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When π and c are constants, then the generator of w_t acts on $\tilde{v} \in C^2$ by

$$(A^{c,\pi}\tilde{v})(w) = \big((r + (\alpha - r)\pi)w - c\big)\tilde{v}'(w) + \frac{1}{2}w^2\pi^2\sigma^2\tilde{v}''(w).$$

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The HJB-equation reads

$$\max_{c,\pi}\{(A^{c,\pi}\tilde{v})(w) + \frac{c^\gamma}{\gamma} - \delta\tilde{v}(w)\} = 0 \quad \text{for all } w > 0.$$

The maxima are achieved at

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It is easy to see that $v(w) = \gamma^{-1}C^{\gamma-1}w^{\gamma}$ is solution of this differential equation.

Step ??:
Let $(c_t, \pi_t) \in \mathcal{U}$ be an arbitrary policy and define the process

$$x_t := \int_0^t \sigma \pi_u \, dz_u.$$

Then w_t is given explicitly (proof: Itô's formula) by

$$w_t = \left(w - \int_0^t c_s f_s \, ds \right) \mathcal{E}(x_t) \exp\left(rt + \int_0^t (\alpha - r)\pi_u \, du \right)$$

where $\mathcal{E}$ is the stochastic exponential of x_t and

$$f_s := \exp\left(-rs - \int_0^s \left((\alpha - r)\pi_u - \frac{1}{2}\sigma^2 \pi_u^2 \right) du - \int_0^s \sigma \pi_u \, dz_u \right).$$

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$\Rightarrow$ w_t has moments of all orders by Holder's inequality and since π_t is bounded.

Step ??:

Define for any policy (c_t, π_t) the process

$$M_t := \int_0^t e^{-\delta s} u(c_s)\, ds + e^{-\delta t} v(w_t),$$

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$$\begin{aligned} M_t &= M_0 + \int_0^t e^{-\delta s} \Big((A^{c,\pi} v)(w_s) + \frac{c_s^{\gamma}}{\gamma} - \delta v(w_s) \Big)\, ds \\ &\quad + \sigma C^{\gamma-1} \int_0^t e^{-\delta s} \pi_s w_s^{\gamma}\, dz_s. \end{aligned}$$

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$\Rightarrow$ M_t is a supermartingale and if $(c_t, \pi_t) = (c_t^*, \pi_t^*)$ it is a martingale. Thus,

$$v(w) = M_0 \geq \mathbb{E}_w[M_t] = \mathbb{E}_w[\int_0^t e^{-\delta s} u(c_s)\, ds] + \mathbb{E}_w[e^{-\delta t} v(w_t)].$$

The proof is complete if we can show that

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$$e^{-\delta t} w_t^\gamma = w_0^\gamma \mathcal{E}(\gamma x_t) \exp\left(\int_0^t a_s \, ds \right),$$

where

$$a_s = \gamma\Big(r + (\alpha - r)\pi_s - \frac{c_s}{w_s} - \frac{1}{2}(1-\gamma)\pi_s^2\sigma^2 \Big) - \delta.$$

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Since $a_s \le -(1-\gamma)C$ the claim follows. This completes the proof.

Guessing solution for problem with transaction costs.

Ansatz: try L and U absolutely continuous with bounded derivatives, that is,

$$L_t = \int_0^t l_s \, ds, \qquad U_t = \int_0^t u_s \, ds, \qquad 0 \le l_s, u_s \le \kappa$$

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The HJB-equation reads

$$\max_{c,l,u} \Big\{ \frac{1}{2}\sigma^2 y^2 \tilde{v}_{yy} + rx\tilde{v}_x + \alpha y \tilde{v}_y + \frac{1}{\gamma} c^\gamma - c\tilde{v}_x$$
$$\big(-(1+\lambda)\tilde{v}_x + \tilde{v}_y \big) l + \big((1-\mu)\tilde{v}_x - \tilde{v}_y \big) u - \delta \tilde{v} \Big\} = 0.$$

Since $\tilde{v}_x$ and $\tilde{v}_y$ are positive (extra wealth gives increased utility), we see that the maxima are attained as follows:

$$\begin{aligned} c &= (\tilde{v}_x)^{1/(\gamma-1)}, \\ l &= \begin{cases} \kappa, & \text{if } \tilde{v}_y \geq (1+\lambda)\tilde{v}_x, \\ 0, & \text{if } \tilde{v}_y < (1+\lambda)\tilde{v}_x, \end{cases} \\ u &= \begin{cases} 0, & \text{if } \tilde{v}_y > (1-\mu)\tilde{v}_x, \\ \kappa, & \text{if } \tilde{v}_y \leq (1-\mu)\tilde{v}_x. \end{cases} \end{aligned}$$

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This indicates that the optimal transaction policies are "bang-bang": buying and selling either take place at maximum rate or not at all, and the solvency region splits into three regions

- B, the region in which stocks are bought,
- S, the region in which stocks are sold,
- NT the region where no transactions take place.

Let us analyse the boundary

$$\tilde{v}_y = (1+\lambda)\tilde{v}_x$$

between S and NT (a similar argument applies for the boundary between NT and B). To this end assume that $\tilde{v} \in C^1$ and that it is homothetic which implies that

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It follows that if $\tilde{v}_y(x,y) = (1+\lambda)\tilde{v}_x(x,y)$ for some point (x,y), then the same is true for all points along the ray through (x,y).

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- the finite transaction in S or B moves the portfolio down or up a line of slope $-1/(1-\mu)$ or $-1/(1+\lambda)$.
- after the initial transaction, all further transactions must take place at the boundaries, and this suggests a “local time” type of transaction policy,
- meanwhile, consumption takes place at rate $(v_x)^{1/(\gamma-1)}$.

