Examples:

(i) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ then $\operatorname{spec}(A) = \{2\}$

$\lambda = 2$ has geometric multiplicity 1 (i.e one l.i. eigenvector)
algebraic multiplicity 2.

(ii) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ then

$$\pi_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 (2-\lambda)$$

$\therefore \operatorname{spec}(A) = \{1, 2\}$ $\quad$ $\lambda = 1$ has alg. multip. 2
$\lambda = 2$ " " " 1

Eigenvectors:

For $\lambda = 1$: $(A - I)v = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{v} = 0$

$$\Rightarrow \left.\begin{matrix} a_3 = 0 \\ 0 = 0 \\ a_3 = 0 \end{matrix}\right\} \Rightarrow v = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\forall a_1, a_2 \in \mathbb{C}$

Thus we have two lin. indep. eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence $\lambda = 1$ has geometric multip. 2

<u>For $\lambda = 2$:</u> $(A-2I)v = 0 \Rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$

Thus $\left.\begin{aligned} -a_1 + a_3 &= 0 \\ -a_2 &= 0 \\ 0 &= 0 \end{aligned}\right\} \; v = \begin{pmatrix} a_1 \\ 0 \\ a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \; \forall a_1 \in \mathbb{C}$

i.e. $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is the associated eigenvector

and so $\lambda = 2$ has geometric multip. 1 (i.e. is a simple eigenvector).

Lemma: Let $\lambda \in \operatorname{spec}(A)$ and v be a generalised eigenvector of order $m \geq 2$. Then for $k = 1, \dots, m-1$, the vector

$$v_{m-k} := (A - \lambda I)^k v$$

is a generalised eigenvector of order $m-k$.
(NB. $v_m = (A-\lambda I)^0 v = Iv = v$)

Proof:

v is a generalised eigenvector of order m

$\iff$ (a) $(A - \lambda I)^k v \neq 0$, $0 \leq k \leq m-1$

(b) $(A-\lambda I)^m v = 0$

$v_{m-k} := (A - \lambda I)^k v$ is a gen. eigenvector of order $m-k$

$\iff$ (i) $(A-\lambda I)^l v_{m-k} \neq 0$, $0 \leq l \leq m-k-1$

(ii) $(A-\lambda I)^{m-k} v_{m-k} = 0$.

For (i): $(A-\lambda I)^l v_{m-k} = (A-\lambda I)^l (A-\lambda I)^k v$

$= (A-\lambda I)^{l+k} v \neq 0$ for $0 \leq l+k \leq m-1$

using (a) i.e. $0 \leq l \leq m-k-1$

For (ii): $(A-\lambda I)^{m-k} v_{m-k} = (A-\lambda I)^{m-k} (A-\lambda I)^k v$

$= (A-\lambda I)^m v = 0$ using (b).

$\square$

Theorem: Let $A \in \mathbb{C}^{n\times n}$ and $v \in \mathbb{C}^n$ be a generalised eigenvector of order m w.r.t. eigenvalue $\lambda \in \operatorname{spec}(A)$. Define m vectors as follows:

$$
\begin{aligned}
v_1 &= (A-\lambda I)^{m-1} v \\
v_2 &= (A-\lambda I)^{m-2} v \\
&\vdots \\
v_k &= (A-\lambda I)^{m-k} v \\
&\vdots \\
v_{m-1} &= (A-\lambda I) v \\
v_m &= v
\end{aligned}
$$

(i) Then $v_1, \dots, v_m$ are lin. indep.

(ii) The functions

$$x_k(t) := e^{\lambda t}\left(v_k + \frac{t}{1!} v_{k-1} + \frac{t^2}{2!} v_{k-2} + \dots + \frac{t^{k-1}}{(k-1)!} v_1\right)$$

for $k = 1, \dots, m$ form a set of m lin. indep. solutions for $\dot{x} = Ax$.

$\ulcorner$ $x_k(t) = e^{\lambda t} \sum_{i=0}^{k-1} \frac{t^i}{i!} v_{k-i}$. $\lrcorner$

Proof:

(i) Argue by contradiction i.e. assume $v_1, \dots, v_m$ are linearly dependent.

