Lecture 5 (Edinburgh): BPRE and Queueing systems

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June 26, 2006

1 Processes with immigration counted by random characteristics

Continuous time branching process with immigration and final product: Consider a BPI in which immigration occurs with rate ρ_0 and the reproduction function of the number of immigrants $g(s) = Es^{\eta}$. That is, at an immigration moment a random number η of children is produced and a final product $\chi^* \geq 0$ which is not changed later on (that is, a random variable which is INDEPENDENT on the moment of immigration),

$$\psi^{\chi}(s,\lambda) = \mathbf{E}s^{\eta}e^{-\lambda\chi^*},\ \psi^{\chi}(s,0) = g(s) = \mathbf{E}s^{\eta}.$$

The aboriginal individuals have the exponential life-time distribution with parameter ρ_1 and the reproduction function $f(s) = Es^{\xi}$. An aboriginal individual produces at the end of the life a random number ξ of children and a final product $\chi \geq 0$ with probability generating function

$$\varphi^{\chi}(s,\lambda) = \mathbf{E}s^{\xi}e^{-\lambda\chi}, \varphi^{\chi}(s,0) = f(s)$$

which is not changed later on. The random variables χ and χ^* are called random characteristics.

Let

$$Z_*^{\chi}(t) = \sum_D \chi_D + \sum_I \chi_I^*$$

where summation is for all particles D which died up to the moment t and all immigrants immigrated up to moment t.

The joint distribution of the components of the vector $(Z_*(t), Z_*^{\chi}(t))$, where $Z_*(t)$ is the number of particles in the process with immigration, is described by the following integral and differential equations. Setting

$$\Phi_*\left(t,s,\lambda\right) = \mathbf{E}\left[s^{Z(t)}e^{-\lambda Z_*^{\chi}(t)}|(Z(t), Z_*^{\chi}(t)) = (0,0)\right]$$

and $G_0(t) = 1 - e^{-\rho_0 t}$ we get

$$\Phi_*\left(t,s,\lambda\right) = 1 - G_0(t) + \int_0^t \psi^\chi(\Phi\left(t-u,s,\lambda\right),\lambda)\Phi_*\left(t-u,s,\lambda\right)dG_0(u)$$

with

$$\Phi_*\left(t,s,\lambda\right) = 1.$$

This leads to

$$\Phi_*\left(t,s,\lambda\right) = \exp\left\{\int_0^t \rho_0(\psi^\chi(\Phi\left(u,s,\lambda\right),\lambda) - 1)du\right\}.$$

Here (RECALL) for

$$\Phi\left(t,s,\lambda\right) = \mathbf{E}\left[s^{Z(t)}e^{-\lambda Z^\chi(t)}|(Z(0),Z^\chi(0)) = (1,0)\right]$$

and $G(t) = 1 - e^{-\rho_1 t}$ we have

$$\Phi\left(t,s,\lambda\right) = s\left(1 - G(t)\right) + \int_0^t \varphi^\chi(\Phi\left(t-u,s,\lambda\right),\lambda)dG(u).$$

In particular, for

$$F^0(t,s) = \mathbf{E}\left[s^{Z_*(t)}|\text{immigration, } Z_*(0) = 0\right] = \Phi_*\left(t,s,0\right), F^0(0,s) = 1$$

and

$$F^1(t,s) = \mathbf{E}\left[s^{Z(t)}|\text{no immigration, } Z(0) = 1\right] = \Phi\left(t,s,0\right),\ F^1(0,s) = s,$$

we get

$$F^0(t,s) = \exp\left\{\int_0^t \rho_0(g(F^1(u,s)) - 1)du\right\}.$$

and, recall,

$$\frac{\partial F^1(t,s)}{\partial t} = \rho_1\left(f(F^1(t,s)) - F^1(t,s)\right),\quad F^1(0,s) = s,$$

and

$$\frac{\partial F^1(t;s)}{\partial t} = \rho_1(f\left(s\right) - s)\frac{\partial F^1(t;s)}{\partial s},\ F^1(0,s) = s.$$

