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Implications of rank one convexity

by

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ABSTRACT. — In this paper we study some of the implications for stability of equilibria in nonlinear elastostatics of assuming that the stored energy function is rank one convex.

Key words : Rank one convexity, nonlinear elasticity, one parameter family of equilibria.

RÉSUMÉ. — Dans cet article nous étudions la stabilité des équilibres en élastostatique non linéaire sous l'hypothèse que la fonction d'énergie emmagasinée est convexe de rang un.

INTRODUCTION

Let the functional E be defined by

$$E(\underline{u}) = \int_\Omega L(\underline{x}, \underline{u}(\underline{x}), \nabla \underline{u}(\underline{x}))\, dx, \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ and $\underline{u} : \Omega \to \mathbb{R}^n$. Then any smooth weak minimiser $\underline{u}_0$ of E satisfies

$$\left.\begin{aligned} &\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathrm{L}}{\partial \mathrm{F}^i_\alpha}(\underline{x}, \underline{u}, \nabla \underline{u})\right) = \frac{\partial \mathrm{L}}{\partial u^i}(\underline{x}, \underline{u}, \nabla \underline{u}) \\ &\qquad i = 1, 2, \ldots, n, \qquad \forall\, \underline{x} \in \Omega \end{aligned}\right\} \tag{0.2}$$

and the Legendre-Hadamard condition

$$\left.\begin{aligned} &\frac{\partial^2 \mathrm{L}}{\partial \mathrm{F}^i_\alpha \partial \mathrm{F}^j_\beta}(\underline{x}, \underline{u}_0(\underline{x}), \nabla \underline{u}_0(\underline{x}))\, \lambda^i \mu^\alpha \lambda^j \mu^\beta \geqq 0, \\ &\qquad \forall\, \underline{x} \in \Omega, \qquad \forall\, \underline{\lambda}, \underline{\mu} \in \mathbb{R}^n. \end{aligned}\right\} \tag{0.3}$$

(for a proof *see* Morrey [11]). The integrand L is said to be rank one convex if

$$\frac{\partial^2 \mathrm{L}}{\partial \mathrm{F}^i_\alpha \partial \mathrm{F}^j_\beta}(\underline{x}, \underline{u}, \mathrm{P})\, \lambda^i \mu^\alpha \lambda^j \mu^\beta \geqq 0, \qquad \forall\, \underline{\lambda}, \underline{\mu} \in \mathbb{R}^n, \tag{0.4}$$

and for all $\underline{x}$, $\underline{u}$, P in the domain of definition of L.

In this paper we examine some of the implications for stability of smooth solutions of (0.2) of assuming that L is rank one convex.

Suppose $\underline{u}_0$ is a smooth solution of (0.2). Roughly speaking, we say that $\underline{u}_0$ imbedded in a one parameter family of equilibria $\{\underline{u}(\alpha, .)\}$ parametrised by α in some interval I around zero if

(i) for each $\alpha \in \mathrm{I}$, $\underline{u}(\alpha, \underline{x})$ is a solution of (0.2) and

(ii) $\underline{u}_0(\underline{x}) \equiv \underline{u}(0, \underline{x})$.

For the precise meaning *see* Definition 1.5.

In the context of nonlinear elastostatics $\mathrm{L}(\underline{x}, \underline{u}, \nabla \underline{u}) = \mathrm{W}(\nabla \underline{u})$, where W is the stored energy function. In section 2, part II, we generate families of equilibria from any smooth equilibrium solution by exploiting the invariances of the underlying Euler-Lagrange system.

Given any integrand $\mathrm{L}(\underline{x}, \underline{u}, \nabla \underline{u})$ and a corresponding one parameter family of equilibria $\{\underline{u}(\alpha, .)\}$ we can generate further maps $\underline{u}_\theta : \Omega \to \mathbb{R}^n$ by taking any function $\theta : \bar{\Omega} \to \mathrm{I}$ and defining

$$\underline{u}_\theta(\underline{x}) = \underline{u}(\theta(\underline{x}), \underline{x}),$$

i. e. we allow the parameter to be a function of position. If $\theta|_{\partial\Omega} = 0$ then $\underline{u}_\theta$ and $\underline{u}_0$ satisfy the same boundary condition. Our main result in section 1 is that $\underline{u}_0$ is the global minimiser in this class i. e.

$$\mathrm{E}(\underline{u}_0) \leqq \mathrm{E}(\underline{u}_\theta)$$

for all functions $\theta(x)$ such that $\theta|_{\partial\Omega}=0$. Our arguments use what appears to be a natural generalisation of the one dimensional field theory of the Calculus of Variations (*see* e. g. Cesari [6], Morrey [11]).

Basically, the problem is to show that $\underline{u}_0$ minimises E on some admissible set $\mathscr{A}$. A necessary and sufficient condition for this to hold is that there exist a functional $\mathscr{F}:\mathscr{A}\to\mathbb{R}$ with the properties

(i) $\mathrm{E}(\underline{u})\geqq\mathscr{F}(\underline{u}),\ \forall\, u\in\mathscr{A}$;
(ii) $\mathscr{F}(\underline{u})\geqq\mathscr{F}(\underline{u}_0),\ \forall\,\underline{u}\in\mathscr{A}$ and
(iii) $\mathscr{F}(\underline{u}_0)=\mathrm{E}(\underline{u}_0)$.

It is clear that the existence of $\mathscr{F}$ satisfying properties (i)-(iii) implies that $\underline{u}_0$ is globally minimising in $\mathscr{A}$. Conversely, if $\underline{u}_0$ is the global minimiser in $\mathscr{A}$ then $\mathscr{F}=\mathrm{E}$ satisfies (i)-(iii).

In the field theory of the Calculus of Variations the approach is to ensure that (i) holds by an appropriate convexity assumption on L and that (ii) and (iii) are satisfied by choosing $\mathscr{F}$ to be a null Lagrangian i. e.

$$\mathscr{F}(\underline{u})=\int_\Omega \mathrm{N}(\underline{x},\underline{u},\nabla\underline{u})\,dx,$$

where N is chosen so that $\mathscr{F}(\underline{u})=\mathscr{F}(\underline{u}_0)=\mathrm{E}(\underline{u}_0),\ \forall\,\underline{u}\in\mathscr{A}$. A central problem is to characterise the set of all N with this property. This question has been studied by a number of authors including Ball, Currie and Olver [4], Edelen [7], Ericksen [8], Landers [10], Olver [12], Olver and Sivaloganathan [13], Rund [14]. In our arguments N is given by the right side of (1.11), this generalises the situation for one dimensional problems where $u:\mathbb{R}\to\mathbb{R}$ and $\mathscr{F}$ is the Hilbert invariant integral of the classical field theory (*see* Cesari [6], p. 70). Examples of the application of the one dimensional field theory to elastostatics are contained in Ball and Marsden [5] and Sivaloganathan [15]. There have been various attempts to extend the field theory to multiple integral problems (*see* e. g. Weyl [17], Morrey [11], Rund [14] and the references therein) however, in contrast to the one dimensional theory, the gap between the known necessary and sufficient conditions is still large.

