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A field theory approach to stability of radial equilibria in nonlinear elasticity

By J. SIVALOGANATHAN
School of Mathematics, University of Bath, Bath BA2 7AY

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1. *Introduction*

In this paper we study the stability of a class of singular radial solutions to the equilibrium equations of nonlinear elasticity, in which a hole forms at the centre of a ball of isotropic material held in a state of tension under prescribed boundary displacements. The existence of such cavitating solutions has been shown by Ball [**1**], Stuart [**11**] and Sivaloganathan [**10**]. Our methods involve elements of the field theory of the calculus of variations and provide a new unified interpretation of the phenomenon of cavitation.

For the displacement boundary-value problem for a ball of elastic material, it is known that (under suitable assumptions on the stored energy function) there exists a critical boundary displacement at which the homogeneous deformation loses stability and a singular stable solution – corresponding to a deformation with a cavity – bifurcates off (see Ball [**1**], Sivaloganathan [**10**]). However, the methods used to obtain these results rely on variational techniques. Stuart [**11**] proves the existence of such singular equilibria using a shooting argument but is able to conclude little as to their stability. Our methods hold under mild hypotheses on the stored energy function (see Section 5) and apply to the cases he considers. In particular, they apply to situations in which the growth of the stored energy function is insufficient to guarantee the compactness of minimizing sequences for the total energy, and where consequently the existence of equilibrium solutions must be proved using methods other than variational ones.

The invariance of the equilibrium equations under rescaling leads to a natural choice for a field of extremals. More specifically, any singular solution may be extended to an infinite domain as a solution of the equilibrium equation (Proposition 5·3). We construct a field by extending the set of all rescalings of such a solution using homogeneous deformations. This construction is central to the arguments of this paper and is shown in Figure 1. Then by applying a modified form of a sufficiency theorem due originally to Weierstrass (see Section 4) we are able to show that, for boundary displacements greater than some critical value, the cavitating deformations are unique and globally stable in the class of radial deformations. For boundary displacements less than or equal to this critical value, a homogeneous deformation is the unique global minimizer (Theorem 6·10).

As noted in Ball [**1**], the homogeneous deformations are isolated from the cavitating ones in the space of smooth functions and thus the bifurcation is not obtainable using standard bifurcation techniques. Our field theory approach has the advantage over previous ones of treating both the bifurcating singular and trivial homogeneous solutions within the same framework.

In this paper we do not consider surface energy effects. However, experimental work on the internal rupture of rubber suggests that this has little effect on the initiation of fracture through cavitation. For further details see Ball [1] and the references therein.

2. *Notation*

We denote the set of all real 3×3 matrices by $M^{3\times 3}$ and we write

$$M_+^{3\times 3} = \{F \in M^{3\times 3};\ \det F > 0\},$$
$$\mathrm{SO}(3) = \{F \in M^{3\times 3};\ \det F = 1\}.$$

The Sobolev space $W^{1,1}(a,b)$ is the Banach space of equivalence classes of Lebesgue measurable functions whose first derivative exists in the sense of distributions and lies in $L^1(a,b)$. We equip $W^{1,1}(a,b)$ with the norm $\|\ \|_{1,1}$ where

$$\|u\|_{1,1} = \int_a^b |u|\,dx + \int_a^b |u'|\,dx.$$

3. *The stored energy function and radial deformations*

We shall consider deformations of a homogeneous ball of elastic material which in its reference configuration occupies the region

$$B = \{\mathbf{X} \in \mathbb{R}^3;\ |\mathbf{X}| < 1\}. \tag{3·1}$$

A deformation of the ball is a function $\mathbf{x}\colon B \to \mathbb{R}^3$ under which a particle with position vector $\mathbf{X}$ moves to a point with position vector $\mathbf{x}(\mathbf{X})$. Henceforth we will be concerned with radial deformations in which case $\mathbf{x}$ may be expressed in the form

$$\mathbf{x}(\mathbf{X}) = \frac{r(R)}{R}\mathbf{X}, \tag{3·2}$$

where $R = |\mathbf{X}|$.

In the displacement boundary-value problem the values of $\mathbf{x}$ are prescribed on the boundary of B, so that

$$\mathbf{x}(\mathbf{X}) = \lambda \mathbf{X} \quad \text{for} \quad \mathbf{X} \in \partial B. \tag{3·3}$$

If $W\colon M_+^{3\times 3} \to \mathbb{R}^+$ is the stored energy function of the material then the total energy E associated with the deformation $\mathbf{x}$ is given by

$$E(\mathbf{x}) = \int_B W(\nabla \mathbf{x}(\mathbf{X}))\,dX, \tag{3·4}$$

whenever $\mathbf{x}$ satisfies the local invertibility condition

$$\det(\nabla \mathbf{x}(\mathbf{X})) > 0 \quad \text{for} \quad \mathbf{X} \in B. \tag{3·5}$$

The equilibrium equations of nonlinear elasticity under zero body force are the Euler–Lagrange equations for the functional E, namely

$$\frac{\partial}{\partial X^\alpha}\left(\frac{\partial W}{\partial x^i_{,\alpha}}(\nabla \mathbf{x}(\mathbf{X}))\right) = 0 \quad (i = 1, 2, 3). \tag{3·6}$$

We assume that W is frame-indifferent, so that

$$W(QF) = W(F) \quad \text{for all} \quad F \in M_+^{3\times 3}, \quad Q \in \mathrm{SO}(3) \tag{3·7}$$

and we suppose further that W is isotropic, so that it satisfies the additional condition

$$W(FQ) = W(F) \quad \text{for all} \quad F \in M_+^{3\times 3}, \quad Q \in \mathrm{SO}(3). \tag{3·8}$$

It is well known that (3·7) and (3·8) hold if and only if there exists a symmetric function $\Phi\colon \mathbb{R}^3_{++} \to \mathbb{R}$ satisfying

$$W(F) = \Phi(v_1, v_2, v_3) \quad \text{for all} \quad F \in M_+^{3\times 3}, \tag{3·9}$$

where $$\mathbb{R}^3_{++} = \{(c_1, c_2, c_3) \in \mathbb{R}^3;\ c_i > 0 \quad i = 1, 2, 3\}$$

and the v_i are the eigenvalues of $(F^T F)^{\frac{1}{2}}$, known as the principal stretches. (For a proof see Truesdell and Noll [**12**].)