Theorem 1 *If $g'(1) < \infty$ and $f^{'}(1) < 1$ then*

$$\begin{aligned}\lim_{t\to\infty} F^0(t,s) &= \exp\left\{\int_0^\infty \rho_0(g(F^1(u,s)) - 1)du\right\} \\ &= \exp\left\{\int_s^1 \frac{\rho_0(g(y) - 1)}{\rho_1\left(f(y) - y\right)}dy\right\}.\end{aligned}$$

Proof. Since

$$\begin{aligned} 0 &\le 1 - g(F^1(u,s)) \le g'(1)\left(1 - F^1(u,s)\right) \\ &\le g'(1)e^{\rho_1\left(f'(1)-1\right)u}(1-s) \end{aligned}$$

the integral converges uniformly in $s \in [0,1]$. Hence

$$\lim_{t\to\infty} F^0(t,s) = \exp\left\{\int_0^\infty \rho_0(g(F^1(u,s))-1)du\right\}.$$

Now

$$\begin{aligned} & \frac{d}{ds}\int_0^\infty \rho_0(g(F^1(u,s))-1)du \\ = {} & \rho_0 \int_0^\infty \frac{dg(F^1(u,s))}{dF^1}\frac{\partial F^1(u,s)}{\partial s}du \\ = {} & \rho_0 \int_0^\infty \frac{dg(F^1(u,s))}{dF^1}\frac{\partial F^1(u,s)}{\partial s}du \\ = {} & \rho_0 \int_0^\infty \frac{dg(F^1(u,s))}{dF^1}\frac{\partial F^1(u,s)}{\partial u}\frac{du}{\rho_1(f(s)-s)} \\ = {} & \frac{\rho_0}{\rho_1(f(s)-s)}\int_0^\infty \frac{\partial g(F^1(u,s))}{\partial u}du = \frac{\rho_0}{\rho_1(f(s)-s)} g(F^1(u,s))|_0^\infty \\ = {} & \frac{\rho_0\,(1-g(s))}{\rho_1(f(s)-s)}. \end{aligned}$$

Differentiation is also justified since convergence

$$\lim_{t\to\infty}\frac{\rho_0}{\rho_1(f(s)-s)}g(F^1(t,s)) = \frac{\rho_0}{\rho_1(f(s)-s)}$$

is uniform in $s \in [0,1)$.

2 System $M|G|1$ with retrials (repeated calls)

2.1 Description of the queueing system

Consider an $M|G|1$ system with Poisson flow of customers having intensity ρ_0 and the following service discipline: a just arriving customer is immediately served if the server is idle else the customer joins the queue and repeats its attempts with exponentially distributed time-intervals with parameter ρ_1 until success.

Assume that the vectors $(\xi_i, \pi_i), i = 1, 2, ...$ are iid and have components which are equal, respectively, to the number of new customers arriving during the service time of the respective customer and its service.

Suppose that initially there were $n+1$ customers, one is marked and the server is idle. The problem is to evaluate the waiting time of the marked customer.

Under our assumptions for the time-interval σ_1,

$$\mathbf{P}\left(\sigma_1 > t\right) = e^{-t(\rho_1(n+1)+\rho_0)}$$

the server is idle then some of the customers comes to the service and is served within the time-interval π_1 and the server remains idle for the time-interval σ_2 whose distribution depends on the number of customers staying in the queue just after the moment $\sigma_1 + \pi_1$ so on. Let N be the number of customers served BEFORE the marked one was taken to the service. In this case the waiting time of the marked customer is

$$V_n = \sigma_1 + \pi_1 + \sigma_2 + \pi_2 + ... + \sigma_N + \pi_N + \sigma_{N+1}$$

while the total number R_n of unsuccessful calls of the marked customer until success equals

$$R_n = r_1 + r_2 + ... + r_N + 1$$

where r_i is the number of attempts to call by the marked customer between the end of the services of the $(i-1)-$ th and i-th customers. Hence, as before, we may assume that at the each, say, $i-$th customer, produces at the end of the service a final product χ_i and the number of new customers arriving to the system during the service time of the i-th customer is just ξ_i. Suppose that $(\xi_i, \chi_i), i = 1, 2, ...$ are iid and set

$$T_n^{*\chi} = \chi_1 + \chi_2 + ... + \chi_N.$$

Clearly,

$$R_n = T_n^{*r} + 1, V_n = T_n^{*\pi} + \sigma_1 + \sigma_2 + ... + \sigma_N + \sigma_{N+1}.$$

2.2 The associated BP with immigration.

Now we construct an associated branching process with immigration. We have two types of particles 0 and 1. The life-lengths of the particles of the respective types are exponential with parameters

$$\rho_0, \rho_1.$$

Each particle, say D, produces at the end of her life (ξ_D, χ_D) and, additionally, a particle of the type 0 produces exactly *one* particle of type 0.