In section 2 we apply the results of section 1 to nonlinear elasticity. In Proposition 2.4 we obtain a partial answer to the question of whether rank one convexity implies quasiconvexity by showing that the homogeneous deformations are globally minimising in various classes of deformations (for further details regarding this open problem *see* Ball [3]). Finally, in section 2, part III, we apply our techniques to prove a uniqueness result originally due to Knops and Stuart [9]. The result, which is contained in

Theorem 2.5, is that if the stored energy functions is rank one convex and strictly quasiconvex then the only smooth solution of the equilibrium equations satisfying affine boundary conditions is the affine (homogeneous) map itself.

The results of this paper are, we hope, a first step towards a field theory for three dimensional problems of nonlinear elasticity under realistic convexity assumptions on the stored energy function.

1. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^n$, $n \geqq 1$ be a bounded domain with strongly Lipschitz boundary and let $L : V \times \mathbb{R}^n \times M^{n \times n} \to \mathbb{R}$ be C^2 on its domain of definition, where $M^{n \times n}$ denotes the space of $n \times n$ matrices and $V \subseteq \mathbb{R}^n$ is some open set with $\Omega \subset V$. Later, in considering applications to elasticity, we allow L to have singularities.

Given $\underline{u} \in W^{1, \infty}(\Omega; \mathbb{R}^n)$ define the corresponding energy $E(\underline{u})$ *by*

$$E(\underline{u}) = \int_\Omega L(\underline{x}, \underline{u}(\underline{x}), \nabla \underline{u}(\underline{x}))\, dx, \tag{1.1}$$

where $(\nabla \underline{u}(\underline{x}))^i_j = \dfrac{\partial u^i}{\partial x^j}$, $i, j = 1, 2, \ldots, n$.

For the purposes of this paper we consider the displacement boundary value problem in which the values of $\underline{u}$ are specified on the boundary of Ω by

$$\underline{u}\,|_{\partial\Omega} = \underline{u}_0\,|_{\partial\Omega} \tag{1.2}$$

for some given function $\underline{u}_0$.

Null Lagrangians

Let $N : U \times I \times \mathbb{R}^n \to \mathbb{R}$ be C^1 on its domain of definition, where $I \subseteq \mathbb{R}$ is some open interval and $U \subset\subset \mathbb{R}^n$ is open and contains Ω.

Definition 1.1. — We say that $N(\underline{x}, \theta, \underline{P})$ is a null lagrangian if and only if the functional

$$\mathscr{L}(\theta) \overset{\text{def}}{=} \int_\Omega N(\underline{x}, \theta(\underline{x}), \nabla \theta(\underline{x}))\, dx$$

satisfies

$$\mathscr{L}(\theta_1)=\mathscr{L}(\theta_2)$$

for all θ_i such that:

(i) $\theta_i \in C^1(\bar{\Omega}; \mathbb{R})$, $i=1, 2$;

(ii) $\theta_1-\theta_2 \in C_0^\infty(\Omega; \mathbb{R})$ and

(iii) $\theta_i(\underset{\sim}{x}) \in I$, $\forall \underset{\sim}{x} \in \bar{\Omega}$, $i=1, 2$.

Remark 1.2. — It follows by approximation that $\mathscr{L}$ is constant on all $\theta \in W^{1,\infty}(\Omega)$ that satisfy (iii) and that agree on $\partial\Omega$.

The next Lemma gives a characterisation of N.

Lemma 1.3. — *$N(\underset{\sim}{x}, \theta, \nabla\theta)$ is a null lagrangian if and only if*

$$\int_\Omega \left\{\left[\frac{\partial N}{\partial \theta_{,\beta}}(\underset{\sim}{x}, \theta(\underset{\sim}{x}), \nabla\theta(\underset{\sim}{x}))\right]\eta_{,\beta}(\underset{\sim}{x}) + \frac{\partial N}{\partial \theta}(\underset{\sim}{x}, \theta(\underset{\sim}{x}), \nabla\theta(\underset{\sim}{x}))\,\eta(\underset{\sim}{x})\right\} dx=0, \quad (1.4)$$

$\forall \theta \in C^1(\bar{\Omega})$ *and* $\forall \eta \in C_0^\infty(\Omega)$ *such that :*

(i) $\theta(\underset{\sim}{x}) \in I$, $\forall \underset{\sim}{x} \in \bar{\Omega}$

and (1.5)

(ii) $\theta(\underset{\sim}{x})+\eta(\underset{\sim}{x}) \in I$, $\forall \underset{\sim}{x} \in \bar{\Omega}$.

Proof. — If N is a null lagrangian then any $\theta \in C^1(\bar{\Omega})$ is a global minimiser of $\mathscr{L}$ against all admissible variations $\eta \in C_0^\infty(\Omega)$ and thus (1.4) holds. Conversely given $\theta \in C^1(\bar{\Omega})$ and $\eta \in C_0^\infty(\Omega)$ satisfying (i) and (ii) then it follows by the dominated convergence theorem that

$$\begin{aligned}\mathscr{L}(\theta+\eta) &= \mathscr{L}(\theta)+\int_0^1 \frac{d}{dt}\mathscr{L}(\theta+t\eta)\,dt \\ &= \mathscr{L}(\theta)+\int_0^1\int_\Omega \left[\frac{\partial N}{\partial \theta_{,\alpha}}(\underset{\sim}{x}, \theta+t\eta, \nabla\theta+t\nabla\eta)\right]\eta_{,\alpha} \qquad (1.6)\\ &+\left[\frac{\partial N}{\partial \theta}(\underset{\sim}{x}, \theta+t\eta, \nabla\theta+t\nabla\eta)\right]\eta\,dx\,dt \\ &= \mathscr{L}(\theta)\end{aligned}$$

this follows on replacing θ by $\theta+t\eta$ in (1.4) and observing the convexity of the domain of definition of N for fixed $\underset{\sim}{x}$.

Remark 1.4. — If $I = \mathbb{R}$ then Lemma 1.3 is simply the statement that N is a null lagrangian if and only if the Euler-Lagrange equation for $\mathscr{L}$ is identically satisfied in the sense of distributions for all $\theta \in C^1(\bar{\Omega})$.

Families of equilibria

DEFINITION 1.5. — We say that the one parameter family of functions $\{\underline{u}(\alpha, \underline{x})\}$ parametrised by $\alpha \in I$ some open interval around zero is a one parameter family of equilibria if

(i) for each $\alpha \in I$, $\underline{u}(\alpha, .) \in C^2(\bar{\Omega}; \mathbb{R}^n)$ and is a solution of (0.2);

(ii) all the partial derivatives up to second order i.e.

$$\frac{\partial u^i}{\partial \alpha}, \quad \frac{\partial u^i}{\partial x^j}, \quad \frac{\partial^2 u^i}{\partial \alpha^2}, \quad \frac{\partial^2 u^i}{\partial \alpha\, \partial x^j}, \quad \frac{\partial^2 u^i}{\partial x^j\, \partial x^k}, \qquad i, j, k = 1, 2, \ldots, n$$

exist and are continuous on $I \times \bar{\Omega}$.

