{x_2} = \frac{4}{s^2}, \quad s\hat{x_2} + 5 + 2\hat{x_2} - 4\hat{x_1} = -\frac{4}{s^2} - \frac{2}{s}.$$

Thus

$$\begin{cases} s\hat{x_1} - 2\hat{x_2} = \dfrac{4 + 2s^2}{s^2} & (1) \\ \\ -4\hat{x_1} + (2+s)\hat{x_2} = -\dfrac{4 + 2s + 5s^2}{s^2} & (2) \end{cases}$$

Now $(2+s) \times (1) + 2 \times (2)$ *gives*

$$(s(s+2) - 8)\hat{x_1} = \frac{(2s^2+4)(s+2)}{s^2} - \frac{10s^2 + 4s + 8}{s^2} = \frac{2s^3 - 6s^2}{s^2} = 2s - 6,$$

so that

$$\hat{x_1} = \frac{2s-6}{s^2 + 2s - 8} = \frac{2s-6}{(s+4)(s-2)} = \frac{7/3}{s+4} - \frac{1/3}{s-2} \Rightarrow x_1(t) = -\frac{1}{3}e^{2t} + \frac{7}{3}e^{-4t},$$

and hence

$$x_2(t) = \frac{1}{2}\dot{x}_1(t) - 2t = -\frac{1}{3}e^{2t} - \frac{14}{3}e^{-4t} - 2t.$$

2.3 The convolution integral.

Suppose $\hat{f}(s)$ and $\hat{g}(s)$ are known and consider the product $\hat{f}(s)\hat{g}(s)$. Of what function is this the Laplace transform? Equivalently, what is the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$, $\mathcal{L}^{-1}\left\{\hat{f}(s)\hat{g}(s)\right\}$?

Firstly,

$$\begin{aligned} \hat{f}(s)\hat{g}(s) &= \hat{f}(s)\int_0^\infty g(\tau)e^{-s\tau}d\tau \\ &= \int_0^\infty \hat{f}(s)g(\tau)e^{-s\tau}d\tau \\ &= \int_0^\infty \int_0^\infty H(t-\tau)f(t-\tau)e^{-st}dt\, g(\tau)d\tau, \end{aligned}$$

by the Delay formula. Therefore,

$$\hat{f}(s)\hat{g}(s) = \int_0^\infty \int_0^\infty H(t-\tau)f(t-\tau)g(\tau)d\tau\, e^{-st}dt,$$

by switching the order of integration. This gives

$$\hat{f}(s)\hat{g}(s) = \int_0^\infty \left[\int_0^t f(t-\tau)g(\tau)d\tau\right] e^{-st}dt.$$

Hence $\hat{f}(s)\hat{g}(s)$ is the Laplace Transform of

$$t \mapsto \int_0^t f(t-\tau)g(\tau)d\tau.$$

This motivates the following definition.

Definition (Convolution)**.** *The function*

$$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)d\tau$$

is called the convolution of f and g.

Examples.

1. *In order to compute*

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\},$$

recall that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t, \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

Hence

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\,\frac{s}{s^2+1}\right\} \\
&= \sin t * \cos t \\
&= \int_0^t \sin(t-\tau)\cos\tau d\tau = \int_0^t (\sin t\cos\tau - \cos t\sin\tau)\cos\tau d\tau \\
&= \sin t\int_0^t \cos^2\tau d\tau - \cos t\int_0^t \sin\tau\cos\tau d\tau \\
&= \frac{1}{2}\sin t\int_0^t (1+\cos 2\tau)d\tau - \frac{1}{2}\cos t\int_0^t \sin 2\tau d\tau \\
&= \frac{1}{2}\sin t\left[\tau + \frac{1}{2}\sin 2\tau\right]_0^t + \frac{1}{2}\cos t\left[\frac{1}{2}\cos 2\tau\right]_0^t \\
&= \frac{1}{2}\sin t\left(t + \frac{1}{2}\sin 2t\right) + \frac{1}{2}\cos t\left(\frac{1}{2}\cos 2t - \frac{1}{2}\right) \\
&= \frac{1}{2}t\sin t + \frac{1}{4}\left(\sin 2t\sin t + \cos t\cos 2t\right) - \frac{1}{4}\cos t \\
&= \frac{1}{2}t\sin t + \frac{1}{4}\left(\cos(2t-t) - \cos t\right) = \frac{1}{2}t\sin t.
\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t \iff \mathcal{L}\left\{\frac{1}{2}t\sin t\right\}(s) = \frac{s}{(s^2+1)^2}.$$

