Proceedings of the Royal Society of Edinburgh, **132A**, 1361–1371, *2002*

On conservation laws and necessary conditions in the calculus of variations

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(MS received 1 October 2001; accepted 1 March 2002)

It is well known from the work of Noether that every variational symmetry of an integral functional gives rise to a corresponding conservation law. In this paper, we prove that each such conservation law arises directly as the Euler–Lagrange equation for the functional on taking suitable variations around a minimizer.

1. Introduction

Classically, since the pioneering work of Noether [9], conservation laws associated to stationary points of an integral functional have been obtained upon writing the invariance of that functional under a suitable one-parameter group of transformations and combining the result with the Euler–Lagrange equations that express stationarity (see, for example, [10]). Such an approach implicitly assumes that the stationary points of the functional are smooth enough for all chain rules to be licit; it also presupposes that the Euler–Lagrange equations are indeed satisfied.

More recently, Morrey (see, for example, [8]) introduced the notion of quasiconvexity, which was later developed by Ball [1] to obtain a variational approach to static hyperelasticity. A difficulty in that variational approach is the current inability to prove that minimizers of the potential energy satisfy the associated Euler–Lagrange equations[1]. A major difficulty is that the usually adopted restriction that kinematically admissible displacement fields $\boldsymbol{u}$ should satisfy

$$\det \nabla \boldsymbol{u} > 0 \quad \text{a.e.}$$

prohibits the usual smooth variations $\boldsymbol{u}_0 + t\boldsymbol{\phi}$, $t \in \mathbb{R}$, $\boldsymbol{\phi} \in C_0^1(\Omega)$, around a minimizer $\boldsymbol{u}_0 \in W^{1,1}(\Omega)$.

In spite of that obstacle, Ball showed in [3] that conservation of the energy-momentum tensor—a seminal conservation law associated to the homogeneous character of the energy density—could be obtained for minimizers of the potential energy, even though the Euler–Lagrange equations might not be satisfied. Ball

[1]Partial regularity results for quasiconvex integrands have been obtained by Evans [7], but under hypotheses that are incompatible with the usual growth requirement (4.3) in nonlinear elasticity.

obtained this result by using 'inner' variations around $\boldsymbol{u}_0$, that is, variations of the form $\boldsymbol{u}_0(\boldsymbol{x}+t\boldsymbol{\phi}(\boldsymbol{x}))$, $t \in \mathbb{R}$, $\boldsymbol{\phi} \in C_0^1(\Omega)$.

In this short paper, we propose to generalize the above quoted result and to show that a suitable choice of more general one-parameter groups of transformations will directly generate a host of conservation laws (in weak form) for minimizers of the energy, the Euler–Lagrange equations not withstanding.

In § 2, we recall the classical Noetherian approach to conservation laws. In § 3, we demonstrate how more general one-parameter groups of transformations permit us to circumvent the Euler–Lagrange equations and directly derive conservation laws for energy minimizers. In § 4, we apply this approach to the specific setting of hyperelasticity and recover, in particular, Ball's results. The final section is very short and points to a few possible extensions of our method.

Finally, we should stress a strong bias in favour of conciseness. We thus chose not to spell out all smoothness and/or growth assumptions on the energy densities, since these may vary with the specific kind of transformations under consideration. The reader is invited to consult the relevant references for a description of the precise assumptions on those densities, given a particular invariance.

For further background and references on conservation laws and applications, we refer to the comprehensive text of Olver [10]. In this paper, we adopt the notation used in [10] and will also refer to key calculations contained therein.

2. Variational symmetries

Let $\Omega \subset \mathbb{R}^m$ be a domain and consider the integral functional

$$E(\boldsymbol{u};\Omega') = \int_{\Omega'} L(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x}),\nabla\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x, \tag{2.1}$$

where $L : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times m} \to \mathbb{R}$ is a given Carathéodory integrand, Ω' is an arbitrary open subdomain of Ω and E is defined on maps $\boldsymbol{u} : \Omega' \to \mathbb{R}^n$, which are (at least) in $W^{1,1}(\Omega';\mathbb{R}^n)$.

Consider a smooth one-parameter group of transformations of independent and dependent variables given by

$$(\boldsymbol{x},\boldsymbol{u}) \to (\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}}) = (\Xi_t(\boldsymbol{x},\boldsymbol{u}),\Phi_t(\boldsymbol{x},\boldsymbol{u})), \tag{2.2}$$

where $t \in \mathbb{R}$ is the group parameter and $t = 0$ corresponds to the identity transformation (so that $\Xi_0(\boldsymbol{x},\boldsymbol{u}) \equiv \boldsymbol{x}$ and $\Phi_0(\boldsymbol{x},\boldsymbol{u}) \equiv \boldsymbol{u}$). We will sometimes write $(\Xi_t(\boldsymbol{x},\boldsymbol{u}),\Phi_t(\boldsymbol{x},\boldsymbol{u})) = (\Xi(\boldsymbol{x},\boldsymbol{u},t),\Phi(\boldsymbol{x},\boldsymbol{u},t))$ to emphasize the role of the parameter t, in which case it will be assumed that Ξ, Φ are smooth with respect to all their arguments.