We next define a family of maps $\mathscr{A}$ that is obtained from the one parameter family of equilibria by replacing the parameter with a function $\theta(\underline{x})$. More specifically

$$\mathscr{A} = \{\underline{u}_\theta(\underline{x}) : \theta \in C^1(\bar{\Omega}; I) \text{ and } \theta|_{\partial\Omega} = 0\}, \tag{1.7}$$

where

$$\underline{u}_\theta(\underline{x}) \stackrel{\text{def}}{=} \underline{u}(\theta(\underline{x}), \underline{x}) \qquad \text{for} \quad \underline{x} \in \bar{\Omega}. \tag{1.8}$$

i.e. $\mathscr{A}$ consists of maps $\underline{u}_\theta$ satisfying the same boundary condition as $\underline{u}_0$, where

$$\underline{u}_0(\underline{x}) \stackrel{\text{def}}{=} \underline{u}(0, \underline{x}). \tag{1.9}$$

The next Theorem is the main result of this section and shows that $\underline{u}_0$ is the global minimiser of E on $\mathscr{A}$.

THEOREM 1.6. — *Let $\{\underline{u}(\alpha, \underline{x})\}$ be a one parameter family of equilibria and let L be rank one convex, then*

$$E(\underline{u}_\theta) \geqq E(\underline{u}_0), \qquad \forall\, \underline{u}_\theta \in \mathscr{A},$$

where $\mathscr{A}$ is defined by (1.7) *and $\underline{u}_0$ is given by* (1.9).

Proof. — A simple calculation gives

$$\nabla \underline{u}_\theta(\underline{x}) = \nabla \underline{u}(\theta(\underline{x}), \underline{x}) = \left(\frac{\partial u^i}{\partial x^j}(\alpha, \underline{x})\Big|_{\alpha = \theta(\underline{x})} + \frac{\partial u^i}{\partial \alpha}(\theta(\underline{x}), \underline{x}) \frac{\partial \theta}{\partial x^j}(\underline{x}) \right)$$

$$= \frac{\partial \underline{u}}{\partial \underline{x}}(\alpha, \underline{x})\Big|_{\alpha = \theta(\underline{x})} + \frac{\partial \underline{u}}{\partial \alpha}(\theta(\underline{x}), \underline{x}) \otimes \frac{\partial \theta}{\partial \underline{x}}.$$

It then follows from the assumption that L is rank one convex (0.4) that

$$\mathrm{L}(\underline{x}, \underline{u}_0(\underline{x}), \nabla \underline{u}_0(\underline{x})) \geqq \mathrm{L}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x})\right)$$

$$+ \frac{\partial \mathrm{L}}{\partial \mathrm{F}^i_\beta}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x})\right) \frac{\partial u^i}{\partial \alpha}(\theta(\underline{x}), \underline{x}) \frac{\partial \theta(\underline{x})}{\partial x^\beta}, \quad (1.11)$$

where

$$\frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x}) = \left(\frac{\partial u^i}{\partial x^j}(\alpha, \underline{x})\right)\bigg|_{\alpha = \theta(\underline{x})}$$

Notice that the right hand side of inequality (1.11) is a function of $\underline{x}$, $\theta(\underline{x})$, $\nabla\theta(\underline{x})$. We next show that it is in fact a null lagrangian (*see* Definition 1.1). By Lemma 1.3 it is sufficient to prove that the expression

$$\int_\Omega \left\{\left[\frac{\partial \mathrm{L}}{\partial \mathrm{F}^i_\beta}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x})\right) \frac{\partial u^i}{\partial \alpha}(\theta(\underline{x}), \underline{x})\right] \eta_{,\beta}(\underline{x})\right.$$

$$+ \left[\frac{\partial \mathrm{L}}{\partial u^i}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x})\right) \frac{\partial u^i}{\partial \alpha}(\theta(\underline{x}), \underline{x})\right.$$

$$+ \frac{\partial \mathrm{L}}{\partial \mathrm{F}^i_\beta}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x})\right) \frac{\partial^2 u^i}{\partial x^\beta\, \partial \alpha}(\theta(\underline{x}), \underline{x})$$

$$+ \frac{\partial^2 \mathrm{L}}{\partial \mathrm{F}^i_\beta\, \partial u^j}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}, \underline{x})\right) \frac{\partial u^j}{\partial \alpha}(\theta(\underline{x}), \underline{x}) \frac{\partial u^i}{\partial \alpha}(\theta(\underline{x}), \underline{x}) \frac{\partial \theta}{\partial x^\beta}(\underline{x})$$

$$+ \frac{\partial^2 \mathrm{L}}{\partial \mathrm{F}^i_\beta\, \partial \mathrm{F}^j_\gamma}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x})\right) \frac{\partial^2 u^j}{\partial x^\gamma\, \partial \alpha}(\theta(\underline{x}), \underline{x}) \frac{\partial u^i}{\partial \alpha}(\theta(\underline{x}), \underline{x}) \frac{\partial \theta}{\partial x^\beta}(\underline{x})$$

$$\left.+ \frac{\partial \mathrm{L}}{\partial \mathrm{F}^i_\beta}\left(\underline{x}, \underline{u}_0(\underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\theta(\underline{x}), \underline{x})\right) \frac{\partial^2 u^i}{\partial \alpha^2}(\theta(\underline{x}), \underline{x}) \frac{\partial \theta}{\partial x^\beta}(\underline{x})\right] \eta(\underline{x})\right\} dx \quad (1.13)$$

is equal to zero. We integrate the first term in square brackets in (1.13) by parts which yields

$$\int_\Omega \Bigg\{ \Bigg[-\left(\frac{\partial^2 u^i}{\partial\alpha^2}(\theta(\underline{x}), \underline{x})\frac{\partial\theta}{\partial x^\beta} + \frac{\partial^2 u^i}{\partial\alpha\,\partial x^\beta}(\theta(\underline{x}), \underline{x})\right)\frac{\partial L}{\partial F^i_\beta}$$

$$-\frac{\partial u^i}{\partial\alpha}(\theta(\underline{x}), \underline{x})\left(\frac{\partial^2 L}{\partial F^i_\beta\,\partial x^\beta} + \frac{\partial^2 L}{\partial F^i_\beta\,\partial u^j}\left[\frac{\partial u^j}{\partial x^\beta}(\theta(\underline{x}), \underline{x}) + \frac{\partial u^j}{\partial\alpha}(\theta(\underline{x}), \underline{x})\frac{\partial\theta}{\partial x^\beta}(\underline{x})\right]\right.$$

$$\left. + \frac{\partial^2 L}{\partial F^i_\beta\,\partial F^j_\gamma}\left[\frac{\partial^2 u^j}{\partial x^\gamma\,\partial x^\beta}(\theta(\underline{x}), \underline{x}) + \frac{\partial^2 u^j}{\partial x^\gamma\,\partial\alpha}(\theta(\underline{x}), \underline{x})\frac{\partial\theta}{\partial x^\beta}(\underline{x})\right]\right) + \frac{\partial L}{\partial u^i}\frac{\partial u^i}{\partial\alpha}(\theta(\underline{x}), \underline{x})$$