In the case of radial deformations

$$v_1 = r'(R), \quad v_2 = v_3 = \frac{r(R)}{R}, \tag{3·10}$$

and by (3·4) and (3·9) the corresponding energy then takes the form

$$E(\mathbf{x}) = 4\pi I(r) \overset{\text{def}}{=} 4\pi \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) dR. \tag{3·11}$$

It is shown in Ball [**1**] that the study of weak solutions to (3·6) of the form (3·2) is equivalent to studying solutions r on $(0, 1]$ of the *radial equilibrium equation*

$$\frac{d}{dR}\left[R^2 \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right)\right] = 2R\Phi_{,2}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right), \tag{3·12}$$

where $\Phi_{,i}$ denotes differentiation of Φ with respect to its ith argument and r satisfies

$$\text{(i)} \quad r(1) = \lambda > 0, \tag{3·13}$$

$$\text{(ii)} \quad r'(R) > 0 \quad \text{for} \quad R \in (0, 1], \tag{3·14}$$

$$\text{(iii)} \quad \text{if} \quad r(0) \overset{\text{def}}{=} \operatorname*{Lim}_{R \to 0} r(R) > 0 \quad \text{then} \quad \operatorname*{Lim}_{R \to 0} T(r(R)) = 0, \tag{3·15}$$

where $$T(r(R)) \overset{\text{def}}{=} \left(\frac{R}{r(R)}\right)^2 \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \tag{3·16}$$

is the radial component of the Cauchy stress. Notice that (3·12) is the Euler–Lagrange equation for the functional I given by (3·11) and that the *homogeneous deformation*

$$r(R) \equiv \lambda R \tag{3·17}$$

is always a solution of (3·12)–(3·15) (this follows from the symmetry of Φ). Condition (i) represents the boundary condition (3·3) and (ii) corresponds to the requirement of local invertibility (3·5). It follows from (3·2) that if $r(0) > 0$ then the deformed ball contains a cavity and (iii) is the natural boundary condition that the cavity is stress-free. We say that $r \in C^2((0, 1])$ is a *cavitating equilibrium solution* if it is a solution of (3·12)–(3·15) satisfying $r(0) > 0$.

4. *Elements of the field theory*

In this section we present elements of the classical Weierstrass theory of the Calculus of Variations. For ease of presentation we restrict attention to conditions under which solutions of the Euler–Lagrange equation for

$$\mathscr{L}(y) \overset{\text{def}}{=} \int_0^1 f(x, y(x), y'(x))\, dx \tag{4·1}$$

namely solutions of

$$\frac{d}{dx}(f_{,3}(x, y(x), y'(x))) = f_{,2}(x, y(x), y'(x)) \tag{4·2}$$

are global minimizers of $\mathscr{L}$ on the set of admissible functions

$$\mathscr{A} \overset{\text{def}}{=} \{y \in W^{1,1}(0,1);\ y(0) \geq \alpha, y(1) = \beta\}, \tag{4·3}$$

where f is a positive C^2 function, $f_{,i}$ denotes differentiation of f with respect to its ith argument, and $\alpha \geq 0$, $\beta > 0$ are constants. (For a statement of the general theory and related results we refer to Cesari [**4**], Gelfand and Fomin [**5**], Morrey [**8**] and the references therein).

Definition 4·1. The *Weierstrass excess function* $\mathscr{E}: \mathbb{R}^4 \to \mathbb{R}$ corresponding to the integral (4·1) is given by

$$\mathscr{E}(x, y; p, q) \overset{\text{def}}{=} f(x, y, q) - f(x, y, p) + (p - q) f_{,3}(x, y, p). \tag{4·4}$$

It is well known that if $y_0 \in C^1([0,1])$ is a strong local minimum of $\mathscr{L}$ on $\mathscr{A}$ then

$$\mathscr{E}(x, y_0(x); y_0'(x), q) \geq 0 \quad \text{for all} \quad q \in \mathbb{R} \quad \text{and} \quad x \in (0,1). \tag{4·5}$$

In higher dimensions the analogous conditions are quasiconvexity and the Legendre-Hadamard condition (see Giaquinta [6], Ball and Marsden [2]) which imply (4·5) in the case of dimension one.

Definition 4·2. If $D \subset \mathbb{R}$ is connected we say that $y \in C^2(D)$ is an extremal of $\mathscr{L}$ on D if y is a solution of (4·2) on D.

Definition 4·3. If $S \subset \mathbb{R}^2$ is an open simply connected region, we say that the one-parameter family of functions $\{y(\cdot, \delta);\ \delta \in \Delta\}$, $y(\cdot, \delta\}: \mathbb{R} \to \mathbb{R}$ with $\Delta \subset \mathbb{R}$ constitutes a *field of extremals* $\mathscr{F}$ of $\mathscr{L}$ over S if

(i) for each $(a, b) \in S$ there exists a unique $\delta \in \Delta$ such that $y(a, \delta) = b$,

(ii) for each $\delta \in \Delta$ $y(\cdot, \delta) \in C^2(D_\delta)$ is a solution of (4·2) on D_δ, where

$$D_\delta \supseteq \{x \in \mathbb{R};\ (x, y(x, \delta)) \in S\}.$$

Definition 4·4. We define the *slope function* $P: S \to \mathbb{R}$ corresponding to the field of extremals $\mathscr{F}$ by

$$P(a, b) = \frac{d}{dx} y(x, \delta_0)\bigg|_{x=a} \quad \text{for} \quad (a, b) \in S,$$

where $\delta_0 \in \Delta$ is the unique element satisfying $y(a, \delta_0) = b$. (i.e. $P(a, b)$ is the slope at (a, b) of the unique extremal of the field passing through that point). For the purposes of this section we require $P \in C^1(S)$.

Definition 4·5. We define the *Hilbert integral* $\mathscr{L}^*$ relative to $\mathscr{L}$ and the field $\mathscr{F}$ by

$$\mathscr{L}^*(z) = \int_0^1 f(x, z(x), P(x, z(x))) + (z'(x) - P(x, z)) f_{,3}(x, z(x), P(x, z(x)))\, dx \tag{4·6}$$

for $z \in \mathscr{A}$, where P is as in Definition 4·4.

Definition 4·6. For each $y \in C([0,1])$ the graph of y, $Gr(y)$, is given by

$$Gr(y) = \{(x, y(x));\, x \in [0,1]\}.$$

Definition 4·7. We say that y_0 is embedded in the field of extremals $\mathscr{F}$ over S if

(i) $Gr(y_0) \subset S$,

(ii) $y_0(x) \equiv y(x, \bar{\delta})$ for $x \in [0,1]$, some $\bar{\delta} \in \Delta$.