An important concept when working with such groups of transformations is the notion of the *infinitesimal generator* of the group, which is the differential operator defined by[2]

$$\boldsymbol{v} = \xi^\alpha(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial x^\alpha} + \phi^i(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial u^i}, \tag{2.3}$$

[2]Here and in the rest of this paper we use the convention of summing over repeated indices.

where

$$\xi(\boldsymbol{x},\boldsymbol{u}) := \frac{\mathrm{d}}{\mathrm{d}t}\Xi(\boldsymbol{x},\boldsymbol{u},t)\bigg|_{t=0} \quad \text{and} \quad \phi(\boldsymbol{x},\boldsymbol{u}) := \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{x},\boldsymbol{u},t)\bigg|_{t=0}$$

(see [10] for further details).

REMARK 2.1. Hence, in particular, given a real-valued function $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u})$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{F}(\Xi_t(\boldsymbol{x},\boldsymbol{u}),\Phi_t(\boldsymbol{x},\boldsymbol{u}))\bigg|_{t=0} = \boldsymbol{v}(\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u})).$$

Given any particular subdomain $\Omega' \subset \Omega$, and any map $\boldsymbol{u} : \Omega' \to \mathbb{R}^n$, the change of variables given by (2.2) induces a new transformed function $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}})$, $\tilde{\boldsymbol{u}} : \tilde{\Omega}' \to \mathbb{R}^n$, where $\tilde{\Omega}' = (\Xi_t \circ (\mathrm{id} \times \boldsymbol{u}))(\Omega)$, which is obtained as follows. First note that $\tilde{\boldsymbol{x}} = \Xi_t(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x})) = (\Xi_t \circ (\mathrm{id} \times \boldsymbol{u}))(\boldsymbol{x})$. By our assumptions on the group of transformations, it follows, upon application of the implicit function theorem, that this relation can be inverted for small t to yield $\boldsymbol{x} = (\Xi_t \circ (\mathrm{id} \times \boldsymbol{u}))^{-1}(\tilde{\boldsymbol{x}})$. The new transformed function $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}})$ depends on t and is given by

$$\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}) = [\Phi_t \circ (\mathrm{id} \times \boldsymbol{u})] \circ [\Xi_t \circ (\mathrm{id} \times \boldsymbol{u})]^{-1}(\tilde{\boldsymbol{x}}). \tag{2.4}$$

Through the above construction, the group of transformations (2.2) naturally extends to a corresponding action on the first derivatives $\nabla_{\boldsymbol{x}}\boldsymbol{u}$ (which are mapped to $\nabla_{\tilde{\boldsymbol{x}}}\tilde{\boldsymbol{u}}$). It can be shown that this extended action

$$(\boldsymbol{x},\boldsymbol{u},\nabla_{\boldsymbol{x}}\boldsymbol{u}) \to (\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}},\nabla_{\tilde{\boldsymbol{x}}}\tilde{\boldsymbol{u}})$$

has infinitesimal generator $\boldsymbol{v}^{(1)}$ (known as the first prolongation of $\boldsymbol{v}$), given by

$$\boldsymbol{v}^{(1)} = \xi^\alpha(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial x^\alpha} + \phi^i(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial u^i} + \pi^i_\alpha(\boldsymbol{x},\boldsymbol{u},\nabla\boldsymbol{u})\frac{\partial}{\partial F^i_\alpha}, \tag{2.5}$$

where

$$\pi^i_\alpha(\boldsymbol{x},\boldsymbol{u},\nabla\boldsymbol{u}) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{u}^i(\tilde{\boldsymbol{x}})}{\partial\tilde{x}^\alpha}\bigg|_{t=0} = \frac{\partial}{\partial x^\alpha}[\phi^i(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x}))] - \frac{\partial u^i}{\partial x^\beta}\frac{\partial}{\partial x^\alpha}[\xi^\beta(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x}))] \tag{2.6}$$

(see [10, p. 114]).

Following standard terminology (see, for example, [10, definition 4.10]), we say that (2.2) is a variational symmetry of (2.1) if, for any subdomain $\Omega' \subset \Omega$ and any map $\boldsymbol{u} : \Omega' \to \mathbb{R}^n$, we have

$$\int_{\Omega'} L(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x}),\nabla\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x = \int_{\tilde{\Omega}'} L(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}),\nabla\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}))\,\mathrm{d}\tilde{x}.$$

