Analysis of FETI methods for multiscale PDEs –
Part II: Interface variation

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ANALYSIS OF FETI METHODS FOR MULTISCALE PDES – PART II: INTERFACE VARIATION

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ABSTRACT. In this article we give a new rigorous condition number estimate of the finite element tearing and interconnecting (FETI) method and a variant thereof, all-floating FETI. We consider the scalar elliptic equation in a two- or three-dimensional domain with a highly heterogeneous (multiscale) diffusion coefficient. This coefficient is allowed to have large jumps not only *across* but also *along* subdomain interfaces and in the interior of the subdomains. In other words, the subdomain partitioning does not need to resolve any jumps in the coefficient. Under suitable assumptions, we can show that the condition numbers of the one-level and the all-floating FETI system are robust with respect to strong variations in the contrast in the coefficient. We get only a dependence on some geometric parameters associated with the coefficient variation. In particular, we can show robustness for so-called face, edge, and vertex islands in high-contrast media. As a central tool we prove and use new weighted Poincaré and discrete Sobolev type inequalities that are explicit in the weight. Our theoretical findings are confirmed in a series of numerical experiments.

Keywords FETI · domain decomposition · finite element method · multiscale problems · preconditioning · varying and high contrast coefficients · weighted Poincaré inequalities

Mathematics Subject Classification (2000) 65F10, 65N22, 65N30, 65N55

1. INTRODUCTION

In recent years, the detailed and fast simulation of biological, physical or engineering processes has become an almost standard demand. Often such problems are posed on complex geometries and involve highly heterogeneous (often non-linear) material parameters. As a consequence, the development of efficient and robust parallel solvers for heterogeneous media has been a very active area of research, specifically in the setting of multiscale solvers, and in the domain decomposition and multigrid communities [2, 3, 7, 8, 13, 14, 15, 16, 17, 29, 30, 31, 32, 33, 34, 35, 40, 41, 42].

In this paper, we are concerned with the convergence of a variant of the finite element tearing and interconnecting (FETI) domain decomposition method in the context of heterogeneous (multiscale) problems, for the particular case that we have large jumps in the coefficient not only across, but also along the subdomain interfaces. As such this paper is a continuation of the work in [29, 28], where we have shown the robustness of one-level FETI methods with respect to coefficient variation in the subdomain interiors, and of [30], where we have extended these results also to dual primal FETI methods. See also [21, 26, 27].

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In the bigger context, the work follows some earlier work on the robustness of two-level Schwarz-type domain decomposition methods for heterogeneous media by Graham et al. [14], Graham and Scheichl [15, 16], as well as Scheichl and Vainikko [35].

FETI methods are robust domain decomposition methods for solving finite element discretisations of partial differential equations (PDEs) with excellent parallel scalability properties. They belong to the class of dual iterative substructuring methods and were introduced by Farhat and Roux [11]. For an extensive literature review on the analysis of FETI methods see our first paper [29].

Assuming that the coefficients of the PDE are constant in each subdomain, Klawonn and Widlund [19] proved (based on pioneering work by Mandel and Tezaur [23]) that the spectral condition number of the preconditioned FETI system is bounded by

$$C\,(1+\log(H/h))^2\,, \tag{1.1}$$

where the constant C in (1.1) is independent of possible jumps in the coefficients across subdomain interfaces when a special scaling of the preconditioner is applied. Here, as usual, H and h denote the subdomain diameter and the mesh width, respectively, and C is independent of H and h, as well. This bound (1.1) was also shown to hold true for FETI-DP methods and for the related balancing Neumann-Neumann and BDDC methods [19, 20, 22]. An excellent account of all these results can be found in the recent monograph [39] by Toselli and Widlund. We also refer to [5] where it is shown that this bound is sharp. Finally, we mention also that until recently, certain regularity assumptions on the subdomains had to be made. In the article [18] (see also [9]) the authors were able to weaken these assumptions significantly and to treat also quite irregular subdomains in two dimensions, as they appear when decomposing unstructured meshes with graph partitioners.

However, all the above mentioned analyses assume that the coefficients of the PDE are piecewise constant with respect to the subdomain partitioning. The main focus of the present work is the analysis of FETI methods for highly heterogeneous multiscale problems, i. e., in the case of coefficient jumps that are not aligned with the subdomain interfaces and/or vary strongly within a subdomain, particularly for the case that jumps occur *along* the interface. It has already been observed numerically by several authors (see e. g. [17, 21, 31, 32]) that a simple generalisation of the scaling employed by Klawonn and Widlund in [19] leads to robustness of the FETI method even in this case. In the following, we restrict ourselves to the model elliptic problem

$$-\nabla\cdot(\alpha\,\nabla u) = f\,, \tag{1.2}$$

in a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, subject to suitable boundary conditions on the boundary $\partial\Omega$. The coefficient $\alpha(x)$ may vary over many orders of magnitude in an unstructured way on Ω. It is not surprising that (1.1) also holds in this case but in general with $C = C(\alpha)$, i. e., with some possible loss of α-robustness. In our recent article [29] we have shown that the dependence on α is restricted to the variation of $\alpha(x)$ in the vicinity of subdomain interfaces (within each subdomain). More precisely, if $\Omega_{i,\eta}$ denotes the boundary layer of width η of any of the subdomains Ω_i, and if $\alpha(x) \simeq \alpha(y)$ for all $x, y \in \Omega_{i,\eta}$, then $C(\alpha) \lesssim (H/\eta)^2$, independent of the variation of $\alpha(x)$ in the remainder $\Omega_i\backslash\Omega_{i,\eta}$ of each subdomain and independent of any jumps of $\alpha(x)$ across subdomain interfaces. The hidden constant depends on the local variation of $\alpha(x)$ in $\Omega_{i,\eta}$. Of course this constant blows up when a coefficient jump appears *along* a subdomain interface, but numerical experiments ([21, 29, 32]) indicate that the condition number of the preconditioned FETI system does not blow up, at least when α jumps only a few times along each subdomain interface.

In the present work, we extend these results and give a rigorous analysis showing that one-level FETI and a variant of one-level FETI, the *all-floating* (or *total*) FETI method [10, 24, 25, 27], are α-robust in the following sense:

(i) Let us assume that each subdomain boundary layer $\Omega_{i,\eta}$ can be subdivided into two connected subregions $\Omega_{i,\eta}^{(1)}$, $\Omega_{i,\eta}^{(2)}$ such that $\alpha(x) \simeq \alpha(y)$ for all $x, y \in \Omega_{i,\eta}^{(k)}$, separately for each $k = 1, 2$. Then the constant in (1.1) is bounded by

$$C(\alpha) \lesssim (H/\eta)^{\beta},$$

where the exponent β equals d or $d-1$, depending on the size of the interface between $\Omega_{i,\eta}^{(1)}$ and $\Omega_{i,\eta}^{(2)}$. The bound is again completely independent of the values of $\alpha(x)$ in the subdomain interiors, but also of the contrast of the value of $\alpha(x)$ in $\Omega_{i,\eta}^{(1)}$ and in $\Omega_{i,\eta}^{(2)}$.

(ii) Let us assume that each subdomain boundary layer $\Omega_{i,\eta}$ can be subdivided into M connected subregions $\Omega_{i,\eta}^{(1)}, \ldots, \Omega_{i,\eta}^{(M)}$ such that $\alpha(x) \simeq \alpha(y)$ for all $x, y \in \Omega_{i,\eta}^{(k)}$, again separately for each $k = 1, \ldots, M$. In addition, we assume that we can extend each of the subregions $\Omega_{i,\eta}^{(k)}$ to the interior of Ω_i such that all the extended subregions $\Omega_i^{(k)}$ have *one common* interface X_i^*, that may be a vertex, a line, or a surface, and such that for each point $x \in \Omega_i^{(k)}$ we have $\alpha(x) \gtrsim \min_{y \in \Omega_{i,\eta}^{(k)}} \alpha(y)$, i. e., $\alpha(x)$ in the interior is essentially larger than $\alpha(x)$ in the boundary layer in each of the extensions. If the interface X_i^* between the extended subregions in each subdomain Ω_i is at least an edge (resp. vertex) in three (resp. two) dimensions, then

$$C(\alpha) \lesssim (H/\eta)^2 \left(1 + \log(H/h)\right).$$

Our central theoretical tools to prove this robustness are new *weighted* Poincaré and discrete Sobolev type inequalities that are explicit in the weight α and in the geometrical parameters H and η. To the best of our knowledge these inequalities have not appeared in the literature before. The difference to many existing inequalities in weighted norms is that our constants are explicit in the weight, and often even independent thereof. Note however that other weighted Poincaré type inequalities have been proved by Xu and Zhu [42] (based on [4]) and in a recent article by Galvis and Efendiev [12].

The remainder of this article is structured as follows. Section 2 starts with some preliminaries. Section 3 is devoted to weighted Poincaré and discrete Sobolev type inequalities. In Section 4 we describe our generalisation of the one-level and the all-floating FETI method to multiscale PDEs and give the statements of our key results. The proofs of these results are then given in Section 5 where we introduce some additional technical tools needed in our analysis. We finish with some numerical experiments that confirm our theoretical results in Section 6.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^d$ (with $d = 2$ or 3) be a connected, open, and bounded domain with Lipschitz boundary $\partial\Omega$. We consider the following model problem: Find $u \in H^1(\Omega)$, $u|_{\partial\Omega} = g_D$ such that

$$\int_\Omega \alpha(x)\, \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x)\, v(x)\, dx \qquad \forall v \in H_0^1(\Omega), \tag{2.1}$$

for given functions $f \in L^2(\Omega)$ and $g_D \in H^{1/2}(\Omega)$. For simplicity, we choose Dirichlet boundary conditions on the whole of $\partial\Omega$. However, all of our work can easily be generalised to Neumann boundary conditions on a part of $\partial\Omega$. The domain Ω decomposes into non-overlapping subdomains $\{\Omega_i\}_{i=1,\ldots,N}$ such that

$$\overline{\Omega} = \bigcup\nolimits_{i=1}^{N} \overline{\Omega}_i\,. \tag{2.2}$$

The subdomain boundaries $\partial\Omega_i$ are assumed to be Lipschitz. We define the skeleton Γ_S, the interface Γ, and the subdomain interfaces Γ_{ij} by

$$\Gamma_S := \bigcup\nolimits_i \partial\Omega_i\,, \qquad \Gamma := \Gamma_S \setminus \partial\Omega\,, \qquad \Gamma_{ij} := (\partial\Omega_i \cap \partial\Omega_j) \setminus \partial\Omega\,. \tag{2.3}$$

The subdomain diameters are denoted by $H_i := \operatorname{diam}\Omega_i$. As in our previous work [29] we need some regularity assumptions on the subdomain partition.

Definition 2.1 (regular domain)**.** For $d = 2$ (or 3), let $D \subset \mathbb{R}^d$ be a bounded contractible domain with a simply-connected Lipschitz boundary. D is called a *regular domain*, if it can be decomposed into a conforming coarse mesh of shape-regular triangles (tetrahedra). Whenever considering a family of regular domains, such as partitions into subdomains, we implicitly assume that the number of simplices forming an individual subdomain is uniformly bounded.

Definition 2.2 (shape parameter)**.** For a simplex T, we define the *shape parameter* $\rho(T)$ to be the radius of the largest inscribed ball. The *shape parameter* of a regular domain D is defined as $\rho(D) := \min\limits_{1\le i\le s} \rho(T_i)$, where $\{T_i\}_{1\le i\le s}$ are the simplices according to Definition 2.1.

Definition 2.3 (shape-regular partition)**.** Let D be an open domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. A partition of D into regular subdomains $\{D_i\}_{i=1,\ldots,N}$, such that $\overline{D} = \bigcup_{i=1}^N \overline{D}_i$, is called *shape-regular*, if

$$\rho(D_i) \simeq \operatorname{diam} D_i\,, \qquad \text{and} \qquad \overline{D}_i \cap \overline{D}_j \neq \emptyset \implies \operatorname{diam} D_i \simeq \operatorname{diam} D_j \quad \forall i,\, j = 1,\ldots,N\,.$$

Assumption A1. The subdomains $\{\Omega_i\}$ form a non-overlapping shape-regular partition of Ω, and the underlying coarse mesh (cf. Definition 2.1) is conforming.

We introduce the following topological sets similar to [39, Definition 4.1].

Definition 2.4. The skeleton Γ_S is the disjoint union of

- *subdomain faces*, regarded as open and connected sets (not necessarily planar), which are shared by two subdomains or by one subdomain and the outer boundary $\partial\Omega$,
- *subdomain edges*, also regarded as open and connected (but not necessarily straight), shared by at least two subdomains, such that the closure of all edges forms the boundaries of the faces,
- *subdomain vertices*, where at least two subdomain edges meet.

We denote by

- $\mathcal{F}_i$ the set of subdomain faces,
- $\mathcal{E}_i$ the set of subdomain edges,
- $\mathcal{V}_i$ the set of subdomain vertices

on $\partial\Omega_i$. These are further subdivided into faces, edges, and vertices on the interface $\partial\Omega_i \cap \Gamma$, denoted by $\mathcal{F}_i^\Gamma$, $\mathcal{E}_i^\Gamma$, and $\mathcal{V}_i^\Gamma$, respectively, and into faces, edges, and vertices on the outer (Dirichlet) boundary $\partial\Omega_i \cap \partial\Omega$, denoted by $\mathcal{F}_i^D$, $\mathcal{E}_i^D$, and $\mathcal{V}_i^D$, respectively.

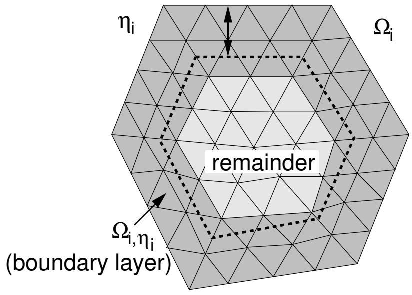

FIGURE 1. Boundary layer Ω_{i,η_i} of the subdomain Ω_i, cf. Definition 2.5.

We consider simplicial triangulations $\mathcal{T}_i$ on Ω_i which are quasi-uniform and shape-regular. The local mesh parameter is denoted by h_i. We require that the triangulations match on the subdomain interfaces. Note that these assumptions (together with Assumption A1) imply that $H_i \simeq H_j$ and $h_i \simeq h_j$ if $\Gamma_{i,j} \neq \emptyset$. The set of nodes of the mesh on the local boundary $\partial\Omega_i$ is denoted by $\partial\Omega_i^h$, and similarly we define Γ^h and Γ_{ij}^h to be the sets of nodes on the interface Γ and on the subdomain interface Γ_{ij}, respectively. A typical node will be denoted by x^h. For the discretisation of (2.1) we use continuous piecewise linear finite elements. We denote by $V^h(\Omega)$, $V^h(\Omega_i)$ and $V^h(\partial\Omega_i)$ the spaces of continuous piecewise linear functions (with respect to the mesh) on the domain Ω, on a subdomain Ω_i and on the local boundary $\partial\Omega_i$, respectively. Note that these spaces do *not* incorporate the essential boundary conditions on $\partial\Omega$. Without loss of generality we assume that the given Dirichlet trace g_D is in $V^h(\partial\Omega)$. Also, without loss of generality we assume that the coefficient α is piecewise constant on the elements of the triangulation.

Our analysis will require some notion of a boundary layer near subdomain interfaces. Therefore we need the following definition which is closely related to the one in [14].

Definition 2.5 (discrete boundary layer)**.** Let $i \in \{1, \ldots, N\}$ and $\eta_i > 0$. We define the *discrete boundary layer* of Ω_i to be the open set Ω_{i,η_i} such that

$$\overline{\Omega}_{i,\eta_i} := \bigcup \left\{ \overline{\tau} : \tau \in \mathcal{T}_i \,,\ \operatorname{dist}(\tau, \partial\Omega_i) \leq \eta_i \right\},$$

i. e., the set of all points which have at most distance η_i from the boundary $\partial\Omega_i$ extended to a union of elements. An illustration of this definition is given in Fig. 1.

Definition 2.6 (η_i-regular)**.** The discrete boundary layer Ω_{i,η_i} is called *η_i-regular* if there exists a shape-regular partition

$$\Xi_{i,\eta_i} := \left\{ \omega_{i,1}, \ldots, \omega_{i,s_i} \right\}$$

of Ω_{i,η_i} into non-overlapping, regular (in the sense of Definition 2.1) *patches* $\omega_{i,j}$ with $\operatorname{diam} \omega_{i,j} \simeq \eta_i$, such that (i) the intersection of $\partial\omega_{i,j}$ with $\partial\Omega_i$ is non-empty and equal to the union of a set of faces of the simplices forming the patch $\omega_{i,j}$, and (ii) the intersection of $\partial\omega_{i,j}$ with an edge $E \in \mathcal{E}_i$ is the union of edges of the simplices forming the patch $\omega_{i,j}$.

Assumption A2. For each $i \in \{1, \ldots, N\}$ the parameter $\eta_i > 0$ is chosen such that

(i) Ω_{i,η_i} is η_i–regular,
(ii) $\eta_i \simeq \eta_j$, if $\Gamma_{ij} \neq \emptyset$, and
(iii) the meshes induced by the patches match on the subdomain interfaces.