The following result is well known (see Cesari [4] for a proof).

PROPOSITION 4·8. *Let $\mathscr{F}$ be a field of extremals of $\mathscr{L}$ over a region $S \subset \mathbb{R}^2$ which is open and simply connected. Then there exists a function $\mathscr{H}^*: \mathbb{R}^3 \to \mathbb{R}$ with the property that*

(i) $\mathscr{H}^*(x, z(x), P(x, z(x))) \in W^{1,1}(0,1)$,

and

(ii) $\mathscr{L}^*(z) = \mathscr{H}^*(x, z(x), P(x, z(x)))|_{x=0}^{x=1}$,

where (i) *and* (ii) *hold for all $z \in W^{1,1}(0,1)$ with $Gr(z) \subset S$.*

Using the proposition we can prove that next result which is a modified version of a theorem found in Cesari [4], p. 73.

THEOREM 4·9. *Suppose that the extremal $y_0 \in C^2([0,1])$ with $y_0 \in \mathscr{A}$ is embedded in a field of extremals $\mathscr{F}$ over S (an open simply connected subset of $\mathbb{R}^2$). Let*

$$\mathscr{E}(x, y; P(x,y), q) > 0 \quad \textit{for all} \quad (x,y) \in S, \quad \textit{all} \quad q \in \mathbb{R} \quad \textit{with} \quad q \neq P(x,y), \tag{4·6}$$

where P is the slope function corresponding to $\mathscr{F}$. Then

$$\mathscr{L}(y_0) < \mathscr{L}(y)$$

for all $y \in \mathscr{A}$, $y \not\equiv y_0$ satisfying $Gr(y) \subset S$ and such that

$$\mathscr{H}^*(0, y(0), P(0, y(0))) = \mathscr{H}^*(0, y_0(0), y_0'(0)). \tag{4·7}$$

Proof. It follows from (4·4) and (4·6) that

$$\begin{aligned}\mathscr{L}(y) > \mathscr{L}^*(y) &= \mathscr{H}^*(x, y(x), P(x, y(x)))|_{x=0}^{x=1} \\ &= \mathscr{H}^*(x, y_0(x), y_0'(x))|_{x=0}^{x=1} = \mathscr{L}^*(y_0) = \mathscr{L}(y_0),\end{aligned}$$

whenever $y \in \mathscr{A}$, $y \not\equiv y_0$ satisfies $Gr(y) \subset S$ and (4·7). The last equality follows from the definition of $\mathscr{L}^*$ and the assumption that y_0 is embedded in $\mathscr{F}$.

Remark 4·10. Notice that this version of theorem 4·9 allows the competing functions $y \in \mathscr{A}$ to satisfy $y(0) > \alpha$ provided that condition (4·7) is satisfied.

5. *Constitutive assumptions and properties of radial equilibria*

For the remainder of this paper we make the following assumptions on the material response. We assume that $\Phi \in C^3(\mathbb{R}^3_{++})$ and that $\Phi_{,i}(1,1,1) = 0$ so that the undeformed configuration is a natural state. We suppose further that

(I) $\Phi_{,11}(v_1, v_2, v_3) > 0$ (the tension-extension inequality),

(II) $\left(\dfrac{v_i \Phi_{,i}(v_1, v_2, v_3) - v_j \Phi_{,j} \Phi_{,j}(v_1, v_2, v_3)}{v_i - v_j}\right) \geq 0 \quad i \neq j,\ v_i \neq v_j$,

(these are the weakened Baker–Ericksen inequalities),

(III) for each $a \in (0, \infty)$ there exist $\bar{v}, \underline{v} \in (0, \infty)$ such that (*a*) $\Phi_{,1}(\bar{v}, a, a) > 0$ and (*b*) $\Phi_{,1}(\underline{v}, a, a) < 0$.

For examples of stored energy functions and their correlation with experimental data we refer to Ogden [9] and the references therein.

We now examine the implications of our assumptions on the properties of solutions of (3·12).

PROPOSITION 5·1. *Let* $r \in C^2((0,1])$ *be a solution of* (3·12) *satisfying* (3·14). If $r(R_0)/R_0 = r'(R_0) \overset{\text{def}}{=} \lambda_0$ *for some* $R_0 \in (0,1]$, $\lambda_0 \in (0,\infty)$, *then* $r(R) \equiv \lambda_0 R$ for $R \in (0,1]$.

Proof. Equation (3·12) is of the form $r'' = f(R, r, r')$ where f is C^1. Standard results for ordinary differential equations then imply that the solution $r(R)$ to the initial value problem with data $r(R_0) = \lambda_0 R_0$, $r'(R_0) = \lambda_0$, is unique. Hence $r(R) \equiv \lambda_0 R$.

COROLLARY 5·2. *If* $r \in C^2((0,1])$ *is a solution of* (3·12)–(3·15) *with*

$$r(0) \overset{\text{def}}{=} \operatorname*{Lim}_{R\to 0} r(R) > 0$$

(*i.e. a cavitating equilibrium solution*), *then* $r(R)/R$ *is a monotone function. Moreover*

(i) $\dfrac{d}{dR}\left(\dfrac{r(R)}{R}\right) = \dfrac{1}{R}\left(r'(R) - \dfrac{r(R)}{R}\right)$ *for* $R \in (0,1]$,

(ii) $r'(R) < \dfrac{r(R)}{R}$ *for* $R \in (0,1]$.

Proof. Statement (i) is immediate. From Proposition 5·1 and (i), $r(R)/R$ is a monotone function. (ii) then follows since $r(R)/R \to \infty$ as $R \to 0$ from the assumption that $r(0) > 0$.

PROPOSITION 5·3. *Let* $r \in C^2((0,1])$ *be a cavitating equilibrium solution. Then* r *is extendable to* $r \in C^2((0,\infty))$ *as a solution of* (3·12) *that satisfies*

(i) $\dfrac{r(R)}{R} > r'(R) > 0$ *for* $R \in (0,\infty)$,

(ii) $\operatorname*{Lim}_{R\to\infty} \dfrac{r(R)}{R} = \operatorname*{Lim}_{R\to\infty} r'(R) = \lambda_c$ *for some* $\lambda_c \in [1,\infty)$.

The proof of this result is analogous to that of Sivaloganathan [**10**], proposition 1·6, and will be omitted.

The following conservation law which is easily verifiable will play an important role in our analysis.