$$+\frac{\partial L}{\partial F^i_\beta}\frac{\partial^2 u^i}{\partial x^\beta\,\partial\alpha}(\theta(\underline{x}), \underline{x}) + \frac{\partial^2 L}{\partial F^i_\beta\,\partial u^j}\frac{\partial u^j}{\partial\alpha}(\theta(\underline{x}), \underline{x})\frac{\partial u^i}{\partial\alpha}(\theta(\underline{x}), \underline{x})\frac{\partial\theta}{\partial x^\beta}(\underline{x})$$

$$+\frac{\partial^2 L}{\partial F^i_\beta\,\partial F^j_\gamma}\frac{\partial^2 u^j}{\partial x^\gamma\,\partial\alpha}(\theta(\underline{x}), \underline{x})\frac{\partial u^i}{\partial\alpha}(\theta(\underline{x}), \underline{x})\frac{\partial\theta}{\partial x^\beta}(\underline{x})$$

$$+\frac{\partial L}{\partial F^i_\beta}\frac{\partial^2 u^i}{\partial\alpha^2}(\theta(\underline{x}), \underline{x})\frac{\partial\theta}{\partial x^\beta}(\underline{x})\Bigg]\eta(\underline{x})\Bigg\}\,dx. \quad (1.14)$$

where we have suppressed the arguments of the derivatives of L. Simplifying (1.14) yields

$$\int_\Omega \Bigg\{ -\Bigg[\frac{\partial u^i}{\partial\alpha}(\theta(\underline{x}),\underline{x})\left(\frac{\partial^2 L}{\partial F^i_\beta\,\partial F^j_\gamma}\frac{\partial^2 u^j}{\partial x^\gamma\,\partial x^\beta}(\theta(\underline{x}), \underline{x})\right.$$

$$\left. + \frac{\partial^2 L}{\partial F^i_\beta\,\partial u^j}\frac{\partial u^j}{\partial x^\beta}(\theta(\underline{x}), \underline{x}) + \frac{\partial^2 L}{\partial F^i_\beta\,\partial x^\beta} - \frac{\partial L}{\partial u^i}\right)\Bigg]\eta(\underline{x})\Bigg\}\,dx$$

$$= \int_\Omega \Bigg\{ -\left(\frac{\partial u^i}{\partial\alpha}(\theta(\underline{x}), \underline{x})\left[\frac{\partial}{\partial x^\beta}\left(\frac{\partial L}{\partial F^i_\beta}(\underline{x}, \underline{u}(\alpha, \underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(\alpha, \underline{x})\right)\Bigg|_{\alpha=\theta(\underline{x})}\right.\right.$$

$$\left.\left. -\frac{\partial L}{\partial u^i}\left(\underline{x}, \underline{u}(\theta(\underline{x}), \underline{x}), \frac{\partial \underline{u}}{\partial x}(\theta(\underline{x}), \underline{x})\right)\right]\right)\eta(\underline{x})\Bigg\}\,dx, \quad (1.15)$$

which is equal to zero by our definition of a one parameter family of equilibria [*see* Definition 1.5 (i)].

Hence by Lemma 1.3 the integral of the right-hand side of (1.11) is constant for all maps in $\mathscr{A}$ and is equal in particular to its value for the map corresponding to $\theta \equiv 0$, $\underline{u}(0, \underline{x})$, i. e.

$$E(\underline{u}_0) = \int_\Omega L(\underline{x}, \underline{u}_0, \nabla \underline{u}_0)\, dx$$

$$\geqq \int_\Omega L\left(\underline{x}, \underline{u}(0, \underline{x}), \frac{\partial \underline{u}}{\partial \underline{x}}(0, \underline{x})\right) dx = \int_\Omega L(\underline{x}, \underline{u}_0, \nabla \underline{u}_0)\, dx = E(\underline{u}_0),$$

proving the result.

Remark 1.7. — The integral of the right-hand side of (1.11) appears to be the natural generalisation of the Hilbert invariant integral of the one dimensional field theory of the calculus of variations (*see* Cesari [6]).

The Theorem has a natural generalisation to the case when

$$\underline{u}: \quad \mathbb{R}^n \to \mathbb{R}^m, \qquad n \neq m.$$

2. APPLICATIONS TO ELASTOSTATICS

1. The stored Energy Function

In this section we describe some of the implications of the results of section 1 for the stability of equilibria, under zero body force, of a homogeneous isotropic elastic body which in its reference state occupies some bounded domain $\Omega \subset \mathbb{R}^n$. In the notation of section 1 this corresponds to the case

$$L(\underline{x}, \underline{u}, \nabla \underline{u}) = W(\nabla \underline{u}), \tag{2.1}$$

where $W : M_+^{n \times n} \to \mathbb{R}^+$ is the stored energy function of the material and

$$M_+^{n \times n} = \{ F \in M^{n \times n} : \det F > 0 \}. \tag{2.2}$$

We will assume for the purposes of this section that $W \in C^2\left(M_+^{n \times n}, \mathbb{R}^+\right)$.

Any deformation of the body $\underline{u} : \Omega \to \mathbb{R}^n$ satisfying the local invertibility condition

$$\det(\nabla \underline{u}(\underline{x})) > 0 \qquad \text{for} \quad \text{a. e. } \underline{x} \in \Omega \tag{2.3}$$

has an associated energy $E(\underline{u})$ given by

$$E(\underline{u})=\int_{\Omega} W(\nabla \underline{u}(\underline{x}))\,dx. \tag{2.4}$$

In order that all maps $\underline{u}$ with finite energy should satisfy (2.3) we require that

$$W(F)\to\infty \quad \text{as} \quad \det F\to 0$$

and extend the domain of W to all of $M^{n\times n}$ by setting

$$W(F)=\infty \qquad \text{if} \quad \det F\leqq 0. \tag{2.5}$$

The equilibrium equations, under zero body force, are the Euler-Lagrange equations for (2.4)

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial W}{\partial F^{i}_{\alpha}}(\nabla \underline{u})\right)=0, \qquad i=1,\,2,\,\ldots,\,n. \tag{2.6}$$

Remark 2.1. — The *homogeneous deformations*, which are the affine maps given by

$$\underline{u}(\underline{x})\equiv F\,\underline{x}+\underline{c} \quad \text{some } F\in M^{n\times n}_{+}, \qquad \underline{c}\in\mathbb{R}^{n}, \tag{2.7}$$

are always solutions of (2.6).

II. Families of equilibria

In the context of nonlinear elasticity we can generate one parameter families of equilibria (*see* Definition 1.5) by taking any smooth solution $\underline{u}_0\in C^2(\bar{\Omega})$ of (2.6) together with any differentiable group that leaves the Eurler-Lagrange system (2.6) invariant. Thus, for example, the invariance of the energy E under rigid body motions implies that

$$W(QF)=W(F), \qquad \forall\,F\in M^{n\times n}_{+}, \quad Q\in SO(n). \tag{2.8}$$

and hence that

$$\underline{u}(\alpha,\,\underline{x})=Q(\alpha)\,\underline{u}_0(\underline{x})+\underline{d}(\alpha) \tag{2.9}$$

is such a family whenever $Q: I\to SO(n)$ and $\underline{d}: I\to\mathbb{R}^n$ are any twice continuously differentiable functions.