It then follows that if (2.2) is a variational symmetry, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde{\Omega}'} L(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{u}},\nabla_{\tilde{\boldsymbol{x}}}\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}))\bigg|_{t=0} = 0 \tag{2.7}$$

for any map $\boldsymbol{u} : \Omega' \to \mathbb{R}^n$. It follows from [10, theorem 4.12] that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde{\Omega}'} L(\tilde{x}, \tilde{u}, \nabla_{\tilde{x}}\tilde{u}(\tilde{x}))\bigg|_{t=0}$$
$$= \int_{\Omega'} \boldsymbol{v}^{(1)} L(\boldsymbol{x}, \boldsymbol{u}, \nabla_{\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x})) + L(\boldsymbol{x}, \boldsymbol{u}, \nabla_{\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{Div}\,\xi(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x, \quad (2.8)$$

where

$$\mathrm{Div}\,\xi(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) = \frac{\partial}{\partial x^\beta}(\xi^\beta(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))).$$

From (2.7) and the arbitrariness of Ω', it now follows that the integrand on the right-hand side must vanish identically, and hence, by (2.5), that

$$0 = \left[\xi^\alpha(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial x^\alpha} + \phi^i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial u^i} + \pi^i_\alpha(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})\frac{\partial}{\partial F^i_\alpha}\right] L(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})$$
$$+ \left[\frac{\partial}{\partial x^\alpha}\xi^\alpha(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\right] L(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u}). \quad (2.9)$$

Using (2.6), we then obtain that

$$\phi^i(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\frac{\partial L}{\partial u^i} + \frac{\partial}{\partial x^\alpha}(\phi^i(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})))\frac{\partial L}{\partial F^i_\alpha} + \xi^\alpha\frac{\partial L}{\partial x^\alpha} + \left(\frac{\partial}{\partial x^\alpha}\xi^\beta(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\right) M^\beta_\alpha = 0 \quad (2.10)$$

for any map $\boldsymbol{u} : \Omega \to \mathbb{R}^n$, where

$$M^\beta_\alpha(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u}) = \left[L(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})\delta^\beta_\alpha - \frac{\partial u^k}{\partial x_\beta}\frac{\partial L}{\partial F^k_\alpha}(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})\right]$$

are the components of the $m \times m$ tensor

$$\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u}) = \left[L(\boldsymbol{x}, \boldsymbol{u}, \nabla, \boldsymbol{u})\boldsymbol{I} - (\nabla\boldsymbol{u})^{\mathrm{T}}\frac{\partial L}{\partial F}(\boldsymbol{x}, \boldsymbol{u}, \nabla\boldsymbol{u})\right],$$

which is known as the *energy–momentum* tensor.

REMARK 2.2. Note that the condition that (2.2) be a variational symmetry is expressible as a pointwise condition in the following way. Since (2.9) holds for all maps $\boldsymbol{u} : \Omega \to \mathbb{R}^n$, given any triple $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$, we can find a map $\boldsymbol{u}_0(\cdot)$ such that $(\boldsymbol{x}, \boldsymbol{u}_0(\boldsymbol{x}), \nabla\boldsymbol{u}_0(\boldsymbol{x})) = (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})$. It then follows that

$$\left[\xi^\alpha(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial x_\alpha} + \phi^i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial u^i} + \pi^i_\alpha(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})\frac{\partial}{\partial F^i_\alpha}\right] L(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})$$
$$+ \left[\frac{\partial}{\partial x^\alpha}\xi^\alpha(\boldsymbol{x}, \boldsymbol{u}) + \frac{\partial}{\partial u^i}\xi^\alpha(\boldsymbol{x}, \boldsymbol{u})F^i_\alpha\right] L(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) = 0$$

pointwise for any choice of $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F}) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$.

REMARK 2.3. Assume that $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ satisfies the Euler–Lagrange equations for (2.1), that is,

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial F^i_\alpha}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\right) = \frac{\partial L}{\partial u^i}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x})) \tag{2.11}$$

for $i = 1, 2, \dots, n$. Further assume that L and $\boldsymbol{u}$ are smooth enough. Then an application of the chain rule yields

$$\frac{\partial}{\partial x^\alpha}(M^\beta_\alpha(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))) = \frac{\partial L}{\partial x^\beta}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x})) \tag{2.12}$$

for $\beta = 1, 2, \dots, m$.

It now follows from (2.11), (2.12) and (2.10) that if (2.2) is a variational symmetry of (2.1), then $\boldsymbol{u}$ satisfies the conservation law

$$\frac{\partial}{\partial x^\alpha}\left[\phi^i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial L}{\partial F^i_\alpha}(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) + \xi^\beta(\boldsymbol{x}, \boldsymbol{u}) M^\beta_\alpha(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\right] = 0 \tag{2.13}$$

for $i = 1, 2, \dots, n$. For later use, we note that the weak form of the above conservation law is

$$\int_\Omega \frac{\partial \theta}{\partial x^\alpha}(\boldsymbol{x})\left[\phi^i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial L}{\partial F^i_\alpha} + \xi^\beta(\boldsymbol{x}, \boldsymbol{u}) M^\beta_\alpha(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\right] \mathrm{d}x = 0 \tag{2.14}$$

for any $\theta \in C^1_0(\Omega)$.