PROPOSITION 5·4. *If* $r \in C^2((0,1])$ *is a solution of* (3·12) *satisfying* (3·14) *then*

$$\frac{d}{dR}\left[R^3\left(\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) + \left(\frac{r}{R} - r'\right)\Phi_{,1}\left(r', \frac{r}{R}, \frac{r}{R}\right)\right)\right] = 3R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \tag{5·1}$$

for $R \in (0,1]$.

As noted in Ball [**1**], (5·1) is the radial version of a general conservation law for finite elastostatics (see concluding remarks). For future reference we introduce the related notation

$$H^*\left(R, \frac{r}{R}, r'\right) \overset{\text{def}}{=} \frac{R^3}{3}\left[\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) + \left(\frac{r}{R} - r'\right)\Phi_{,1}\left(r', \frac{r}{R}, \frac{r}{R}\right)\right]. \tag{5·2}$$

PROPOSITION 5·5. *Let* $r \in C^2((0,1])$ *be a cavitating equilibrium solution. Then*

(i) $\operatorname*{Lim}_{R\to 0} H^*\left(R, \dfrac{r}{R}, r'\right) = 0$,

(ii) $I(r) = H^*(1, r(1), r'(1))$,

in particular any cavitating equilibrium solution has finite energy. The proof of this proposition follows the lines of Sivaloganathan [**10**], proposition 1·13, and will be omitted.

Our next proposition concerns the invertibility of the relation $v = r(R)/R$.

PROPOSITION 5·6. *If $r \in C^2((0,\infty))$ is a cavitating equilibrium solution then there exists $g \in C^2((\lambda_c, \infty))$ satisfying*

(i) $g\left(\frac{r(R)}{R}\right) = R$ *for* $R \in (0,\infty)$,

(ii) $\lim\limits_{v\to\infty} g(v) = 0$,

(iii) $\lim\limits_{v\to\lambda_c} g(v) = \infty$,

where λ_c is as given in Proposition 5·3.

Proof. This is an easy consequence of Propositions 5·2, 5·3 and the inverse function theorem.

COROLLARY 5·7. *Let $r \in C^2((0,1])$ be a cavitating equilibrium solution. Then*

$$\lim_{v\to\infty} H^*(g(v), v, r'(g(v))) = 0,$$

where g is as given in Proposition 5·6.

Proof. This follows from Proposition 5·6 and Proposition 5·5 (i).

6. *Interpretation of cavitation using the field theory*

6.1. *Construction of a field*

We now combine the ideas and results of Sections 4 and 5 to study the stability of solutions to the radial equilibrium equation (3·12) on the set of admissible deformations

$$A_\lambda = \{r \in W^{1,1}(0,1);\ r(1) = \lambda, r' > 0 \text{ a.e.}, r(0) \geqslant 0\}. \tag{6·1}$$

We assume throughout this section the existence of at least one cavitating equilibrium solution $r_c \in C^2((0,1])$, which by Proposition 5·3 may be extended to $r_c \in C^2((0,\infty))$ as a solution of (3·12) with

$$\frac{r_c(R)}{R} \searrow \lambda_c \quad \text{as} \quad R \to \infty \tag{6·2}$$

for some $\lambda_c \in [1,\infty)$. (In particular we make no assumption on the uniqueness of r_c.)

PROPOSITION 6·1. *Let*

$$y(R,\delta) \overset{\text{def}}{\equiv} \delta r_c\left(\frac{R}{\delta}\right) \quad \textit{for} \quad R \in (0,\infty) \quad \textit{and} \quad \delta \in (0,\infty). \tag{6·3}$$

Then

$$\mathscr{F}_c \overset{\text{def}}{=} \{y(\cdot,\delta);\ \delta \in (0,\infty)\} \tag{6·4}$$

is a field of extremals of I over D_{λ_c}, where

$$D_{\lambda_c} = \{(R,r) \in \mathbb{R}^2;\ r > \lambda_c R, r > 0, R > 0\} \tag{6·5}$$

and I is given by (3·11).

Proof. The set $\mathscr{F}_c$ consists of extremals because of the invariance of (3·12) under the scaling $(r,R) \to (dr, dR)$ for any $d \in (0,\infty)$. By Definition 4·3 it is now sufficient to show that through any point of D_{λ_c} there passes only one element of $\mathscr{F}_c$. Let $(R_0, r_0) \in D_{\lambda_c}$; by (6·2), (6·3) and as $r_c(R)/R \to \infty$ as $R \to 0$, it follows that

$$y(R_0,\delta) \to \lambda_c R_0 \quad \text{as} \quad \delta \to 0$$

and

$$y(R_0,\delta) \to \infty \quad \text{as} \quad \delta \to \infty.$$

Hence there exists $\delta_0 \in (0,\infty)$ such that $y(R_0,\delta_0) = r_0$; so the extremals cover D_{λ_c}. The uniqueness of δ_0 is a consequence of Corollary 5·2.

PROPOSITION 6·2. *Let $P_c\colon D_{\lambda_c} \to \mathbb{R}$ be defined by*

$$P_c(R_0,r_0) = r_c'\left(g_c\left(\frac{r_0}{R_0}\right)\right) \quad \textit{for} \quad (R_0,r_0)\in D_{\lambda_c}, \tag{6·6}$$

where g_c is the function of Proposition 5·6 corresponding to r_c. Then P_c is the slope function corresponding to the field of extremals $\mathscr{F}_c$ given by (6·4).

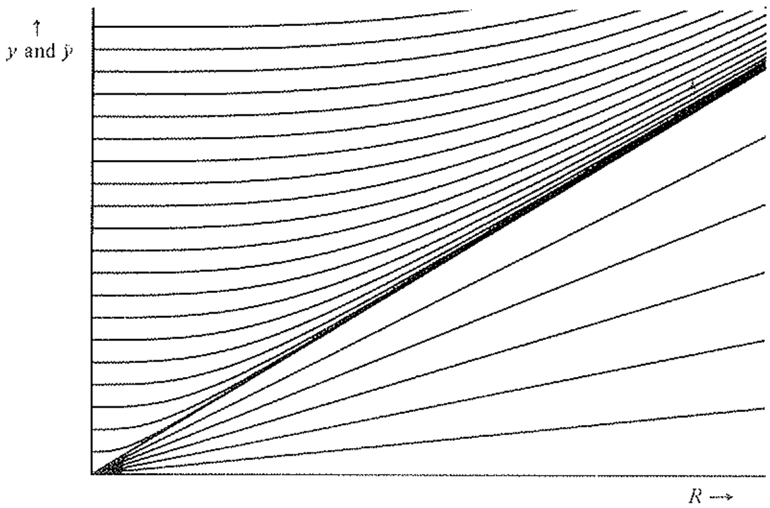

Fig. 1. A typical field.