Similarly, the assumption of isotropy (*see* Truesdell and Noll [16]) implies that

$$W(FQ)=W(F), \qquad \forall\,F\in M^{n\times n}_{+}, \quad Q\in SO(n) \tag{2.10}$$

and hence that

$$\underline{u}(\alpha, \underline{x}) = \underline{u}_0 (Q(\alpha)\, \underline{x} + \underline{d}(\alpha)) \qquad (2.11)$$

is also a one parameter family of equilibria. However, in this case, it is clear that in general $\underline{u}(\alpha, .)$ will not be defined on all of $\bar{\Omega}$ so that Lemma 1.3 and consequently Theorem 1.6 do not apply directly. [We would certainly need to further restrict the class $\mathscr{A}$, given by (1.7), by requiring that the functions $\theta : \bar{\Omega} \to I$ satisfy $Q(\theta(\underline{x}))\, \underline{x} + \underline{d}(\theta(\underline{x})) \in \bar{\Omega}$, $\forall\, \underline{x} \in \bar{\Omega}$. Other difficulties would arise in ensuring that a version of Lemma 1.3 holds as this relies on the convexity of the domain of definition of the putative null Lagrangian.] Notice that in proving a version of Theorem 1.6 the requirement that $\underline{u}_0 \in \mathscr{A}$ should satisfy (2.3) causes no difficulties as (2.5) guarantees that $E(\underline{u}_0) = \infty$ whenever (2.3) fails on a set of non zero measure.

Another family of equilibria, which we will use later in this section, is generated by the invariance of (2.6) under the scaling

$$(\underline{x}, \underline{u}) \to (\delta\, \underline{x}, \delta \underline{u}), \qquad \delta > 0,$$

$$\underline{u}(\alpha, \underline{x}) = (1-\alpha)\, \underline{u}_0 \left(\frac{\underline{x}}{1-\alpha} \right) \qquad \text{where} \quad \alpha < 1, \qquad (2.12)$$

is a solution of (2.6) whenever $\underline{u}_0$ is. [The parameter $\delta = 1 - \alpha$ is chosen so that $\underline{u}(0, \underline{x}) \equiv \underline{u}_0(\underline{x})$.] Again $\underline{u}(\alpha, .)$ will not in general be defined on all of $\bar{\Omega}$. Notice also that the family defined by (2.12) is contained in that described by (2.9) in the case when $\underline{u}_0$ is an affine map (*see* Remark 2.1).

Finally, using Remark 2.1, we can generate another family of equilibria which is different in character from (2.9), (2.11) and (2.12):

$$\underline{u}(\alpha, \underline{x}) = K(\alpha)\, \underline{x} + \underline{d}(\alpha) \qquad (2.13)$$

is a family of equilibria defined on $\mathbb{R}^n$ for any choice of smooth functions $K : I \to M_+^{n \times n}$, $\underline{d} : I \to \mathbb{R}^n$.

Remark 2.2. — Any combination of the families (2.9), (2.11) and (2.12) will also generate a family of equilibria.

DEFINITION 2.3. — We say that W is quasiconvex at $F \in M_+^{n \times n}$ if

$$\int_D W(F + \nabla \underline{\varphi}(\underline{x}))\, dx \geqq \int_D W(F)\, dx = \text{mes}(D)\, W(F) \qquad (2.14)$$

for all bounded open sets $D \subset \mathbb{R}^n$, $\forall\, \underline{\varphi} \in W_0^{1,\infty}(D)$, and is strictly quasiconvex at F if (2.14) holds with strict inequality whenever $\underline{\varphi} \not\equiv 0$. We say that W is quasiconvex if (2.14) holds $\forall\, F \in M_+^{n \times n}$.

It is well known that if W is quasiconvex then it satisfies (0.4) (*see* e.g. Morrey [11], Ball [2]). The converse question, first posed by Morrey, is still open. A partial answer is furnished by the next proposition.

PROPOSITION 2.4. — *Let the family of equilibria* $\underline{u}(\alpha, .)$, $\alpha \in \mathrm{I}$ *(some open interval around zero) be defined by* (2.13), *where* $\mathrm{K}: \mathrm{I} \to \mathrm{M}_+^{n \times n}$ *and* $\underline{d}: \mathrm{I} \to \mathbb{R}^n$ are twice continuously differentiable functions satisfying $\mathrm{K}(0) = \mathrm{F}$, $\underline{d}(0) = \underline{c}$ *for some* $\mathrm{F} \in \mathrm{M}_+^{n \times n}$, $\underline{c} \in \mathbb{R}^n$.

Then $\mathrm{E}(\underline{u}_0) \geqq \mathrm{E}(\underline{u}_0)$, $\forall \underline{u}_0 \in \mathscr{A}$, *where* $\mathscr{A}$ *is defined by* (1.7) *and* $\underline{u}_0(\underline{x}) \equiv \mathrm{F}\,\underline{x} + \underline{c}$.

Proof. — This follows by a straightforward application of Theorem 1.6.

III. Uniqueness of equilibria

In this section we show how the uniqueness result of Knops and Stuart [9] may be derived by the methods of section 1. Their result is the following.

THEOREM 2.5. — *Let* Ω *be star-shaped and let* $\mathrm{W} \in \mathrm{C}^2(\mathrm{M}_+^{n \times n}, \mathbb{R})$ *be rank one convex and strictly quasiconvex at* $\mathrm{F} \in \mathrm{M}_+^{n \times n}$. *If* $\underline{u}_0 \in \mathrm{C}^2(\bar{\Omega})$ *is a solution of* (2.6) *satisfying*

$$\underline{u}_0(\underline{x}) = \mathrm{F}\,\underline{x} + \underline{c}, \qquad \forall\, \underline{x} \in \partial\Omega, \quad \textit{for some } \underline{c} \in \mathbb{R}^n, \tag{2.15}$$

then

$$\underline{u}_0(\underline{x}) \equiv \mathrm{F}\,\underline{x} + \underline{c}, \qquad \forall\, \underline{x} \in \bar{\Omega}.$$

The proof of this Theorem is deferred until the end of this section. The idea of it and the main difficulties are most easily demonstrated by considering the case $\Omega = \mathrm{B}$ the unit ball in $\mathbb{R}^n$ and setting $\underline{c} = \underline{0}$. Assume the existence of an equilibrium $\underline{u}_0(\underline{x}) \not\equiv \mathrm{F}\,\underline{x}$ and to try to apply a version of Theorem 1.6 with the family $\{\underline{u}(\alpha, \underline{x})\}$ defined by (2.12) to conclude that

$$\mathrm{E}(\underline{u}_0) \geqq \mathrm{E}(\underline{u}_0), \qquad \forall\, \theta \ \text{ s.t. } \ \theta\big|_{\partial \mathrm{B}} = 0. \tag{2.16}$$