Thus, in our previous setting this is a Markov process with immigration rate ρ_0 and the reproduction function

$$\psi^{\chi}(s,\lambda) = \varphi^{\chi}(s,\lambda), \ \psi^{\chi}(s,0) = f(s)$$

(since in the case under consideration there is no difference in the service times of immigrants and aboriginal individuals).

Denote by $\sigma_1^{'}, \sigma_2^{'}, ...$ the splitting moments of the BPI and let the process start by $n+1$ particles of type 1 (with one of them marked) and 1 particle of type 0.

Thus, we have interpretation - a particle of type zero - a customer from outside, a particle of type 1 - a customer from the queue.

We follow the evolution of the queue at the moments $\sigma_1, \sigma_1 + \pi_1 + \sigma_2, \sigma_1 + \pi_1 + \sigma_2 + \pi_2 + \sigma_3, ...$

We record how many new customers come to the system, which customer is served (from outside or from the queue) and what is the amount of the final product it produces. Thus, the final product produced up to the service moment of the marked customer coincides with the amount of the final product in the BPI up to the splitting moment of the marked particle,

$$\mathbf{P}\,(\tau \leq x) = 1 - e^{-\rho_1 x}.$$

One can see that

$$\sigma_1 \stackrel{d}{=} \sigma_1', \sigma_2 \stackrel{d}{=} \sigma_2', ...$$

and using this relation check that the following statement is valid.

Theorem 2 *the BPI and the queueing system can be specified on a common probability space in such a way that*

$$T_n^{*\chi} = Z_1^{\chi}(\tau) + ... + Z_n^{\chi}(\tau) + Z_*^{\chi}(\tau) \;\; a.s.$$

and

$$\sigma_1 = \sigma_1', \sigma_2 = \sigma_2', ...a.s.$$

where τ *and the rv* $Z_i^{\chi}(t), i = 1, 2, ..., n$ *are independent and* $P(\tau \leq x) = 1 - e^{-\rho_1 x}$ *and*

$$Z_i^{\chi}(\tau) \stackrel{d}{=} Z^{\chi}(\tau)$$

and $Z_*^{\chi}(t)$ *is the final product produced in a BPI up to moment* t *which starts by one individual of type zero at time 0.*

Since

$$\sigma_1 + \sigma_2 + ... + \sigma_N + \sigma_{N+1} \stackrel{a.s.}{=} \sigma_1' + \sigma_2' + ... + \sigma_N' + \sigma_{N+1}' = \tau$$

we get the following

Corollary 3

$$\begin{aligned} R_n &= r_1 + r_2 + ... + r_N + 1 \\ &= Z_1^r(\tau) + ... + Z_n^r(\tau) + Z_*^r(\tau) + 1 \\ &= T_n^{*r} + 1 \end{aligned}$$

and

$$\begin{aligned} V_n &= \sigma_1 + \pi_1 + \sigma_2 + \pi_2 + ... + \sigma_N + \pi_N + \sigma_{N+1} \\ &= T_n^{*\pi} + \sigma_1 + \sigma_2 + ... + \sigma_N + \sigma_{N+1} \\ &= Z_1^{\pi}(\tau) + ... + Z_n^{\pi}(\tau) + Z_*^{\pi}(\tau) + \tau. \end{aligned}$$

Now we recall that

$$\Phi(t,s,\lambda) = \mathbf{E}\left[s^{Z(t)}e^{-\lambda Z^{\chi}(t)}|(Z(0),Z^{\chi}(0))=(1,0)\right],$$

$$\Phi(t,\lambda) = \mathbf{E}\left[e^{-\lambda Z^{\chi}(t)}|(Z(0),Z^{\chi}(0))=(1,0)\right],$$

and

$$\Phi_*(t,\lambda) = \mathbf{E}\left[e^{-\lambda Z^{\chi}(t)}|\text{ immigration; } Z(0)=0\right].$$