Elaborating on the last remark, our goal in the next section is to show that if (2.2) is a variational symmetry of E given by (2.1), and if further $\boldsymbol{u}_0$ is a minimizer for $E(\,\cdot\,; \Omega)$—for any kind of appropriate boundary conditions and under any kind of constraint, provided these are preserved under composition of $\boldsymbol{u}_0$ by a small and smooth perturbation of the identity—then the conservation law (2.14) still holds true.

3. Variational symmetry and minimizers

Suppose that $\boldsymbol{u}_0$ is a minimizer of the functional $E(\,\cdot\,; \Omega)$ and let $\theta \in C^1_0(\Omega)$. Let (2.2) be a variational symmetry of E and define the one-parameter family of variations by replacing the parameter t in the definition of the group by the function $t\theta(\boldsymbol{x})$. Thus we set

$$(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}}) = (\tilde{\Xi}_t(\boldsymbol{x}, \boldsymbol{u}), \tilde{\Phi}_t(\boldsymbol{x}, \boldsymbol{u})) = (\Xi(\boldsymbol{x}, \boldsymbol{u}, t\theta(\boldsymbol{x})), \Phi(\boldsymbol{x}, \boldsymbol{u}, t\theta(\boldsymbol{x}))).$$

Note that, by (2.3), the infinitesimal generator corresponding to the above group of transformations is given by

$$\tilde{\boldsymbol{v}} = \tilde{\xi}^\alpha(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial x^\alpha} + \tilde{\phi}^i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial u^i},$$

where

$$\tilde{\xi}(\boldsymbol{x}, \boldsymbol{u}) = \theta(\boldsymbol{x})\xi(\boldsymbol{x}, \boldsymbol{u}), \qquad \tilde{\phi}(\boldsymbol{x}, \boldsymbol{u}) = \theta(\boldsymbol{x})\phi(\boldsymbol{x}, \boldsymbol{u}) \tag{3.1}$$

and (by (2.5) and (2.6)) its first prolongation is given by

$$\tilde{v}^{(1)} = \tilde{\xi}^{\alpha}(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial x^{\alpha}} + \tilde{\phi}^{i}(\boldsymbol{x}, \boldsymbol{u})\frac{\partial}{\partial u^{i}} + \tilde{\pi}^{i}_{\alpha}(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u})\frac{\partial}{\partial F^{i}_{\alpha}}, \tag{3.2}$$

where

$$\begin{aligned}\tilde{\pi}^{i}_{\alpha}(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{u}^{i}(\tilde{\boldsymbol{x}})}{\partial \tilde{x}^{\alpha}}\bigg|_{t=0} \\ &= \frac{\partial}{\partial x^{\alpha}}[\theta(\boldsymbol{x})\phi^{i}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))] - \frac{\partial u^{i}}{\partial x^{\beta}}\frac{\partial}{\partial x^{\alpha}}[\theta(\boldsymbol{x})\xi^{\beta}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))]. \end{aligned} \tag{3.3}$$

In analogy with (2.4), we define the corresponding family of variations around $\boldsymbol{u}_0$ by

$$\tilde{\boldsymbol{u}}_t(\tilde{\boldsymbol{x}}) = [\Phi \circ (\mathrm{id} \times \boldsymbol{u}_0 \times t\theta)] \circ [\Xi \circ (\mathrm{id} \times \boldsymbol{u}_0 \times t\theta)]^{-1}(\tilde{\boldsymbol{x}}).$$

Note that, since $\theta \in C^1_0(\Omega)$, $\tilde{\boldsymbol{x}} = \boldsymbol{x}$ if $\boldsymbol{x} \in \partial\Omega$ and thus $\tilde{\Omega} = \Omega$ for $|t|$ small. Further, $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}) = \boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{x} \in \partial\Omega$, so that $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}})$ is an admissible test function for $E(\,\cdot\,;\Omega)$. In other words, since $\boldsymbol{u}_0$ is a minimizer for $E(\,\cdot\,;\Omega)$, $E(\tilde{\boldsymbol{u}}_0;\Omega) \geqslant E(\boldsymbol{u}_0;\Omega)$, from which it easily follows that (2.7) holds for $\tilde{\Omega} = \Omega$, and then, using (2.8), that

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde{\Omega}} L(\tilde{\boldsymbol{x}}_0, \tilde{\boldsymbol{u}}_0, \nabla_{\tilde{\boldsymbol{x}}}\tilde{\boldsymbol{u}}_0(\tilde{\boldsymbol{x}}))\bigg|_{t=0} \\ &= \int_{\Omega} \tilde{v}^{(1)}L(\boldsymbol{x}, \boldsymbol{u}_0, \nabla_{\boldsymbol{x}}\boldsymbol{u}_0(\boldsymbol{x})) + L(\boldsymbol{x}, \boldsymbol{u}_0, \nabla_{\boldsymbol{x}}\boldsymbol{u}_0(\boldsymbol{x}))\,\mathrm{Div}\,\tilde{\xi}(\boldsymbol{x}, \boldsymbol{u}_0(\boldsymbol{x}))\,\mathrm{d}x. \end{aligned} \tag{3.4}$$