2. *Consider the initial value problem*

$$\ddot{y}(t) + y(t) = f(t), \quad y(0) = 0, \quad \dot{y}(0) = 1$$

where

$$f(t) = \begin{cases} 1, & t \in [0,1] \\ 0, & t > 1. \end{cases}$$

Write $f(t) = H(t) - H(t-1)$. Taking Laplace Transforms of both sides gives

$$s^2\hat{y} - sy(0) - y'(0) + \hat{y} = \mathcal{L}\{H(t)\}(s) - \mathcal{L}\{H(t-1)\}(s),$$

whence

$$(s^2+1)\hat{y} = \frac{1}{s} - \frac{e^{-s}}{s} + 1,$$

and so

$$\hat{y} = \frac{1}{s(s^2+1)} + \frac{1}{s^2+1} - \frac{e^{-s}}{s(s^2+1)} = \frac{1}{s}\frac{1}{s^2+1} + \frac{1}{s^2+1} - \frac{e^{-s}}{s}\frac{1}{s^2+1}.$$

Inverting Laplace transforms,

$$y(t) = 1 * \sin t + \sin t - H(t-1) * \sin t.$$

Now

$$\begin{aligned} H(t-1) * \sin t &= \int_0^t \sin(t-\tau)H(\tau-1)d\tau \\ &= H(t-1)\int_1^t \sin(t-\tau)d\tau \\ &= H(t-1)\left[\cos(t-\tau)\right]_1^t \\ &= H(t-1)[1-\cos(t-1)]. \end{aligned}$$

Thus

$$y(t) = 1 - \cos t + \sin t + H(t-1)[\cos(t-1) - 1].$$

3. *In order to solve*

$$\ddot{x}(t) + \omega^2 x(t) = g(t), \quad \dot{x}(0) = x(0) = 0,$$

taking Laplace transforms gives

$$(s^2+\omega^2)\hat{x} = \hat{g} \quad \Longrightarrow \quad \hat{x} = \frac{\hat{g}}{s^2+\omega^2},$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin(\omega t).$$

Hence

$$x(t) = \mathcal{L}^{-1}\left\{\frac{\hat{g}}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin(\omega t) * g(t) = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))g(\tau)d\tau,$$

the solution for arbitrary $g(t)$*.*

4. *Consider*

$$\mathcal{L}^{-1}\left\{\frac{s+9}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+9}{(s+3)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4} + \frac{6}{(s+3)^2+4}\right\}$$

(completing the square in the denominator). Now

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt, \quad \mathcal{L}^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = e^{-at}\sin bt.$$

This gives

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s+9}{s^2+6s+13}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\} \\ &= e^{-3t}\cos 2t + 3e^{-3t}\sin 2t \\ &= e^{-3t}\left(\cos 2t + 3\sin 2t\right). \end{aligned}$$

□

Note that the convolution is commutative:

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau=-\int_t^0 f(\sigma)g(t-\sigma)d\sigma \text{ for } \sigma=t-\tau.$$

Hence

$$(f*g)(t)=\int_0^t f(\sigma)g(t-\sigma)d\sigma=(g*f)(t).$$

Laplace transform can be used for computing matrix exponentials $\exp(tA)$. Consider the initial value problem

$$\dot{\Phi}(t)=A\Phi(t), \quad \Phi(0)=I.$$

The solution is $\Phi(t)=\exp(tA)$. Taking Laplace Transforms gives

$$\mathcal{L}\left\{\dot{\Phi}(t)\right\}(s)=s\hat{\Phi}(s)-\Phi(0)=\mathcal{L}\left\{A\Phi(t)\right\}(s)=A\mathcal{L}\left\{\Phi(t)\right\}(s)=A\hat{\Phi},$$

by linearity. Thus

$$s\hat{\Phi}-I=A\hat{\Phi} \iff (sI-A)\hat{\Phi}=I.$$

Finally, (provided that $\operatorname{Re} s$ is greater than the real part of each eigenvalue of A)

$$\hat{\Phi}=(sI-A)^{-1}.$$

Now

$$\mathcal{L}\left\{\exp(tA)\right\}(s)=(sI-A)^{-1},$$

so

$$\exp(tA)=\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}(t).$$

Thus, in principle, $\exp(tA)$ can be computed using the Laplace Transform as follows:

- Compute $(sI-A)^{-1}$;
- Invert the entries of $(s\,I-A)^{-1}$, that is, find functions which have Laplace Transforms equal to the entries of $(sI-A)^{-1}$.

