Utility Maximisation: Wealth & Consumption Optimisation.

$$\text{Th}(6.6) \begin{cases} dS^0(t) = S^0(t)\, r(t)\, dt \\ dS^i(t) = S^i(t) \left(b_i(t)\, dt + \sum_{j=1}^{d} \sigma_{ij}\, dB_t^j \right) \quad i=1,\dots,d. \end{cases}$$

is complete $(N = d)$.

(H) i) r, b, σ meas., adapted, uniformly bounded, $r \geq 0$

ii) $\forall t$, $\sigma(t)$ invertible, $\sigma^{-1}(t)$ bounded, σ predictable.

iii) $c \geq 0$, adapted process,

$$\int_0^T c(t)\, dt < \infty \quad \text{a.s.}$$

iv) π predictable $\int_0^T \|\pi(t)\|^2 dt < \infty$ a.s.

$$(8.1)\ \mathbb{P}^x. \begin{cases} dX^{\pi,c}(t) = \left[X^{\pi,c}(t)\, r(t) - c(t) \right] dt + \sum_{i=1}^{d} \pi_i(t) \left(b_i(t) - r(t) \right) dt \\ \qquad + \sum_{i,j=1}^{d} \pi_i(t)\, \sigma_{ij}\, dB_t^j \qquad t \geq 0. \\ X^{\pi,c}(0) = x. \end{cases}$$

$X^{\pi,c}$ wealth process, $\mathbb{P}^x$ initial wealth x.

U is a <u>utility function</u> if :

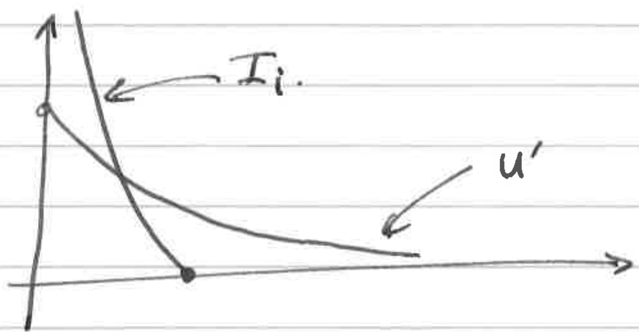
- $U : \mathbb{R}_+ \to \mathbb{R}$
- strictly concave, strictly increasing, in C^1.
- $\lim_{x \to \infty} U'(x) = 0$.

$$U^{(p)}(x) = \begin{cases} x^p/p & p \in (0,1) \\ \log(x) & p = 0. \end{cases}$$

A <u>preference structure</u> is two utility functions (U_1, U_2)

U_i' is strictly decreasing, so $\exists$ ~~left increasing~~ left-inverse I_i, ~~cont.~~ cts

$$I_i : (0, U_i'(0)) \longrightarrow \mathbb{R}$$



$$J(x; \pi, c) = \mathbb{E}^x\left[\int_0^T U_1(c(t))\,dt + U_2(X_T^{\pi,c})\right]$$

$$\mathcal{A}(x) = \{(\pi, c) : X^{\pi,c}(t) \geq 0 \text{ for all } 0 \leq t \leq T\}$$

Maximise J in $\mathcal{A}(x)$.

Recall the risk-neutral measure:

Let $\eta(t) = -\sigma(t)^{-1}\left[b(t) - r(t)\underline{1}\right]$

$$L_t = \exp\left\{\int_0^t \eta(s)\, dB_s - \frac{1}{2}\int_0^t \|\eta(s)\|^2 ds\right\}$$

Let $\left.\frac{d\mathbb{Q}^x}{d\mathbb{P}^x}\right|_{\mathcal{F}_T} = L_T$; $\mathbb{P}^x \sim \mathbb{Q}^x$

$$\tilde{B}_t = B_t - \int_0^t \eta(s)\, ds.$$

$\tilde{B}$ an $(\mathcal{F}_t)$-BM.

Under $\mathbb{Q}^x$,

$$dS^i(t) = S^i(t)\left[r(t)\,dt + \sum \sigma_{ij}\, d\tilde{B}^j_t\right] \quad t \geq 0$$

$$dX^{\pi,c}(t) = \left[X^{\pi,c}(t)\, r(t) - c(t)\right]dt + \sum_{i,j=1}^{d} \pi_i(t)\, \sigma_{ij}(t)\, d\tilde{B}^j_t \quad t \geq 0$$

$$X^{\pi,c}(0) = x.$$

$$X^{\pi,c}(t)\, R(t) = x - \int_0^t c(s)\, R(s)\, ds + \int_0^t R(s)\, \pi^T(s)\sigma(s)\, d\tilde{B}_s$$

where $R(t) = e^{-\int_0^t r(s)\, ds}$

$$\text{Let } M_t = X^{\pi,c}(t)\,R(t) + \int_0^t c(s)\,R(s)\,ds$$

$$= x + \int_0^t R(s)\,\pi(s)^T \sigma(s)\,d\tilde{B}_s^i \, .$$

$\Rightarrow$ M_t a local $\mathbb{Q}^x$-martingale.