Next, using (2.10), (3.2), (3.1) and (3.3), it follows that (3.4) is equal to

$$\begin{aligned}&\int_{\Omega}\bigg[\tilde{\xi}^{\alpha}(\boldsymbol{x}, \boldsymbol{u}_0)\frac{\partial}{\partial x^{\alpha}} + \tilde{\phi}^{i}(\boldsymbol{x}, \boldsymbol{u}_0)\frac{\partial}{\partial u^{i}} + \tilde{\pi}^{i}_{\alpha}(\boldsymbol{x}, \boldsymbol{u}_0, \nabla \boldsymbol{u}_0)\frac{\partial}{\partial F^{i}_{\alpha}}\bigg]L \\ &\qquad\qquad + L(\boldsymbol{x}, \boldsymbol{u}_0, \nabla_{\boldsymbol{x}}\boldsymbol{u}_0(\boldsymbol{x}))\,\mathrm{Div}\,\tilde{\xi}(\boldsymbol{x}, \boldsymbol{u}_0(\boldsymbol{x}))\,\mathrm{d}x \\ &= \int_{\Omega}\bigg[\theta\xi^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \theta\phi^{i}\frac{\partial}{\partial u^{i}}\bigg]L \\ &\qquad + \bigg[\frac{\partial}{\partial x^{\alpha}}(\phi^{i}(\boldsymbol{x}, \boldsymbol{u}_0(\boldsymbol{x}))\theta(\boldsymbol{x})) - \frac{\partial u^{i}_0}{\partial x^{\beta}}\frac{\partial}{\partial x^{\alpha}}(\xi^{\beta}(\boldsymbol{x}, \boldsymbol{u}_0(\boldsymbol{x}))\theta(\boldsymbol{x}))\bigg]\frac{\partial}{\partial F^{i}_{\alpha}}L \\ &\qquad\qquad + L\,\mathrm{Div}[\theta\xi] \\ &= \int_{\Omega}[v^{(1)}L(\boldsymbol{x}, \boldsymbol{u}_0, \nabla \boldsymbol{u}_0(\boldsymbol{x})) + L(\boldsymbol{x}, \boldsymbol{u}_0, \nabla \boldsymbol{u}_0(\boldsymbol{x}))\,\mathrm{Div}\,\xi(\boldsymbol{x}, \boldsymbol{u}_0(\boldsymbol{x}))]\theta(\boldsymbol{x}) \\ &\qquad + \bigg[\phi^{i}(\boldsymbol{x}, \boldsymbol{u}_0)\frac{\partial L}{\partial F^{i}_{\alpha}}(\boldsymbol{x}, \boldsymbol{u}_0, \nabla \boldsymbol{u}_0) + \xi^{\beta}(\boldsymbol{x}, \boldsymbol{u}_0)M^{\beta}_{\alpha}(\boldsymbol{x}, \boldsymbol{u}_0, \nabla \boldsymbol{u}_0)\bigg]\frac{\partial\theta}{\partial x^{\alpha}}(\boldsymbol{x})\,\mathrm{d}x \\ &= 0.\end{aligned}$$

Using (2.9), it now follows that the first term in square brackets vanishes identically (since (2.2) is a variational symmetry), and hence $\boldsymbol{u}_0$ satisfies (2.14), which is the weak form of the conservation law (2.13).

We have thus recovered (2.14) for any variational symmetry, provided that $\boldsymbol{u}_0$ is a minimizer for $E(\,\cdot\,;\Omega)$. In the next section, we apply this to the specific setting of hyperelasticity and recover known results in a rather straightforward manner.

4. Hyperelasticity and conservation laws

In nonlinear elasticity, $m = n = 2$ or 3 and the integral functional

$$E(\boldsymbol{u}) = \int_{\Omega} L(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x \tag{4.1}$$

is the energy stored by a deformation $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ of an elastic body occupying the region Ω in its reference configuration. The integrand $L : \Omega \times M_+^{n\times n} \to \mathbb{R}^n$ is known as the stored-energy function of the material (where $M_+^{n\times n}$ denotes the set of real $n \times n$ matrices with positive determinant).

Typically, deformations are required to satisfy the local invertibility condition

$$\det \nabla \boldsymbol{u}(\boldsymbol{x}) > 0 \quad \text{a.e.}, \tag{4.2}$$

which is usually incorporated variationally by requiring that

$$L(\boldsymbol{x}, \boldsymbol{F}) \to \infty \quad \text{as } \det \boldsymbol{F} \to 0. \tag{4.3}$$

Assumption (4.2) prevents local interpenetration of matter. (If $\boldsymbol{u}$ is C^1, this result follows directly from the inverse function theorem, see also [5]. If $\boldsymbol{u}$ only lies in a Sobolev space, then results on the invertibility properties of $\boldsymbol{u}$ are contained, for example, in [2, 6, 12].)