For the sake of simplicity, we make no difference between functions on discrete spaces and their vector representations with respect to the standard nodal basis, as well as between operators and their matrix representations with respect to the same basis. Similarly, we identify any discrete space X with its dual space X^*.

On the subdomain Ω_i, we can assemble the local finite element stiffness matrix K_i and group it with respect to the unknowns on the subdomain boundary (subscript B) and the interior (subscript I),

$$K_i = \begin{pmatrix} K_{i,BB} & K_{i,BI} \\ K_{i,IB} & K_{i,II} \end{pmatrix} . \tag{2.4}$$

Since none of the spaces $V^h(\Omega_i)$ incorporates essential boundary conditions, each of the local operators K_i is only positive semi-definite with $\ker K_i = \mathrm{span}\{\mathbf{1}_{\Omega_i}\}$, where $\mathbf{1}_{\Omega_i}$ denotes the constant function 1 on Ω_i. We define the Schur complement S_i of $K_{i,II}$ in K_i by

$$S_i = K_{i,BB} - K_{i,BI}[K_{i,II}]^{-1}K_{i,IB} . \tag{2.5}$$

Note, that the application of S_i means actually solving a Dirichlet boundary value problem on the subdomain Ω_i. Since S_i is symmetric positive semidefinite, it defines a seminorm,

$$|v|_{S_i} := \langle S_i\, v,\, v\rangle^{1/2} \qquad \text{for } v \in V^h(\partial\Omega_i) , \tag{2.6}$$

that obeys the minimising property

$$|v|_{S_i}^2 = \min\Big\{ \int_{\Omega_i} \alpha(x)\, |\nabla\widetilde{v}(x)|^2\, dx \;:\; \widetilde{v} \in V^h(\Omega_i) \,,\; \widetilde{v}_{|\partial\Omega_i} = v \Big\} . \tag{2.7}$$

We denote by $\mathcal{H}_{i,\alpha}v$ the function $\widetilde{v} \in V^h(\Omega_i)$ for which the minimum is attained, and we call this function the *discrete α-harmonic extension* of v from $V^h(\partial\Omega_i)$ to $V^h(\Omega_i)$.

The Galerkin projection of (2.1) onto the space $V^h(\Omega)$ (which does not include the essential boundary conditions) leads to the following constrained linear system. Find $\widetilde{u} \in V^h(\Omega)$, $u|_{\partial\Omega} = g_D$ such that (2.1) (substituting $\widetilde{u}$ for u and $\widetilde{v}$ for v) holds for all test functions $\widetilde{v} \in V^h(\Omega)$, $\widetilde{v}|_{\partial\Omega} = 0$, in short

$$\widetilde{K}\, \widetilde{u} = \widetilde{f} . \tag{2.8}$$

The global stiffness matrix $\widetilde{K}$ and the load vector $\widetilde{f}$ can be assembled from (parts of) the local contributions K_i and f_i, respectively. Non-homogeneous Dirichlet boundary conditions can be treated with standard homogenisation techniques. FETI methods are special domain decomposition methods to solve system (2.8) in parallel. The common idea of these methods is to decouple the system subdomain-wise and to enforce the continuity of $\widetilde{u}$ across the subdomain interfaces by Lagrange multipliers λ. There are various strategies to eliminate the primal variables and to design parallel preconditioners for the dual system in λ; these are the one-level, dual-primal, and all-floating or total FETI methods, see [10, 39], as well as [24, 25] for further results developed in the closely related area of boundary elements (BETI).

To simplify the presentation and the proofs we will follow mainly the all-floating approach where the Dirichlet boundary conditions are also incorporated via Lagrange multipliers. However, the results carry over also to the more classical one-level FETI approach, albeit with one additional assumption. We will come back to this below. See [27, 30] for details on how the proof techniques can also be extended to dual-primal methods.

3. Weighted Poincaré and discrete Sobolev inequalities

The crucial new theoretical tools needed for our analysis below are weighted Poincaré and discrete Sobolev inequalities. As we have seen in [29], the robustness of FETI methods is not affected by variations of the coefficient in the interior of each subdomain Ω_i. This is the reason why in [29] we generalised the Poincaré and discrete Sobolev inequalities to the discrete boundary layer Ω_{i,η_i}. Here we go one step further and prove certain weighted versions of Poincaré and discrete Sobolev inequalities on the discrete boundary layer Ω_{i,η_i} in order to allow also for some coefficient variation along the interfaces between subdomains.

3.1. **Weighted Poincaré inequality – two coefficient regions.** Let us consider a single subdomain Ω_i and let us fix $\eta_i > 0$ according to Assumption 2. To state our weighted Poincaré inequality we further decompose the discrete boundary layer Ω_{i,η_i} into two non-overlapping connected subregions $\Omega^{(1)}_{i,\eta_i}$ and $\Omega^{(2)}_{i,\eta_i}$ such that

$$\overline{\Omega}_{i,\eta_i} = \overline{\Omega}^{(1)}_{i,\eta_i} \cup \overline{\Omega}^{(2)}_{i,\eta_i}\,, \tag{3.1}$$

see also Figure 2. Let $\Gamma^{(12)}_{i,\eta_i}$ be the larger of the connected components of the interface $\partial\Omega^{(1)}_{i,\eta_i} \cap \partial\Omega^{(2)}_{i,\eta_i}$.

Definition 3.1. If Ω_{i,η_i} is η_i–regular, we say that the partitioning (3.1) is *compatible*, if each of the subregions $\Omega^{(k)}_{i,\eta_i}$ can be partitioned into a union of the patches $\omega_{i,j}$ in Definition 2.6, such that the interface $\Gamma^{(12)}_{i,\eta_i}$ is the union of faces of the patches.

On each of the subregions $\Omega^{(k)}_{i,\eta_i}$ we set

$$\underline{\alpha}^{(k)}_{i,\eta_i} := \min_{x\in\Omega^{(k)}_{i,\eta_i}} \alpha(x)\,, \qquad \overline{\alpha}^{(k)}_{i,\eta_i} := \max_{x\in\Omega^{(k)}_{i,\eta_i}} \alpha(x)\,, \tag{3.2}$$

such that $\underline{\alpha}^{(k)}_{i,\eta_i} \le \alpha(x) \le \overline{\alpha}^{(k)}_{i,\eta_i}$ for all $x \in \Omega^{(k)}_{i,\eta_i}$.

Lemma 3.2 (weighted Poincaré inequality)**.** *Let $\eta_i > 0$ and let Ω_{i,η_i} be η_i–regular. Suppose that the partitioning* (3.1) *is compatible and $u \in H^1(\Omega_{i,\eta_i})$ with $\int_{\Gamma^{(12)}_{i,\eta_i}} u\,ds = 0$. Then,*

$$\frac{1}{\eta_i^2}\int_{\Omega_{i,\eta_i}} \alpha(x)\,|u(x)|^2\,dx \;\lesssim\; C^*_{i,\eta_i} \int_{\Omega_{i,\eta_i}} \alpha(x)\,|\nabla u(x)|^2\,dx$$

with

$$C^*_{i,\eta_i} := \Big(\frac{H_i}{\eta_i}\Big)^d \max_{k=1,2} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}}\,,$$

in general. If $d = 3$ and the area of the interface fulfils $|\Gamma^{(12)}_{i,\eta_i}| \gtrsim H_i\,\eta_i$ the inequality holds with

$$C^*_{i,\eta_i} = \Big(\frac{H_i}{\eta_i}\Big)^2 \max_{k=1,2} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}}\,.$$

Proof. Let $k \in \{1,2\}$. Since Ω_{i,η_i} is η_i–regular and the partitioning (3.1) is compatible, we can partition $\Omega^{(k)}_{i,\eta_i}$ into a union of patches from the set Ξ_{i,η_i} (cf. Definition 2.6) denoted by $\{\omega^{(k)}_{i,j}\}$, and

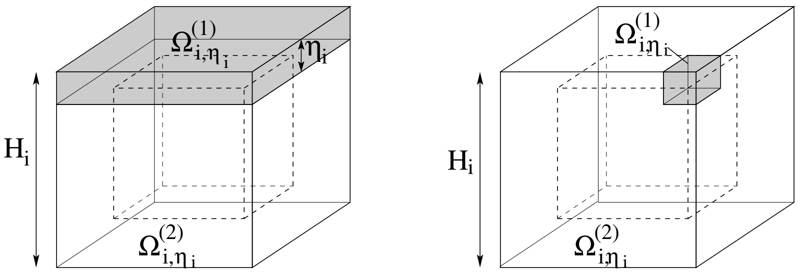

FIGURE 2. Different settings of the subregions $\Omega^{(k)}_{i,\eta_i}$ in three dimensions (*Left:* $|\Gamma^{(12)}_{i,\eta_i}| \simeq H_i\,\eta_i$; *Right:* $|\Gamma^{(12)}_{i,\eta_i}| \simeq \eta_i^2$).

$$\frac{1}{\eta_i^2}\,\|u\|^2_{L^2(\Omega^{(k)}_{i,\eta_i})} = \sum_j \frac{1}{\eta_i^2}\,\|u\|^2_{L^2(\omega^{(k)}_{i,j})}\,.$$

Let $\Lambda_i^{(k)} := \partial\Omega^{(k)}_{i,\eta_i} \cap \partial\Omega_i$. Then, by Definition 2.6, $\Lambda_i^{(k)}$ is the union of faces $\gamma^{(k)}_{i,j}$ of the patches $\{\omega^{(k)}_{i,j}\}$. Hence, by a standard Friedrichs type inequality on each patch $\omega^{(k)}_{i,j}$, we have

$$\frac{1}{\eta_i^2}\,\|u\|^2_{L^2(\Omega^{(k)}_{i,\eta_i})} \;\lesssim\; \sum_j \Big\{ |u|^2_{H^1(\omega^{(k)}_{i,j})} + \frac{1}{\eta_i}\,\|u\|^2_{L^2(\gamma^{(k)}_{i,j})} \Big\} \;=\; |u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})} + \frac{1}{\eta_i}\,\|u\|^2_{L^2(\Lambda_i^{(k)})}\,. \tag{3.3}$$

Now, on each of the parts $\Lambda_i^{(k)}$ we can apply a generalised Poincaré inequality of the type proved in [29], namely

$$\frac{1}{\eta_i}\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \Big(\frac{H_i}{\eta_i}\Big)^\beta\,|u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})} \qquad \forall u \in H^1(\Omega^{(k)}_{i,\eta_i}),\ \int_{\Gamma^{(12)}_{i,\eta_i}} u\,ds = 0\,, \tag{3.4}$$

with $\beta = d$ in general, and $\beta = 2$ if $d = 3$ and $|\Gamma^{(12)}_{i,\eta_i}| \gtrsim H_i\,\eta_i$. A proof of (3.4) is given in the Appendix. Substituting (3.4) into (3.3) and using (3.2) we finally get

$$\begin{aligned}\frac{1}{\eta_i^2}\int_{\Omega_{i,\eta_i}} \alpha(x)\,|u(x)|^2\,dx \;&\lesssim\; \sum_{k=1,2} \overline{\alpha}^{(k)}_{i,\eta_i}\Big(\frac{H_i}{\eta_i}\Big)^\beta |u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})}\\ &\lesssim\; \Big(\frac{H_i}{\eta_i}\Big)^\beta \max_{k=1,2}\frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}} \int_{\Omega_{i,\eta_i}} \alpha(x)\,|\nabla u(x)|^2\,dx\,.\end{aligned}$$

□

Remark 3.3. (i) First of all note that the Poincaré constant in Lemma 3.2 only depends on the local variation of α on each subregion $\Omega^{(k)}_{i,\eta_i}$ of Ω_{i,η_i}, and is completely independent of the values and the variation of α in the interior of Ω_i, or of the contrast between the two regions $\Omega^{(1)}_{i,\eta_i}$ and $\Omega^{(2)}_{i,\eta_i}$. In particular, if $\alpha(x)$ is constant on each of the regions $\Omega^{(k)}_{i,\eta_i}$ then $C^*_{i,\eta_i} = (H_i/\eta_i)^\beta$ (with β depending on the size of the interface $\Gamma^{(12)}_{i,\eta_i}$) and the Poincaré constant is completely independent of the range of α.

(ii) Note that if $\eta_i > H_i/2$ then $\Omega_{i,\eta_i} = \Omega_i$ and so Lemma 3.2 also holds on all of Ω_i. In this case the inequality reduces to

$$\frac{1}{H_i^2}\int_{\Omega_i} \alpha(x)\,|u(x)|^2\,dx \;\lesssim\; C^*_{i,H_i}\int_{\Omega_i} \alpha(x)\,|\nabla u(x)|^2\,dx\,.$$

(iii) Also if one of the subregions, say $\Omega_{i,\eta_i}^{(2)}$, is empty, we can set $\Gamma_{i,\eta_i}^{(12)} := \partial\Omega_i$, and the inequality reduces to

$$\frac{1}{\eta_i^2}\int_{\Omega_{i,\eta_i}} \alpha(x)\,|u(x)|^2\,dx \;\lesssim\; \Big(\frac{H_i}{\eta_i}\Big)^2 \frac{\overline{\alpha}_{i,\eta_i}^{(1)}}{\underline{\alpha}_{i,\eta_i}^{(1)}} \int_{\Omega_{i,\eta_i}} \alpha(x)\,|\nabla u(x)|^2\,dx\,.$$

(iv) Note that our new proof idea (see Section 5) uses only arguments on *patches* instead of subdomains. Therefore we can even generalise Lemma 3.2 to non-regular subdomains whose boundary layers are η_i-regular and where each subdomain face (resp. edge) still touches $\mathcal{O}((H_i/\eta_i)^{d-1})$ patches. Then, in order to prove (3.4) (see the Appendix) we need the following assumptions for each $k = 1, 2$. (a) There exists a shape-regular partition of $\Omega_{i,\eta_i}^{(k)}$ into regular patches of diameter $\mathcal{O}(\eta_i)$, (b) each pair of such patches must be connectible via a path containing at most $\mathcal{O}(H_i/\eta_i)$ patches, and (c) in three dimensions, the overlapping argument as stated in the proof of [29, Lemma 4.3] must be applicable. In fact these assumptions can only be violated if the regions $\Omega_{i,\eta_i}^{(k)}$ are rather badly shaped.

Obviously, the above concept can be extended in a straightforward way to $M_i > 2$ subregions $\Omega_{i,\eta_i}^{(k)}$ by introducing $M_i - 1$ functionals.However, since we will only have one degree of freedom per subdomain in our coarse space (see Lemma 5.2 below), this will be of no use here. Instead we prove a weighted discrete Sobolev type inequality in the next subsection.

Weighted Poincaré inequalities with more than one functional (based on [4]) were used by Xu and Zhu [42] recently to prove coefficient–robustness of geometric multigrid in the case of piecewise constant coefficients that are resolved by the coarse meshes. A very similar weighted Poincare inequality to the one we gave here (with one functional) was also recently proved by Galvis and Efendiev [12] and used in the analysis of two-level overlapping Schwarz.

3.2. **Weighted discrete Sobolev inequality – multiple coefficient regions.** Let us again consider a single subdomain Ω_i, but now let $\{\Omega_i^{(k)}\}_{k=1,\dots,M_i}$ be a non-overlapping shape-regular partitioning of (all of) Ω_i into M_i connected subregions, i. e.

$$\overline{\Omega}_i = \overline{\Omega}_i^{(1)} \cup \ldots \cup \overline{\Omega}_i^{(M_i)}\,. \tag{3.5}$$

such that the intersection of all subregions

$$X_i^* := \bigcap_{k=1}^{M_i} \overline{\Omega}_i^{(k)} \tag{3.6}$$

is a vertex, an edge, or a face of all the subregions $\Omega_i^{(k)}$. Note that this restricts us to $M_i = \mathcal{O}(1)$ and that the diameter of each of the subregions $\Omega_i^{(k)}$ is $\mathcal{O}(H_i)$, see Figure 3, left.

Now, let $\eta_i > 0$ and let $\Omega_{i,\eta_i}^{(k)} := \Omega_i^{(k)} \cap \Omega_{i,\eta_i}$, i. e. the part of the discrete boundary layer in $\Omega_i^{(k)}$. For each $\Omega_{i,\eta_i}^{(k)}$, let $\underline{\alpha}_{i,\eta_i}^{(k)}$ and $\overline{\alpha}_{i,\eta_i}^{(k)}$ be as defined in (3.2). Note, however, that α may take larger values in the interior of Ω_i.

