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CAVITATION, THE INCOMPRESSIBLE LIMIT, AND MATERIAL INHOMOGENEITY

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Motivated by work of Gent and Lindley [3], a rigorous treatment of cavitation in finite elasticity was first given by Ball [2] in a fundamental paper, and subsequently by a number of authors [4, 10, 12, 16] (see also [1] for aelotropic materials and [9] for the dynamic problem). The setting for these works is a ball of initially perfect material which is held in a state of tension under prescribed radial loads or displacements on the boundary. It is known that under appropriate assumptions on the stored energy function, there exist weak solutions to the equilibrium equations of elasticity in which a cavity forms at the centre of the ball (see, e.g., [2, 12, 16]). For the displacement boundary value problem, if λ represents the radius of the deformed ball, it can be shown that these cavitating solutions bifurcate from the initially stable homogeneous deformation at a critical boundary displacement $\lambda = \lambda_{\text{crit}}$ (at which point the homogeneous deformation loses stability). Analogously, for the traction problem, bifurcation to a deformation with a cavity occurs at a critical value of the applied Cauchy load P_{crit} (see [2] for a discussion of stability).

In Secs. 1 and 2 of this paper we prove convergence of these critical loads and displacements in the incompressible limit.

Consider a hyperelastic body occupying the region $B = \{\mathbf{X} \in \mathbf{R}^3 : |\mathbf{X} < 1\}$. With any deformation $\mathbf{u}: B \to \mathbf{R}^3$ of the ball, we associate an energy

$$E(\mathbf{u}) = \int_B W(\nabla \mathbf{u}(\mathbf{X}))\, dX\,, \tag{1}$$

where W is the stored energy function of the material and characterises the material response. We consider, initially, a class of stored energy functions of the form

$$W^k(F) = W^{\text{inc}}(F) + h(k\,, \det F - 1) \quad \forall F \in M^{3\times 3}\,,\ k \in (0\,, k_0)\,, \tag{2}$$

where W^{inc} is the stored energy function of an incompressible material and h is a compressibility term with the property that

$$h(k\,, \delta) \to \infty \quad \text{as } k \to 0 \text{ if } \delta \neq 0\,. \tag{3}$$

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The energy corresponding to a deformation $\mathbf{u}$ of a ball of elastic material characterised by W^k is then

$$E^k(\mathbf{u}) = \int_B W^k(\nabla \mathbf{u}(\mathbf{X}))\, dX . \tag{4}$$

Clearly $E^k(\mathbf{u}) \to \infty$ as $k \to 0$ unless

$$\det \nabla \mathbf{u}(\mathbf{X}) = 1 \quad \text{for almost every } \mathbf{X} \in B .$$

Thus, k represents the compressibility of the material and in the limit $k \to 0$ only incompressible deformations have finite energy: we call this the *incompressible limit.*

In Theorem 3 we show that the critical displacements λ^k_{crit} at which cavitation occurs for materials characterised by W^k satisfy

$$\lambda^k_{\text{crit}} \to 1 \quad \text{as } k \to 0 . \tag{5}$$

The corresponding Cauchy loads at which bifurcation occurs are given by

$$P^k_{\text{crit}} = \frac{1}{(\lambda^k_{\text{crit}})^2} \Phi, {}^k_1(\lambda^k_{\text{crit}}, \lambda^k_{\text{crit}}, \lambda^k_{\text{crit}}) \tag{6}$$

(see Sec. 2.2), where $\Phi^k(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) = W^k(F) \ \ \forall F \in M^{3\times 3}_+$, the v_i being the eigenvalues of $\sqrt{F^{\mathrm{T}}F}$.

To show convergence of these critical loads to the incompressible critical load $P^{\text{inc}}_{\text{crit}}$ poses a more difficult problem as it involves passing to the limit in (6) as $\lambda^k_{\text{crit}} \to 1$ and $k \to 0$ simultaneously. We overcome this by an alternative characterisation of the critical load as the "stress at infinity" in an infinite body. This relies on the invariance of the equilibrium equations under rescaling which, as noted in [2], is such that an infinitesimal hole in a finite expanse of material behaves as a finite hole in an infinite expanse. If $P(c)$ is the radial Cauchy stress on the boundary of a ball for a cavitating equilibrium solution with cavity of size c, then the critical load at which cavitation occurs is the limit of $P(c)$ as $c \to 0$. Under the rescaling this may be replaced by the limiting value of the stress on the outer boundary of a finite ball that contains a cavity of fixed size, the limit now being taken as the size of the ball tends to infinity (with the hole size remaining constant). Using this approach in Sec. 2, we prove in Theorem 6 that

$$P^k_{\text{crit}} \to P^{\text{inc}}_{\text{crit}} \quad \text{as } k \to 0 .$$

An explicit example of convergence of critical loads and displacements, which motivated this work, is given in Ball [2, Example 7.6].