If the stored-energy function is explicitly independent of $\boldsymbol{x}$, so that $L = L(\boldsymbol{F})$, then the material is said to be *homogeneous.*

Frame indifference of the stored energy requires that

$$L(QF) = L(F)$$

for all $F \in M_+^{n\times n}$ and all $Q \in SO(n)$ (the $n \times n$ special orthogonal matrices, i.e. orthogonal matrices with determinant $+1$). If, in addition, the material is *isotropic*, then

$$L(FQ) = L(F)$$

for all $F \in M_+^{n\times n}$ and all $Q \in SO(n)$ (see [5] for further background on nonlinear elasticity). In the context of nonlinear elasticity, the structure and properties mentioned above give rise to the following variational symmetries: homogeneity of the material yields

$$(\boldsymbol{x}, \boldsymbol{u}) \to (\boldsymbol{x} + t\boldsymbol{c}, \boldsymbol{u}); \tag{4.4}$$

translational invariance yields

$$(\boldsymbol{x}, \boldsymbol{u}) \to (\boldsymbol{x}, \boldsymbol{u} + t\boldsymbol{c}); \tag{4.5}$$

rotational invariance of the energy yields

$$(\boldsymbol{x}, \boldsymbol{u}) \to (\boldsymbol{x}, \boldsymbol{Q}(t)\boldsymbol{u});$$

and isotropy of the material yields

$$(\boldsymbol{x}, \boldsymbol{u}) \to (\boldsymbol{Q}(t)\boldsymbol{x}, \boldsymbol{u}),$$

where $\boldsymbol{c} \in \mathbb{R}^n$ is a constant vector and $\boldsymbol{Q} : (-\delta, \delta) \to SO(n)$, $\delta > 0$, satisfies $\boldsymbol{Q}(0) = I$. Each of the above variational symmetries immediately gives rise to corresponding conservation laws via the formula (2.13) (see [10, pp. 281–283]).

Of notable interest is the conservation law

$$\frac{\partial}{\partial x^\alpha}\left[L(\nabla \boldsymbol{u}_0(\boldsymbol{x}))\delta^\beta_\alpha - \frac{\partial u^k}{\partial x^\beta}\frac{\partial L}{\partial F^k_\alpha}(\nabla \boldsymbol{u}_0(\boldsymbol{x}))\right] = 0, \quad \beta = 1, 2, \ldots, n, \tag{4.6}$$

which we obtain here from the use of the family of symmetries given by (4.4) and upon application of (2.13). This conservation law was derived by Ball in [3] using 'inner' variations of the form $\boldsymbol{u}_0(\boldsymbol{x} + t\boldsymbol{\phi}(\boldsymbol{x}))$ (with $\boldsymbol{\phi} \in C^1_0(\Omega)$ and $t \in \mathbb{R}$ to palliate the difficulty encountered in using usual variations of the form $\boldsymbol{u}_0(\boldsymbol{x}) + t\boldsymbol{\phi}(\boldsymbol{x})$ because these may violate (4.2) for arbitrarily small $t \neq 0$, even though $\boldsymbol{u}_0$ satisfies (4.2)).

Ball also considered 'outer' variations of the form

$$\boldsymbol{u}_0(\boldsymbol{x}) + t\boldsymbol{\phi}(\boldsymbol{u}_0(\boldsymbol{x})), \quad \boldsymbol{\phi} \in C^1_0(\boldsymbol{u}_0(\Omega)),$$

from which he derived the weak form of the Cauchy equilibrium equations satisfied by $\boldsymbol{u}_0$, namely,

$$(\mathrm{Div}_y\, \boldsymbol{T})_i = \frac{\partial}{\partial y^\alpha}[T_{i\alpha}] = 0 \tag{4.7}$$

in $\boldsymbol{u}_0(\Omega)$, where

$$\boldsymbol{T}(\boldsymbol{y}) = \frac{\partial L}{\partial \boldsymbol{F}}(\nabla \boldsymbol{u}_0(\boldsymbol{x}))[\nabla \boldsymbol{u}_0(\boldsymbol{x})]^{\mathrm{T}}(\det \nabla \boldsymbol{u}_0(\boldsymbol{x}))^{-1}$$

is called the Cauchy stress tensor and $\boldsymbol{x} = \boldsymbol{u}_0^{-1}(\boldsymbol{y})$. In the context of this paper (and the weak form of the conservation laws (2.14)), this same result obtained in [3] follows from the use of the family of symmetries (4.5) together with test functions $\theta(\boldsymbol{x}) = \tilde{\theta}(\boldsymbol{u}_0(\boldsymbol{x}))$, where $\tilde{\theta} \in C^1_0(\boldsymbol{u}_0(\Omega))$ (see [3, 4] for further details).

Remark 4.1. As a further example, we now suppose that $L(\boldsymbol{F})$ is homogeneous of degree p, so that $L(\lambda \boldsymbol{F}) = \lambda^p L(\boldsymbol{F})$ for all $\boldsymbol{F} \in M^{n\times n}_+$ for any $\lambda > 0$. Then

$$(\boldsymbol{x}, \boldsymbol{u}) \to ((1+t)\boldsymbol{x}, (1+t)^{(p-n)/p}\boldsymbol{u}) = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{u}})$$

is a further variational symmetry of (4.1).