Lemma 3.4 (weighted discrete Sobolev inequality)**.** *Let $\eta_i > 0$ and let Ω_{i,η_i} be η_i–regular. Suppose that the partitioning* (3.5) *is compatible and that*

$$\alpha(x) \;\gtrsim\; \underline{\alpha}_{i,\eta_i}^{(k)} \qquad \forall x \in \Omega_i^{(k)} \quad \forall k = 1,\ldots,M_i\,. \tag{3.7}$$

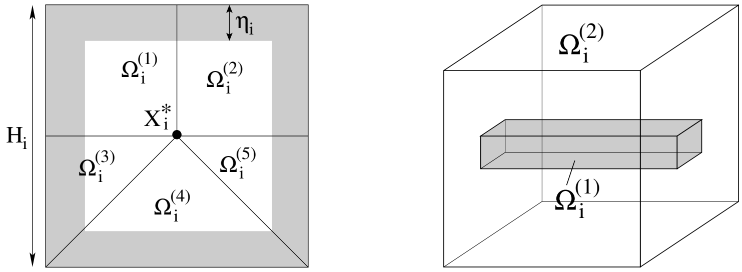

FIGURE 3. *Left* multiple coefficient regions, admissible for Lemma 3.4, in grey: $\Omega_{i,\eta_i}^{(k)}$, *right* beam type example, see Remark 3.5(iii)

Then, for all $u \in V^h(\Omega_i)$ *with* $\int_{X_i^*} u\, ds = 0$*, if* X_i^* *is a face or an edge, and* $u(X_i^*) = 0$*, if* X_i^* *is a vertex, we have*

$$\frac{1}{\eta_i^2} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |u(x)|^2\, dx \;\lesssim\; C_{i,\eta_i}^* \int_{\Omega_i} \alpha(x)\, |\nabla u(x)|^2\, dx$$

with

$$C_{i,\eta_i}^* \;:=\; \sigma(H_i, h_i)\, \frac{H_i}{\eta_i}\, \max_{k=1}^{M_i} \frac{\overline{\alpha}_{i,\eta_i}^{(k)}}{\underline{\alpha}_{i,\eta_i}^{(k)}}$$

and

$$\sigma(H_i, h_i) := \begin{cases} 1 & \textit{if } X_i^* \textit{ is a face in three dimensions or an edge in two dimensions,} \\ (1 + \log(H_i/h_i)) & \textit{if } X_i^* \textit{ is an edge in three dimensions or a vertex in two dimensions,} \\ H_i/h_i & \textit{if } X_i^* \textit{ is a vertex in three dimensions.} \end{cases}$$

Proof. The proof is identical to that of Lemma 3.2. However, instead of (3.4) we use the following discrete Sobolev inequality on each of the subregions $\Omega_i^{(k)}$,

$$\frac{1}{H_i}\, \|u\|_{L^2(\Lambda_i^{(k)})}^2 \;\lesssim\; \sigma(H_i, h_i)\, |u|_{H^1(\Omega_i^{(k)})}^2\,. \tag{3.8}$$

This inequality follows from [39, Lemma 4.15, Lemma 4.21, and Sect. 4.6.1]. □

Remark 3.5. (i) Note that as long as the coefficient α fulfils condition (3.7) the constant in Lemma 3.2 only depends on the local variation of α on each subregion $\Omega_i^{(k)}$, but not on the contrast between the regions $\Omega_i^{(k)}$. In particular if α is constant on each of the regions $\Omega_i^{(k)}$ then the Poincaré constant is completely independent of α.

(ii) Next we would like to introduce the concept of an *artificial coefficient* in order to establish a better understanding of Lemma 3.4. Suppose that we can find an artificial coefficient $\alpha_{\mathrm{art}}(x)$ and M_i regions $\Omega_i^{(k)}$ with the common interface X_i^*, such that α_{art} restricted to $\Omega_i^{(k)}$ stays in the same range as α_{art} restricted to $\Omega_{i,\eta_i}^{(k)}$. Then, Lemma 3.4 holds for $\alpha = \alpha_{\mathrm{art}}$. However, it still holds, if α is made larger in the subdomain interior, i. e. if $\alpha(x) \geq \alpha_{\mathrm{art}}(x)$ for all $x \in \Omega_i \setminus \Omega_{i,\eta_i}$.

(iii) The beam type example in Figure 3, right, cannot be treated with the weighted Poincaré inequality in Lemma 3.2 because the beam is not connected in the boundary layer. However, it can be treated with Lemma 3.4 since the intersection of the two subregions contains a face. If the thickness of the beam is $\mathcal{O}(\eta_i)$ one can even derive analogous estimates being explicit in the aspect ratio H_i/η_i using the framework given in the Appendix.

The following stronger result follows immediately from the proof of Lemma 3.4.

Corollary 3.6. *The statement of Lemma 3.4 still holds if the subregions $\Omega_i^{(k)}$ overlap and condition* (3.7) *holds. Also, their union does not have to be the whole of Ω_i as long as the entire boundary layer Ω_{i,η_i} remains covered and inequality* (3.8) *holds.*

Remark 3.7. (i) The overlapping case can be seen as follows. For each subregion $\Omega_i^{(k)}$ we can find an artificial coefficient $\alpha_{\text{art}}^{(k)}$ which stays in the same range as when restricted to $\Omega_{i,\eta_i}^{(k)}$. The real coefficient α must now be larger than each $\alpha_{\text{art}}^{(k)}$, i. e., for a point x common to more than one subregion, $\alpha(x)$ must be larger than the maximum of the artificial coefficients at x. Then by carefully adding up the separate inequalities (3.8) on these subregions, one easily concludes the weighted discrete Sobolev inequality of Lemma 3.4 by *lifting* the artificial coefficients to α. Note that we can even choose artificial subregions $\Omega_{i,\text{art}}^{(k)}$ that overlap in the boundary layer. This way we can treat the plate type example, shown in Figure 4, left, where a region $\Omega_i^{(1)}$ with a larger coefficient cuts through a region with a smaller coefficient. To see this we set $\Omega_{i,\text{art}}^{(1)} = \Omega_i^{(1)}$ and $\Omega_{i,\text{art}}^{(2)} = \Omega_i$. Then the inequality follows again by a lifting argument.

(ii) In contrast to the beam example in Remark 3.5(iii) it makes a big difference whether the coefficient in the plate is larger or smaller than in the surrounding region. In the latter case our theory does not apply and we cannot guarantee that the Poincaré constant in Lemma 3.4 is independent of the contrast.

(iii) To discuss the second assertion in the Corollary 3.6, suppose now that the union of the subregions $\Omega_i^{(k)}$ forms just a part of Ω_i and let $\Omega_i^{(R)}$ denote the remainder that is located in the interior of Ω_i, see Figure 4, right. Then the coefficient in $\Omega_i^{(R)}$ can be chosen arbitrarily, in particular arbitrarily *small*, and the Poincaré constant in Lemma 3.4 is independent of the values of α in $\Omega_i^{(R)}$. The difference compared to (ii) in this remark is that in the current setting the inclusions do not *separate* regions of larger coefficients from each other. We point out that the shapes of $\Omega_i^{(k)}$ may of course influence the Poincaré constant as well. If we generalise to subregions $\Omega_i^{(k)}$ that can be partitioned into regular patches of diameter $\mathcal{O}(\eta_i)$, we can work out similar results using the framework given in the Appendix, but possibly with a different dependency on H_i/η_i. We would also like to point to the recent articles [9, 18] where inequalities related to (3.8) were proved for quite irregular subregions in two dimensions with only weak dependence on their shapes.

4. FETI methods for multiscale elliptic PDEs

Let us now describe the classical one-level and the all-floating FETI method. Following [39, Sect. 6.3], we introduce separate unknowns $u_i \in V^h(\Omega_i)$ on the subdomains and denote by $u = [u_1, \ldots, u_N]^\top$ the discontinuous approximation of $\widetilde{u}$ in $\prod_{i=1}^N V^h(\Omega_i)$. The continuity of the solution is enforced by constraints of the form

$$u_i(x^h) - u_j(x^h) = 0 \qquad \text{for } x^h \in \Gamma_{ij}^h \,. \tag{4.1}$$

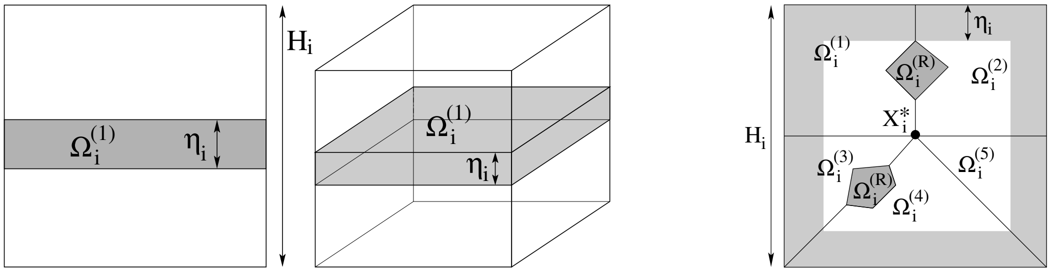

FIGURE 4. Coefficient distributions related to Corollary 3.6. *Left* plate type example, see Remark 3.5(i), *right* inclusions $\Omega_i^{(R)}$,

Note that at nodes where more than two subdomains meet this introduces redundancies. In this work we consider only fully redundant constraints, i. e., the full set of possible constraints is used. In the all-floating formulation (cf. [10, 25]), we incorporate the Dirichlet boundary conditions by additional constraints of the form

$$u_i(x^h) - g_D(x^h) = 0 \qquad \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega\,. \tag{4.2}$$

Note that unlike the continuity constraints (4.1), the Dirichlet constraints (4.2) act completely locally on each subdomain. Let N_C denote the total number of constraints in (4.1) and (4.2) and set $U := \mathbb{R}^{N_C}$. Then we can write (4.1) and (4.2) compactly as

$$\sum_{i=1}^N B_i\, u_i = b \;\in\; U\,, \qquad \text{or equivalently,} \qquad B\, u = b\,. \tag{4.3}$$

The operators $B_i : V^h(\Omega_i) \to U$ can be represented by signed Boolean matrices. The full jump operator $B : \prod_{i=1}^N V^h(\Omega_i) \to U$ is defined as $B := [B_1, \ldots, B_N]$. The entries of the vector $b \in U$ that correspond to the constraints in (4.2) contain the values $g_D(x^h)$. All other entries of b are zero. By a simple energy minimisation argument, it follows from the above that solving (2.8) is equivalent to finding $u = [u_1, \ldots, u_N]^\top$ and $\lambda \in U$ satisfying the saddle point system

$$\begin{pmatrix} K_1 & & 0 & B_1^\top \\ & \ddots & & \vdots \\ 0 & & K_N & B_N^\top \\ B_1 & \cdots & B_N & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \\ \lambda \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \\ b \end{pmatrix}. \tag{4.4}$$

System (4.4) is uniquely solvable up to adding elements from $\ker B^\top$ to λ if and only if the block $K := \operatorname{diag}(K_i)$ is SPD on $\ker B$, or equivalently, $\ker K \cap \ker B = \{0\}$. This condition is true whenever the Dirichlet boundary is non-empty. We refer to U as the space of Lagrange multipliers. The more classical one-level FETI formulation leads to a very similar saddle point system. See [29, 39] for details.

Recall that each of the operators K_i is only positive semi-definite and $\ker K_i = \operatorname{span}\{\mathbf{1}_{\Omega_i}\}$. We introduce operators $R_i : \mathbb{R} \to V^h(\Omega_i) : \xi_i \mapsto \xi_i\, \mathbf{1}_{\Omega_i}$ such that $\operatorname{range} R_i = \ker K_i$. Let $K_i^\dagger$ denote some pseudoinverse of K_i. Then, under the compatibility condition

$$f_i - B_i^\top \lambda \;\in\; \operatorname{range} K_i\,,$$

we can eliminate the unknowns u_i from (4.4), i. e.

$$u_i \;=\; K_i^\dagger [f_i - B_i^\top \lambda] + R_i\, \xi_i\,, \tag{4.5}$$

for some $\xi = [\xi_i]_{i=1}^N$. Finally, using that range $K_i = \ker R_i^\top$ and with the abbreviations

$$\begin{aligned} &K := \operatorname{diag}(K_i)\,, && f := [f_1, \ldots, f_N]^\top\,, && R := \operatorname{diag}(R_i)\,, && K^\dagger := \operatorname{diag}(K_i^\dagger)\,,\\ &F := B\, K^\dagger\, B^\top\,, && G := B\, R\,, && d := B\, K^\dagger f - b\,, && e := R^\top f\,, \end{aligned}$$

we arrive at the dual formulation (see [39] for details): Find $(\lambda, \xi) \in U \times \mathbb{R}^N$ such that

$$\begin{pmatrix} F & -G \\ G^\top & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \xi \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix}. \tag{4.6}$$

In practice, this saddle point system is solved using a projection

$$P : U \to \ker G^\top \subset U \qquad \text{such that} \qquad P := I - Q\, G\, (G^\top Q\, G)^{-1}\, G^\top\,, \tag{4.7}$$

where a careful choice of the SPD scaling operator $Q : U \to U$ will be crucial to render the method robust to coefficient variation (see (4.14)–(4.15) below). Introducing the subspace

$$V := \{\lambda \in U : \langle B\, z\,, \lambda\rangle = 0 \quad \forall z \in \ker K\} = \ker G^\top = \operatorname{range} P \tag{4.8}$$

and using the projection operator the saddle point system (4.6) can be reduced to solving

$$P^\top F\, \widetilde{\lambda} = P^\top (d - F\, \lambda_0)\,, \tag{4.9}$$

for $\widetilde{\lambda} \in V$, where $\lambda_0 = Q\, G (G^\top Q\, G)^{-1} e$. The original variables λ and ξ can then be recovered from the relations $\lambda = \lambda_0 + \widetilde{\lambda}$ and $\xi = (G^\top Q\, G)^{-1} G^\top Q\, (F\, \lambda - d)$.

Several things are worth mentioning. First note that $G^\top Q\, G$ is the Galerkin projection of $B^\top Q\, B$ onto $\ker K$. Since $\ker K \cap \ker B = \{0\}$, the operator $G^\top Q\, G$ is invertible as long as Q is SPD on range G. Furthermore, since equation (4.9) is SPD on the subspace V modulo $\ker B^\top$, it can be solved using a projected preconditioned conjugate gradient method. The actual solution u can finally be recovered using (4.5). Note that even if λ is only unique up to an element from $\ker B^\top$, the solution u is always unique (see e. g. [27] for a more detailed discussion). The crucial ingredients that will make the method robust with respect to varying coefficients are the choice of Q and of the preconditioner M^{-1}. In the sequel we present suitable choices for Q and M^{-1}, generalising the method analysed by Klawonn and Widlund and building on the results in [29, §5.2 & 5.3].

4.1. **Choice of Q and M^{-1}.** We follow [29, §5.2]. In order to define M^{-1} we need to introduce scaling operators D_i for the Boolean matrices B_i on the space U of Lagrange multipliers. For each subdomain Ω_i and for each $x^h \in \partial\Omega_i$, we define the pointwise weight

$$\widehat{\alpha}_i(x^h) := \max_{\tau \subset \omega_i(x^h)} \alpha_{|\tau}\,, \tag{4.10}$$

where $\omega_i(x^h) \subset \overline{\Omega}_i$ is the (local) patch of all elements in $\mathcal{T}_i$ that contain node x^h (see [29, Fig. 5]). Furthermore, we define the weighted counting functions

$$\delta_i^\dagger(x^h) := \begin{cases} \widehat{\alpha}_i(x^h) \Big[\sum\limits_{k \in \mathcal{N}(x^h)} \widehat{\alpha}_k(x^h) \Big]^{-1} & \text{for } x^h \in \partial\Omega_i^h\,,\\ 0 & \text{for } x^h \in \Gamma_S^h \setminus \partial\Omega_i^h\,, \end{cases} \tag{4.11}$$

where $\mathcal{N}(x^h) := \{k : x^h \in \partial\Omega_k\}$, the index set of the subdomains sharing node $x^h \in \Gamma^h$. Each function $\delta_i^\dagger(\cdot)$ can be interpreted as a finite element function on the skeleton Γ_S, and the union of all these functions provides a partition of unity on the skeleton, cf. [39, Section 6.2.1]. Now, to define D_i, let $\lambda_{ij}(x^h)$ denote the component of $\lambda \in U$ which corresponds to the constraint (of type (4.1)) at an interface node $x^h \in \Gamma_{ij}^h$ and let $\lambda_{iD}(x^h)$ denote the component of λ corresponding to the constraint (of type (4.2)) at a Dirichlet node $x^h \in \partial\Omega_i^h \cap \partial\Omega$. Let $D_i : U \to U$ be the diagonal matrix such that

$$\begin{cases} (D_i\lambda)_{ij}(x^h) := \delta_j^\dagger(x^h)\lambda_{ij}(x^h) & \text{for } x^h \in \Gamma_{ij}^h\,, \\ (D_i\lambda)_{iD}(x^h) := \lambda_{iD}(x^h) & \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega\,. \end{cases} \tag{4.12}$$

Then our FETI preconditioner is chosen to be

$$M^{-1} := \sum_{i=1}^N D_i\, B_i \begin{pmatrix} S_i & 0 \\ 0 & 0 \end{pmatrix} B_i^\top D_i\,. \tag{4.13}$$

where the S_i are the Schur complements of the stiffness matrices K_i defined in (2.5).