Proof. This follows from Definition 4·4 and (6·4) on noting that if $\delta\in(0,\infty)$ satisfies

$$\delta r_c\left(\frac{R_0}{\delta}\right) = r_0$$

then

$$\frac{r_c\left(\dfrac{R_0}{\delta}\right)}{\left(\dfrac{R_0}{\delta}\right)} = \frac{r_0}{R_0}$$

and thus

$$g_c\left(\frac{r_0}{R_0}\right) = \frac{R_0}{\delta}$$

by Proposition 5·6. Hence

$$\frac{d}{dR}\left(\delta r_c\left(\frac{R}{\delta}\right)\right)\bigg|_{R=R_0} = r_c'\left(g_c\left(\frac{r_0}{R_0}\right)\right).$$

We now construct a field of extremals $\mathscr{F}$ over $\mathbb{R}^2_{++}$ by extending $\mathscr{F}_c$ using homogeneous deformations.

Define $\mathscr{F}$ by

$$\mathscr{F} = \mathscr{F}_c \cup \mathscr{F}_h \tag{6·7}$$

where

$$\mathscr{F}_h = \{\bar{y}(\cdot,\delta);\ \delta\in(-\lambda_c,0]\} \tag{6·8}$$

and

$$\bar{y}(R,\delta) \equiv (\lambda_c+\delta)R \quad \text{for} \quad R\in(0,\infty). \tag{6·9}$$

The following proposition is an easy consequence.

PROPOSITION 6·3. *The set $\mathscr{F}$ as defined by (6·7) is a field of extremals of I over $\mathbb{R}^2_{++}$. Moreover the corresponding slope function P is given by*

$$P(R_0, r_0) = \begin{cases} P_c(R_0, r_0) & \text{if} \quad r_0 > \lambda_c R_0 \\ \dfrac{r_0}{R_0} & \text{if} \quad r_0 \leqslant \lambda_c R_0, \end{cases} \tag{6·10}$$

P_c being given by (6·6).

Proof. This is an easy consequence of Propositions 6·1, 6·2, Definition 4·3 and (6·7)–(6·9).

6·2. *Application of the sufficiency theorem*

In this section we show how the phenomenon of cavitation may be viewed within the framework of the theory developed in Section 4 and in particular how it may be regarded as a modified application of Theorem 4·9 using the field $\mathscr{F}$ constructed above. The main difficulties arise because of the singular nature of $\mathscr{F}$ at the origin and because the corresponding slope function P (though continuous) is not continuously differentiable across the line $\lambda_c R$.

Remark 6·4. Our constitutive assumption (I) implies that the Weierstrass excess function $\mathscr{E}$ (see Definition 4·1) corresponding to the integral (3·11) is given by

$$\mathscr{E}(R, r; p, q) = R^2\left[\Phi\left(q, \frac{r}{R}, \frac{r}{R}\right) - \Phi\left(p, \frac{r}{R}, \frac{r}{R}\right) - (q-p)\,\Phi_{,1}\left(p, \frac{r}{R}, \frac{r}{R}\right)\right]$$

and satisfies

$$\mathscr{E}(R, r; p, q) > 0 \quad \text{for all} \quad R, r, p, q \in (0, \infty) \quad \text{with} \quad p \neq q.$$

Remark 6·5. The Hilbert integral I^* relative to I and the field $\mathscr{F}$ is given by

$$I^*(r) = \int_0^1 R^2\left[\Phi\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right) + (r' - P(R,r))\,\Phi_{,1}\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right)dR, \tag{6·11}$$

where P is given by (6·10) and I is given by (3·11).

PROPOSITION 6·6. *Let r_c be a cavitating equilibrium solution. Let $\mathscr{F}$ be defined by* (6·7) *and H^* by* (5·2). *Then for each $\lambda \in (0, \infty)$ and $r \in A_\lambda$,*

$$\frac{d}{dR}\left\{H^*\left(R, \frac{r}{R}, P(R, r)\right\} = R^2\left[\Phi\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right) + (r' - P(R,r))\,\Phi_{,1}\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right)\right] \tag{6·12}$$

for almost every $R \in D$, where $D \subset (0, 1)$ is any nonempty interval on which $r(R)$ satisfies either

(i) $\dfrac{r(R)}{R} > \lambda_c$ *for* $R \in D$ *or*

(ii) $\dfrac{r(R)}{R} < \lambda_c$ *for* $R \in D$.

Proof. We first consider the case when (i) holds.

Case (i)

From the definition of P (6·10) it is sufficient to show that

$$R^2\left[\Phi\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right), \frac{r}{R}, \frac{r}{R}\right) + \left(r' - r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\Phi_{,1}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right), \frac{r}{R}, \frac{r}{R}\right)\right]$$
$$= \frac{d}{dR}\left[H^*\left(R, \frac{r}{R}, r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\right] \quad \text{for a.e.} \quad R \in D. \tag{6·13}$$

The expression on the right-hand side of (6·13) is equal almost everywhere to

$$R^2\left[\Phi+\left(\frac{r}{R}-r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\Phi_{,1}\right]+\frac{R^3}{3}\left[\Phi_{,1}\frac{d}{dR}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)+2\Phi_{,2}\frac{1}{R}\left(r'-\frac{r}{R}\right)\right.$$
$$\left.+\left(\frac{r}{R}-r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\frac{d}{dR}\Phi_{,1}+\left(\frac{1}{R}\left(r'-\frac{r}{R}\right)-\frac{d}{dR}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\Phi_{,1}\right],$$

which in turn is equal everywhere to

$$R^2\left[\Phi+\left(r'-r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\Phi_{,1}\right]+\frac{R^3}{3}\left[2(\Phi_{,2}-\Phi_{,1})\frac{1}{R}\left(r'-\frac{r}{R}\right)\right.$$
$$\left.+\left(\frac{r}{R}-r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\frac{d}{dR}\Phi_{,1}\right], \quad (6·14)$$

where the arguments of Φ and its derivatives are $r_c'(g_c(r/R)), r/R, r/R$. To prove (6·13) it is therefore sufficient to show that the second expression in square brackets in (6·14) is equal to zero almost everywhere, i.e. that

$$\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right)-\frac{r}{R}\right)\frac{d}{dR}\Phi_{,1}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right),\frac{r}{R},\frac{r}{R}\right)=2\left[\Phi_{,2}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right),\frac{r}{R},\frac{r}{R}\right)\right.$$
$$\left.-\Phi_{,1}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right),\frac{r}{R},\frac{r}{R}\right)\right]\frac{1}{R}\left(r'-\frac{r}{R}\right), \quad (6·15)$$

a.e. $R\in D$.