Application of (2.13) immediately yields the following conservation law:

$$\frac{\partial}{\partial x^\alpha}\left[M^\beta_\alpha(\nabla \boldsymbol{u}_0(\boldsymbol{x}))x^\beta + \frac{p-n}{p}\frac{\partial L}{\partial F^k_\alpha}(\nabla \boldsymbol{u}_0(\boldsymbol{x}))u^k_0(\boldsymbol{x})\right] = 0, \quad \beta = 1, 2, \ldots, n$$

(see [10, example 4.32]). Note, however, that the requirement of p-homogeneity of L is incompatible with condition (4.3), which must therefore be dropped if such a setting is adopted. The next example illustrates such a situation and demonstrates that minimizers may satisfy the weak form of some conservation laws but not others.

Example 4.2. Let $A \subset \mathbb{R}^2$ be the 2D annulus $A = \{\boldsymbol{x} \in \mathbb{R}^2 : a < |\boldsymbol{x}| < b\}$, where $0 < a < b$. Consider minimizing

$$E(\boldsymbol{u}) = \int_A |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x \tag{4.8}$$

on $W^{1,2}(A)$ subject to the boundary condition

$$u|_{\partial A} = \mathrm{id}. \tag{4.9}$$

The Euler–Lagrange equations for (4.8) and (4.9) are formally given by

$$\left.\begin{aligned} \Delta u &= 0, \\ u|_{\partial A} &= \mathrm{id} \end{aligned}\right\} \tag{4.10}$$

and, by the strict convexity of (4.8), the unique global minimizer is given by the homogeneous map

$$u^h \equiv x.$$

There are clearly no other weak solutions for this Dirichlet problem in $W^{1,2}(A)$. However, the results of [11] show that, for each $N \in \mathbb{N}$, there exists a minimizer u^N of E on

$$\mathcal{A}_N = \{u \in W^{1,2}(A) : \det \nabla u \geqslant 0 \text{ a.e.}, \\ u : A \to \bar{A},\ u = x \text{ on } \partial A,\ u \text{ satisfies } (H_N)\},$$

where (H_N) is a homotopy condition that the map u twists the annulus through $2\pi N$ (see [11, definition 2.8])[3].

The arguments of [3] apply to show rigorously that each u^N must satisfy the weak form of the corresponding energy–momentum equations (4.6), namely,

$$\operatorname{Div} M = \operatorname{Div}[|\nabla u|^2 I - 2\nabla u^{\mathrm{T}} \nabla u] = 0,$$

which are given in component form by

$$\frac{\partial}{\partial x^\alpha} M_\alpha^\beta = \frac{\partial}{\partial x^\alpha}\left[|\nabla u|^2 \delta_\alpha^\beta - 2\frac{\partial u^k}{\partial x^\beta}\frac{\partial u^k}{\partial x^\alpha}\right] = 0.$$

Note that u^N cannot satisfy the weak form of the Euler–Lagrange equations (4.10) because of the uniqueness result alluded to above.

It is interesting to note that the integrand in (4.8) is homogeneous of degree two and so, by remark 4.1 ($p = n = 2$), each u^N also satisfies the weak form of the conservation law

$$\frac{\partial}{\partial x^\alpha}[M_\alpha^\beta x^\beta] = 0.$$

We anticipate that, in this example, each u^N will be degenerate in the sense that $\det \nabla u^N = 0$ on a set of non-zero measure. In this case, the corresponding Cauchy stress tensor will be undefined and so the weak form of the Cauchy equations (4.7) will not hold.

[3]More precisely, using polar coordinates, we write

$$u(x) = u(r, \theta), \quad r \in [a, b], \quad \theta \in [0, 2\pi).$$

Then, for a.e. $\theta \in [0, 2\pi)$,

$$\gamma_\theta(r) = \frac{u(r, \theta)}{|u(r, \theta)|}, \quad r \in [a, b],$$

is a closed continuous curve in the plane with a well-defined winding number around the origin, denoted $\mathrm{wind}(\gamma_\theta)$. The map u is said to satisfy (H_N), provided that $\mathrm{wind}(\gamma_\theta) = N$ for a.e. $\theta \in [0, 2\pi)$.