The linear operator Q which appears in the projection P in (4.7) is (usually) also set to be a diagonal matrix with

$$\begin{cases} (Q\,\lambda)_{ij}(x^h) := \min(\widehat{\alpha}_i(x^h), \widehat{\alpha}_j(x^h))\, q_i(x^h)\, \lambda_{ij}(x^h) & \text{for } x^h \in \Gamma_{ij}^h\,, \\ (Q\,\lambda)_{iD}(x^h) := \widehat{\alpha}_i(x^h)\, q_i(x^h)\, \lambda_{iD}(x^h) & \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega\,, \end{cases} \tag{4.14}$$

where, in three dimensions,

$$q_i(x^h) := \begin{cases} (1 + \log(H_i/h_i))\,\frac{h_i^2}{H_i} & \text{if } x^h \text{ lies on a subdomain face}\,, \\ h_i & \text{if } x^h \text{ lies on a subdomain edge or vertex}\,. \end{cases} \tag{4.15}$$

In two dimensions, $q_i(x^h) = (1 + \log(H_i/h_i))\, h_i/H_i$ for edges and $q_i(x^h) = 1$ for vertices. Note that $q_i(x^h) \simeq q_j(x^h)$ for neighbouring subdomains since $H_i \simeq H_j$ and $h_i \simeq h_j$. If the coefficient α is piecewise constant with respect to the subdomains, our choices of M^{-1} and Q coincide with the ones given in [25] (which builds on [19]).

In each step of the projected preconditioned conjugate gradient method, we have to apply $P^\top F$ and $P\,M^{-1}$. Therefore, the main ingredients of FETI methods are local Dirichlet and regularised Neumann solves on the subdomains Ω_i as well as a coarse solve with the operator $G^\top Q\, G$, which is sparse, see [29, 39]. We assume that these types of problems can be handled by direct solvers.

We remark that if we do not impose the Dirichlet boundary conditions by Lagrange multipliers but incorporate them in the local spaces $V(\Omega_i)$, we obtain the standard one-level FETI method, cf. [29, 39]. There, some of the operators K_i are regular, thus some of the kernel spanning operators R_i are not needed, and the dimension of the coarse space gets smaller, but otherwise the setup and the choice of Q and M^{-1} is the same.

4.2. **Main result.** We are now ready to state the main result of this paper and present new condition number bounds for the standard one-level and for the all-floating FETI method. The proofs are postponed to Section 5. Before we give these results we need one final technical assumption on the local variation of $\alpha(x)$ (see also Remark 4.4 below).

Assumption A3. Let $i \in \{1, \ldots, N\}$ and let $\omega_{i,j} \in \Xi_{i,\eta_i}$ be one of the patches covering Ω_{i,η_i} in Definition 2.6. We assume that $\widehat{\alpha}_i(x^h)$ is constant on each face f and on each edge e of $\partial\omega_{i,j} \cap \partial\Omega_i$, i.e. we assume no coefficient variation locally on any of the patch (boundary) faces or edges in any of the subdomain boundary layers. (Recall that faces and edges are always considered to be open, i.e. they do not contain their boundary nodes.)

Theorem 4.1 (all-floating FETI). *Let $\{\eta_i\}$ and $\alpha(x)$ be such that Assumptions A1, A2 and A3 hold. On each subdomain let C^*_{i,η_i} be the minimum of the values of C^*_{i,η_i} in Lemma 3.2 and in Lemma 3.4 with compatible partitionings* (3.1) *and* (3.5), *respectively, such that* (3.7) *holds. Then the condition number κ of the preconditioned all-floating FETI system described above satisfies*

$$\kappa \ \lesssim\ \max_{j=1}^{N} \frac{H_j}{\eta_j}\ \max_{i=1}^{N} \left(C^*_{i,\eta_i}\, (1 + \log(H_i/h_i))^2 \right) .$$

The hidden constant is independent of H_i, η_i, h_i and N, as well as of the contrast in the coefficient α. It does depend on the local variations of α on each patch $\omega_{i,j}$ (see also Remark 4.4). The constants C^*_{i,η_i} depend on (i) $\max_k \overline{\alpha}^{(k)}_{i,\eta_i}/\underline{\alpha}^{(k)}_{i,\eta_i}$, (ii) a power of H_i/η_i, and possibly (iii) a logarithmic or linear term in H_i/h_i (see Section 3 for details).

Proof. Postponed to Section 5. □

Corollary 4.2 (one-level FETI). *Let the assumptions of Theorem 4.1 hold. In addition, let us assume that, for each subdomain Ω_i that touches the Dirichlet boundary, the subregions $\Omega^{(k)}_{i,\eta_i}$ in* (3.1) *or $\Omega^{(k)}_i$ in* (3.5) *all touch the Dirichlet boundary at least in a subdomain edge (resp. vertex) in three (resp. two) dimensions. Then the condition number of the one-level FETI system also satisfies*

$$\kappa \ \lesssim\ \max_{j=1}^{N} \frac{H_j}{\eta_j}\ \max_{i=1}^{N} \left(C^*_{i,\eta_i}\, (1 + \log(H_i/h_i))^2 \right) ,$$

Proof. Follows from Theorem 4.1. For some details see the final paragraph Section 5. □

Remark 4.3.
- (i) First, we would like to illustrate this result for a special case. If (a) η_i can be chosen in the order of H_i, (b) $\alpha(x)$ is constant (or only mildly varying) in the subregions $\Omega^{(k)}_{i,\eta_i}$, and (c) the interface X^*_i from Lemma 3.4 can be chosen to be at least an edge in three dimensions, then
$$\kappa \ \lesssim\ (1 + \log(H_i/h_i))^3$$
at worst. Moreover, if the number of subregions $M_i = 2$ for all subdomains Ω_i, then Lemma 3.2 applies and the cubic dependence is reduced to a quadratic one.
- (ii) We would like to emphasise once more that in case we use Lemma 3.2 the estimates are totally independent of the values of $\alpha(x)$ in the subdomain interiors $\Omega_i \setminus \Omega_{i,\eta_i}$, see also [29, Remark 3.5].
- (iii) Condition (3.7) in Lemma 3.4 is in accordance with the theory given in [29] that better estimates can be achieved, if the coefficient in the interior of each subdomain is larger than near its boundary.
- (iv) It can be seen from the proof of Theorem 4.1 in Section 5 that Assumption A1 is only needed for the weighted Poincaré and discrete Sobolev type inequalities in Section 3. In that sense, we could relax A1 to the weaker assumption that the boundary layer Ω_{i,η_i} of each subdomain is η_i-regular, see also Remark 3.3(iv). In view of [18], even weaker assumptions might be possible.
- (v) The extra factor $\max_j \frac{H_j}{\eta_j}$ in Theorem 4.1 and Corollary 4.2 can be eliminated, i.e.

$$\kappa \ \lesssim\ \max_{i=1}^{N} \left(C^*_{i,\eta_i}\, (1 + \log(H_i/h_i))^2 \right) , \tag{4.16}$$

if we set $Q = M^{-1}$ or if we assume a priori information on $\alpha(x)$ and redefine q_i in (4.15) to be

$$q_i(x^h) := \begin{cases} (1+\log(H_i/h_i))\frac{h_i^2}{\eta_i} & \text{if } x^h \text{ lies on a subdomain face}\,, \\ h_i & \text{if } x^h \text{ lies on a subdomain edge or vertex}\,, \end{cases} \tag{4.17}$$

in three dimensions (with suitable modifications in two dimensions). Note that this choice requires an a priori knowledge of the parameter η_i for each subdomain Ω_i. However, we believe that using technical tricks and a discrete weighted Sobolev type inequality for edges, one should be able to get estimate (4.16) also for the original choice of Q, but we did not pursue this issue further.

Remark 4.4. Assumption A3 ensures that the functions $\delta_j^\dagger$ in (4.11) are constant on boundary edges and faces of the boundary layer patches $\omega_{i,j}$ in Definition 2.6. We can drop Assumption A3, if we choose

$$\widehat{\alpha}_i(x^h) := \max_{j:\, x^h \in \partial\omega_{i,j}} \|\alpha\|_{L^\infty(\omega_{i,j})}$$

instead of (4.10). This choice for $\widehat{\alpha}_i(x^h)$ requires even more a priori knowledge on the coefficient than in Remark 4.3(v) and is not very suitable for implementations.

However, we believe that the statements of Theorem 4.1 and Corollary 4.2 can also be proved for the original choice of $\widehat{\alpha}_i(x^h)$ in (4.10) under a suitable smoothness assumption on $\widehat{\alpha}_i(x^h)$, i.e.

$$\frac{|\widehat{\alpha}_i(x^h) - \widehat{\alpha}_i(y^h)|}{|x^h - y^h|} \lesssim \frac{1}{\eta_i}\,\widehat{\alpha}_i(x^h)\,,$$

where x^h and y^h are two neighbouring nodes on a boundary face (resp. edge) of one of the boundary patches $\omega_{i,j}$. This assumption basically excludes rapid oscillations in $\alpha(x)$ on the boundary of any of the patches, but allows for large jumps between patches.

5. Proof of Theorem 4.1

As in our previous work [29] we give the proof for the three-dimensional case; the two-dimensional case is analogous. Section 5.1 introduces an abstract framework on the operator level, similar to usual FETI proofs, cf. [39, Sect. 6.3]. Section 5.2 introduces some technical tools which we need specifically for the case of varying coefficients. Finally, we prove the crucial estimate for the projection operator defined in (5.2) below. In contrast to the proof idea outlined in [29, Sect. 4] our following proof reduces the crucial bounds to standard results on *patches* $\omega_{i,j}$ (cf. Definition 2.6) rather than on subdomains.

5.1. **Abstract framework.** First, we define the spaces

$$W_i := V^h(\partial\Omega_i)\,, \qquad W = \prod_{i=1}^N W_i\,,$$

in which we will carry out the analysis, we write $S_i : W_i \to W_i$, and we define $S : W \to W$ by $S := \operatorname{diag}(S)$. We recall that each S_i induces the seminorm $|w_i|_{S_i} := \langle S_i\, w_i,\, w_i\rangle^{1/2}$, cf. (2.6), and that $|w_i + c|_{S_i} = |w_i|_{S_i}$ for any constant $c \in \mathbb{R}$. On the product space W we define

$$|w|_S := \Big(\sum_{i=1}^N |w_i|_{S_i}^2\Big)^{1/2} \qquad \text{for } w \in W\,.$$

Furthermore, we define the space

$$V' := \{\mu \in U : \langle B\,z,\, Q\,\mu\rangle = 0 \quad \forall z \in \ker K\} \;=\; \operatorname{range} P^\top\,, \tag{5.1}$$

which can be shown to be isomorphic to the space V defined in (4.8). In the following we assume that $U = \operatorname{range} B$, i.e., $\ker B^\top = \{0\}$. The general case, where we have to work in the factor spaces modulo $\ker B^\top$, follows then from this special case; for details see e.g. [27, 39]. Finally we define the operator

$$P_D := \left[B_1^\top D_1\, B_1 \,\middle|\, \ldots \,\middle|\, B_N^\top D_N B_N\right], \tag{5.2}$$

which can be shown to be a projection fulfilling $B\,P_D = B$. In other words, $I - P_D$ is a projection to the functions that are continuous across the subdomain interfaces and that fulfil the homogeneous Dirichlet conditions on $\partial\Omega$. Analogously to [19] we can show that

$$(P_D\, w)_i(x^h) = \begin{cases} \sum_{j\in\mathcal{N}(x^h)} \delta_j^\dagger(x^h)\left[w_i(x^h) - w_j(x^h)\right] & \text{for } x^h \in \partial\Omega_i^h \cap \Gamma\,, \\ w_i(x^h) & \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega\,. \end{cases} \tag{5.3}$$

In the following we will regard P_D as an operator mapping W to W, because B acts only on degrees of freedom on the subdomain boundaries. Similarly, we will occasionally regard B as a mapping from W to U, and $B^\top : U \to W$. Note also that $\ker S_i = \operatorname{span}\{\mathbf{1}_{\partial\Omega_i}\}$. As a second important identity we have

$$B^\top M^{-1} B = P_D^\top S\, P_D\,. \tag{5.4}$$

Using the fact that P_D is a projection, it can be shown that the preconditioner $P\,M^{-1}$ is SPD as a mapping from V' to V as long as Q is SPD on $\operatorname{range} G = B\,(\ker K)$; for details see e.g. [19, 27]. Therefore, $P\,M^{-1}$ has a well-defined SPD inverse $M : V \to V'$. To show our bound on κ we show the spectral bounds

$$\langle M\,\lambda,\, \lambda\rangle \;\leq\; \langle F\,\lambda,\, \lambda\rangle \;\lesssim\; C^*\,\langle M\,\lambda,\, \lambda\rangle \qquad \forall \lambda \in V\,, \tag{5.5}$$

with

$$C^* := \left(\max_{j=1}^{N} \frac{H_j}{\eta_j}\right)\; \max_{i=1}^{N}\left(C^*_{i,\eta_i}\,(1+\log(H_i/h_i))^2\right).$$

The lower bound in (5.5) can be shown by algebraic arguments independently of our particular choices of Q and D_i following [39, Theorem 6.15]. Using similar algebraic arguments the upper bound can be reduced to an estimate in the space W. Before we give that estimate we need the following lemma.

Lemma 5.1. *For any $w \in W$, there exists a unique $z_w \in \ker S$ such that $B(w + z_w) \in V'$. Moreover,*

$$\|B\,z_w\|_Q \leq \|B\,w\|_Q\,,$$

where $\|\mu\|_Q := \langle \mu,\, Q\,\mu\rangle^{1/2}$. The unique element z_w is explicitly given by

$$z_w = \operatorname*{argmin}_{z\in\ker K} \|B(w+z_w)\|_Q = -R\,(G^\top Q\,G)^{-1} G^\top Q\,B\,w\,,$$

which shows that the mapping $w \mapsto z_w$ is linear. Furthermore, $w \mapsto -z_w$ is a projection onto $\ker S$*, i.e., the piecewise constant functions, and this projection is orthogonal with respect to the inner product induced by $B^\top Q\,B$.*

Proof. The proof follows directly from [39, Lemma 6.12]. □

As shown in [39, Sect. 6.3], the crucial estimate

$$|P_D(w + z_w)|_S^2 \lesssim C^* \, |w|_S^2 \qquad \forall w \in W \, . \tag{5.6}$$

implies the upper bound in (5.5). In order to make our weighted Poincaré and Sobolev type inequalities from Lemma 3.2 and Lemma 3.4 applicable, we define for each $i = 1, \ldots, N$ the linear functional $g_i : W_i \to \mathbb{R}$ by

$$g_i(w_i) := \begin{cases} \frac{1}{|\Gamma_{i,\eta_i}^{(12)}|} \int_{\Gamma_{i,\eta_i}^{(12)}} \mathcal{H}_{i,\alpha} w_i \, ds \, , & \text{if Lemma 3.2 is applied,} \\ \frac{1}{|X_i^*|} \int_{X_i^*} \mathcal{H}_{i,\alpha} w_i \, ds \, , & \text{if Lemma 3.4 is applied and } X_i^* \text{ is a face or an edge,} \\ (\mathcal{H}_{i,\alpha} \, w_i)(X_i^*) \, , & \text{if Lemma 3.4 is applied and } X_i^* \text{ is a vertex,} \end{cases}$$

depending on which lemma we wish to use, and the subspaces

$$W_i^\perp := \big\{ w_i \in W_i : g_i(w_i) = 0 \big\} \, , \qquad W^\perp := \prod_{i=1}^N W_i^\perp \, . \tag{5.7}$$

Above, $\mathcal{H}_{i,\alpha} w_i$ denotes the discrete α-harmonic extension of w_i, see equation (2.7).

Lemma 5.2. *Inequality* (5.6) *holds for all* $w \in W$ *if*

$$|P_D(w + z_w)|_S^2 \lesssim C^* \, |w|_S^2 \qquad \forall w \in W^\perp \, . \tag{5.8}$$

Proof. First, by Lemma 5.1, $z_y = -y$ for all $y \in \ker S$, and using the linearity we obtain the invariance

$$w + z_w = (w + y) + z_{w+y} \qquad \forall w \in W \, , \; y \in \ker S \, .$$

Secondly, we have the invariance $|w + y|_S = |w|_S$ for all $w \in W$ and $y \in \ker S$. Thus both sides of inequality (5.6) are invariant if we add a y from $\ker S$ to w. Finally, for all $w \in W$ we can find a unique $y_w \in \ker S$ such that $w - y_w \in W^\perp$ by choosing $(y_w)_i := g_i(\mathcal{H}_{i,\alpha} w_i)$. □

5.2. **Technical tools.** To conclude our proof we only need to show inequality (5.8). If $\alpha(x)$ is constant on each subdomain Ω_i, this inequality is shown using an additive splitting into terms corresponding to subdomain faces, edges, and vertices, which is motivated from formula (5.3). In our case, however, the functions $\delta_j^\dagger$ are in general no longer constant on such subdomain faces or edges, indeed they can have arbitrary large jumps. Therefore, we need a finer splitting into terms corresponding to faces, edges, and vertices of the patches forming Ω_{i,η_i}, cf. Definition 2.6. We denote by

- $\mathcal{F}_i = \{F\}$ the set of *subdomain* faces,
- $\mathcal{E}_i = \{E\}$ the set of *subdomain* edges,
- $\mathcal{V}_i = \{V\}$ the set of *subdomain* vertices

of Ω_i, and by

- $\mathbb{F}_i = \{f\}$ the set of *patch* faces,
- $\mathbb{E}_i = \{e\}$ the set of *patch* edges,
- $\mathbb{V}_i = \{v\}$ the set *patch* of vertices

with respect to patches from Ξ_{i,η_i} which are part of the subdomain boundary $\partial\Omega_i$. Recall that $\mathcal{F}_i^\Gamma$, $\mathcal{E}_i^\Gamma$, $\mathcal{V}_i^\Gamma$, and $\mathcal{F}_i^D$, $\mathcal{E}_i^D$, $\mathcal{V}_i^D$ denote the subsets of faces, edges, and vertices lying on Γ and $\partial\Omega$, respectively. Correspondingly, we define the subsets $\mathbb{F}_i^\Gamma$, $\mathbb{E}_i^\Gamma$, $\mathbb{V}_i^\Gamma$ and $\mathbb{F}_i^D$, $\mathbb{E}_i^D$, and

$\mathbb{V}_i^D$. For convenience we also define $\mathcal{X}_i := \mathcal{F}_i \cup \mathcal{E}_i \cup \mathcal{V}_i$, and $\mathbb{X}_i := \mathbb{F}_i \cup \mathbb{E}_i \cup \mathbb{V}_i$. For each $\mathrm{x} \in \mathbb{X}_i$ we define the union of touching patches $\omega_i(\mathrm{x})$ by

$$\overline{\omega_i(\mathrm{x})} := \bigcup_{\mathrm{x} \subset \partial\omega_{i,j}} \overline{\omega_{i,j}}\,. \tag{5.9}$$

Note, that by Definition 2.1, $\omega_i(\mathrm{x})$ is a regular domain. Without loss of generality we assume that the patches are aligned with the triangulation $\mathcal{T}_i$. If they are not we can use the Scott-Zhang quasi-interpolation operator [37] as we did in [29].