If π, c is in ~~{d~~ $\mathcal{A}(x)$, $M_t \geq 0$.

So a $\mathbb{Q}^x$-supermartingale.

$$(8.2). \quad \mathbb{E}^{\mathbb{Q}^x}\left[X^{\pi,c}(T)\,R(T) + \int_0^T c(s)\,R(s)\,ds\right] = \mathbb{E}^{\mathbb{Q}^x} M_T \leq \mathbb{E}^{\mathbb{Q}^x} M_0 = x.$$

<u>Prop 8.1</u> : If c is a consumption strategy and $Z \in \mathcal{F}_T$, $Z \geq 0$, s.t.

$$(8.3) \qquad \mathbb{E}^{\mathbb{Q}^x}\left[Z R(T) + \int_0^T R(s)\,c(s)\,ds\right] = x,$$

then $\exists\, \pi$: $(\pi, c) \in \mathcal{A}(x)$ and $X_T^{\pi,c} = Z$, $\mathbb{Q}^x$-a.s.

<u>Proof</u> :

$$\text{Let } M_t = \mathbb{E}^{\mathbb{Q}^x}\left[Z R(T) + \int_0^T R(s)\,c(s)\,ds \,\Big|\, \mathcal{F}_t\right],$$

a $\mathbb{Q}^x$-mg.

So $\exists\, \phi$ predictable, $\int_0^T \|\phi_t\|^2\,dt < \infty$ $\mathbb{Q}^x$-a.s.,

s.t. $M_t = x + \int_0^t \phi(s)\,d\tilde{B}_s$

Let $\quad \pi(t) = \frac{1}{R(t)} (\sigma^T(t))^{-1} \phi^T(t)$, then

$$\phi(s)\, d\tilde{B}_s = R(s)\, \pi(s)^T \sigma(s)\, d\tilde{B}_s .$$

So $\quad X^{\pi,c}(t) R(t) = M_t - \int_0^t R(s) c(s)\, ds.$

Is $\quad (\pi, c) \in \mathcal{A}(x)$?

$\forall t < T: \quad X^{\pi,c}(t) R(t) = \mathbb{E}^{\mathbb{Q}^x}\left[\{ Z R(T) \} + \left(\int_0^T - \int_0^t \right) R(s) c(s)\, ds \,\middle/ \mathcal{F}_t \right] \geq 0$

and $\quad X^{\pi,c}(T) R(T) = M(T) = Z R(T) \geq 0.$

So $\quad (\pi, c) \in \mathcal{A}(x).$

$\square$.

Set $\quad \mathcal{A}'(x) = \left\{ (c, Z) : Z \geq 0,\ \mathbb{E}^{\mathbb{Q}^x}\left[Z R(T) + \int_0^T c(s) R(s)\, ds \right] \leq x \right\}$

For a utility function U, let $\tilde{U}(y) = \sup_{x \in \mathbb{R}_+} \{ U(x) - xy \}$ $\quad y \in \mathbb{R}$

It holds that $\forall y \in \mathbb{R}$

$$\tilde{U}(y) = U(I(y)) - y\, I(y).$$

Let $\quad \mathcal{L}(c, Z, \lambda) = \mathbb{E}\left[\int_0^T U_1(c(t))\, dt + U_2(Z) + \lambda \left[x - \left(\int_0^T L_t R(t) c(t)\, dt + L_T R(T) Z \right) \right] \right]$

See then that (c^*, z^*) is optimal if $\exists \lambda^* > 0$ s.t. (c^*, z^*, λ^*) is a saddle point of $\mathcal{L}$:

$$\mathcal{L}(c, z; \lambda^*) \overset{(8.4)}{\leq} \mathcal{L}(c^*, z^*; \lambda^*) \leq \mathcal{L}(c^*, z^*; \lambda)$$

~~So~~ Suppose $\exists (c^*, z^*, \lambda^*)$

For (8.4), want to maximise $\int_0^T (U_1(c(t)) - \lambda^* L_t R(t) c(t))\,dt$

and: ~~+~~ $U_2(z) - L_T R(T) z$

Now $$\sup_{c(t)} \left[U_1(c(t)) - \lambda^* L_t R(t) c(t) \right] = \tilde{U}_1(\lambda^* L_t R(t))$$
$$= U_1(I_1(\lambda^* L_t R(t))) - \lambda^* L_t R(t) I_1(\lambda^* L_t R(t))$$

Let $\zeta_t = L_t R(t)$ and set: $c^*(t) = I_1(\lambda^* \zeta_t)$.

& $z^* = I_2(\lambda^* \zeta_T)$.

If λ^* is the correct multiplier:

$$(8.6). \quad \mathbb{E}^x \left[\int_0^T \zeta_t I_1(\lambda^* \zeta_t)\,dt + \zeta_T I_2(\lambda^* \zeta_T) \right] = x$$

Recall again that $U(I(y)) - y I(y) \geq U(x) - xy$, all x, y.