Then

$$\mathbf{E}e^{-\lambda T_n^{*\chi}} = \int_0^\infty e^{-t}\Phi^n(t,\lambda)\Phi_*(t,\lambda)\,dt$$

where

$$\frac{\partial\Phi(t,\lambda)}{\partial t} = \rho_1\left(\varphi^{\chi}(\Phi(t,\lambda),\lambda) - \Phi(t,\lambda)\right)$$

$$\Phi(0,\lambda) = 1.$$

and

$$\Phi_*(t,\lambda) = \exp\left\{\int_0^t \rho_0(\varphi^{\chi}(\Phi(u,\lambda),\lambda)-1)du\right\}.$$

Let $m = \mathbf{E}\xi - 1$. In the above situation we have as before (the proof is omitted)

Theorem 4 *As* $n\to\infty$

$$\frac{T_{*n}^{\chi}}{\rho_1 n\mathbf{E}\chi} \xrightarrow{d} \zeta$$

where the distribution function of ζ *is*

$$F_m(x) = 1-(1+mx)^{-1/m},\ 0\le x\le x_m,$$

where

$$x_m = -\frac{1}{m},\ m<0,\ x_m=\infty,\ m\ge 0,$$

and

$$F_0(x) = 1-e^{-x}.$$

3 Crump-Mode-Jagers process counted by random characteristics

We give here only an informal description of the Crump-Mode-Jagers process counted by random characteristics or, what is the same, of the general branching process counted by random characteristics. A particle, say, x, of this process is characterised by three random processes

$$(\lambda_x, \xi_x(\cdot), \chi_x(\cdot))$$

which are iid copies of a triple $(\lambda, \xi(\cdot), \chi(\cdot))$ and whose components have the following sense:

if a particle was born at moment σ_x then

$\lambda_x -$ is the life-length of the particle;

$\xi_x(t-\sigma_x)$ - is the number of children produced by the particle within the time-interval $[\sigma_x, t)$; $\xi_x(t-\sigma_x) = 0$ if $t - \sigma_x < 0$;

$\chi_x(t-\sigma_x) \geq 0-$ is a stochastic process subject to changes ONLY within the time-interval $[\sigma_x, \sigma_x + \lambda_x)$ while outside the interval it has the form

$$\chi_x(t-\sigma_x) = \begin{cases} 0 & \text{if} \quad t - \sigma_x < 0 \\ \\ \chi_x(\lambda_x) & \text{if} \quad t - \sigma_x \geq \lambda_x \end{cases}$$

(it is NOT assumed that $\chi_x(t)$ is a nondecreasing function in $t \geq 0$).

The stochastic process

$$Z^{\chi}(t) = \sum_x \chi_x(t - \sigma_x)$$

where summation is taken over all particles x born in the process up to moment t is called the general branching process counted by random characteristics.

Examples:

1) $\chi(t) = I\{t \in [0, \lambda)\} -$ in this case $Z^{\chi}(t) = Z(t)$ is the number of particles existing in the process up to moment t;

2)

$$\chi(t) = tI\{t \in [0, \lambda)\} + \lambda I\{\lambda < t\}$$

then

$$Z^{\chi}(t) = \int_0^t Z(u)du;$$

3) $\chi(t) = I\{t \geq 0\}$ then $Z^{\chi}(t)$ is the total number of particles born up to moment t.

Classification. $E\xi(\infty) <,=,> 1$ - subcritical, critical and supercritical, respectively.