Similar to [39, Section 4.6] we define the finite element cut-off functions

- $\vartheta_{\mathrm{v}} \in V^h(\Omega_i)$ as being 1 at the vertex v, and zero on all other nodes.
- $\vartheta_e \in V^h(\Omega_i)$ as being 1 at the nodes on the (open) edge e, and zero on all other nodes,
- $\vartheta_f \in V^h(\Omega_i)$ as being 1 at the nodes on the (open) face f, zero on all nodes in $\overline{\Omega}_i \setminus (\omega_i(f) \cup f)$, and discrete harmonic inside of $\omega_i(f)$.
- θ_{v}, θ_e, and $\theta_f \in V^h(\partial\Omega_i)$ as the traces of ϑ_{v}, ϑ_e, and ϑ_f, respectively.

To be more exact we would have to write $\vartheta_{i,\mathrm{v}}$, $\theta_{i,\mathrm{v}}$, etc., but the domain index i will always be clear from the context and is therefore skipped.

Throughout the whole section we make use of the nodal interpolator I^h onto $V^h(\Omega_i)$ (resp. $V^h(\partial\Omega_i)$) which is continuous in the H^1-seminorm (resp. $H^{1/2}$-seminorm) and in the L^2-norm for quadratic functions. See [39, Lemma 3.9].

By definition the functions θ_{x} provide a partition of unity on $\partial\Omega_i$ in the sense that

$$\sum_{\mathrm{x} \in \mathbb{X}_i} I^h(\theta_{\mathrm{x}} u) = u \qquad \forall u \in W_i\,. \tag{5.10}$$

The following lemma states that cutting a function u by one of the functions ϑ_{x} costs only "low" energy.

Lemma 5.3. *For $\mathrm{x} \in \mathbb{X}_i$ let $\omega_{i,j} \in \Xi_{i,\eta_i}$ be an arbitrary patch such that $\mathrm{x} \subset \partial\omega_{i,j}$. Then,*

$$|I^h(\vartheta_{\mathrm{x}} u)|^2_{H^1(\omega_i(\mathrm{x}))} \;\lesssim\; (1 + \log(\eta_i/h_i))^2 \left\{ |u|^2_{H^1(\omega_{i,j})} + \frac{1}{\eta_i^2} \|u\|^2_{L^2(\omega_{i,j})} \right\}.$$

Proof. For a face f there is only one such patch, i. e., $\omega_i(f) = \omega_{i,j}$. Due to [39, Lemma 4.24],

$$|I^h(\vartheta_f\, u)|^2_{H^1(\omega_i(f))} \;\lesssim\; (1 + \log(\eta_i/h_i))^2 \left\{ |u|^2_{H^1(\omega_{i,j})} + \frac{1}{\eta_i^2} \|u\|^2_{L^2(\omega_{i,j})} \right\}.$$

For an edge e, [39, Lemma 4.16 and Lemma 4.19] yield

$$|I^h(\vartheta_e u)|^2_{H^1(\omega_i(e))} \;\lesssim\; \|u\|^2_{L^2(e)} \;\lesssim\; (1 + \log(\eta_i/h_i)) \left\{ |u|^2_{H^1(\omega_{i,j})} + \frac{1}{\eta_i^2} \|u\|^2_{L^2(\omega_{i,j})} \right\}.$$

For a vertex v recall that ϑ_{v} is the nodal basis function associated with v. We easily obtain

$$|I^h(\vartheta_{\mathrm{v}} u)|^2_{H^1(\omega_i(\mathrm{v}))} \;\lesssim\; h_i\, |u(\mathrm{v})|^2 \;\lesssim\; \|u\|^2_{L^2(e)}\,,$$

for some edge $e \in \mathbb{E}_i$ with $\mathrm{v} \in \overline{e}$ and $e \subset \overline{\omega_{i,j}}$. From here we can continue as above. □

The next lemma states that the cut-off functions ϑ_{x} can be used to estimate the energy norm by a decomposition into patches.

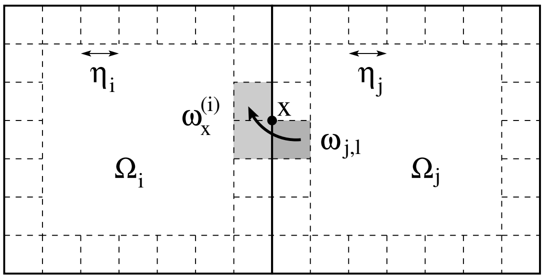

FIGURE 5. Illustration of the operator $\Pi_{\mathrm{x}}^{j\to i}$ from Lemma 5.5.

Lemma 5.4. *For a function $u \in W_i$, we have*

$$|u|_{S_i}^2 \;\lesssim\; \sum_{x\in\mathbb{X}_i} \int_{\omega_i(x)} \alpha(x)\, |\nabla(I^h(\vartheta_{\mathrm{x}}\widetilde{u}))(x)|^2\, dx\,,$$

where $\widetilde{u}$ is an arbitrary extension of u from $\partial\Omega_i$ to $V^h(\Omega_{i,\eta_i})$. The hidden constant is independent of the number of patches in Ω_{i,η_i}.

Proof. First, we convince ourselves that the function

$$v := \sum_{x\in\mathbb{X}_i} I^h(\vartheta_{\mathrm{x}}\widetilde{u})$$

is an extension of u from $\partial\Omega_i$ to Ω_i with its support in Ω_{i,η_i}, and that $v \in V^h(\Omega_i)$. This is true because each node in $\partial\Omega_i^h$ belongs to a unique $\mathrm{x} \in \mathbb{X}_i$ and ϑ_{x} vanishes on $\partial\Omega_i^h \setminus \mathrm{x}$, and the supports of the ϑ_{x} lie entirely in Ω_{i,η_i}. Using the minimum property (2.7) of the Schur complement, we have

$$|u|_{S_i}^2 \le \int_{\Omega_{i,\eta_i}} \alpha(x)\, |\nabla v(x)|^2\, dx\,.$$

Since (i) each ϑ_{x} is only supported in $\omega_i(\mathrm{x})$ which consists only of a finite number of patches, (ii) each patch is contained in finitely many supports $\omega_i(\mathrm{x})$ and (iii) the supports $\omega_i(\mathrm{x})$ have finite overlap, we can conclude that

$$|u|_{S_i}^2 \lesssim \sum_{x\in\mathbb{X}_i} \int_{\omega_i(x)} \alpha(x) |\nabla v(x)|^2\, dx\,,$$

which proves the desired statement. □

The last tool is inspired by [18]. For an illustration see Figure 5.

Lemma 5.5. *Let $\mathrm{x} \in \mathbb{X}_i \cap \mathbb{X}_j$ and let $\omega_{j,\ell}$ be a patch with $\mathrm{x} \in \overline{\omega_{j,\ell}}$. Then there exists an operator $\Pi_{\mathrm{x}}^{j\to i} : V^h(\omega_{j,\ell}) \to V^h(\omega_i(\mathrm{x}))$ such that for all $u \in V^h(\omega_{j,\ell})$,*

$$(\Pi_{\mathrm{x}}^{j\to i} u)_{|\mathrm{x}} \;=\; u_{|\mathrm{x}}\,,$$

$$|\Pi_{\mathrm{x}}^{j\to i} u|_{H^1(\omega_i(\mathrm{x}))}^2 + \frac{1}{\eta_i^2}\, \|\Pi_{\mathrm{x}}^{j\to i} u\|_{L^2(\omega_i(\mathrm{x}))}^2 \;\lesssim\; |u|_{H^1(\omega_{j,\ell})}^2 + \frac{1}{\eta_i^2}\, \|u\|_{L^2(\omega_{j,\ell})}^2\,, \tag{5.11}$$

the operator is even continuous in the L^2-norm and the H^1-seminorm, separately.

Proof. Let $\mathcal{U}$ be the open connected union of $\omega_i(\mathrm{x})$ and $\omega_{j,\ell}$ such that $\operatorname{diam}\mathcal{U} \simeq \eta_i$. In a first step we construct a continuous extension operator $\mathcal{E} : H^1(\omega_{j,\ell}) \to H^1(\mathcal{U})$ making use of the fact that $\omega_{j,\ell}$ is Lipschitz, and for the time being assuming that $\operatorname{diam}\omega_{j,\ell} = 1$. Following Stein [38] (see e. g. also [1, p. 146ff]), there exists an extension operator $\widetilde{\mathcal{E}} : H^1(\omega_{j,\ell}) \to H^1(\mathbb{R}^d)$ with $(\widetilde{\mathcal{E}}u)_{|\omega_{j,\ell}} = u$ and

$$\|\widetilde{\mathcal{E}}u\|_{L^2(\mathbb{R}^d)} \lesssim \|u\|_{L^2(\omega_{j,\ell})}\,, \qquad \|\widetilde{\mathcal{E}}u\|_{H^1(\mathbb{R}^d)} \lesssim \|u\|_{H^1(\omega_{j,\ell})}\,.$$

Using the mean value $\overline{u} := |\omega_{j,\ell}|^{-1} \int_{\omega_{j,\ell}} u\,dx$ we define

$$\mathcal{E} : H^1(\omega_{j,\ell}) \to H^1(\mathcal{U}) : u \mapsto \big[\widetilde{\mathcal{E}}(u-\overline{u})\big]_{|\mathcal{U}} + \overline{u}\,.$$

Obviously, $(\mathcal{E}u)_{|\omega_{j,\ell}} = u$. This operator is stable in L^2, as we have

$$\|\mathcal{E}u\|_{L^2(\mathcal{U})} \;\leq\; \|\widetilde{\mathcal{E}}(u-\overline{u})\|_{L^2(\mathbb{R}^d)} + \|\overline{u}\|_{L^2(\mathcal{U})} \;\lesssim\; \|u-\overline{u}\|_{L^2(\omega_{j,\ell})} + \|\overline{u}\|_{L^2(\mathcal{U})} \;\lesssim\; \|u\|_{L^2(\omega_{j,\ell})}\,,$$

where in the last step we used Cauchy's inequality and the fact that $|\mathcal{U}| \simeq |\omega_{j,\ell}|$. By using Poincaré's inequality on $\omega_{i,\ell}$ we can also conclude the stability in the H^1-seminorm,

$$|\mathcal{E}u|_{H^1(\mathcal{U})} \;\leq\; \|\widetilde{\mathcal{E}}(u-\overline{u})\|_{H^1(\mathbb{R}^d)} + |\overline{u}|_{H^1(\mathcal{U})} \;\lesssim\; \|u-\overline{u}\|_{H^1(\omega_{j,\ell})} \;\lesssim\; |u-\overline{u}|_{H^1(\omega_{j,\ell})}\,.$$

Using a simple dilation argument the two above equations remain also valid when the diameter of $\omega_{i,\ell}$ differs from one, and the constants involved depend only on the shapes of $\omega_{j,\ell}$ and $\mathcal{U}$.

In a second step we use the quasi-interpolation operator $\Pi^h : H^1(\mathcal{U}) \to V^h(\mathcal{U})$ introduced by Scott and Zhang [37] (see also [6] and [18]). This operator is similar to the one by Clément but averages on manifolds instead of patches, which makes it eventually possible to preserve boundary values. In our current setting we can choose the averaging manifolds such that

$$\begin{aligned} && (\Pi^h u)_{|\mathrm{x}} = u_{|\mathrm{x}} \qquad & \forall u \in H^1(\mathcal{U}),\ u|_{\omega_{j,\ell}} \in V^h(\omega_{j,\ell})\,,\\ |\Pi^h u|^2_{H^1(\mathcal{U})} \lesssim |u|^2_{H^1(\mathcal{U})}\,, && \|\Pi^h u\|^2_{L^2(\mathcal{U})} \lesssim \|u\|^2_{L^2(\mathcal{U})} \qquad & \forall u \in H^1(\mathcal{U})\,. \end{aligned}$$

Defining $\Pi^{j\to i}_{\mathrm{x}} u := (\Pi^h\,\mathcal{E}u)_{|\omega_{j,\ell}}$ we meet the requirements of the lemma. ☐

5.3. **The P_D estimates.** In this subsection we show inequality (5.8) by estimating $|P_D\,w|^2_S$ and $|P_D\,z_w|^2_S$ separately. For compact notation we define the α-weighted (semi)norms

$$\|u\|_{L^2(D),\alpha} := \Big(\int_D \alpha(x)\,|u(x)|^2\,dx\Big)^{1/2}, \qquad |u|_{H^1(D),\alpha} := \Big(\int_D \alpha(x)\,|\nabla u(x)|^2\,dx\Big)^{1/2} \tag{5.12}$$

for a generic domain D.

Lemma 5.6. *Let the assumptions of Theorem 4.1 hold. Then,*

$$|P_D\,w|^2_S \;\lesssim\; \max_{j=i}^{N}\Big\{C^*_{i,\eta_i}\,(1+\log(\eta_i/h_i))^2\Big\}\,|w|^2_S \qquad \forall w \in W^\perp\,.$$

Proof. Due to Assumption A3, the functions $\widehat{\alpha}_i$ defined on $\partial\Omega^h_i$ are piecewise constant with respect to (patch) faces $f \in \mathbb{F}^\Gamma_i$ and edges $e \in \mathbb{E}^\Gamma_i$, and are thus equal to a constant $\widehat{\alpha}_{i\,|\mathrm{x}}$ on each $\mathrm{x} \in \mathbb{X}^\Gamma_i$. As a consequence, the functions $\delta^\dagger_i$ share the same property, and we define

$\delta^\dagger_{i\,|\mathrm{x}}$ analogously. Let i now be fixed. Using identity (5.3) and the partition of unity property (5.10) we obtain that

$$|(P_D\, w)_i|^2_{S_i} = \Big| \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}} \delta^\dagger_{j\,|\mathrm{x}}\, I^h\big(\theta_\mathrm{x}(w_i - w_j)\big) \;+\; \sum_{\mathrm{x}\in\mathbb{X}_i^D} I^h(\theta_\mathrm{x} w_i)\Big|^2_{S_i} \tag{5.13}$$

Let Ω_j be one of the neighbours of Ω_i. For a fixed $\mathrm{x}\in\mathbb{X}_j$, large jumps in $\alpha(x)$ can occur in $\omega_i(\mathrm{x})$. We can, however, always find one patch $\omega_{i,k}\in\Xi_{i,\eta_i}$ such that $\omega_{i,k}\subset\omega_i(\mathrm{x})$ and

$$\|\alpha\|_{L^\infty(\omega_i(\mathrm{x}))} \;\le\; \|\alpha\|_{L^\infty(\omega_{i,k})} \;\le\; \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\,\widehat{\alpha}_{i\,|\mathrm{x}}\,. \tag{5.14}$$