Setting $W=r/R$ in (6·15), this is equivalent to showing that

$$\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right)-\frac{r}{R}\right)\left(r'-\frac{r}{R}\right)\left[\frac{d}{dW}\Phi_{,1}(r_c'(g_c(W),W,W)\right]\Big|_{W=\frac{r}{R}}$$
$$=2\left[\Phi_{,2}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right),\frac{r}{R},\frac{r}{R}\right)-\Phi_{,1}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right),\frac{r}{R},\frac{r}{R}\right)\right]\left(r'-\frac{r}{R}\right), \quad (6·16)$$

for a.e. $R\in D$.

But r_c is a solution of (3·12) and setting $v=r_c/R$ gives

$$R\frac{d}{dR}=(r_c'(g_c(v))-v)\frac{d}{dv}.$$

Hence

$$[r_c'(g_c(v))-v]\frac{d}{dv}\Phi_{,1}(r_c'(g_c(v)),v,v)=2[\Phi_{,2}(r_c'(g_c(v)),v,v)-\Phi_{,1}(r_c'(g_c(v)),v,v)]$$

for $v\in(\lambda_c,\infty)$.

Comparison with (6·16) then yields the result.

Case (ii)

In this case (6·12) follows immediately from the definition of P (6·10) on noting that

$$\frac{d}{dR}\left[H^*\left(R,\frac{r}{R},P(R,r)\right)\right]=\frac{1}{3}\frac{d}{dR}\left[R^3\Phi\left(\frac{r}{R},\frac{r}{R},\frac{r}{R}\right)\right]$$
$$=R^2\left[\Phi\left(\frac{r}{R},\frac{r}{R},\frac{r}{R}\right)+\left(r'-\frac{r}{R}\right)\Phi_{,1}\left(\frac{r}{R},\frac{r}{R},\frac{r}{R}\right)\right] \quad \text{for a.e.} \quad R\in D, \quad (6·17)$$

where in the last step we have used the symmetry of Φ. Equations (6·17) and (6·13) together with (6·10) prove the proposition.

PROPOSITION 6·7. *Let $r \in A_\lambda$ satisfy $\underline{\mathrm{Lim}}_{R\to 0}\, r(R)/R > 0$ and let the field $\mathscr{F}$ and corresponding slope function P be given by* (6·7) *and* (6·10) *respectively. Then*

$$\operatorname*{Lim}_{R\to 0} H^*\left(R, \frac{r}{R}, P(R,r)\right) = 0, \qquad (6\cdot 18)$$

where H^* is defined by (5·2).

Proof. We consider the two cases

(i) $\overline{\operatorname*{Lim}}_{R\to 0} \dfrac{r(R)}{R} < +\infty$

and

(ii) $\overline{\operatorname*{Lim}}_{R\to 0} \dfrac{r(R)}{R} = +\infty.$

Case (i)

In this case there exist constants $M_1, M_2 > 0$ such that

$$M_1 \leqslant \frac{r(R)}{R} \leqslant M_2 \quad \text{for} \quad R \text{ sufficiently small.}$$

On setting $v = r/R$ it follows from Proposition 5·6 and the smoothness of Φ and r_c that

$$|\Phi(r_c'(g_c(v)), v, v) + (v - r_c'(g_c(v)))\,\Phi_{,1}(r_c'(g_c(v)), v, v)| \leqslant \text{Const.}$$

for $v \in [\lambda_c, M_2]$.

Hence

$$\left|\Phi\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right), \frac{r}{R}, \frac{r}{R}\right) + \left(\frac{r}{R} - r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\Phi_{,1}\left(r_c'\left(g_c\left(\frac{r}{R}\right)\right), \frac{r}{R}, \frac{r}{R}\right)\right| \leqslant \text{Const.} \qquad (6\cdot 19)$$

for R sufficiently small and such that $r/R \in [\lambda_c, M_2]$. For values of R for which $r/R \in [M_1, \lambda_c)$ it follows from the smoothness of Φ that

$$\left|\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right)\right| \leqslant \text{Const.} \qquad (6\cdot 20)$$

Statement (6·18) is now a consequence of (6·19), (6·20), (6·10) and (5·2).

Case (ii)

Again by assumption there exists $M_1 > 0$ such that $M_1 \leqslant r(R)/R$ for R sufficiently small.

It follows from Corollary 5·7 that for each $\epsilon > 0$ there exists $\tilde{M}$ with the property that

$$|H^*(g_c(v), v, r_c'(g_c(v)))| < \epsilon \quad \text{for} \quad v \in (\tilde{M}, \infty);$$

thus

$$\left|H^*\left(R, \frac{r}{R}, r_c'\left(g_c\left(\frac{r}{R}\right)\right)\right)\right| < \epsilon \quad \text{for any } R \text{ for which } \quad \frac{r}{R} \in (\tilde{M}, \infty). \qquad (6\cdot 21)$$

Finally, applying the arguments of case (i) for values of R for which $r/R \in [M_1, \tilde{M}]$ and using (6·21), we obtain (6·18).

We next prove one of the main results of this paper.

THEOREM 6·8. *Let $r_c \in C^2((0,\infty))$ be a cavitating equilibrium solution and let the field $\mathscr{F}$ be defined by* (6·7). *Then for each $\lambda \in (0,\infty)$, if $y \in \mathscr{F}$ is the unique element satisfying $y(1) = \lambda$, then*

$$I(y) < I(r) \qquad (6\cdot 22)$$

for all $r \in A_\lambda$, $r \not\equiv y$ with $\underline{\mathrm{Lim}}_{R\to 0}\, r(R)/R > 0$.

Proof. Our proof proceeds in three steps.

Step 1

First notice that by Proposition 5·5 (6·22) holds trivially if $I(r) = +\infty$. It is a consequence of Remark 6·4 that

$$R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \geqslant R^2\left[\Phi\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right) + (r' - P(R,r))\Phi_{,1}\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right)\right] \quad \text{for} \quad R \in (0,1], \tag{6·23}$$

where P is defined by (6·10) and where strict inequality holds in (6·23) if $r' \neq P$.