Let

$$0 \leq v(1) \leq v(2) \leq ... \leq v(n) \leq ...$$

be the birth moments of the children of the initial particle. Then

$$\xi_0(t) = \#\{n : v(n) \leq t\}$$

is the number of children born by the initial particle up to moment t. We have

$$Z^{\chi}(t) = \chi_0(t) + \sum_{x \neq 0} \chi_x(t - \sigma_x) = \chi_0(t) + \sum_{v(n) \leq t} Z_n^{\chi}(t - v(n))$$

where $Z_n^\chi(\cdot)$, $n = 1, 2, ...$ are iid copies of $Z^\chi(\cdot)$. Hence it follows that

$$\begin{aligned}
\mathbf{E}Z^\chi(t) &= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{v(n)\le t} Z_n^\chi(t - v(n))\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{v(n)\le t} \mathbf{E}\left[Z_n^\chi(t - v(n)) \,|v(1), v(2), ..., v(n), ...\right]\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{v(n)\le t} \mathbf{E}\left[Z_n^\chi(t - v(n)) \,|v(n)\right]\right] \\
&= \mathbf{E}\chi(t) + \mathbf{E}\left[\sum_{u\le t} \mathbf{E}\left[Z^\chi(t-u)\right](\xi_0(u) - \xi_0(u-))\right] \\
&= \mathbf{E}\chi(t) + \int_0^t \mathbf{E}Z^\chi(t-u)\,\mathbf{E}\xi(du).
\end{aligned}$$

Thus, we get the following renewal-type equation for $A^\chi(t) = \mathbf{E}Z^\chi(t)$ and $\mu(t) = \mathbf{E}\xi(t)$:

$$A^\chi(t) = \mathbf{E}\chi(t) + \int_0^t A^\chi(t-u)\mu(du). \tag{1}$$

Malthusian parameter: a number α is called the Malthusian parameter of the process if

$$\int_0^\infty e^{-\alpha t}\mu(dt) = 1 \tag{2}$$

(such a solution not always exists). For the critical processes $\alpha = 0$, for the supercritical processes $\alpha > 0$, for the subcritical processes $\alpha < 0$ (if exists).

If the Malthusian parameter exists we can rewrite (1) as

$$C^\chi(t) = e^{-\alpha t}\mathbf{E}\chi(t) + \int_0^t C^\chi(t-u)d\left(\int_0^u e^{-\alpha y}\mu(dy)\right)$$

where $C^\chi(t) = e^{-\alpha t}A^\chi(t)$. In view of (2) and given that, say, $e^{-\alpha t}\mathbf{E}\chi(t)$ is directly Riemann integrable and

$$\int_0^\infty e^{-\alpha t}\mathbf{E}\chi(t)dt < \infty, \; \int_0^\infty te^{-\alpha t}\mu(dt) < \infty$$

we can apply the key renewal theorem to conclude that if the measure

$$M(t) = \int_0^t e^{-\alpha y}\mu(dy)$$

is non-lattice then

$$\lim_{t\to\infty} C^\chi(t) = \lim_{t\to\infty} e^{-\alpha t}A^\chi(t) = \int_0^\infty e^{-\alpha t}\mathbf{E}\chi(t)dt\left(\int_0^\infty te^{-\alpha t}\mu(dt)\right)^{-1}.$$

In particular, if $G(t)$ is the life-length distribution of particles and $\chi(t) = I\{t \in [0,\lambda)\}$ we get

$$\mathbf{E}\chi(t) = \mathbf{P}(\lambda > t) = 1 - G(t)$$

and

$$\lim_{t\to\infty} e^{-\alpha t}\mathbf{E}Z(t) = \frac{\int_0^\infty e^{-\alpha t}(1-G(t))\,dt}{\int_0^\infty te^{-\alpha t}\mu(dt)}$$

if the respective integrals converge.

4 $M|G|1$ system with processor sharing discipline

The model: a Poisson flow of customers with intensity Λ comes to a system with one server which has unit service intensity. The service time distribution of a particular customer is (if there are no other customers in the queue) $B(u)$. If there are M customers in the system at some moment T they are served simultaneously with intensity M^{-1} each.

Let

$$W_{l_1,\dots,l_{N-1}}(l_N)$$

be the waiting time for the end of service of a customer with service time l_N which arrived to the queue at the moment when the queue had $N-1$ customers with remaining service times $l_1,\dots,l_{N-1}$.

The question is to study the properties of the random variable $W_{l_1,\dots,l_{N-1}}(l_N)$ when $l_N \to \infty$.

To solve this problem we construct an auxiliary general branching process.

Construction of the branching process.