In other words, $\omega_{i,k}$ is the patch in the union $\omega_i(\mathrm{x})$ of patches touching x where the maximum is attained (locally). Obviously, the estimate above depends only on the *local* variation in $\alpha(x)$ but not on the contrast of $\alpha(x)$ between the subregions $\Omega^{(k)}_{i,\eta_i}$. Let the operator $\Pi^{j\to i}_\mathrm{x} : V^h(\omega_{j,\ell}) \to V^h(\omega_i(\mathrm{x}))$ be defined according to Lemma 5.5. With the definitions

$$\begin{aligned} \widetilde{w}_j &:= \mathcal{H}_{j,\alpha} w_j && \text{for } j=1,\ldots,N\,,\\ \widetilde{w}^\mathrm{x}_j &:= \Pi^{j\to i}_\mathrm{x}\widetilde{w}_j && \text{for } \mathrm{x}\in\mathbb{X}_i\cap\mathbb{X}_j\,, \end{aligned} \tag{5.15}$$

we have by construction that $(\widetilde{w}_j)_{|\partial\Omega_j} = w_j$ and $(\widetilde{w}^\mathrm{x}_j)_{|\mathrm{x}} = (w_j)_{|\mathrm{x}}$ for all j. From the definition (5.13), we see that the function

$$v_i := \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}} \delta^\dagger_{j\,|\mathrm{x}}\, I^h\big(\vartheta_\mathrm{x}(\widetilde{w}_i - \widetilde{w}^\mathrm{x}_j)\big) \;+\; \sum_{\mathrm{x}\in\mathbb{X}_i^D} I^h(\vartheta_\mathrm{x}\widetilde{w}_i) \tag{5.16}$$

is an extension of $(P_D\,w)_i$ from W_i to $V^h(\Omega_i)$. By Lemma 5.4 we can conclude that

$$\begin{aligned} |(P_D\,w)_i|^2_{S_i} \lesssim{}& \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}} \underbrace{(\delta^\dagger_{j,\mathrm{x}})^2 \int_{\omega_i(\mathrm{x})} \alpha\,\big|\nabla\big[I^h\big(\vartheta_\mathrm{x}(\widetilde{w}_i - \widetilde{w}^\mathrm{x}_j)\big)\big]\big|^2\,dx}_{=:\psi_{ij,\mathrm{x}}} \;+\\ &+\; \sum_{\mathrm{x}\in\mathbb{X}_i^D} \underbrace{\int_{\omega_i(\mathrm{x})} \alpha\,|\nabla(I^h(\vartheta_\mathrm{x}\widetilde{w}_i))|^2\,dx}_{=:\psi_{i,\mathrm{x}}}\,. \end{aligned} \tag{5.17}$$

Using (5.14), we have

$$\psi_{i,\mathrm{x}} \;\le\; \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\,\widehat{\alpha}_{i\,|\mathrm{x}}\,|I^h(\vartheta_\mathrm{x}\widetilde{w}_i)|^2_{H^1(\omega_i(\mathrm{x}))} \qquad \forall \mathrm{x}\in\mathbb{X}_i^D\,. \tag{5.18}$$

With the same arguments and the elementary inequality

$$\widehat{\alpha}_i(x^h)\,\big(\delta^\dagger_j(x^h)\big)^2 \;\le\; \min\big(\widehat{\alpha}_i(x^h),\,\widehat{\alpha}_j(x^h)\big) \qquad \forall x^h\in\Gamma^h_{ij}\,, \tag{5.19}$$

cf. [39, Sect. 6.2.3], we obtain that for all $\mathrm{x}\in\mathbb{X}_i^\Gamma$ and $j\in\mathcal{N}_\mathrm{x}$,

$$\begin{aligned} \psi_{ij,\mathrm{x}} &\lesssim \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\,\big(\delta^\dagger_{j\,|\mathrm{x}}\big)^2\,\widehat{\alpha}_{i\,\mathrm{x}}\,|I^h(\vartheta_x(\widetilde{w}_i - \widetilde{w}^\mathrm{x}_j))|^2_{H^1(\omega_i(\mathrm{x}))}\\ &\lesssim \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\,\Big\{\widehat{\alpha}_{i\,|\mathrm{x}}\,|I^h(\vartheta_x\widetilde{w}_i)|^2_{H^1(\omega_i(\mathrm{x}))} + \widehat{\alpha}_{j\,|\mathrm{x}}\,|I^h(\vartheta_\mathrm{x}\widetilde{w}^\mathrm{x}_j)|^2_{H^1(\omega_i(\mathrm{x}))}\Big\}\,. \end{aligned} \tag{5.20}$$

From here on, for a simpler presentation, we will not make the dependency on the local variations $\sup_{x,y\in\omega_{i,k}} \alpha(x)/\alpha(y)$ explicit anymore, but hide them using the $\lesssim$ symbolism. The combination of (5.17), (5.18), and (5.20) then yields

$$|(P_D\, w)_i|_{S_i}^2 \lesssim \sum_{\mathrm{x}\in\mathbb{X}_i} \widehat{\alpha}_{i\,|\mathrm{x}}\, |I^h(\vartheta_x \widetilde{w}_i)|^2_{H^1(\omega_i(\mathrm{x}))} + \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}\setminus\{i\}} \widehat{\alpha}_{j\,|\mathrm{x}}\, |I^h(\vartheta_\mathrm{x} \widetilde{w}_j^\mathrm{x})|^2_{H^1(\omega_i(\mathrm{x}))}\,. \tag{5.21}$$

For later purposes we introduce the following construction. For a fixed $\mathrm{x} \in \mathbb{X}_i \cap \mathbb{X}_j$, we choose patches $\omega_{i,m} \in \Xi_{i,\eta_i}$ and $\omega_{j,\ell} \in \Xi_{j,\eta_j}$ such that

$$\begin{aligned} \widehat{\alpha}_{i\,|\mathrm{x}} \;&\leq\; \Big(\sup_{x,y\in\omega_{i,m}} \frac{\alpha(x)}{\alpha(y)}\Big)\, \alpha(x) \qquad \forall x\in\omega_{i,m}\,,\\ \widehat{\alpha}_{j\,|\mathrm{x}} \;&\leq\; \Big(\sup_{x,y\in\omega_{j,\ell}} \frac{\alpha(x)}{\alpha(y)}\Big)\, \alpha(x) \qquad \forall x\in\omega_{j,\ell}\,, \end{aligned} \tag{5.22}$$

i. e., we choose the patches where $\widehat{\alpha}_{i\,|\mathrm{x}}$ and $\widehat{\alpha}_{j\,|\mathrm{x}}$ are attained. Note that m and ℓ depend on x.

We continue now to further estimate (5.21). The terms in the first sum of this expression are estimated using Lemma 5.3, which yields

$$\begin{aligned} \widehat{\alpha}_{i\,|\mathrm{x}}\, |I^h(\vartheta_\mathrm{x} \widetilde{w}_i)|^2_{H^1(\omega_i(\mathrm{x}))} \;&\lesssim\; \widehat{\alpha}_{i\,|\mathrm{x}}\, (1+\log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_i|^2_{H^1(\omega_{i,m})} + \frac{1}{\eta_i^2} \|\widetilde{w}_i\|^2_{L^2(\omega_{i,m})}\Big\}\\ &\lesssim\; (1+\log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_i|^2_{H^1(\omega_{i,m}),\alpha} + \frac{1}{\eta_i^2}\, \|\widetilde{w}_i\|^2_{L^2(\omega_{i,m}),\alpha}\Big\}\,, \end{aligned} \tag{5.23}$$

where in the last line we used (5.22). The second sum in (5.21) is estimated using Lemma 5.3 and Lemma 5.5. For a fixed $\mathrm{x} \in \mathbb{X}_i^\Gamma$ and $j \in \mathcal{N}_\mathrm{x} \setminus \{i\}$ we obtain (using that $\widetilde{w}_j^\mathrm{x} = \Pi_\mathrm{x}^{j\to i} \widetilde{w}_j$)

$$\begin{aligned} &\widehat{\alpha}_{j\,|\mathrm{x}}\, |I^h(\vartheta_\mathrm{x} \widetilde{w}_j^\mathrm{x})|^2_{H^1(\omega_i(\mathrm{x}))}\\ &\quad\lesssim\; \widehat{\alpha}_{j\,|\mathrm{x}}\, (1+\log(\eta_i/h_i))^2 \Big\{ |\Pi_\mathrm{x}^{j\to i} \widetilde{w}_j|^2_{H^1(\omega_i(\mathrm{x}))} + \frac{1}{\eta_i^2}\, \|\Pi_\mathrm{x}^{j\to i}\widetilde{w}_j\|^2_{L^2(\omega_i(\mathrm{x}))}\Big\}\\ &\quad\lesssim\; \widehat{\alpha}_{j\,|\mathrm{x}}\, (1+\log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_j|^2_{H^1(\omega_{j,\ell})} + \frac{1}{\eta_j^2}\, \|\widetilde{w}_j\|^2_{L^2(\omega_{j,\ell})}\Big\}\\ &\quad\lesssim\; (1+\log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_j|^2_{H^1(\omega_{j,\ell}),\alpha} + \frac{1}{\eta_j^2}\, \|\widetilde{w}_j\|^2_{L^2(\omega_{j,\ell}),\alpha}\Big\}\,, \end{aligned} \tag{5.24}$$

where in the last line we used again (5.22). Combining estimates (5.21), (5.23), and (5.24), we obtain with a finite summation argument that

$$|(P_D\, w)_i|^2_{S_i} \;\lesssim\; \sum_{j\in\mathcal{N}_i} (1+\log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_j|^2_{H^1(\Omega_{j,\eta_j}),\alpha} + \frac{1}{\eta_i^2}\, \|\widetilde{w}_j\|^2_{L^2(\Omega_{j,\eta_j}),\alpha}\Big\}\,,$$

where $\mathcal{N}_i = \{j : \overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset\}$ is index set of neighbours of Ω_i (including i). Since $w_j \in W_j^\perp$, the function $\widetilde{w}_j = \mathcal{H}_{j,\alpha} w_j$ satisfies $g_j(\widetilde{w}_j) = 0$. Hence, Lemma 3.2 and Lemma 3.4 yield

$$|(P_D\, w)_i|^2_{S_i} \;\lesssim\; \sum_{j\in\mathcal{N}_i} C^*_{j,\eta_j}\, (1+\log(\eta_i/h_i))^2\, |\widetilde{w}_j|^2_{H^1(\Omega_j),\alpha}\,. \tag{5.25}$$

Finally, using that $h_j \simeq h_i$, $\eta_j \simeq \eta_i$, $|\widetilde{w}_j|_{H^1(\Omega_j),\alpha} = |w_j|_{S_j}$, and that each subdomain has only finitely many neighbours, we obtain the statement of Lemma 5.6. □

Lemma 5.7. *Let the assumptions of Theorem 4.1 hold and suppose that Q is chosen according to* (4.14)–(4.15)*. Then,*

$$|P_D\, z_w|_S^2 \;\lesssim\; \Big(\max_{k=1}^N \frac{H_k}{\eta_k}\Big)\, \max_{j=1}^N \Big\{ C^*_{j,\eta_j}\,(1+\log(H_k/h_k))^2 \Big\}\, |w|_S^2 \qquad \forall w \in W^\perp\,.$$

Proof. Using equation (5.4) and Lemma 5.1 we have

$$|P_D\, z_w|_S^2 \;=\; \|B\, z_w\|_Q^2 \;\leq\; \|B\, w\|_Q^2 \;=\; |P_D\, w|_S^2\,,$$

and the previous Lemma 5.6 implies the desired statement.

For the other two cases, recall that $z_w \in \ker S$, i. e., z_w is constant on each subdomain. We denote these constant components by z_i. For a moment, let i be fixed. With the same arguments as in the proof of Lemma 5.6, the function

$$v_i := \sum_{x\in\mathbb{X}_i^\Gamma}\sum_{j\in\mathcal{N}_x} \delta_{j\,|x}^\dagger\, I^h(\vartheta_x(z_i-z_j)) + \sum_{x\in\mathbb{X}_i^D} I^h(\vartheta_x z_i)$$

is an extension of $(P_D\, z_w)_i$ from W_i to $V^h(\Omega_i)$. An application of Lemma 5.4 yields

$$\begin{aligned}
|(P_D z_w)_i|_{S_i}^2 &\lesssim \sum_{x\in\mathbb{X}_i^\Gamma}\sum_{j\in\mathcal{N}_x} (\delta_{j\,|x}^\dagger)^2\, |I^h(\vartheta_x(z_i-z_j))|^2_{H^1(\omega_i(x)),\alpha} + \sum_{x\in\mathbb{X}_i^D} |I^h(\vartheta_x z_i)|^2_{H^1(\omega_i(x)),\alpha}\\
&= \sum_{x\in\mathbb{X}_i}\sum_{j\in\mathcal{N}_x} (\delta_{j\,|x}^\dagger)^2\, |\vartheta_x|^2_{H^1(\omega_i(x)),\alpha}\, |z_i-z_j|^2 + \sum_{x\in\mathbb{X}_i^D} |\vartheta_x|^2_{H^1(\omega_i(x)),\alpha} |z_i|^2\,.
\end{aligned}$$

As in the proof of Lemma 5.6, we do not make explicit the dependency on the local variations $\sup_{x,y\in\omega_{i,k}} \alpha(x)/\alpha(y)$. Using (5.14), i. e., $\|\alpha\|_{L^\infty(\omega_i(x))} \lesssim \widehat{\alpha}_{i\,|x}$, and due to [39, Lemma 4.16, Lemma 4.25] we have

$$\begin{aligned}
|\vartheta_f|^2_{H^1(\omega_i(x)),\alpha} &\lesssim \varphi_{i,f} := \widehat{\alpha}_{i\,|f}\,(1+\log(\eta_i/h_i))\,\eta_i && \forall f\in\mathbb{F}_i\,,\\
|\vartheta_e|^2_{H^1(\omega_i(x)),\alpha} &\lesssim \varphi_{i,e} := \widehat{\alpha}_{i\,|e}\,\eta_i && \forall e\in\mathbb{E}_i\,,\\
|\vartheta_v|^2_{H^1(\omega_i(x)),\alpha} &\lesssim \varphi_{i,v} := \widehat{\alpha}_{i\,|v}\,h_i && \forall v\in\mathbb{V}_i\,.
\end{aligned}$$

Obviously, for each edge e (and for each vertex v) we can find a face f with $e \subset \overline{f}$ (resp. an edge e with $v \in \overline{e}$) such that

$$\varphi_{i,e} \;\leq\; \varphi_{i,f} \qquad \text{and} \qquad \varphi_{i,v} \;\leq\; \varphi_{i,e}\,,$$

by choosing the one touching the patch with the effectively largest coefficient. Since each patch face contains only a finite number of patch edges and vertices, and due to the elementary inequality (5.19) we obtain

$$\begin{aligned}
|(P_D z_w)_i|_{S_i}^2 \lesssim &\sum_{X\in\mathcal{X}_i^\Gamma\cap\mathcal{X}_j^\Gamma}\;\sum_{x\in\mathbb{X}_i,\,x\subset X} \min(\varphi_{i,x},\,\varphi_{j,x})\,|z_i-z_j|^2 \;+\\
&+\sum_{X\in\mathcal{X}_i^D}\;\sum_{x\in\mathbb{X}_i,\,x\subset X} \varphi_{i,x}\,|z_i|^2\,,
\end{aligned} \tag{5.26}$$

where the x in the inner sums are of the same kind (face/edge/vertex) as the X in the outer sums. We treat subdomain face, edge, and vertex contributions separately.

- Since each face $f \in \mathbb{F}_i$ contains $\mathcal{O}((\eta_i/h_i)^2)$ nodes, we can bound the subdomain face contributions in (5.26) from above by

$$\begin{aligned}&\sum_{F\in\mathcal{F}_i^\Gamma\cap\mathcal{F}_j^\Gamma}\ \sum_{f\in\mathbb{F}_i,\, f\subset F}\ \sum_{x^h\in f^h} \min(\widehat{\alpha}_{i\,|f},\, \widehat{\alpha}_{j\,|f})\ \underbrace{(1+\log(\eta_i/h_i))\,\frac{h_i^2}{\eta_i}}_{\le\ q_i(x^h)\,H_i/\eta_i}\ |z_i - z_j|^2 \\ &+ \sum_{F\in\mathbb{F}_i^D}\ \sum_{f\in\mathbb{F}_i\, f\subset F}\ \sum_{x^h\in f^h} \widehat{\alpha}_{i\,|f}\ \underbrace{(1+\log(\eta_i/h_i))\,\frac{h_i^2}{\eta_i}}_{\le\ q_i(x^h)\,H_i/\eta_i}\ |z_i|^2\,,\end{aligned} \tag{5.27}$$

where f^h is the set of interior nodes on f.
- Since each edge $e \in \mathbb{E}_i$ contains $\mathcal{O}(\eta_i/h_i)$ nodes, we can bound the subdomain edge contributions in (5.26) from above by

$$\begin{aligned}&\sum_{E\in\mathcal{E}_i^\Gamma\cap\mathcal{E}_j^\Gamma}\ \sum_{e\in\mathbb{E}_i,\, e\subset E}\ \sum_{x^h\in e^h} \min(\widehat{\alpha}_{i\,|e},\, \widehat{\alpha}_{j\,|e})\ \underbrace{h_i}_{=\ q_i(x^h)}\ |z_i - z_j|^2 \\ &+ \sum_{E\in\mathcal{E}_i^D}\ \sum_{e\in\mathbb{E}_i,\, e\subset E}\ \sum_{x^h\in e^h} \widehat{\alpha}_{i\,|e}\ \underbrace{h_i}_{=q_i(x^h)}\ |z_i|^2\,,\end{aligned}$$

where e^h is the set of interior nodes on e.
- Similarly, the subdomain vertex contributions from (5.26) can be bounded by

$$\sum_{V\in\mathcal{V}_i^\Gamma\cap\mathcal{V}_j^\Gamma} \min(\widehat{\alpha}_i(V),\, \widehat{\alpha}_j(V))\ \underbrace{h_i}_{=q_i(V)}\ |z_i - z_j|^2 + \sum_{V\in\mathcal{V}_i^D} \widehat{\alpha}_i(V)\ \underbrace{h_i}_{=q_i(V)}\ |z_i|^2\,.$$

According to [19] we observe that the expressions $|z_i - z_j|$ and $|z_i|$ are components of $B\,z_w$. Collecting all terms appropriately and using the definition (4.14) of Q, we can conclude by Lemma 5.1 that

$$|(P_D\, z_w)|_S^2 \ \lesssim\ \Big(\max_{k=1}^{N}\frac{H_k}{\eta_k}\Big)\,\|B\,z_w\|_Q^2 \ \lesssim\ \Big(\max_{k=1}^{N}\frac{H_k}{\eta_k}\Big)\,\|B\,w\|_Q^2\,.$$

In the following we split $\|B\,w\|_Q^2$ into face, edge, and vertex terms,

$$\|B\,w\|_Q^2 \ =\ \sum_{i=1}^{N}\Big\{\sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma\cap\mathbb{X}_j^\Gamma} r_{ij}^{\mathrm{x}} + \sum_{\mathrm{x}\in\mathbb{X}_i^D} r_i^{\mathrm{x}}\Big\}\,,$$

with

$$r_{ij}^{\mathrm{x}} := \min(\widehat{\alpha}_{i\,|\mathrm{x}},\, \widehat{\alpha}_{j\,|\mathrm{x}})\, q_{i\,|\mathrm{x}} \sum_{x^h\in\mathrm{x}^h}\big(w_i(x^h) - w_j(x^h)\big)^2\,, \qquad r_i^{\mathrm{x}} := \widehat{\alpha}_{i\,|\mathrm{x}}\ q_{i\,|\mathrm{x}} \sum_{x^h\in\mathrm{x}^h} w_i(x^h)^2\,,$$

and treat patch face, edge, and vertex terms again separately.

