$$\therefore \quad \hat{y} = \frac{1}{(s^2+1)} + \frac{1}{s}\cdot\frac{1}{(s^2+1)} - \frac{e^{-s}}{s}\cdot\frac{1}{(s^2+1)}$$

Inverting gives

$$y(t) = \sin t + (1 * \sin t) - (H(t-1) * \sin t)$$

$$= \sin t + \int_{\tau=0}^{t} \sin\tau \, d\tau - \int_{\tau=0}^{t} \sin(t-\tau) H(\tau-1) \, d\tau$$

$$\stackrel{(*)}{=} \sin t + \left[-\cos\tau\right]_{\tau=0}^{t} - H(t-1)\int_{\tau=1}^{t} \sin(t-\tau)\, d\tau$$

$$= \sin t + 1 - \cos t - H(t-1)\left[\cos(t-\tau)\right]_{\tau=1}^{t}$$

$$= 1 - \cos t + \sin t - H(t-1)\left(1 - \cos(t-1)\right).$$

(*) Note: $\displaystyle\int_{\tau=0}^{t} \sin(t-\tau) H(\tau-1)\, d\tau = \begin{cases} 0 & \text{if } t \leq 1 \\ \displaystyle\int_{\tau=1}^{t} \sin(t-\tau)\, d\tau & \text{if } t > 1 \end{cases}$

□

3. Solve $\ddot{x}(t) + w^2 x(t) = g(t)$, $\quad x(0) = \dot{x}(0) = 0$

Taking L.T. of ODE:

$$s^2\hat{x} - s\overset{=0}{x(0)} - \overset{=0}{\dot{x}(0)} + w^2\hat{x} = \hat{g}$$

$$\therefore \quad \hat{x}(s) = \frac{\hat{g}(s)}{(s^2+w^2)} \equiv \hat{f}(s)\hat{g}(s), \quad \text{where } \hat{f}(s) = \frac{1}{(s^2+w^2)}.$$

Since $\mathcal{L}^{-1}\{\hat{f}(s)\} = \frac{1}{\omega}\sin\omega t$, then

$$x(t) = \mathcal{L}^{-1}\{\hat{f}(s)\,\hat{g}(s)\}(t) = (f*g)(t)$$

$$= \frac{1}{\omega}\int_{\tau=0}^{t} \sin\omega(t-\tau)\, g(\tau)\, d\tau$$

is the solution for arbitrary $g(t)$ (and $\omega \in \mathbb{R}$ constant).

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4. Determine $\mathcal{L}^{-1}\left\{\dfrac{s+9}{s^2+6s+13}\right\}$

Answer: $\mathcal{L}^{-1}\left\{\dfrac{s+9}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s+9}{(s+3)^2+4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{(s+3)^2+4}\right\}.$$

$$= e^{-3t}\cos 2t + 3e^{-3t}\sin 2t = e^{-3t}(\cos 2t + 3\sin 2t)$$

using $\mathcal{L}^{-1}\left\{\dfrac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$

$\mathcal{L}^{-1}\left\{\dfrac{b}{(s+a)^2+b^2}\right\} = e^{-at}\sin bt$ (by damping formula)

□

- Use of LT to calculate matrix exponentials $\exp(tA)$.

Consider the IVP: $\dot{\Phi}(t) = A\,\Phi(t)$, $\Phi(0) = I$ with $A \in \mathbb{C}^{n\times n}$ constant matrix. The solution we know is $\Phi(t) = \exp(tA)$, which we will calculate using LT.

Take LT of ODE:

$$s\,\hat{\Phi}(s) - \Phi(0) = \mathcal{L}\{A\,\Phi(t)\}(s) = A\,\hat{\Phi}(s)$$

$$\therefore\ (sI - A)\,\hat{\Phi} = I$$

$$\therefore\ \hat{\Phi}(s) = (sI-A)^{-1}\,I \qquad \text{provided } \mathrm{Re}(s) > \mathrm{Re}(\lambda) \text{ for each } \lambda \in \operatorname{spec}(A).$$

$$\therefore\ \mathcal{L}\{\exp(tA)\}(s) = (sI-A)^{-1}$$

$$\therefore\ \boxed{\exp(tA) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}(t)}\ .$$

In principle, $\exp(tA)$ can be computed by LT as follows:

(a) Compute $(sI-A)^{-1}$

(b) Invert each of the entries of $(sI-A)^{-1}$
(inverse LT)

Example: Consider $A = \begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix}$.