If in particular we could choose $\theta(\underline{x}) \equiv \bar{\theta}(\underline{x}) = 1 - |\underline{x}|$ $\left(\text{i.e. } \dfrac{\underline{x}}{1 - \theta(\underline{x})} \text{ retracts}\right.$ each point $\underline{x}$ radially onto the boundary $\left.\partial \mathrm{B}\right)$ then by (2.16), (2.15), (1.8) and (2.12) it follows that

$$\underline{u}_{\bar{0}}(\underline{x}) = |\underline{x}|\,\underline{u}_0\left(\frac{\underline{x}}{|\underline{x}|}\right) \equiv \mathrm{F}\,\underline{x}, \qquad \forall\,\underline{x}\in\bar{\Omega}\setminus\{\underline{0}\},$$

and that

$$\mathrm{E}(\underline{u}^h) \geqq \mathrm{E}(\underline{u}_0) \quad \text{where } \underline{u}^h(\underline{x}) \stackrel{\text{def}}{=} \mathrm{F}\,\underline{x}, \quad \forall\,\underline{x}\in\bar{\Omega}.$$

(Knops and Stuart [9] conclude this by the use of a divergence identity for equilibrium solutions). But if W is strictly quasiconvex then $\mathrm{E}(\underline{u}_0) > \mathrm{E}(\underline{u}^h)$, a contradiction. However, in this case, even though $\bar{\theta}\in \mathrm{W}^{1,\infty}(\mathrm{B})$, Lemma 1.3 and Remark 1.2 do not apply directly. There are two main problems associated with our putative null Lagrangian [the right-hand side of (2.26)]: firstly it is defined for those functions $\theta(\underline{x})$ for which $\dfrac{\underline{x}}{1-\theta(\underline{x})}\in\bar{\Omega}$, $\forall\,\underline{x}\in\bar{\Omega}$ i.e. for a given $\underline{x}\in\bar{\Omega}$ the admissible values $\theta(\underline{x})$ must lie in a closed set. Hence when trying to obtain the analogue of Remark 1.2 by approximating θ by smoother functions care has to be taken to ensure that the approximating functions respect the constraint. Secondly, Lemma 1.3 does not apply directly since it requires the integrand to be in C^1 in order to use the dominated convergence theorem to obtain (1.6) but the differentiability properties of (2.26) in $\underline{x}$ and $1-\theta$ as $\underline{x}$, $(1-\theta)\to 0$ are not clear. We circumvent these difficulties in Proposition 2.8 and Corollary 2.9 using a procedure which allows us to work away from the singular point $\theta = 1$.

Hypotheses on Ω. — For the remainder of this paper we will assume further that Ω is star-shaped with respect to the origin, i.e. that $\underline{0}\in\Omega$ and that for each $\underline{x}\in\Omega\setminus\{\underline{0}\}$ there exists a unique $t>0$ such that $t\underline{x}\in\partial\Omega$. We will also assume that Ω has a C^1 boundary in the sense that there exists an open neighbourhood U of $\partial\Omega$ and $\varphi\in C^1(\mathrm{U}, \mathbb{R})$, with $\nabla\varphi$ non vanishing on U, such that

$$\partial\Omega = \{\underline{x}\in\mathrm{U} : \varphi(\underline{x}) = 0\} \tag{2.17}$$

The assumption that Ω is star-shaped with respect to the origin then implies that

$$\underline{Y}\cdot\nabla\varphi(\underline{Y}) \neq 0, \qquad \forall\,\underline{Y}\in\partial\Omega. \tag{2.18}$$

DEFINITION 2.6. — We define the scaling function $\varepsilon : \bar{\Omega} \to \mathbb{R}$ for each $\underset{\sim}{x} \in \bar{\Omega} \setminus \{0\}$ by assigning $\varepsilon(\underset{\sim}{x})$ to be the unique positive number such that

$$\frac{\underset{\sim}{x}}{\varepsilon(\underset{\sim}{x})} \in \partial\Omega \tag{2.19}$$

and set $\varepsilon(\underset{\sim}{0}) = 0$.

LEMMA 2.7. — *The scaling function* ε *satisfies*

$$\varepsilon \in W^{1,\infty}(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega} \setminus \{\underset{\sim}{0}\}, \mathbb{R}).$$

Proof. — Let α, β be respectively $\inf_{\underset{\sim}{x} \in \partial\Omega} \|\underset{\sim}{x}\|$, $\sup_{\underset{\sim}{x} \in \partial\Omega} \|\underset{\sim}{x}\|$, where $\|.\|$ denotes the usual Euclidean metric. Since $\underset{\sim}{0} \in \Omega$ by assumption it follows that α, $\beta > 0$ and that

$$\alpha \leqq \left\| \frac{\underset{\sim}{x}}{\varepsilon(\underset{\sim}{x})} \right\| \leqq \beta, \qquad \forall \underset{\sim}{x} \in \bar{\Omega} \setminus \{\underset{\sim}{0}\}.$$

As $\varepsilon > 0$ it follows that

$$\frac{1}{\beta}\|\underset{\sim}{x}\| \leqq \varepsilon(\underset{\sim}{x}) \leqq \frac{1}{\alpha}\|\underset{\sim}{x}\|, \qquad \forall \underset{\sim}{x} \in \bar{\Omega} \setminus \{\underset{\sim}{0}\} \tag{2.20}$$

By definition

$$\varphi\left(\frac{\underset{\sim}{x}}{\varepsilon(\underset{\sim}{x})}\right) = 0, \qquad \forall \underset{\sim}{x} \in \bar{\Omega} \setminus \{\underset{\sim}{0}\} \tag{2.21}$$

where φ is the function of (2.17). We now extend the definition of ε to $t\Omega$ where $t > 1$ is chosen so that $t\Omega \subset \Omega \cup U$ and $\underset{\sim}{Y} . \nabla\varphi(\underset{\sim}{Y}) \neq 0$, this is always possible by (2.18). It then follows from (2.18) and the implicit function theorem that $\varepsilon \in C^1(t\Omega \setminus \{\underset{\sim}{0}\}, \mathbb{R})$ and hence $\varepsilon \in C(\bar{\Omega}, \mathbb{R})$ by (2.20). Differentiation of (2.21) gives

$$\frac{1}{\varepsilon(\underset{\sim}{x})}\nabla\varphi\left(\frac{\underset{\sim}{x}}{\varepsilon(\underset{\sim}{x})}\right) - \frac{1}{\varepsilon^2(\underset{\sim}{x})}\left[\underset{\sim}{x} . \nabla\varphi\left(\frac{\underset{\sim}{x}}{\varepsilon(\underset{\sim}{x})}\right)\right]\nabla\varepsilon(\underset{\sim}{x}) = 0, \qquad \forall \underset{\sim}{x} \in t\Omega \setminus \{\underset{\sim}{0}\}$$

$$\Rightarrow \quad \nabla\varepsilon(\underset{\sim}{x}) = \frac{\nabla\varphi(\underset{\sim}{x}/\varepsilon)}{(\underset{\sim}{x}/\varepsilon) . \nabla\varphi(\underset{\sim}{x}/\varepsilon)}, \qquad \forall \underset{\sim}{x} \in t\Omega \setminus \{\underset{\sim}{0}\}. \tag{2.22}$$

As $\varepsilon \in C^1(t\Omega \setminus \{\underset{\sim}{0}\}, \mathbb{R})$ clearly $\varepsilon \in C^1(\bar{\Omega} \setminus \{\underset{\sim}{0}\}, \mathbb{R})$ and the boundedness of $\nabla\varepsilon$ now follows from the continuity of the right-hand side of (2.22) and the compactness of $\partial\Omega$.