- Recall that $q_{i\,|f} = (1+\log(H_i/h_i))\,h_i^2/H_i$. Due to the quasi-uniformity assumption on the mesh on Ω_i we obtain for the patch face terms that

$$\begin{aligned} r_{ij}^f &\lesssim\ \min(\widehat{\alpha}_{i\,|f},\, \widehat{\alpha}_{j\,|f})\,(1+\log(H_i/h_i))\,\frac{1}{H_i}\ \underbrace{\|w_i - w_j\|_{L^2(f)}^2}_{\lesssim\|w_i\|_{L_2(f)}^2+\|w_j\|_{L_2(f)}^2}\,, \\ r_i^f &\lesssim\ \widehat{\alpha}_{i\,|f}\,(1+\log(H_i/h_i))\,\frac{1}{H_i}\,\|w_i\|_{L^2(f)}^2\,.\end{aligned}$$

In the following we use construction (5.22) for $x = f$, i.e., $\widehat{\alpha}_{i\,|f} \lesssim \alpha_{|\omega_{i,m}}$ and $\widehat{\alpha}_{j\,|f} \lesssim \alpha_{|\omega_{j,\ell}}$. A standard Sobolev norm equivalence (cf. [39, Sect. A.4]) yields

$$\frac{1}{\eta_i}\|w_i\|^2_{L^2(f)} \;\lesssim\; |\mathcal{H}_{i,\alpha} w_i|^2_{H^1(\omega_{i,m})} + \frac{1}{\eta_i^2}\,\|\mathcal{H}_{i,\alpha} w_i\|^2_{L^2(\omega_{i,m})}\,,$$

and analogously, the L^2-norm of w_j on f is bounded in terms of the scaled H^1-norm on $\omega_{j,\ell}$. Combining these with the estimate before we obtain after summation that

(5.28)

$$\sum_{f\in\mathbb{F}_i^\Gamma\cap\mathbb{F}_j^\Gamma} r_{ij}^f + \sum_{f\in\mathbb{F}_i^D} r_i^f \;\lesssim\; \frac{\eta_i}{H_i}\,(1+\log(H_i/h_i)) \sum_{j\in\mathcal{N}_i} \Big\{ |\mathcal{H}_{j,\alpha} w_j|^2_{H^1(\Omega_{j,\eta_j}),\alpha} + \frac{1}{\eta_j^2}\,\|\mathcal{H}_{j,\alpha} w_j\|^2_{L^2(\Omega_{j,\eta_j}),\alpha} \Big\}\,.$$

- For the patch edge terms we obtain by similar arguments that

$$r_{ij}^e \;\lesssim\; \widehat{\alpha}_{i\,|e}\,\|w_i\|^2_{L^2(e)} + \widehat{\alpha}_{j\,|e}\,\|w_j\|^2_{L^2(e)}\,, \qquad r_i^e \;\lesssim\; \widehat{\alpha}_{i\,|e}\|w_i\|^2_{L^2(e)}\,.$$

Due to [39, Lemma 4.16] and construction (5.22) for $x = e$, we have

$$\|w_j\|^2_{L^2(e)} \;\lesssim\; (1+\log(\eta_j/h_j)) \Big\{ |\mathcal{H}_{j,\alpha} w_j|^2_{H^1(\omega_{j,\ell})} + \frac{1}{\eta_j^2}\,\|\mathcal{H}_{j,\alpha} w_j\|^2_{L^2(\omega_{j,\ell})} \Big\}\,,$$

and the analogous estimate for $\|w_i\|^2_{L^2(e)}$. Collecting and summing all terms we obtain

$$\sum_{e\in\mathbb{E}_i^\Gamma\cap\mathbb{E}_j^\Gamma} r_{ij}^e + \sum_{e\in\mathbb{E}_i^D} r_i^e \;\lesssim\; (1+\log(\eta_i/h_i)) \sum_{j\in\mathcal{N}_i} \Big\{ |\mathcal{H}_{j,\alpha} w_j|^2_{H^1(\Omega_{j,\eta_j}),\alpha} + \frac{1}{\eta_j^2}\,\|\mathcal{H}_{j,\alpha} w_j|^2_{L^2(\Omega_{j,\eta_j}),\alpha} \Big\}\,.$$

- The patch vertex terms can be estimated trivially by patch edge terms.

Combining all the estimates, noticing that

$$|\mathcal{H}_{i,\alpha} w_i|^2_{H^1(\Omega_{i,\eta_i}),\alpha} \;\le\; |w_i|^2_{S_i}\,, \qquad \eta_i/H_i \;\le\; 1\,,$$

and using Lemma 3.2 we obtain

$$|(P_D\, z_w)_i|^2_{S_i} \;\lesssim\; \Big(\max_{k=1}^N \frac{H_k}{\eta_k}\Big) \sum_{j\in\mathcal{N}_i} C^*_{j,\eta_j}\,(1+\log(H_j/\eta_j))\,|w_j|^2_{S_j} \qquad \forall w\in W^\perp,\ i=1,\ldots,N\,,$$

which directly implies the statement of Lemma 5.7. □

Combining Lemma 5.6 and Lemma 5.7, we finally obtain inequality (5.8).

To obtain the improved result in Remark 4.3(v) for the choice $Q = M^{-1}$ we can use the proof of [39, Lemma 6.14] instead of Lemma 5.7 and avoid to introduce the extra factor $\max_{k=1}^N H_k/\eta_k$. On the other hand, if Q is chosen according to (4.17), i.e. using some a priori information on η_i, then the factors H_i/η_i and η_i/H_i in (5.27) and (5.28) (resp.) disappear and again we can avoid to introduce the extra factor $\max_{k=1}^N H_k/\eta_k$.

The proof of Corollary 4.2 requires no new ideas. For non-floating subdomains, one can use standard Friedrichs or discrete Poincaré-Friedrichs inequalities on the subregions $\Omega_i^{(k)}$ using that the function w_i vanishes at least on an edge. If $M_i = 2$ for all subdomains Ω_i, the quadratic bound is obtained using the standard trick of introducing a face average, as it is described e. g. in [19, 29].

6. Numerical results

In this section we would like to confirm our new theoretical results. We point out that results for interior "island" coefficients as well as for interface variation are already contained in [29, Section 5] and in [30]. Note in particular, that our new theory explains the robustness of one-level FETI in case of the nonlinear magnetostatics with large interface variation in [29, Section 5.4], see also [28].

In all our computations we used PARDISO [36] as sparse direct solver for the subdomain problems.

6.1. **Edge islands.** In Example 1 we choose Ω to consist of 16 squares with an island coefficient cutting through a subdomain edge (cf. Fig. 6, left). In order to rule out symmetries we have shifted the centre of the island to the right of the subdomain interface. In Figure 6 and in what follows, H denotes the subdomain width and η denotes the characteristic geometric scale of the coefficient island. We set the coefficient to 1 outside the island (shaded region) and to a constant α_I inside. We impose Dirichlet boundary conditions on the entire boundary $\partial\Omega$ and choose a piecewise constant right-hand side.

In Tables 1 and 2 we display number of PCG iterations (to achieve a relative residual reduction of 10^{-8}) and condition numbers (estimated by Lanczos' method) for the cases $\alpha_I = 10^{+5}$ and $\alpha_I = 10^{-5}$.

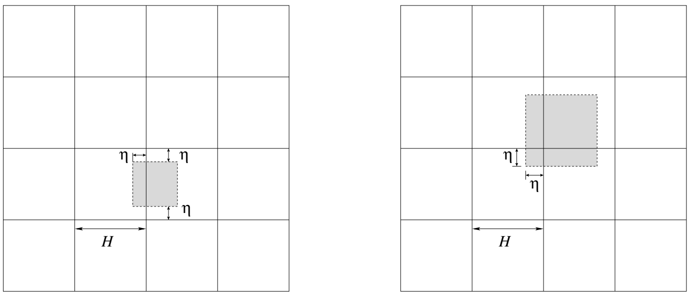

FIGURE 6. Coefficient distribution. *Left* Example 1 (edge island), *right* Example 2 (crosspoint island)

	$\frac{H}{h} = 64$		128		256		512	
$\frac{H}{\eta} = 4$	17	(19)	17	(19)	20	(22)	22	(23)
8	22	(19)	23	(24)	25	(25)	26	(26)
16	22	(23)	24	(25)	26	(28)	28	(30)
32	23	(25)	25	(27)	27	(29)	29	(31)
64	26	(27)	27	(30)	30	(32)	31	(33)
128	–		33	(31)	36	(36)	37	(37)
256	–		–		43	(39)	46	(42)
512	–		–		–		52	(49)

TABLE 1. Example 1: number of CG iterations; island coefficient with $\alpha_I = 10^{+5}$, in brackets: $\alpha_I = 10^{-5}$.

	$\frac{H}{h} = 64$		128		256		512	
$\frac{H}{\eta} = 4$	6.9	(7.0)	6.9	(6.9)	10.1	(10.1)	11.9	(12.0)
8	6.9	(11.0)	8.4	(13.3)	10.1	(15.7)	11.9	(18.3)
16	7.7	(20.8)	9.0	(25.0)	10.5	(29.4)	12.2	(22.5)
32	10.8	(39.0)	11.7	(47.3)	12.8	(55.7)	14.1	(64.3)
64	18.4	(72.0)	19.1	(88.2)	19.9	(104.9)	20.8	(121.7)
128	–		35.0	(163.2)	35.7	(195.8)	36.5	(229.1)
256	–		–		67.9	(363.6)	68.8	(429.0)
512	–		–		–		133.2	(800.5)

TABLE 2. Example 1: estimated condition numbers; island coefficient with $\alpha_I = 10^{+5}$, in brackets: $\alpha_I = 10^{-5}$.

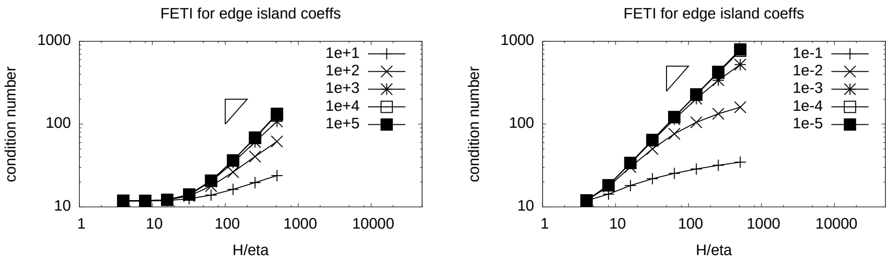

FIGURE 7. Example 1: estimated condition numbers; $H/h = 512$, varying ratio H/η and varying magnitude of the jump α_I. *Left* $\alpha_I > 1$, *right* $\alpha_I < 1$

6.2. **Cross point islands.** In Example 2 we choose the coefficient distribution sketched in Fig 6, right. Again we set α to a constant α_I in the island (the shaded square), and 1 elsewhere. Note that here, the width/height of the island remains of fixed size H.

	$\frac{H}{h} = 64$		128		256		512	
$\frac{H}{\eta} = 4$	17	(17)	19	(18)	20	(20)	21	(22)
8	19	(19)	21	(20)	22	(21)	22	(23)
16	20	(19)	21	(22)	23	(23)	24	(25)
32	23	(22)	24	(24)	26	(25)	27	(28)
64	28	(22)	29	(26)	30	(28)	32	(30)
128	–		34	(29)	36	(32)	39	(36)
256	–		–		47	(37)	50	(39)
512	–		–		–		69	(47)

TABLE 3. Example 2: number of CG iterations; island coefficient with $\alpha_I = 10^{+5}$, in brackets: $\alpha_I = 10^{-5}$.

	$\frac{H}{h} = 64$		128		256		512	
$\frac{H}{\eta} = 4$	7.36	(8.06)	8.92	(9.77)	10.64	(11.63)	12.51	(13.64)
8	8.74	(11.99)	10.45	(14.44)	12.31	(17.03)	14.34	(19.78)
16	10.83	(19.83)	12.62	(23.95)	14.60	(28.19)	16.75	(32.58)
32	15.30	(34.19)	16.85	(41.71)	18.63	(49.38)	20.66	(57.18)
64	26.19	(59.91)	27.45	(74.11)	28.81	(88.67)	30.30	(103.38)
128	–		50.98	(132.61)	52.32	(160.58)	53.60	(189.14)
256	–		–		101.48	(291.42)	103.36	(346.84)
512	–		–		–		221.25	(635.45)

TABLE 4. Example 2: estimated condition numbers; island coefficient with $\alpha_I = 10^{+5}$, in brackets: $\alpha_I = 10^{-5}$.

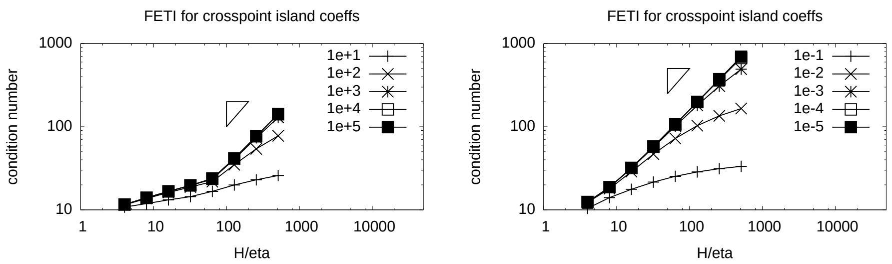

FIGURE 8. Example 2: estimated condition numbers; $H/h = 512$, varying ratio H/η and varying magnitude of the jump α_I. *Left* $\alpha_I > 1$, *right* $\alpha_I < 1$

6.3. **Standard one-level vs. all-floating FETI.** In Example 3 we consider a coefficient island cutting through a domain that touches the Dirichlet boundary, cf. Fig. 9, left. As before, we impose Dirichlet boundary conditions on the whole of $\partial\Omega$ and choose $\alpha = \alpha_I =$ const inside the shaded square, and $\alpha = 1$ elsewhere. As one can see in Table 5, the standard one-level FETI method is not robust when $\alpha_I > 1$, whereas the all-floating method remains robust. The reason why the number of PCG iterations stays small in all cases (even when the condition number blows up) is probably due to the fact that we have only considered one coefficient island. This is related to spectral clustering effects for domain decomposition preconditioners, which are explained in [13].

The fact that one-level FETI is fully robust for $\alpha_I < 1$ is not contradicting our theory and is perfectly explained by Corollary 3.6 and Remark 3.7. To see this let Ω_i be any of the subdomains that supports the coefficient island and let $\Omega_i^{(I)}$ be the part of the island in Ω_i. Then, choosing $\Omega_{i,\mathrm{art}}^{(1)} := \Omega_i \backslash \Omega_i^{(I)}$, $\Omega_{i,\mathrm{art}}^{(2)} := \Omega_i$ and $\eta_i := H_i/2$ Corollary 3.6 applies with $\sigma(H_i, h_i) = 1$ (since the interface X_i^* is an edge, cf. Remark 3.7), and therefore we have robustness due to Corollary 4.2.