Step 2

We claim that

$$\int_a^1 R^2\left[\Phi\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right) + (r' - P(R,r))\,\Phi_{,1}\left(P(R,r), \frac{r}{R}, \frac{r}{R}\right)\right]dR = H^*\left(R, \frac{r}{R}, P(R,r)\right)\Big|_{R=a}^{R=1} \quad \text{for any} \quad a \in (0,1]. \tag{6·24}$$

To show this, we first prove that whenever $r \in A_\lambda$ then

$$H^*\left(R, \frac{r}{R}, P(R,r)\right) \in W^{1,1}(a,1) \quad \text{for each} \quad a \in (0,1). \tag{6·25}$$

If we fix $a \in (0,1)$ and if $r \in A_\lambda$, $\lambda > 0$ then

$$\text{(i)}\ \frac{r(R)}{R} \in W^{1,1}(a,1), \quad \text{(ii)}\ \frac{r(R)}{R} \in \left[r(a), \frac{\lambda}{a}\right]. \tag{6·26}$$

To prove (6·25) it is therefore sufficient, by (6·26) and the definitions of H^* and P, to prove that the function $G\colon (0,1] \times \mathbb{R}^+ \to \mathbb{R}$ defined for $R \in [a,1]$ by

$$G(R,v) = \begin{cases} \dfrac{R^3}{3}(\Phi(r_c'(g_c(v)), v, v) + (v - r_c'(g_c(v)))\,\Phi_{,1}(r_c'(g_c(v)), v, v)) & \text{if} \quad v > \lambda_c \quad (6·27) \\[2mm] \dfrac{R^3}{3}\Phi(v,v,v) & \text{if} \quad v \leqslant \lambda_c \quad (6·28) \end{cases}$$

satisfies the condition that $G(R,\cdot)$ is a Lipschitz function on $[r(a), \lambda/a]$ uniformly for $R \in [a,1]$. To this end we calculate $\partial G/\partial v$ and show that it is bounded uniformly for $R \in [a,1]$ in the two cases

$$v \in \left[r(a), \frac{\lambda}{a}\right] \cap (\lambda_c, \infty) \quad \text{and} \quad v \in \left[r(a), \frac{\lambda}{a}\right] \cap (0, \lambda_c]. \tag{6·29}$$

By using (6·28) and the smoothness of Φ, it is clearly sufficient to consider the first case only; $\partial G/\partial v$ is then given by

$$\begin{aligned} \frac{\partial G(R,v)}{\partial v} &= \frac{R^3}{3}\{\Phi_{,1} r_c''(g_c(v))\,g_c'(v) + 2\Phi_{,2} + \Phi_{,1} - \Phi_{,1} r_c''(g_c(v))\,g_c'(v) \\ &\quad + (v - r_c'(g_c(v)))\,(\Phi_{,11} r_c''(g_c(v))\,g_c'(v) + 2\Phi_{,12})\} \\ &= \frac{R^3}{3}\{2\Phi_{,2} + \Phi_{,1} + (v - r_c'(g_c(v)))\,(\Phi_{,11} r_c''(g_c(v))\,g_c'(v) + 2\Phi_{,12})\}, \end{aligned} \tag{6·30}$$

where the arguments of the derivatives of Φ are $r_c'(g_c(v)), v, v$. Since r_c is a solution of (3·12), on setting $v = r_c/R$ we obtain

$$(r_c'(g_c(v))-v)\frac{d}{dv}\Phi_{,1}(r_c'(g_c(v)),v,v) = 2[\Phi_{,2}(r_c'(g_c(v)),v,v)-\Phi_{,1}(r_c'(g_c(v)),v,v)] \quad \text{for} \quad v\in(\lambda_c,\infty), \qquad (6\cdot31)$$

and hence

$$(v-r_c'(g_c(v)))(\Phi_{,11}r_c''(g_c(v))g_c'(v)+2\Phi_{,12}) = 2[\Phi_{,1}-\Phi_{,2}] \quad \text{for} \quad v\in(\lambda_c,\infty), \qquad (6\cdot32)$$

where the arguments of the derivatives of Φ are $r_c'(g_c(v)), v, v$.

Our claim on the boundedness of $\partial G/\partial v$ is then a consequence of (6·30), (6·32), Propositions 5·3, 5·6, the smoothness of Φ and the assumption that v satisfies the first of the two conditions of (6·29). This proves (6·25), and the claim (6·24) will follow by the fundamental theorem of calculus once we show that (6·12) holds for almost every $R\in(a,1)$. By Proposition 6·6, it is sufficient to show that (6·12) holds almost everywhere on A, where $A = \{R\in(a,1);\, r(R)/R = \lambda_c\}$, whenever mes $(A) > 0$. By (6·26) the derivative of r/R exists for almost every $R\in A$. We may assume without loss of generality that every point of A is an accumulation point of A (by disregarding a countable number of points of A). Thus for any $R\in A$ at which the derivative of $r(R)/R$ exists there exists a sequence $\{R_n\}\in A$ with $R_n\to R$ as $n\to\infty$ and the derivative is given by

$$\operatorname*{Lim}_{n\to\infty}\left(\frac{\dfrac{r(R_n)}{R_n}-\dfrac{r(R)}{R}}{R_n-R}\right) = 0.$$

But the derivative of r/R is given almost everywhere by

$$\frac{1}{R}\left(r'(R)-\frac{r(R)}{R}\right);$$

thus $r'(R) = r(R)/R = \lambda_c$ for almost every $R\in A$. Then an exactly analogous argument to that used in proving Proposition 6.6 (ii), together with (6·25), implies that (6·12) is satisfied for almost every $R\in A$. Together with Proposition 6·6, this proves the claim (6·24).

Step 3

Thus by (6·23), (6·24), Proposition 6·7 and the assumption that $r\not\equiv y$, it follows that

$$I(r) = \operatorname*{Lim}_{a\to 0}\int_a^1 R^2\Phi\left(r',\frac{r}{R},\frac{r}{R}\right)dR > H^*(1,\lambda,P(1,\lambda)) - \operatorname*{Lim}_{a\to 0}H^*\left(a,\frac{r(a)}{a},P(a,r(a))\right). \qquad (6\cdot33)$$

On using Proposition 6·7 the right hand side of (6·33) is equal to

$$H^*(1,\lambda,P(1,\lambda)) = H^*(1,\lambda,y'(1)) = I^*(y) = I(y), \qquad (6\cdot34)$$

where we have used the fact that $I^*(y) = I(y)$ as y is embedded in $\mathscr{F}$.

