Exercise Sheet 3— Stochastic Calculus

3.1: The mean-reverting Ornstein-Uhlenbeck process satisfies the equation:

$$\mathrm{d}X_t = \lambda(\mu - X_t)\,\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where $\lambda, \mu, \sigma > 0$ are fixed parameters.

(a) By considering the process $Y_t = \mathrm{e}^{\lambda t} X_t$, find an expression for X_t involving only $(W_s)_{s \geq 0}$.

(b) Considering $X_0 = x_0 \in \mathbb{R}$ a fixed constant, find the distribution of X_t. [Hint: Recall Example 5.4]

(c) Suppose $X_0 \sim N(\mu, \frac{\sigma^2}{2\lambda})$ is independent of W. Show that X_t is stationary, that is X_t has the same distribution for all $t \geq 0$.

3.2: Recall Question Q2.4. Prove this result for arbitrary n.

[Hint: You may find the relations $h_n'(x) = nh_{n-1}(x)$ and $h_{n+1}(x) - xh_n(x) + nh_{n-1}(x) = 0$ useful.]

3.3: Suppose $\boldsymbol{Y}_t = (Y_t^1, Y_t^2)$ solves:

$$\begin{cases} \mathrm{d}Y_t^1 &= -\frac{1}{2}Y_t^1\,\mathrm{d}t - Y_t^2\,\mathrm{d}W_t \\ \mathrm{d}Y_t^2 &= -\frac{1}{2}Y_t^2\,\mathrm{d}t + Y_t^1\,\mathrm{d}W_t \end{cases}$$

with $\boldsymbol{Y}_0 = (1, 0)$.

(a) Show that $(Y_t^1)^2 + (Y_t^2)^2$ is constant.

(b) Find an Itô process Z_t such that $\boldsymbol{Y}_t = (\cos(Z_t), \sin(Z_t))$.

(c) Let N_t be the 'winding number' of $\boldsymbol{Y}_t$ around the origin — that is, the total number of complete anti-clockwise rotations around the origin (subtracting complete clockwise rotations), made by $\boldsymbol{Y}_t$. Find an expression for $\mathbb{P}(N_t \geq n)$, where $n \in \mathbb{N}$.

3.4: Let X_t be a Geometric Brownian motion, that is, a solution to

$$\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \qquad X_0 = x_0,$$

for $\sigma > 0, \mu \in \mathbb{R}$. Find an ODE satisfied by $m(t) := \mathbb{E}[X_t]$, and hence find $\mathbb{E}[X_t]$.

For a standard Brownian motion W, it is well known that if $a > 0$ then $\lim_{t\to\infty}[W_t - at] = -\infty$. Suppose $0 < \mu < \sigma^2/2$. What can be said about $\lim_{t\to\infty} \mathbb{E}[X_t]$, and $\lim_{t\to\infty} X_t$?

3.5: Fix $x, y \in \mathbb{R}$, and consider the equation

$$\mathrm{d}X_t = \frac{y - X_t}{1 - t}\,\mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = x.$$

Show that

$$X_t = x + (y - x)t + (1 - t)\int_0^t \frac{\mathrm{d}W_s}{1 - s},$$

for $0 \leq t < 1$, and prove that $X_t \to y$ as $t \uparrow 1$ in $\mathcal{L}^2(\mathbb{P})$.

3.6: In a similar manner to solving ODEs, we can solve SDEs by introducing 'integrating factors'.

Suppose we want to solve the SDE

$$\mathrm{d}X_t = \alpha X_t \,\mathrm{d}B_t + r\mathrm{d}t, \quad X_0 = x_0,$$

where r and α are real constants. Let F_t be our 'integrating factor':

$$F_t = \exp\{-\alpha B_t + \frac{1}{2}\alpha^2 t\}.$$

What is $\mathrm{d}F_t$? Let $Y_t = F_t X_t$ and find $\mathrm{d}Y_t$. Solve the resulting equation to find X_t.

3.7: Let $\boldsymbol{W}$ be n-dimensional Brownian motion, for $n \geq 2$, started at $x_0 \neq \mathbf{0}$. Set

$$R_t = |\boldsymbol{W}_t| = \sqrt{(W_t^1)^2 + \cdots + (W_t^n)^2}.$$

Show that

$$\mathrm{d}R_t = \sum_{i=1}^{n} \frac{W_t^i \,\mathrm{d}W_t^i}{R_t} + \frac{n-1}{2R_t}\,\mathrm{d}t.$$

(You may assume that Itô's Lemma holds even though $f(x) = |x|$ is not C^2 at the origin, since the process will almost surely never reach the origin).

Hence find (i) $\mathrm{d}(\ln(R_t))$ for $n = 2$ and (ii) $\mathrm{d}(R_t^{2-n})$ for $n \geq 3$.

3.8: Let W be a standard Brownian motion with $W_0 = 0$, and write $H_{a,b} = \inf\{t \geq 0 : W_t \in \{a, b\}\}$, the first hitting time of either a or b. For the sequence $\phi_u^N = \mathbf{1}_{u \leq H_{-1,N}}$, find $I^N := \int_0^\infty \phi_u^N \,\mathrm{d}W_u$, and hence show that $I^N \to I$ in probability, but not in $\mathcal{L}^2(\mathbb{P})$, for some I which you should identify. What are $\mathbb{E}\left[I^N\right], \mathbb{E}\left[I\right]$?