Example. *Consider*

$$A=\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix}.$$

Then

$$\begin{aligned}(sI-A)^{-1}&=\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}-\begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix}\right)^{-1}\\ &=\begin{pmatrix} s & -2 \\ -4 & s+2 \end{pmatrix}^{-1}\\ &=\frac{1}{s^2+2s-8}\begin{pmatrix} s+2 & 2 \\ 4 & s \end{pmatrix}\\ &=\frac{1}{(s+4)(s-2)}\begin{pmatrix} s+2 & 2 \\ 4 & s \end{pmatrix}.\end{aligned}$$

Continuing using partial fractions,

$$(s\,I-A)^{-1}=\begin{pmatrix} \dfrac{2/3}{s-2}+\dfrac{1/3}{s+4} & \dfrac{1/3}{s-2}-\dfrac{1/3}{s+4} \\ \dfrac{2/3}{s-2}-\dfrac{2/3}{s+4} & \dfrac{1/3}{s-2}+\dfrac{2/3}{s+4} \end{pmatrix}$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{(s\,I-A)^{-1}\right\}=\begin{pmatrix} \frac{2}{3}e^{2t}+\frac{1}{3}e^{-4t} & \frac{1}{3}e^{2t}-\frac{1}{3}e^{-4t} \\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-4t} & \frac{1}{3}e^{2t}+\frac{2}{3}e^{-4t} \end{pmatrix}=\exp(tA).$$

Let us derive the variation of parameters formula via the Laplace transform.

Consider the initial value problem

$$\dot{x}(t) = Ax(t) + g(t), \quad x(0) = x_0.$$

Taking Laplace transforms gives

$$\mathcal{L}\{\dot{x}\}(s) = A\mathcal{L}\{x\}(s) + \mathcal{L}\{g\}(s) \Rightarrow s\hat{x} - x_0 = A\hat{x} + \hat{g}.$$

Hence

$$(sI - A)\hat{x} = \hat{g} + x_0,$$

so that

$$\hat{x} = (sI - A)^{-1}x_0 + (sI - A)^{-1}\hat{g} = \mathcal{L}\{\exp(tA)\}(s)x_0 + \mathcal{L}\{\exp(tA)\}(s)\hat{g},$$

which gives

$$\hat{x} = \mathcal{L}\left\{e^{tA}\right\}(s)x_0 + \mathcal{L}\left\{e^{tA}\right\}(s)\hat{g}.$$

Taking Laplace inverses on both sides yields

$$x(t) = \exp(tA)x_0 + \exp(tA) * g = \exp(tA)x_0 + \int_0^t \exp((t-\tau)A)g(\tau)d\tau,$$

so that

$$x(t) = \exp(tA)x_0 + \exp(tA)\int_0^t \exp(-\tau A)g(\tau)d\tau,$$

as required.

□

2.4 The Dirac Delta function.

What is the inverse Laplace transform of $f(t) \equiv 1$, i.e. $\mathcal{L}^{-1}\{1\}$?

Let $\epsilon > 0$ and consider the piecewise constant function

$$\delta_\epsilon(t) = \begin{cases} \dfrac{1}{\epsilon}, & t \in (0, \epsilon) \\ 0, & \text{otherwise.} \end{cases}$$

If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then

$$\int_{-\infty}^{\infty} f(t)\delta_\epsilon(t)dt = \frac{1}{\epsilon}\int_0^\epsilon f(t)dt.$$

The Mean Value Theorem for integrals states that for continuous functions $f : [a, b] \to \mathbb{R}$, there exists $\xi \in (a, b)$ such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(t)dt.$$

Hence $\exists \xi \in (0, \epsilon)$ such that

$$\int_{-\infty}^{\infty} f(t)\delta_\epsilon(t)dt = \frac{1}{\epsilon}f(\xi)(\epsilon - 0) = f(\xi).$$

Therefore,

$$\lim_{\epsilon \to 0}\int_{-\infty}^{\infty} f(t)\delta_\epsilon(t)dt = f(0),$$

by continuity of f.