As an application of Theorem 6·8 we obtain the following.

THEOREM 6·9. *For each* $\lambda\in(0,\infty)$ *there exists at most one cavitating equilibrium solution* $r_c\in C^2((0,1])$ *satisfying* $r_c(1) = \lambda$.

Proof. We suppose for a contradiction that for some $\bar{\lambda} \in (0, \infty)$ there exist two distinct cavitating equilibrium solutions $r_c, r_{\tilde{c}} \in C^2((0, 1])$ with $r_c(1) = r_{\tilde{c}}(1) = \bar{\lambda}$. These would then give rise to two distinct fields of extremals $\mathscr{F}$ and $\tilde{\mathscr{F}}$. Application of Theorem 6·8 with each of these fields in turn would first yield $I(r_c) < I(r_{\tilde{c}})$ and then $I(r_{\tilde{c}}) < I(r_c)$, a contradiction.

Finally, we indicate how the requirement $\underline{\text{Lim}}_{R\to 0}\, r(R)/R > 0$ may be dropped in the statement of Theorem 6·8. To do this we first note that the conclusions of Theorem 6·8 hold for every $r \in A_\lambda$ with the property that

$$H^*\left(R_n, \frac{r(R_n)}{R_n}, P(R_n, r(R_n))\right) \to 0 \quad \text{as} \quad n \to \infty \tag{6·35}$$

for some sequence $R_n \to 0$ as $n \to \infty$. Again by the arguments of Theorem 6·8, (6·35) is satisfied provided

$$\underset{n\to\infty}{\text{Lim}} \frac{r(R_n)}{R_n} \geqslant \delta > 0 \tag{6·36}$$

for some sequence $R_n \to 0$ as $n \to \infty$ and some constant $\delta > 0$. If there does not exist a sequence satisfying either (6·35) or (6·36), then

$$\underset{R\to 0}{\text{Lim}} \frac{r(R)}{R} = 0 \tag{6·37}$$

and there exists a constant $k > 0$ such that

$$R^3\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) \geqslant k \quad \text{for} \quad R \in (0, 1], \tag{6·38}$$

where we have used (5·2), (6·10) and the continuity of Φ.

We now assume that Φ satisfies the further condition

(IV) $$\Phi(v_1, v_2, v_3) = \sum_{i=1}^{3} \psi(v_i) + \tilde{\Phi}(v_1, v_2, v_3), \tag{6·39}$$

where $$\psi, \tilde{\Phi} > 0 \quad \text{and} \quad \overline{\text{Lim}}_{v\to 0}\, \tilde{\Phi}(v, v, v) < +\infty. \tag{6·40}$$

(i.e. we assume that singular behaviour of Φ for v close to zero is contained purely in the ψ term).

Now suppose that $r \in A_\lambda$ satisfies $I(r) < +\infty$. It follows from (6·39) and (3·11) that

$$R^2\psi\left(\frac{r}{R}\right) \in L^1(0, 1). \tag{6·41}$$

Thus if conditions (6·35) and (6·36) do not hold for any sequence $R_n \to 0$ then by (6·38)–(6·40)

$$0 < \frac{K}{R} < R^2\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) \leqslant 3R^2\psi\left(\frac{r}{R}\right) + \text{const.},$$

contradicting (6·41). So for stored energy functions of the structure indicated the conclusions of Theorem 6·8 hold without the restriction $\underline{\text{Lim}}_{R\to 0}\, r(R)/R > 0$.

There are hypotheses other than (IV) that we may impose on the stored energy function to obtain the same result. However these are of a technical nature and do not shed further light on our interpretation.

Our results also show that under hypotheses (I)–(IV) on the stored energy function the following theorem holds.

THEOREM 6·10. *Suppose there exists a cavitating equilibrium solution* $r_c \in C^2((0,1])$ *with* $r_c(1) = \bar{\lambda} > 0$. *Then*

(i) r_c *is unique and extendable to* $r_c \in C^2((0,\infty))$ *as a solution of* (3·12),

(ii) $\lim_{R\to\infty} \frac{r_c(R)}{R} = \lambda_c$ *for some* $\lambda_c \in [1,\infty)$,

(iii) *if* $\lambda \leqslant \lambda_c$ *then* $r(R) \equiv \lambda R$ *is the unique global minimizer of* I *on* A_λ,

(iv) *if* $\mu > \lambda_c$ *then the global minimizer* r_μ *of* I *on* A_μ *exists, is unique and satisfies* $r_\mu(0) > 0$. *Moreover*

$$r_\mu(R) \equiv \delta r_c\left(\frac{R}{\delta}\right) \quad \textit{for} \quad R \in (0,1],$$

where δ *is the unique root of*

$$\delta r_c\left(\frac{1}{\delta}\right) = \mu.$$

Proof. The theorem is an easy consequence of the above arguments, Proposition 5·3, Theorem 6·8 and the definition of our field of extremals $\mathscr{F}$ (6·7). (The above theorem is proved using variational methods in Sivaloganathan [**10**] though under different hypotheses on the stored energy function).

7. *Concluding remarks*

Exactly analogous results to those given in this paper apply to the two-dimensional problem in which B is the unit disc in $\mathbb{R}^2$.

In a recent paper Horgan and Abeyaratne [7] demonstrate the existence of a cavitating equilibrium solution in the case of a Blatz and Ko material by implicitly integrating the radial equilibrium equation. The corresponding two-dimensional stored energy function is given by

$$\Phi(v_1, v_2) = \mu(v_1^{-2} + v_2^{-2} + 2v_1 v_2 - 4).$$

It is easily seen that Φ satisfies the two-dimensional versions of hypotheses (I)–(IV) and thus that the cavitating solution is the global minimizer of the energy amongst radial deformations (by Theorem 6·10).

In closing we remark that the natural way in which the conservation law (5·1) arises in the expression for the Hilbert integral I^* (see Remark 6·5 and (6·24)) indicates that the general n-dimensional version of this law, namely

$$\frac{\partial}{\partial X^\alpha}\left[X^\alpha W - \frac{\partial W}{\partial x^j_{,\alpha}}(X^\beta x^j_{,\beta} - x^j)\right] = nW,$$

may be of relevance in extending the field theory to higher dimensions (within the context of nonlinear elasticity). However, serious difficulties may arise in extending the Weierstrass theory, as indicated in Ball and Marsden [**2**].

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