PROPOSITION 2.8. — *Suppose that* W *is rank one convex and that* $\underset{\sim}{u}_0 \in C^2(\bar{\Omega})$ *is a solution of* (2.6) *that satisfies* $\underset{\sim}{u}_0|_{\partial\Omega} = F\underset{\sim}{x}$ *for some*

$F \in M_{+}^{n \times n}$. *Let* $\varepsilon_0 \in (0,1)$ *and define*

$$\bar{\theta}(\underset{\sim}{x}) = \begin{cases} 1-\varepsilon(\underset{\sim}{x}) & \textit{if} \quad x \in \bar{\Omega} \setminus \varepsilon_0 \Omega \\ 1-\varepsilon_0 & \textit{if} \quad \underset{\sim}{x} \in \varepsilon_0 \Omega, \end{cases} \tag{2.23}$$

where $\varepsilon(\underset{\sim}{x})$ *is the scaling function corresponding to* Ω, *given by Definition* 2.6. *Then*

$$E(\underset{\sim}{u}_{\bar{\theta}}) \geqq E(\underset{\sim}{u}_0), \tag{2.24}$$

where $\underset{\sim}{u}_{\bar{\theta}}$ *is defined by* (2.12) *and* (1.8).

Notice that

$$\underset{\sim}{u}_{\bar{\theta}}(\underset{\sim}{x}) = \begin{cases} F\underset{\sim}{x} & \textit{if} \quad \underset{\sim}{x} \in \bar{\Omega} \setminus \varepsilon_0 \Omega \\ \varepsilon_0 \underset{\sim}{u}_0\left(\dfrac{\underset{\sim}{x}}{\varepsilon_0}\right) & \textit{if} \quad \underset{\sim}{x} \in \varepsilon_0 \Omega \end{cases} \tag{2.25}$$

Proof. — Our proof proceeds in 2 stages.

Step 1. — We first show the existence of a sequence $\theta_n \in C^1(\bar{\Omega})$ with the following properties

(i) $\dfrac{\underset{\sim}{x}}{1-\theta_n(\underset{\sim}{x})} \in \bar{\Omega}, \quad \forall \underset{\sim}{x} \in \bar{\Omega}, \quad \forall n;$

(ii) $\theta_n \xrightarrow{C(\bar{\Omega})} \bar{\theta} \quad$ as $\quad n \to \infty;$

(iii) $\nabla \theta_n \xrightarrow{L^1(\Omega)} \nabla \bar{\theta} \quad$ as $\quad n \to \infty$ and

(iv) $\theta_n(\underset{\sim}{x}) = \varepsilon(\underset{\sim}{x}), \quad \forall \underset{\sim}{x} \in \bar{\Omega} \setminus \varepsilon_0 \Omega, \quad \forall n.$

To obtain the existence of such a sequence let $\psi_n \in C^1([0,1])$ be another sequence satisfying

(i)′ $\psi_n(t) \geqq t, \quad \forall t \in [0,1], \quad \forall n;$

(ii)′ $\psi_n \xrightarrow{C([0,1])} \bar{\psi} \quad$ as $\quad n \to \infty;$

where $\bar{\psi}(t) = \begin{cases} t & \text{if} \quad t \in (\varepsilon_0, 1) \\ \varepsilon_0 & \text{if} \quad t \in [0, \varepsilon_0], \end{cases}$

(iii)′ $\psi_n' \xrightarrow{L^1(0,1)} \bar{\psi}' \quad$ as $\quad n \to \infty$ and

(iv)′ $\psi_n(t) = \bar{\psi}(t), \quad \forall t \in (\varepsilon_0, 1], \quad \forall n.$

The smoothness of $\varepsilon(\underset{\sim}{x})$, shown in Lemma 2.7, then implies that

$$\theta_n(\underset{\sim}{x}) \overset{\text{def}}{=} \psi_n(1-\varepsilon(\underset{\sim}{x})), \qquad \forall n$$

is a sequence satisfying (i)-(iv).

Step 2. — By (2.12), (1.10) and (1.8) it follows that

$$\nabla \underline{u}_0(\underline{x}) = \nabla \underline{u}_0\left(\frac{\underline{x}}{1-\theta(\underline{x})}\right) - \left[\underline{u}_0\left(\frac{\underline{x}}{1-\theta(\underline{x})}\right) - \nabla \underline{u}_0\left(\frac{\underline{x}}{1-\theta(\underline{x})}\right)\frac{\underline{x}}{1-\theta(\underline{x})}\right] \otimes \nabla \theta(\underline{x}). \quad (2.26)$$

The assumption that W is rank one convex implies that $W(F+\underline{\lambda} \otimes \underline{\mu})$ is a convex function of $\underline{\mu}$ for $F \in M_+^{n\times n}$, $\underline{\lambda}$, $\underline{\mu} \in \mathbb{R}^n$. Thus

$$W(F+\underline{\lambda} \otimes \underline{\mu}) \geqq W(F) + \frac{\partial W}{\partial F_j^i}(F)\,\lambda^i \mu^j \quad (2.27)$$

and hence

$$W(\nabla \underline{u}_0) \geqq W\left(\nabla \underline{u}_0\left(\frac{\underline{x}}{1-\theta(\underline{x})}\right)\right) - \frac{\partial W}{\partial F_j^i}\left(\nabla \underline{u}_0\left(\frac{\underline{x}}{1-\theta(\underline{x})}\right)\right) \times \left[u_0^i\left(\frac{\underline{x}}{1-\theta(\underline{x})}\right) - u_{0,\,k}^i\left(\frac{\underline{x}}{1-\theta(\underline{x})}\right)\frac{x^k}{1-\theta(\underline{x})}\right]\theta_{,j}(\underline{x}), \qquad \forall\, \underline{x} \in \Omega. \quad (2.28)$$

The arguments of Theorem 1.6 show that the integral of the right-hand side of (2.28) is constant for all functions $\theta \in C^1(\bar{\Omega})$ that vanish on $\partial\Omega$ and that satisfy $\frac{\underline{x}}{1-\theta(\underline{x})} \in \bar{\Omega}$, $\forall\, \underline{x} \in \bar{\Omega}$, and is equal in particular to its value for $\theta \equiv 0$, i.e. $E(\underline{u}_0)$.

Now using the approximating sequence θ_n and the dominated convergence theorem we see that the integral of the right-hand side of (2.28) at $\theta = \bar{\theta}$ is equal to its value at $\theta = \theta_n$ for any n and hence (2.24) holds.

