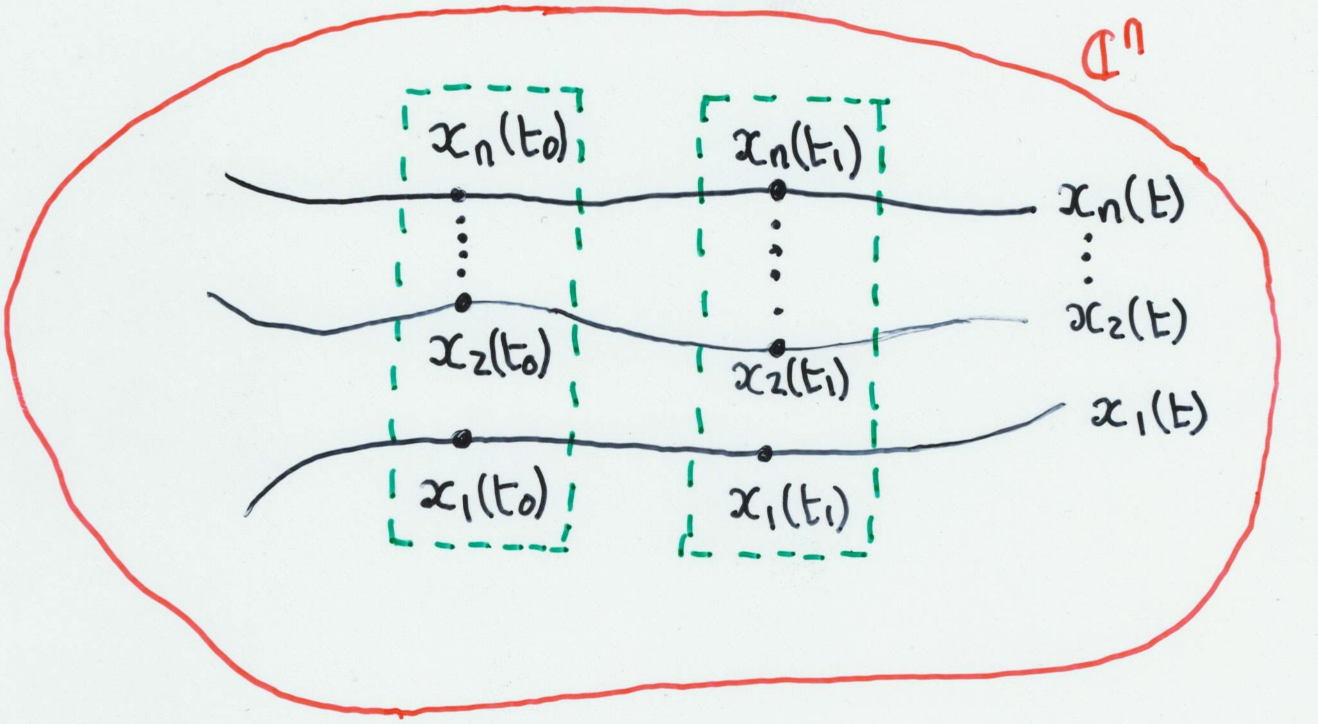
Note: Intuitive geometric description:



Sheet 1 4(a):

If $\{x_i(t_0)\}_{i=1}^n$ are lin. indep. vectors

then $\{x_i(t)\}_{i=1}^n$ are lin indep. vector functions.

<u>NB</u>. If $\{x_i(t_0)\}_{i=1}^n$ are lin. dep. vectors then it does <u>NOT</u> follow that $\{x_i(t)\}_{i=1}^n$ are lin. dep. vector functions.

eg. Consider $y_1(t) = e^t y_2(t)$ from Example at end §1.2

$t=0$: $y_1(0) = y_2(0) \Rightarrow$ lin. dep. vectors

or $t=1$: $y_1(1) = e\, y_2(1) \Rightarrow$ lin. dep. vectors

But we showed $\{y_i(t)\}_{i=1}^2$ were lin. indep. vector functions.