Thus $\exists$ constants $c_1, \dots, c_m \in \mathbb{C}$ not all zero s.t.

$$c_1 v_1 + \dots + c_m v_m = 0$$

Let $c_1, \dots, c_j$ be non-zero $1 < j \leq m$ (i.e. $c_j \neq 0$ is "last" one)

$$\therefore \quad c_1 v_1 + \dots + c_j v_j = 0$$

$$\therefore \quad \sum_{k=1}^{j} c_k v_k = \sum_{k=1}^{j} c_k (A-\lambda I)^{m-k} v = 0$$

$$\therefore \quad (A-\lambda I)^{j-1} \sum_{k=1}^{j} c_k (A-\lambda I)^{m-k} v = 0$$

$$\therefore \quad \sum_{k=1}^{j} c_k (A-\lambda I)^{m+j-k-1} v = 0$$

$$\therefore \quad \sum_{k=1}^{j-1} c_k (A-\lambda I)^{m+j-k-1} v + c_j (A-\lambda I)^{m-1} v = 0$$

For $k=1, \dots, j-1$ we have $m+j-k-1 \geq m$

and so $(A-\lambda I)^{m+j-k-1} v = 0$ (since v is a gen. eigenvector of order m)

$$\therefore \quad c_j (A-\lambda I)^{m-1} v = 0.$$

But $(A-\lambda I)^{m-1} v \neq 0$ and thus $c_j = 0$ ※.

Hence $v_1, \dots, v_m$ must be lin. indep.

(ii) Since $v_1, \dots, v_m$ are lin. indep.

$\lceil x_k(0) = v_k \rfloor$ then $x_1(0), \dots, x_m(0)$ are Lin. indep.

Hence $x_1(t), \dots, x_m(t)$ are lin. indep. $\begin{pmatrix} \text{Sheet 1} \\ \text{Qn 4a} \end{pmatrix}$

Remains to show that $\dot{x}_k = A x_k$ for $k = 1, \dots, m$.

$$\dot{x}_k = \lambda e^{\lambda t}\left(v_k + t v_{k-1} + \frac{t^2}{2} v_{k-2} + \dots + \frac{t^{k-2}}{(k-2)!} v_2 + \frac{t^{k-1}}{(k-1)!} v_1\right)$$

$$+ e^{\lambda t}\left(v_{k-1} + t v_{k-2} + \dots + \frac{t^{k-3}}{(k-3)!} v_2 + \frac{t^{k-2}}{(k-2)!} v_1\right)$$

$$= e^{\lambda t}\Big[(\lambda v_k + v_{k-1}) + (\lambda v_{k-1} + v_{k-2}) t + \dots$$

$$\dots + (\lambda v_2 + v_1) \frac{t^{k-2}}{(k-2)!} + \lambda v_1 \frac{t^{k-1}}{(k-1)!}\Big] \quad (*)$$

But $v_{j-1} = (A-\lambda I)^{m-j+1} v = (A-\lambda I)(A-\lambda I)^{m-j} v$

$$= (A-\lambda I)\, v_j$$

i.e. $\lambda v_j + v_{j-1} = A v_j$ for $2 \leq j \leq m$

Also $(A-\lambda I)^m v = 0 \Rightarrow (A-\lambda I)(A-\lambda I)^{m-1} v = 0$

$$\Rightarrow (A-\lambda I) v_1 = 0 \Rightarrow \lambda v_1 = A v_1$$

Using these results in (*) gives

$$\dot{x}_k = e^{\lambda t}\left[A v_k + A v_{k-1} t + \cdots + A v_2 \frac{t^{k-2}}{(k-2)!} + A v_1 \frac{t^{k-1}}{(k-1)!} \right]$$

$$= A e^{\lambda t}\left[v_k + v_{k-1} t + \cdots + v_2 \frac{t^{k-2}}{(k-2)!} + v_1 \frac{t^{k-1}}{(k-1)!} \right]$$

$$= A x_k \quad \text{as required.} \qquad \square$$