Now $(sI-A)^{-1} = \begin{pmatrix} s & -2 \\ -4 & s+2 \end{pmatrix}^{-1} = \dfrac{1}{(s(s+2)-8)} \begin{pmatrix} s+2 & 2 \\ 4 & s \end{pmatrix}$

$$= \frac{1}{(s+4)(s-2)} \begin{pmatrix} s+2 & 2 \\ 4 & s \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{2/3}{s-2} + \dfrac{1/3}{s+4} & \dfrac{1/3}{s-2} - \dfrac{1/3}{s+4} \\[2ex] \dfrac{2/3}{s-2} - \dfrac{2/3}{s+4} & \dfrac{1/3}{s-2} + \dfrac{2/3}{s+4} \end{pmatrix}$$ by partial fractions

$$\therefore \mathcal{L}^{-1}\{(sI-A)^{-1}\}(t) = \begin{pmatrix} \frac{2}{3}e^{2t} + \frac{1}{3}e^{-4t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-4t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-4t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-4t} \end{pmatrix}$$

$$= \exp(tA) = \exp\begin{pmatrix} 0 & 2t \\ 4t & -2t \end{pmatrix}$$

- Variation of parameters formula derived using LT.

Consider the IVP: $\dot{x}(t) = A\,x(t) + g(t), \quad x(0) = x_0$

Take LT: $\quad s\hat{x} - x(0) = A\hat{x} + \hat{g}$

$$\therefore (sI - A)\,\hat{x}(s) = x_0 + \hat{g}(s)$$

$$\therefore \hat{x}(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}\hat{g}(s)$$

$$\therefore x(t) = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}(t)\,x_0 + \mathcal{L}^{-1}\left\{(sI-A)^{-1}\hat{g}(s)\right\}(t)$$

$$= \exp(tA)\,x_0 + \int_{\tau=0}^{t} \exp((t-\tau)A)\,g(\tau)\,d\tau$$

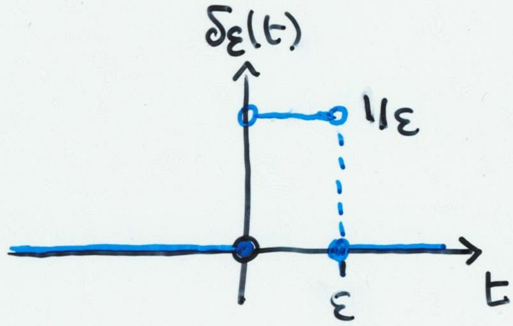
$$\therefore \boxed{x(t) = \underbrace{\exp(tA)}_{\Phi(t)}\,x_0 + \underbrace{\exp(tA)}_{\Phi(t)}\int_{\tau=0}^{t} \underbrace{\exp(-\tau A)}_{\Phi^{-1}(\tau)}\,g(\tau)\,d\tau}$$

$\Phi(0) = I$

2.4. The Dirac Delta function

Let $\varepsilon > 0$ and consider the piecewise constant function

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & t \in (0, \varepsilon) \\ 0, & \text{o.w.} \end{cases}$$



If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on an interval containing $t = 0$, then

$$\int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt = \frac{1}{\varepsilon} \int_0^\varepsilon f(t)\,dt$$

The Mean Value Theorem for integrals states:
For continuous functions $f : [a, b] \to \mathbb{R}$, $\exists$ $\xi \in (a, b)$ s.t.

$$\int_a^b f(t)\,dt = f(\xi)\,(b - a).$$

Hence $\exists$ $\xi \in (0, \varepsilon)$ s.t. $\int_0^\varepsilon f(t)\,dt = f(\xi)\,\varepsilon$.

$$\therefore \int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt = f(\xi)$$

$$\therefore \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt = f(0)$$ by continuity of f.

Introduce the Dirac Delta function $\delta(t)$ as a "limit" of the sequence of functions $\{\delta_\varepsilon(t)\}$ as $\varepsilon \to 0$.

Definition: The Dirac Delta "function" $\delta(t)$ is characterised by the following properties:

(i) $\delta(t) = 0 \quad \forall\ t \in \mathbb{R} \setminus \{0\}$

(ii) For any function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on an open interval containing 0,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0).$$

Notes:

1. $\delta(t)$ is not a usual "function" but technically a "generalised function" or "distribution".

2. There many choices for the approximating sequences $\delta_\varepsilon(t)$. Other examples include:

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2\varepsilon}, & t \in (-\varepsilon, \varepsilon), \\ 0, & \text{o.w.} \end{cases}$$

$$\delta_\varepsilon(t) = \frac{1}{\sqrt{2\pi\varepsilon^2}} \exp\left(-\frac{t^2}{2\varepsilon^2}\right) \qquad t \in (-\infty, \infty).$$