Consider a general branching process in which initially at time $t=0$ there are N particles with remaining life-lengths $l_1,\dots,l_{N-1},l_N$ and which constitute the zero generation of this process. The life-length distribution of any newborn particle λ_x is $P(\lambda_x \le u) = B(u)$, the reproduction process $\xi_x(t)$ of the number of children produced by a particle up to moment t has the probability generating function

$$\mathbf{E}s^{\xi_x(t)} = \int_0^t e^{\Lambda(s-1)u}dB(u) + e^{\Lambda(s-1)t}(1-B(t))$$

that is, this is an ordinary Poisson flow with intensity Λ stopped when the particle dies:

$$\mathbf{E}s^{\xi_x(t)} = \mathbf{E}s^{Poi_\Lambda(t\wedge\lambda_x)}.$$

Let $Z(t;l_1,\dots,l_{N-1},l_N)$ denote the number of particles in the process at moment t with the mentioned initial conditions. We use a simplified notation $Z(t)$ if at moment $t=0$ there is only one particle of zero age in the process.

We will consider also the process with immigration $X(t;l_1,\dots,l_{N-1},l_N)$ which has the same initial conditions and development as $Z(t;l_1,\dots,l_{N-1},l_N)$ but, in addition, given $X(t;l_1,...,l_{N-1},l_N)=0$ it starts again by *one* individual of zero age after a random time r_i having distribution $P\left(r_i\leq u\right)=1-e^{-\Lambda u}$ (if the process dies out for the $i-$th time). $X(t)$ is used if we initially start by the process $Z(t)$.

Now let $\sigma_{x_1}\leq\sigma_{x_2}\leq$ ...be the sequential moments of jumps of the process $X(t;l_1,...,l_{N-1},l_N)$. We construct by the general branching process the following queueing system with $S(T)$ being the number of customers in the queue at moment T:

1) the queue has N customers at $T=0$ with remaining service times $l_1,...,l_{N-1},l_N$;

2) the moment T_i of the $i-$th jump of the queue size $S(\cdot)$ is specified as

$$T_i=\int_0^{\sigma_{x_i}}X(y;l_1,...,l_{N-1},l_N)dy+\int_0^{\sigma_{x_i}}I\left\{X(y;l_1,...,l_{N-1},l_N)=0\right\}dy.$$

3) the service discipline is such that at each moment T the number of customers in the queue and their remaining service times coincide with the number of individuals and the remaining life-lengths of individuals in the branching process at moment $t(T)$ where

$$T=\int_0^{t(T)}X(y;l_1,...,l_N)dy+\int_0^{t(T)}I\left\{X(y;l_1,...,l_N)=0\right\}dy.$$

Thus, $t\leftrightarrow T$ is a random change of time.

Theorem. The described queueing system is a processor-sharing system with service time distribution of customers $B(u)$ and a Poisson flow of customers with intensity of arrivals Λ.

Proof. Let $S(T)$ be the number of customers in the queue at time T and let $\Theta_1,\Theta_2,...$ be the moments of changes the size of the queue. Let us show that the evolution of the constructed queue coincides with the evolution of a queueing system with processor sharing discipline. It is enough to show that this is true for $T\in[0,\Theta_1]$ and then, using the memoryless property of the Poisson flow to show in a similar way that this is true for $T\in[\Theta_1,\Theta_2]$ and so on.

To demonstrate this it is enough to check that:

1) $\Theta_1=Nl_1\wedge...\wedge Nl_N\wedge d$ where $P(d\leq u)=1-e^{-\Lambda u}$;

2) If $\Theta_1=Nl_i$ then at this moment the i-th customer comes out of the queue; if $\Theta_1=d$ then *one* new customer arrives;

3) at any moment $T\in[0,\Theta_1]$ the remaining service times of the initial N customers are $l_1-N^{-1}T,...,l_N-N^{-1}T$.

Let θ_1 be the first moment of change of $X(t;l_1,...,l_N)$. Clearly,

$$\theta_1=l_1\wedge...\wedge l_N\wedge d_1\wedge...\wedge d_N$$

where $P(d_i\leq u)=1-e^{-\Lambda u}$ and where the sense of d_i is the birth of an individual by the initial particle labelled i. On the interval $t\in[0,\theta_1]$ the processing time of the queueing system T and the time t passed from the start of the evolution of the general branching process are related by $T=Nt$. Hence 3) is valid.