COROLLARY 2.9:

$$E(\underline{u}^h) \geqq E(\underline{u}_0), \quad (2.29)$$

where

$$\underline{u}^h(\underline{x}) \equiv F\,\underline{x}, \qquad \underline{x} \in \bar{\Omega}. \quad (2.30)$$

Proof. — By (2.25)

$$E(\underline{u}_{\bar{\lambda}}) = \int_{\Omega \setminus \varepsilon_0 \Omega} W(F)\,dx + \int_{\varepsilon_0 \Omega} W\left(\nabla \underline{u}_0\left(\frac{\underline{x}}{\varepsilon_0}\right)\right)dx$$

$$= (1-\varepsilon_0^n)\,\mathrm{mes}(\Omega)\,W(F) + \varepsilon_0^n \int_\Omega W(\nabla \underline{u}_0(\underline{Y}))\,dY. \quad (2.31)$$

Since

$$E(\underline{u}^h)=\int_{\Omega} W(F)\,dx=\text{mes}(\Omega)\,W(F),$$

inequality (2.29) follows from (2.31) and (2.24).

Proof of Theorem 2.5. — If $\underline{c}=0$ and $\underline{u}_0 \not\equiv F\underline{x}$ by using Corollary 2.9 we obtain a contradiction of the strict quasiconvexity of W at F, and the result follows. If $\underline{c}\neq\underline{0}$ then we apply the arguments of Proposition 2.8 with the family of equilibria (2.12) replaced by a combination of (2.12) and (2.13), more specifically we choose

$$\underline{u}(\alpha, \underline{x})=(1-\alpha)\,\underline{u}_0\left(\frac{\underline{x}}{1-\alpha}\right)+\alpha\,\underline{c}. \tag{2.32}$$

Again, as in Proposition 2.8 we choose $\bar{\theta}$ given by (2.23) then

$$\underline{u}_{\bar{\theta}}(\underline{x})=\begin{cases} F\,\underline{x}+\underline{c} & \text{if} \quad \underline{x}\in\bar{\Omega}\setminus\varepsilon_0\,\Omega \\ \varepsilon_0\,\underline{u}_0\left(\dfrac{\underline{x}}{\varepsilon_0}\right)+(1-\varepsilon_0)\,\underline{c} & \text{if} \quad \underline{x}\in\varepsilon_0\,\Omega \end{cases}$$

Now exactly analogous arguments to those contained in the proof of Proposition 2.8 show that $E(\underline{u}_{\bar{\theta}})\geqq E(\underline{u}_0)$ and hence by the arguments of Corollary 2.9 that $E(\underline{u}^h)\geqq E(\underline{u}_0)$, where $\underline{u}^h(\underline{x})\equiv F\,\underline{x}+\underline{c}$. The remainder of the proof follows as in the case $\underline{c}=\underline{0}$.

Remark 2.10. — The use of different combinations of the families (2.9), (2.11) and (2.12) may permit a proof of Theorem 2.5 for some domains that are not star-shaped.

3. CONCLUDING REMARKS

The scaling family of deformations (2.12) offers an approach to answering the question of whether rank one convexity of W implies quasiconvexity of W. We first remark that, by a scaling argument, in order to show that W is quasiconvex, it is necessary and sufficient to check that the quasiconvexity condition (2.14) holds for *one* bounded open set $D\subset\mathbb{R}^n$, mes $D\neq 0$ (*see* e.g. Ball [1] p. 205). We choose $D=B$ the unit ball in $\mathbb{R}^n$. Now let $\underline{u}: B\to\mathbb{R}^n$ be *any deformation* (not necessarily an equilibrium solution) satisfying

$$\underline{u}\big|_{\partial B} = F\,\underline{x} \tag{3.1}$$

and set

$$\underline{u}_\varphi(\underline{x}) = \varphi(\underline{x})\,\underline{u}\left(\frac{\underline{x}}{\varphi(\underline{x})}\right). \tag{3.2}$$

Then

$$\nabla \underline{u}_\varphi(\underline{x}) = \nabla \underline{u}\left(\frac{\underline{x}}{\varphi(\underline{x})}\right) + \left[\underline{u}\left(\frac{\underline{x}}{\varphi(\underline{x})}\right) - \nabla \underline{u}\left(\frac{\underline{x}}{\varphi(\underline{x})}\right)\frac{\underline{x}}{\varphi(\underline{x})}\right] \otimes \nabla \varphi(\underline{x}) \tag{3.3}$$

and the corresponding energy is given by

$$I(\varphi) \stackrel{\text{def}}{=} E(\underline{u}_\varphi) = \int_B \left(\nabla \underline{u}\left(\frac{\underline{x}}{\varphi}\right) + \left[\underline{u}\left(\frac{\underline{x}}{\varphi}\right) - \nabla \underline{u}\left(\frac{\underline{x}}{\varphi}\right)\frac{\underline{x}}{\varphi}\right] \otimes \nabla \varphi\right) dx. \tag{3.4}$$

The assumption that W is rank one convex implies that the integrand in (3.4) is convex in $\nabla\varphi$. The question of whether rank one convexity implies quasiconvexity is now replaced by the question of whether $\varphi_0(\underline{x}) \equiv |\underline{x}|$ is the global minimiser of I on the set

$$\mathscr{A}_{\underline{u}} = \{\varphi \in W^{1,\infty}(B) : (\underline{u}_\varphi - F\,\underline{x}) \in W_0^{1,\infty}(B),\ \varphi\big|_{\partial B} = 1\}, \tag{3.5}$$

where, in order that $\underline{u}_\varphi$ be well defined, we require that $\dfrac{\underline{x}}{\varphi(\underline{x})} \in \bar{B}$, $\forall\, \underline{x} \in \bar{B}$.

If $\underline{u} \in C^2(\bar{B})$ and $\bar{\varphi} \in \mathscr{A}_{\underline{u}}$ is a smooth solution of the Euler-Lagrange equation for I namely

$$\left(u^i\left(\frac{\underline{x}}{\varphi}\right) - u^i_k\left(\frac{\underline{x}}{\varphi}\right)\frac{x^k}{\varphi}\right)\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}\left(\nabla \underline{u}\left(\frac{\underline{x}}{\varphi}\right) + \left[\underline{u}\left(\frac{\underline{x}}{\varphi}\right) - \nabla \underline{u}\left(\frac{\underline{x}}{\varphi}\right)\frac{\underline{x}}{\varphi}\right] \otimes \nabla\varphi\right)\right] = 0 \tag{3.6}$$

then it is possible to prove an analogue of Corollary 2.9 for the functional I, i.e.

$$I(\bar{\varphi}) \leq I(\varphi_0).$$

This is equivalent to

$$E(\underline{u}_{\bar{\varphi}}) \leq E(\underline{u}_0), \qquad \text{where} \quad \underline{u}_0(\underline{x}) \equiv F\,\underline{x},$$

ie $\underline{u}_{\bar{\varphi}}$ is a deformation of B satisfying the same boundary conditions as the homogeneous deformation with no more energy than it. This result follows by arguments analogous to those contained in the proof of Theorem 1.6 (*see* Remark 1.7), Proposition 2.8 and Corollary 2.9 on observing that (3.6) inherits the scaling invariance of (2.6), ie that $\delta\varphi\left(\frac{\underset{=}{x}}{\delta}\right)$ is a solution of (3.6) whenever φ is.

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