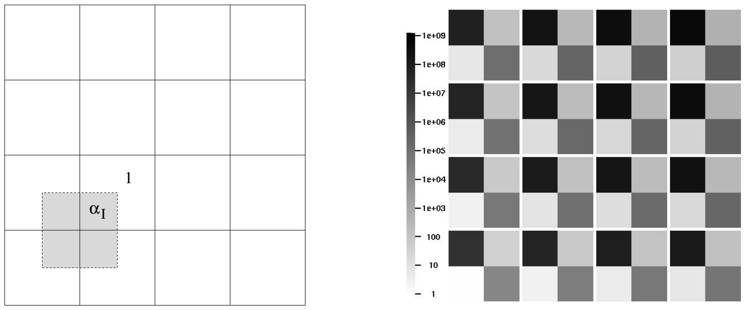

FIGURE 9. *Left* coefficient distribution in Example 3, *right* coefficient distribution and subdomain partitioning in Example 4

	std. one-level		all-floating			std. one-level		all-floating	
α_I	it	cond	it	cond	α_I	it	cond	it	cond
1	17	8.32	19	7.11	1	17	8.32	19	7.11
10^{-1}	21	8.40	22	7.32	10^{+1}	24	$2.59 \cdot 10^1$	22	7.25
10^{-2}	21	8.42	22	7.49	10^{+2}	27	$2.05 \cdot 10^2$	21	7.31
10^{-3}	21	8.43	22	7.51	10^{+3}	31	$2.00 \cdot 10^3$	21	7.32
10^{-4}	21	8.43	22	7.51	10^{+4}	31	$1.99 \cdot 10^4$	21	7.32
10^{-5}	21	8.43	22	7.51	10^{+5}	36	$1.99 \cdot 10^5$	21	7.32
10^{-6}	21	8.43	22	7.51	10^{+6}	38	$1.99 \cdot 10^6$	22	7.32
10^{-7}	21	8.43	22	7.51	10^{+7}	42	$1.99 \cdot 10^7$	21	7.32

TABLE 5. Example 3: iteration numbers and condition number estimates, standard one-level vs. all-floating FETI, $H/h = 128$

6.4. **"Multi-valued" coefficients.** In Example 4 we choose 16 subdomains and the coefficient distribution shown in Fig. 9, right. Again we choose Dirichlet boundary conditions on the whole of $\partial\Omega$. On the lower left subdomain the coefficient takes *four* different values (1, 10^3, 10^5, and 10^7) and varies in a non-quasimonotone way. We repeat this pattern on the other subdomains but slightly increase the coefficient values in order to rule out periodicity. Table 6 shows the iteration numbers and estimated condition numbers for standard one-level and all-floating FETI. As we expect, the all-floating method is robust in α. The reason for the robustness of one-level FETI is that we can always choose artificial subregions and coefficients in order to connect any of the four subregions to the Dirichlet boundary (see Corollary 3.6 and Remark 3.7).

H/h	32	64	128	256	512
std. one-level	25 (28.27)	27 (36.30)	29 (43.94)	30 (50.69)	32 (56.49)
all-floating	26 (20.38)	30 (28.90)	33 (37.07)	34 (44.18)	37 (50.13)

TABLE 6. Example 4: number of PCG iterations, estimated condition numbers in brackets

Appendix A. Proof of the generalised Poincaré inequality (3.4)

We prove that for all $u \in H^1(\Omega^{(k)}_{i,\eta_i})$ with $\int_{\Gamma^{(12)}_{i,\eta_i}} u \, ds = 0$ we have

$$\frac{1}{\eta_i} \, \|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \Big(\frac{H_i}{\eta_i}\Big)^{\beta} \, |u|^2_{H^1(\Omega_{i,\eta_i})} \,, \tag{A.1}$$

with $\beta = d$ in general, and $\beta = 2$ if $d = 3$ and $|\Gamma^{(12)}_{i,\eta_i}| \gtrsim H_i \, \eta_i$.

Recall that Ξ_{i,η_i} is the partition of Ω_{i,η_i} and that the partition $\Omega^{(1)}_{i,\eta_i} \cup \Omega^{(2)}_{i,\eta_i}$ is compatible. Then $\Xi^{(k)}_{i,\eta_i} := \{\omega_{i,j} \in \Xi_{i,\eta_i} : \omega_{i,j} \subset \Omega_i^{(k)}\}$ decomposes $\Omega^{(k)}_{i,\eta_i}$ into patches for each $k \in \{1, 2\}$.

Definition A.1. Let $\omega_{i,j}$, $\omega_{i,\ell} \in \Xi^{(k)}_{i,\eta_i}$. We call $P_{j\ell}$ a *path* of length $M_{j\ell}$ connecting the patches $\omega_{i,j}$ and $\omega_{i,\ell}$, iff it is a connected union of $M_{j\ell}$ patches from $\Xi^{(k)}_{i,\eta_i}$.

Definition A.2. In two (resp. three) dimensions, let $\gamma^{(k)}_{i,j}$ denote the faces (resp. edges) of the patches from $\Xi^{(k)}_{i,\eta_i}$ that are contained in $\partial\Omega_i \cup \Gamma^{(12)}_{i,\eta_i}$.

Similar to the proof of Lemma 4.3 in [29] we integrate the identity

$$u(x)^2 + u(y)^2 - 2\, u(x)\, u(y) \;=\; \big[u(x) - u(y)\big]^2 \,,$$

over $\Lambda_i^{(k)}$ with respect to x and over $\Gamma^{(12)}_{i,\eta_i}$ with respect to y. From our assumption that $\int_{\Gamma^{(12)}_{i,\eta_i}} u \, ds = 0$ and since $0 \le |\Lambda_i^{(k)}| \int_{\Gamma^{(12)}_{i,\eta_i}} |u(y)|^2 \, ds_y$ we obtain

$$|\Gamma^{(12)}_{i,\eta_i}| \, \|u\|^2_{L^2(\Lambda_i^{(k)})} \;\le\; \int_{\Lambda_i^{(k)}} \int_{\Gamma^{(12)}_{i,\eta_i}} |u(x) - u(y)|^2 \, ds_y \, ds_x \,. \tag{A.2}$$

In order to bound the right hand side of this expression, we use the following lemma, which follows from [29, Lemma A.2(i)].

Lemma A.3. *Let $\gamma^{(k)}_{i,\ell}$ and $\gamma^{(k)}_{i,j}$ be faces of the patches $\omega^{(k)}_{i,j}$, $\omega^{(k)}_{i,\ell} \in \Xi^{(k)}_{i,\eta_i}$ according to Definition A.2 and let $P_{j\ell}$ be a path of length $M_{j\ell}$ connecting the two patches. Then*

$$\frac{1}{\eta_i^d} \int_{\gamma_{i,j}} \int_{\gamma_{i,\ell}} |u(x) - u(y)|^2 \, ds_x \, ds_y \;\lesssim\; M_{j\ell} \, |u|^2_{H^1(P_{ij})} \qquad \forall u \in H^1(P_{ij}) \,.$$

Combining (A.2) and Lemma A.3 yields

$$\begin{aligned} |\Gamma^{(12)}_{i,\eta_i}| \, \|u\|^2_{L^2(\Lambda_i^{(k)})} \;&\le\; \sum_{j : \omega_{i,j} \subset \Lambda_i^{(k)}} \; \sum_{\ell : \omega_{i,j} \subset \Gamma^{(12)}_{i,\eta_i}} \int_{\gamma_{i,j}} \int_{\gamma_{i,\ell}} |u(x) - u(y)|^2 \, ds_x \, ds_y \\ &\lesssim\; \sum_{j : \omega_{i,j} \subset \Lambda_i^{(k)}} \; \sum_{\ell : \omega_{i,j} \subset \Gamma^{(12)}_{i,\eta_i}} \eta_i^d \, M_{j\ell} \, |u|^2_{H^1(P_{ij})} \,. \end{aligned}$$

Using the regularity of Ω_i and the η_i-regularity of Ω_{i,η_i}, it is easily shown that (i) the first sum contains $\mathcal{O}(|\Lambda_i^{(k)}|/\eta_i^{d-1})$ terms, (ii) the second sum contains $\mathcal{O}(|\Gamma^{(12)}_{i,\eta_i}|/\eta_i^{d-1})$ terms, and (iii) $M_{j\ell} \lesssim H_i/\eta_i$. Therefore, we can conclude that

$$|\Gamma^{(12)}_{i,\eta_i}| \, \|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \frac{|\Lambda_i^{(k)}|}{\eta_i^{d-1}} \, \frac{|\Gamma^{(12)}_{i,\eta_i}|}{\eta_i^{d-1}} \, \eta_i^d \, \frac{H_i}{\eta_i} \, |u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})}$$

and so

$$\frac{1}{\eta_i}\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \frac{|\Lambda_i^{(k)}|H_i}{\eta_i^d}\,|u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})}\,. \tag{A.3}$$

Since, $|\Lambda_i^{(k)}| \lesssim H_i^{d-1}$ we obtain inequality (A.1) with $\beta = d$. Suppose now that $d = 3$ and $|\Gamma^{(12)}_{i,\eta_i}| \gtrsim H_i\,\eta_i$. Using an analogous overlapping argument as in [29, Lemma A.3], one can show that the paths $P_{j\ell}$ connecting the boundary patches with the interface patches can be grouped in such a way that we save one power of H_i/η_i in (A.3), i. e.,

$$\frac{1}{\eta_i}\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \frac{|\Lambda_i^{(k)}|}{\eta_i^2}\,|u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})}\,,$$

from which one easily deduces (A.1).

References

[1] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics*. Academic Press, Amsterdam, Boston, second edition, 2003.

[2] B. Aksoylu, I. G. Graham, H. Klie, and R. Scheichl. Towards a rigorously justified algebraic preconditioner for high-contrast diffusion problems. *Comput. Visual. Sci.*, 2008. DOI 10.1007/s00791-008-0105-1.

[3] R. E. Alcouffe, A. Brandt, J. J. E. Dendy, and J. W. Painter. The multi-grid method for the diffusion equation with strongly discontinuous coefficients. *SIAM J. Sci. Comput.*, 2(4):430–454, 1981.

[4] J. H. Bramble and J. Xu. Some estimates for a weighted l^2 projection. *Math. Comp.*, 56:463–476, 1991.

[5] S. C. Brenner. Analysis of two-dimensional FETI-DP preconditioners by the standard additive Schwarz framework. *Electron. Trans. Numer. Anal.*, 16:165–185, 2003.

[6] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, second edition, 2002.

[7] T. F. Chan and T. Mathew. Domain decomposition methods. In *Acta Numerica 1994*. Cambridge University Press, 1994.

[8] K. A. Cliffe, I. G. Graham, R. Scheichl, and L. Stals. Parallel computation of flow in heterogeneous media modelled by mixed finite elements. *J. Comput. Phys.*, 164(2):258–282, 2000.

[9] C. R. Dohrmann, A. Klawonn, and O. B. Widlund. Domain decomposition for less regular subdomains: Overlapping Schwarz in two dimensions. *SIAM J. Numer. Anal.*, 46(4):2153–2168, 2008.

[10] Z. Dostál, D. Horák, and R. Kučera. Total FETI – An easier implementable variant of the FETI method for numerical solution of elliptic PDE. *Commun. Numer. Methods Eng.*, 22(12):1155–1162, 2006.

[11] C. Farhat and F.-X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Int. J. Numer. Meth. Engrg.*, 32:1205–1227, 1991.

[12] J. Galvis and Y. Efendiev. Domain decomposition preconditioners for multiscale flows in high contrast media. Technical report, Department of Mathematics, Texas A&M University, College Station, TX, 2009.

[13] I. G. Graham and M. J. Hagger. Unstructured additive Schwarz-conjugate gradient method for elliptic problems with highly discontinuous coefficients. *SIAM J. Sci. Comput.*, 20(6):2041–2066, 1999.

[14] I. G. Graham, P. O. Lechner, and R. Scheichl. Domain decomposition for multiscale PDEs. *Numer. Math.*, 106(4):589–626, 2007.

[15] I. G. Graham and R. Scheichl. Robust domain decomposition algorithms for multiscale PDEs. *Numerical Methods for Partial Differential Equations*, 23:859–878, 2007.

[16] I. G. Graham and R. Scheichl. Coefficient-explicit condition number bounds for overlapping additive Schwarz. In U. Langer, M. Discacciati, D. E. Keyes, O. B. Widlund, and W. Zulehner, editors, *Domain Decomposition Methods in Science and Engineering XVII*, volume 60 of *Lecture Notes in Computational Science and Engineering*, pages 365–372, Berlin Heidelberg, 2008. Springer.

[17] A. Klawonn and O. Rheinbach. Robust FETI-DP methods for heterogeneous three dimensional elasticity problems. *Comput. Methods Appl. Mech. Engrg.*, 196:1400–1414, 2007.

[18] A. Klawonn, O. Rheinbach, and O. B. Widlund. An analysis of a FETI-DP algorithm on irregular subdomains in the plane. *SIAM J. Numer. Anal.*, 46(5):2484–2504, 2008.

[19] A. Klawonn and O. B. Widlund. FETI and Neumann-Neumann iterative substructuring methods: connections and new results. *Comm. Pure Appl. Math.*, 54(1):57–90, 2001.

[20] A. Klawonn, O. B. Widlund, and M. Dryja. Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. *SIAM J. Numer. Anal.*, 40(1):159–179, 2002.
[21] U. Langer and C. Pechstein. Coupled finite and boundary element tearing and interconnecting solvers for nonlinear potential problems. *ZAMM Z. Angew. Math. Mech.*, 86(12), 2006.
[22] J. Mandel, C. R. Dohrmann, and R. Tezaur. An algebraic theory for primal and dual substructuring methods by constraints. *Appl. Numer. Math.*, 54:167–193, 2005.
[23] J. Mandel and R. Tezaur. Convergence of a substrucuring method with Lagrange multipliers. *Numer. Math.*, 73:473–487, 1996.
[24] G. Of. *BETI-Gebietszerlegungsmethoden mit schnellen Randelementverfahren und Anwendungen.* PhD thesis, Universität Stuttgart, Germany, January 2006. (in German).
[25] G. Of. The all-floating BETI method: Numerical results. In U. Langer, M. Discacciati, D. E. Keyes, O. B. Widlund, and W. Zulehner, editors, *Domain Decomposition Methods in Science and Engineering XVII*, volume 60 of *Lecture Notes in Computational Science and Engineering*, pages 295–302, Berlin Heidelberg, 2008. Springer.
[26] C. Pechstein. Boundary element tearing and interconnecting methods in unbounded domains. *Appl. Num. Math.*, 2008. in press; `DOI:10.1016/j.apnum.2008.12.031`.
[27] C. Pechstein. *Finite and boundary element tearing and interconnecting methods for multiscale elliptic partial differential equations.* PhD thesis, Institute of Computational Mathematics, Johannes Kepler University, Linz, Austria, December 2008.
[28] C. Pechstein and R. Scheichl. Robust FETI solvers for multiscale elliptic PDEs. In *Proceedings of the 7th International Conference on Scientific Computing in Electrical Engineering (SCEE).* Helsinki, Finland, September 2008; to appear.
[29] C. Pechstein and R. Scheichl. Analysis of FETI methods for multiscale PDEs. *Numer. Math.*, 111(2):293–333, 2008.
[30] C. Pechstein and R. Scheichl. Scaling up through domain decomposition. BICS Preprint 1/09, Bath Institute for Complex Systems, University of Bath, 2009. submitted.
[31] D. Rixen and C. Farhat. Preconditioning the FETI method for problems with intra- and inter-subdomain coefficient jumps. In P. E. Bjørstad, M. Espedal, and D. Keyes, editors, *Ninth International Conference on Domain Decomposition Methods*, pages 472–479, 1997. `http://www.ddm.org/DD9/Rixen.pdf`.
[32] D. Rixen and C. Farhat. A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. *Internat. J. Numer. Methods Engrg.*, 44:489–516, 1999.
[33] J. Ruge and K. Stüben. Efficient solution of finite difference and finite element equations by algebraic multigrid (AMG). In D. J. Paddon and H. Holstein, editors, *Multigrid Methods for Integral and Differential Equations*, IMA Conference Series, pages 169–212, Oxford, 1985. Clarendon Press.
[34] M. Sarkis. Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using non-conforming elements. *Numer. Math.*, 77(3):383–406, 1997.
[35] R. Scheichl and E. Vainikko. Additive Schwarz and aggregation-based coarsening for elliptic problems with highly variable coefficients. *Computing*, 80(4):319–343, 2007.
[36] O. Schenk and K. Gärtner. On fast factorization pivoting methods for sparse symmetric indefinite systems. *Elec. Trans. Numer. Anal*, 23:158–179, 2006.
[37] L. R. Scott and S. Zhang. Finite element interpolation of non-smooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.
[38] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*, volume 30 of *Princeton Math Series.* Princeton University Press, Princeton, New Jersey, 1970.
[39] A. Toselli and O. Widlund. *Domain Decoposition Methods – Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer, Berlin, Heidelberg, 2005.
[40] J. Van lent, R. Scheichl, and I. G. Graham. Energy minimizing coarse spaces for two-level Schwarz methods for multiscale PDEs. *Numer. Linear Algebra Appl.* published online March 12th 2009, pp. 1–25, `DOI:10.1002/nla.643`.
[41] P. Vanek, J. Mandel, and M. Brezina. Algebraic multigrid by smoothed aggregation for 2nd and 4th order elliptic problems. *Computing*, 56(3):179–196, 1996.
[42] J. Xu and Y. Zhu. Uniform convergent multigrid methods for elliptic problems with strongly discontinuous coefficients. *Math. Models Methods Appl. Sci.*, 18(1):77–105, 2008.