This motivates introducing the Dirac Delta–function $\delta(t)$ as an appropriate "limit" of $\delta_\epsilon(t)$ as $\epsilon \to 0$:

Definition (Dirac Delta Function)**.** *The Dirac Delta "function" $\delta(t)$ is characterised by the following two properties:*

- $\delta(t) = 0$, $\forall t \in \mathbb{R} \setminus \{0\}$*;*
- *For any function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on an open interval containing* 0

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

(In fact, $\delta(t)$ is not a usual "function", and its rigorous definition would require using more advances mathematical tools known as "Distribution theory" or theory of "generalised functions", which is beyond the scope of this course.)

Immediate Consequences

1. Setting $f(t) = 1$,

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous in an interval containing $a \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a).$$

3. For $f(t) = e^{-st}$, where $s \in \mathbb{R}$ is fixed,

$$\int_{-\infty}^{\infty} e^{-st}\delta(t)dt = e^{-s0} = 1.$$

   Hence, formally,

$$\mathcal{L}\left\{\delta(t)\right\}(s) = \int_{0}^{\infty} \delta(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1,$$

   so that

$$\mathcal{L}\left\{\delta(t)\right\}(s) = 1, \quad \mathcal{L}^{-1}\left\{1\right\} = \delta(t).$$

4. Further,

$$\left(f * \delta\right)(t) = \int_{0}^{t} f(t-\tau)\delta(\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau = f(t).$$

   Also, by commutativity, $f(t) * \delta(t) = \delta(t) * f(t) = f(t)$.

5. Moreover, for $T \geq 0$,

$$\mathcal{L}\left\{\delta(t-T)\right\}(s) = \int_{0}^{\infty} \delta(t-T)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t-T)e^{-st}dt = e^{-sT}.$$

   Hence

$$\mathcal{L}\left\{\delta(t-T)\right\}(s) = e^{-sT}, \quad \mathcal{L}^{-1}\left\{e^{-sT}\right\} = \delta(t-T).$$

6. Finally,

$$\delta(t-T) * f(t) = \int_{0}^{t} \delta(t-T-\tau)f(\tau)d\tau = \begin{cases} 0 & \text{if } t < T; \\ f(t-T) & \text{if } t \geq T. \end{cases}$$

   Thus

$$\boxed{\delta(t-T) * f(t) = f(t-T)H(t-T).}$$

Example. *Solve*
$$\ddot{y} + 2\dot{y} + y = \delta(t-1), \quad y(0) = 2, \quad \dot{y}(0) = 3.$$

Solution. Taking the Laplace transform of both sides gives
$$s^2\hat{y} - sy(0) - \dot{y}(0) + 2s\hat{y} - 2y(0) + \hat{y} = e^{-s},$$
so using the initial conditions,
$$s^2\hat{y} - 2s - 3 + 2s\hat{y} - 4 + \hat{y} = e^{-s} \Rightarrow (s^2 + 2s + 1)\hat{y} = e^{-s} + 2s + 7,$$
so that
$$\hat{y} = \frac{e^{-s} + 2s + 7}{s^2 + 2s + 1}.$$
Hence
$$\hat{y} = \frac{e^{-s}}{(s+1)^2} + \frac{5}{(s+1)^2} + \frac{2}{s+1},$$
so
$$y(t) = \delta(t-1) * te^{-t} + 5te^{-t} + 2e^{-t} = (t-1)e^{-(t-1)}H(t-1) + 5te^{-t} + 2e^{-t}.$$
□

2.5 Final value theorem.

Theorem (Final Value Theorem)**.** *Let* $g : [0, \infty) \to \mathbb{R}$ *satisfy*
$$|g(t)| \leq Me^{-\alpha t},$$
for some $\alpha, M > 0$ *(the function* g *is said to be* exponentially decaying*). Then*
$$\int_0^\infty g(t)dt = \lim_{t\to\infty} (g * H)(t) = \mathcal{L}\{g\}(0).$$

Proof.
$$\mathcal{L}\{g\}(0) := \int_0^\infty g(\tau)d\tau = \lim_{t\to\infty} \int_0^t g(\tau)d\tau \Rightarrow \mathcal{L}\{g\}(0) = \lim_{t\to\infty} \int_0^t g(t-\sigma)d\sigma,$$
by setting $\sigma = t - \tau$. Hence
$$\mathcal{L}\{g\}(0) = \lim_{t\to\infty} \int_0^t g(t-\sigma)H(\sigma)d\sigma = \lim_{t\to\infty} (g * H)(t).$$
□