Example of a 2×2 matrix which does not have 2 lin. indep. eigenvectors :

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \text{ then } \Pi_A(\lambda) = (2-\lambda)^2 = 0$$
$$\iff \lambda = 2$$

i.e. $\lambda = 2$ is an eigenvalue of (algebraic) multiplicity 2.

Corresponding eigenvector $v = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{(A-\lambda I)} \underbrace{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}}_{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} d_2 = 0 \\ 0 = 0 \end{matrix} \quad \text{so } v = \begin{pmatrix} d_1 \\ 0 \end{pmatrix}$$

wlog take $d_1 = 1$ so $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (to within a multiplicative constant)

Only one lin. indep. eigenvector

Hence $x_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ solves $\dot{x} = Ax$

How do we find a second lin. indep. solution ?

Try $x_2(t) = u(t)\, e^{\lambda t}$ $\quad u(t) \in \mathbb{C}^2$ and $\lambda = 2$

Then $\dot{x}_2 = \dot{u}\, e^{\lambda t} + u \lambda e^{\lambda t} = A \underbrace{u\, e^{\lambda t}}_{x_2}$

$\Rightarrow \quad \dot{u} = (A - \lambda I)\, u$

Try $u(t) = a + t b$ with $a, b \in \mathbb{C}^2$

Then $\dot{u} = b = (A - \lambda I)(a + t b)$

$\Rightarrow \quad b = (A - \lambda I) a + t (A - \lambda I) b \quad \forall t \in I$

Compare coeffs of t^0 and t^1:

$$(A - \lambda I) a = b, \quad (A - \lambda I) b = 0.$$

Thus b is the eigenvector already found, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

which enables us to determine a:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{(A - \lambda I)} \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{a} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{b} \Rightarrow \left. \begin{matrix} a_2 = 1 \\ 0 = 0 \end{matrix} \right\} \; a = \begin{pmatrix} a_1 \\ 1 \end{pmatrix} \quad \forall a_1 \in \mathbb{C}.$$

$$\therefore \; x_2(t) = \left(\begin{pmatrix} a_1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{2t} = \left(\begin{pmatrix} a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{2t}$$

$$= \underbrace{a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}}_{\text{multiple of } x_1(t)} + \begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t}$$

taking $a_1 = 0$ wlog.

Since $x_1(0) = \begin{pmatrix}1\\0\end{pmatrix}$, $x_2(0) = \begin{pmatrix}0\\1\end{pmatrix}$ are lin. indep.

then $x_1(t)$ and $x_2(t)$ are lin. indep. solutions of $\dot{x} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix} x$ and the general solution is

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} te^{2t} \\ e^{2t} \end{pmatrix}$$

$\square$.

We shall consider the case when an $n \times n$ matrix A has less than n lin. indep. eigenvectors in full generality later.

Note in the above example that

$$(A - \lambda I) b = 0, \quad (A - \lambda I) a = b \neq 0$$

$$(A - \lambda I)^2 a = (A - \lambda I) b = 0.$$

Definition: For $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathrm{spec}(A)$ a vector $v \in \mathbb{C}^n$ is called a generalised eigenvector wrt λ of order $m \in \mathbb{N}$ if the following hold:

(i) $(A - \lambda I)^k v \neq 0, \quad 0 \leq k \leq m-1$

(ii) $(A - \lambda I)^m v = 0$.

$\ulcorner (A-\lambda I)^0 = I \lrcorner$

This definition includes $m = 1$. Thus ordinary eigenvectors are generalised eigenvectors of order 1.

In the above example, a is a generalised eigenvector wrt $\lambda = 2$ of order 2.

Definition: For $A \in \mathbb{C}^{n \times n}$, an eigenvalue $\lambda \in \mathrm{spec}(A)$ has geometric multiplicity $m \in \mathbb{N}$ if m is the largest number for which m lin. indep. eigenvectors exist. If $m = 1$, λ is said to be a simple eigenvalue.

The algebraic multiplicity is the multiplicity of λ as a root of $\pi_A(\lambda)$.