Further, $\Theta_1 = N(l_1 \wedge ... \wedge l_N \wedge d_1 \wedge ... \wedge d_N) = Nl_1 \wedge ... \wedge Nl_N \wedge (N(d_1 \wedge ... \wedge d_N))$ and

$$P(N(d_1 \wedge ... \wedge d_N) \geq y) = \left(e^{-y\Lambda/N}\right)^N = e^{-y\Lambda}.$$

This proves 1). Point 2) is evident.

Corollary 1.

$$S(T) = X(t(T); l_1, ..., l_N).$$

Corollary 2.

$$W_{l_1,...,l_{N-1}}(l_N) = \int_0^{l_N} Z(y; l_1, ..., l_N)dy.$$

More detailed construction:

Let L be the life-length of a particle and let $0 \leq \delta(1) \leq \delta(2) \leq ...$ be the birth moments of her children. Denote

$$\xi(t, L) = \#\{n : \delta(n) \leq t\}.$$

Then the process generated by this particle can be treated as a process with immigration stopped at moment L where

$$Es^{\xi(t,L)} = e^{\Lambda(s-1)\min(t,L)},$$

and, since each newborn particle generates an *ordinary* process without immigration, we see that the offspring size of new particles at moment t in the process is

$$\int_0^t Z_{\xi(u,L)}(t-u)\xi(du, L)$$

where $Z_i(y)$ are independent branching processes initiated by one individual of zero age. Thus,

$$\begin{aligned} Z(y; l_1, ..., l_N) &= I\{l_1 \geq y\} + \int_0^y Z_{\xi(u,l_1)}(y-u)\xi(du, l_1) \\ &\quad + ... + I\{l_N \geq y\} + \int_0^y Z_{\xi(u,l_N)}(y-u)\xi(du, l_N) \end{aligned}$$

and, in particular, we have

$$\begin{aligned} W_{l_1,...,l_{N-1}}(l_N) &= \int_0^{l_N} Z(y; l_1, ..., l_N)dy \\ &= \sum_{k=1}^N \min(l_N, l_k) + \sum_{k=1}^N \int_0^{l_N} dy \int_0^y Z_{\xi(u,l_k)}(y-u)\xi(du, l_k). \end{aligned}$$

Since the birth moments of new particles constitute a Poisson flow with intensity Λ we have $\mathbf{E}\left[\xi(u,l)|l\right] = \min\left(u,l\right)$. Hence

$$\begin{aligned}
& \mathbf{E}\left[\int_0^{l_k} dy \int_0^y Z_{\xi(u,l_k)}(y-u)\xi(du,l_k)\right] \\
= \; & \mathbf{E}\left[\int_0^{l_k} dy \mathbf{E}\left[\int_0^y Z_{\xi(u,l_k)}(y-u)\xi(du,l_k)\,|\xi(u,l_k), 0 \le u \le l_k\right]\right] \\
= \; & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z_{\xi(u,l_k)}(y-u)\,|\xi(u,l_k), 0 \le u \le l\right]\xi(du,l_k)\right] \\
= \; & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(y-u)\right]\xi(du,l_k)\right] \\
= \; & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(y-u)\right]\mathbf{E}\left[\xi(du,l_k)|l_k\right]\right] \\
= \; & \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(y-u)\right]\Lambda du\right] = \Lambda \mathbf{E}\left[\int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(u)\right]du\right].
\end{aligned}$$

Hence

$$\mathbf{E}W_{l_1,\ldots,l_{N-1}}(l_N) = \mathbf{E}\left[\sum_{k=1}^{N} \min(l_N,l_k) + \Lambda \sum_{k=1}^{N} \int_0^{l_k} dy \int_0^y \mathbf{E}\left[Z(u)\right]du\right].$$

One can prove also that if

$$\beta_1 = \mathbf{E}l_N = \int_0^\infty u dB(u) < \infty$$

and $\Lambda\beta_1 < 1$ then for *fixed* $l_1, ..., l_{N-1}$

$$\lim_{l_N \to \infty} W_{l_1,\ldots,l_{N-1}}(l_N) = \frac{1}{1-\Lambda\beta_1}$$

almost surely (in particular, if it comes to an empty system).