In NT the value function $v(x, y)$ satisfies the HJB-equation with $l = u = 0$:

$$\max_c \left\{ \frac{1}{2}\sigma^2 y^2 v_{yy} + (rx - c)v_x + \alpha y v_y + \frac{1}{\gamma} c^\gamma - \delta v \right\} = 0,$$

i.e.,

$$\frac{1}{2}\sigma^2 y^2 v_{yy} + (rx - c)v_x + \alpha y v_y + \frac{1 - \gamma}{\gamma} v_x^{-\gamma/(1-\gamma)} - \delta v = 0.$$

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The final step now consists of reducing this equation to an equation in one variable. In order to do so, define

$$\psi(x) := v(x, 1).$$

By the homothetic property it follows that $v(x, y) = y^\gamma \psi(x/y)$.

If our conjectured optimal policy is correct then v is constant along lines of slope $(1-\mu)^{-1}$ in S and along lines of slope $(1+\lambda)^{-1}$ in B, and this implies by homothetic property that

$$\psi(x) = \frac{1}{\gamma}(x+1-\mu)^{\gamma}, \quad x \le x_0,$$
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for some constants A, B and x_0 and x_T as in the picture.

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for some constants A, B and x_0 and x_T as in the picture. Using the homothetic property again, one can show that ψ satisfies for $x \in [x_0, x_T]$,

$$\beta_3 \psi''(x) + \beta_2 x \psi'(x) + \beta_1 \psi(x) + \frac{1-\gamma}{\gamma}(\psi'(x))^{-\gamma/(1-\gamma)} = 0,$$

where $\beta_1 = -\frac{1}{2}\sigma^2\gamma(1-\gamma+\alpha\gamma) - \delta$, $\beta_2 = \sigma^2(1-\gamma) + r - \alpha$, $\beta_3 = \frac{1}{2}\sigma^2$.

Theorem (4.1, follows from [?])

Take $0 < x_0 < x_T$ *and let* NT *be the closed wedge shown in the picture, with upper and lower boundaries* $\partial S, \partial B$ *respectively. Let* $c : NT \to [0, \infty)$ *be any Lipschitz continuous function and let* $(x, y) \in NT$*. Then there exists a unique process* s_0, s_1 *and continuous increasing processes* L, U *such that for* $t < \tau = \inf\{t \geq 0 : (s_0(t), s_1(t)) = 0\}$

$$\begin{aligned} ds_0(t) &= \big(rs_0(t) - c(s_0(t), s_1(t))\big)dt \\ &\quad -(1+\lambda)dL_t + (1-\mu)dU_t, \quad s_0(0) = x, \\ ds_1(t) &= \alpha s_1(t)dt + \sigma s_1(t)dz_t - dU_t, \quad s_1(0) = y, \\ &\quad L_t = \int_0^t 1_{\{(s_0(\xi), s_1(\xi)) \in \partial B\}} dL_\xi, \\ &\quad U_t = \int_0^t 1_{\{(s_0(\xi), s_1(\xi)) \in \partial S\}} dU_\xi. \end{aligned}$$

The process $\tilde{c}_t := c(s_0(t), s_1(t))$ *satisfies condition* $(2.1)(i)$*.*

Define the set of policies that do not involve short selling:

$$\mathfrak{U}' = \{(c, L, U) \in \mathfrak{U} : (s_0(t), s_1(t)) \in \mathscr{S}'_\mu \text{ for all } t \geq 0\},$$

where $\mathscr{S}'_\mu = \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } x + (1 - \mu)y \geq 0\}$.

Theorem (4.2, proof in [?])

Le $0 < \gamma < 1$ and assume Condition A holds. Suppose there are constants A, B, x_0, x_T and a function $\psi : [-1(1-\mu), \infty) \to \mathbb{R}$ such that

$$
\begin{aligned}
&0 < x_0 < x_T < \infty, \\
&\psi \text{ is } C^2 \text{ and } \psi'(x) > 0 \text{ for all } x, \\
&\psi(x) = \frac{1}{\gamma} A (x + 1 - \mu)^\gamma \text{ for } x \le x_0, \\
&\beta_3 \psi''(x) + \beta_2 x \psi'(x) + \beta_1 \psi(x) \\
&+ \frac{1-\gamma}{\gamma} (\psi'(x))^{-\gamma/(1-\gamma)} = 0 \text{ for } x \in [x_0, x_T], \\
&\psi(x) = \frac{1}{\gamma} B (x + 1 + \lambda)^\gamma \text{ for } x \ge x_T.
\end{aligned}
$$

Theorem

Let N_T *denote the closed wedge*

$$\{(x,y) \in \mathbb{R}^2_+ : x_T^{-1} \leq yx^{-1} \leq x_0^{-1}\}$$

and let B *and* S *denote the regions below and above* NT *as in the picture. For* $(x,y) \in NT \setminus \{(0,0)\}$ *define*

$$c^*(x,y) = y\psi'(x/y)^{-1/(1-\gamma)}.$$

Let $\tilde{c}^*_t = c^*(s_0(t), s_1(t))$ *where* (s_0, s_1, L^*, U^*) *is the unique solution of* (4.1) *with* $c := c^*$*. Then the policy* $(\tilde{c}^*(t), L^*(t), U^*(t))$ *is optimal in the class* $\mathscr{U}'$ *for any initial endowment* $(x,y) \in NT$*. If* $(x,s) \notin NT$ *then an immediate transaction to the closest point in* NT *followed by application of this policy is optimal in* $\mathscr{U}'$*. The maximal expected utility is*

$$v(x,s) = y^\gamma \psi(x/y).$$

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