5. Concluding remarks

For sufficiently regular deformations $\boldsymbol{u} : \Omega \to \mathbb{R}^n$, it is possible to change variables to rewrite the energy functional (4.1) as an integral over the deformed configuration $\boldsymbol{u}(\Omega)$,

$$E(\boldsymbol{u}) = \int_{\Omega} L(\nabla \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x = \int_{\boldsymbol{u}(\Omega)} \hat{L}(\nabla_{\boldsymbol{u}} \boldsymbol{x}(\boldsymbol{u}))\,\mathrm{d}u,$$

where $\hat{L}(\boldsymbol{F}) = \det \boldsymbol{F} L(\boldsymbol{F}^{-1})$ and $\boldsymbol{x} : \boldsymbol{u}(\Omega) \to \mathbb{R}^n$ is the inverse of $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ (see, for example, [1]). In the context of this paper, it is interesting to note that the energy momentum tensor for $\hat{L}$ is equal to the Cauchy stress tensor for L and vice versa, i.e.

$$\frac{\partial \hat{L}}{\partial \boldsymbol{F}} \frac{\boldsymbol{F}^{\mathrm{T}}}{\det \boldsymbol{F}} = \left[L(\boldsymbol{F}^{-1})\boldsymbol{I} - (\boldsymbol{F}^{-1})^{\mathrm{T}} \frac{\partial L}{\partial \boldsymbol{F}}(\boldsymbol{F}^{-1}) \right] = \boldsymbol{M}(\boldsymbol{F}^{-1})$$

and

$$\hat{\boldsymbol{M}}(\boldsymbol{F}) = \left[\hat{L}(\boldsymbol{F}) - \boldsymbol{F}^{\mathrm{T}} \frac{\partial \hat{L}}{\partial \boldsymbol{F}}(\boldsymbol{F}) \right] = \frac{\partial L}{\partial \boldsymbol{F}}(\boldsymbol{F}^{-1}) \frac{(\boldsymbol{F}^{-1})^{\mathrm{T}}}{\det(\boldsymbol{F}^{-1})}.$$

Hence these two tensors are, in a precise sense, dual to one another.

Finally, we note that in this paper we have only considered variational point symmetries of the integral functional (2.1) and the corresponding conservation laws (2.13). However, Noether's theorem also applies to more general symmetries, known as divergence symmetries, for which the infinitesimal invariance criterion (2.10) is replaced by the requirement that

$$\phi^i(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \frac{\partial L}{\partial u^i} + \frac{\partial}{\partial x^\alpha}(\phi^i(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))) \frac{\partial L}{\partial F^i_\alpha} + \xi^\alpha \frac{\partial L}{\partial x^\alpha} + \left(\frac{\partial}{\partial x^\alpha} \xi^\beta(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \right) M^\beta_\alpha = \operatorname{Div} \boldsymbol{B},$$

where $\boldsymbol{B}$ is a vector function of $\boldsymbol{x}$, $\boldsymbol{u}$ and the derivatives of $\boldsymbol{u}$ (see [10, p. 283]). The only modification that arises in our initial derivation of the conservation law (2.13) is the inclusion of a term involving $\boldsymbol{B}$. The modification to our derivation of the weak form of this (more general) conservation law as a necessary condition for a minimizer is straight forward and we leave this as an exercise for the interested reader. For further details of the general version of Noether's theorem we refer to [10].

Acknowledgments

The work of J.S. was supported by a Leverhulme fellowship. The authors also acknowledge the support of the Isaac Newton Institute (Cambridge), where this work was initiated.

References

1 J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Ration. Mech. Analysis* **63** (1977), 337–403.

2 J. M. Ball. Global invertibility of Sobolev functions and the interpenetration of matter. *Proc. R. Soc. Edinb.* A **88** (1981), 315–328.

3 J. M. Ball. Minimisers and the Euler–Lagrange equations. In *Trends and applications of pure mathematics to mechanics* (ed. P. G. Ciarlet and M. Roseau), pp. 1–4 (Springer, 1984).

4 P. Baumann, N. C. Owen and D. Phillips. Maximum principles and *a priori* estimates for a class of problems from nonlinear elasticity. *Analyse Nonlin.* **8** (1991), 119–157.

5 P. G. Ciarlet. *Mathematical elasticity,* vol. 1. *Three-dimensional elasticity* (Amsterdam: North-Holland, 1988).

6 P. G. Ciarlet and J. Necas. Unilateral problems in nonlinear three-dimensional elasticity. *Arch. Ration. Mech. Analysis* **87** (1985), 319–338.

7 L. C. Evans. Quasiconvexity and partial regularity in the calculus of variations. *Arch. Ration. Mech. Analysis* **95** (1986), 227–252.

8 C. B. Morrey. *Multiple integrals in the calculus of variations* (Springer, 1966).

9 E. Noether. Invariante Variationsprobleme. *Nachr. Konig. Gesell. Wissen. Gottingen Math. Phys. Kl.* (1918), 235–257. (Transl. *Transport Theory Statist. Phys.* **1** (1971), 186–207.)

10 P. J. Olver. *Applications of Lie groups to differential equations* (Springer, 1986).

11 K. D. E. Post and J. Sivaloganathan. On homotopy conditions and the existence of multiple equilibria in finite elasticity. *Proc. R. Soc. Edinb.* A **127** (1997), 595–614.

12 V. Sverak. Regularity properties of deformations with finite energy. *Arch. Ration. Mech. Analysis* **100** (1988), 105–127.

(*Issued 13 December 2002*)