$(c,z) \in \mathcal{A}'(x)$:

$$J(x,c,z) = \mathbb{E}^x\left[\int_0^T U_1(c(s))\,ds + U_2(z)\right]$$

$$\le \mathbb{E}^x\left[\int_0^T U_1(I_1(y_s))\,ds + U_2(I_2(y_T))\right]$$

$$+ \mathbb{E}^x\left[\int_0^T y_s\,(c(s) - I_1(y(s)))\,ds + y_T(z - I_2(y_T))\right]$$

Let $y_s = \lambda^* \zeta_s$. Then 1^{st} term is $J(x;c^*,z^*)$.

Now 2^{nd} term:

$$\mathbb{E}^x\left[\int_0^T \lambda^*\zeta_s\,(c(s) - I_1(\lambda^*\zeta_s))\,ds + \lambda^*\zeta_T(z - I_2(\lambda^*\zeta_T))\right]$$

$$= \lambda^*\left(\mathbb{E}^x\left[\int_0^T \zeta_s\, c(s)\,ds + \zeta_T z\right] - x\right)$$

$$= \lambda^*\left(\mathbb{E}^x\left[\int_0^T L_s R(s) c(s)\,ds + L_T R(T) z\right] - x\right)$$

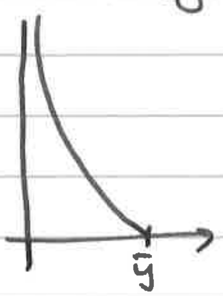
≤ 0 by admissibility

$$\Rightarrow \quad J(x;c,z) \le J(x,c^*,z^*).$$

Assume also (U3): $\cdot$) $\forall \lambda > 0$, $\mathbb{E}\int_0^T |L_t R(t) I_1(\lambda\zeta_t)|\,dt < \infty$

$\cdot$) $\forall \lambda > 0$, $\mathbb{E}|L_T R(T) I_2(\lambda\zeta_T)| < \infty$.

Let $\mathcal{X}(y) = \mathbb{E}\left[\int_0^T \cancel{\zeta_t}\, \zeta_t I_1(y\zeta_t)\,dt + \zeta_T I_2(y\zeta_T)\right]$.



$\mathcal{X}(0+) = \infty$, $\mathcal{X}(\infty) = 0$

$\mathcal{X}$ strictly decreasing on $[0,\bar{y}]$

$\mathcal{X}$ continuous.

So $\exists\, \mathcal{Y} : [0,\infty) \longrightarrow (0, \bar{y}]$,

inverse to $\mathcal{X}$.

Let $\lambda^* = \mathcal{Y}(x)$.

Summary:

$$c^*(t) = I_1(\mathcal{Y}(x)\zeta_t), \quad Z^* = I_2(\mathcal{Y}(x)\zeta_T)$$

$\pi^*(t)$ is associated via martingale representation,

$\zeta_T = L_t R(t)$, L exponential m'gale.

Theorem 7.3: Let $\mathcal{H} = \{ H \in C^{1,2},\ H : [0,T] \times (0,\infty) \to \mathbb{R}$, $\exists\, k, \beta > 0$: $\sup |H(t,y)| \le k(1 + y^{\beta} + y^{-\beta})$, all $y > 0 \}$

Suppose $h_1, h_2 \in \mathcal{H}$, let

$$H(\alpha, y) = \mathbb{E}\left[\int_\alpha^T h_1(y\, \zeta_t^\alpha)\, dt + h_2(\zeta_T^\alpha y) \right]$$

Then H is the unique sol^n in $\mathcal{H}$ of:

$$(8.7)\quad \begin{cases} \dfrac{\partial H}{\partial t} - r(t)\, y\, \dfrac{\partial H}{\partial y} + \frac{1}{2} \|\eta(t)\|^2 y^2 \dfrac{\partial^2 H}{\partial y^2} = -h_1 \\ H(T,y) = h_2(y) \end{cases}$$

$y \longmapsto u_1(I_1(y))$, $y \longmapsto y\, I_1(y) \in \mathcal{H}$.

Prop 7.4 (Deterministic Coefficients — Dana + Jeanblanc; pp 145+)

$$V(x) = \mathbb{E}^{x}\left[\int_0^T U_1(c^*(s))\,ds + U_2(Z)\right]$$

$$\parallel$$

$$V(0,x).$$

Then $V(\alpha, x) = G(\alpha, \mathcal{Y}(\alpha, x))$, G sol$^{\underline{n}}$ to (8.7)

with $h_i = U_i \circ I_i$,

$\mathcal{Y}(\alpha, \cdot)$ inverse to $\mathcal{X}(\alpha, \cdot)$,

$$\mathcal{X}(\alpha, y) = \frac{\mathcal{T}(\alpha, y)}{y}$$

$\mathcal{T}$ sol to (8.7) w/ $h_i(y) = y\, I_i(y)$.