In Sec. 3 we consider some consequences of the arguments of Sec. 2 for cavitation in radially inhomogeneous materials. We show that the mathematical phenomenon of cavitation depends crucially on the nature of the material present at the origin of the ball. In particular, for a class of *inhomogeneous* incompressible materials, the critical load for cavitation is the same as the critical load for a *homogeneous* incompressible ball composed entirely of the material found at the origin of the inhomogeneous ball (this result appears in [11]: the related problem of a composite

sphere made of two homogeneous incompressible materials was subsequently studied, independently, in Horgan and Pence [5, 6] and analogous conclusions drawn). For the inhomogeneous compressible case we give sufficient conditions for cavitation to occur and demonstrate that any deformation which keeps the ball intact is unstable if the radial stretch at the origin exceeds $\lambda_{\text{crit}}^{\text{hom}}$, the critical boundary displacement for a ball composed entirely of the material present at the origin of the inhomogeneous ball. The argument used in the proof of this result is motivated by a result in [7]. In this interesting paper, James and Spector [7] examine the stability of cavitating solutions under general three-dimensional perturbations. They find that, for certain stored energy functions, the energy of cavitating solutions can be further lowered through the introduction of line cracks and they relate this to the phenomenon of crazing.

Other results concerning stability of equilibria with respect to general three-dimensional variations are contained in [8, 14, 15, 17].

1. Preliminaries. Consider a ball of *compressible* homogeneous hyperelastic material which occupies the region $B = \{\mathbf{X} \in \mathbf{R}^3 : |\mathbf{X}| < 1\}$ in its reference state. Any deformation $\mathbf{u}: B \to \mathbf{R}^3$ satisfying the local invertibility condition

$$\det \nabla \mathbf{u}(\mathbf{X}) > 0 \quad \text{a.e. } X \in B \tag{1.1}$$

has the associated energy

$$E(\mathbf{u}) = \int_B W(\nabla \mathbf{u}(\mathbf{X}))\, dX, \tag{1.2}$$

where $W: M_+^{3\times 3} \to \mathbf{R}^+$ is the stored energy function of the material and $M_+^{3\times 3}$ denotes the set of 3×3 matrices with positive determinant.

The equilibrium equations under zero body force are the Euler-Lagrange equations for (1.2),

$$\frac{\partial}{\partial X^\alpha}\left[\frac{\partial W}{\partial F_\alpha^i}(\nabla \mathbf{u})\right] = 0 \quad \forall \mathbf{X} \in B, \ i = 1, 2, 3. \tag{1.3}$$

We will assume that W is *isotropic* and *frame indifferent* so that

$$W(QF) = W(FQ) = W(F) \quad \forall F \in M_+^{3\times 3}, \ \forall Q \in \mathrm{SO}(3).$$

In this case there exists a symmetric function $\Phi: \mathbf{R}_{++}^3 \to \mathbf{R}^+$ satisfying

$$W(F) = \Phi(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) \quad \forall F \in M_+^{3\times 3}, \tag{1.4}$$

where the v_i are the eigenvalues of $\sqrt{F^{\mathrm{T}}F}$, known as the principal stretches, and $\mathbf{R}_{++}^3 = \{(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) : \mathrm{v}_i > 0, \ i = 1, 2, 3\}$.

For *incompressible* materials the admissible deformations $\mathbf{u}$ must satisfy the constraint

$$\det \nabla \mathbf{u}(\mathbf{X}) = 1 \quad \text{a.e. } \mathbf{X} \in B, \tag{1.5}$$

and hence, for such a material, the corresponding stored energy function W^{inc} is only defined on $M_1^{3\times 3} = \{F \in M_+^{3\times 3} : \det F = 1\}$. However, as noted in [2], such an

energy function can be extended to all of $M_+^{3\times 3}$ for example by setting

$$\overline{W}(F) = W^{\text{inc}}\left(\frac{F}{(\det F)^{1/3}}\right) \quad \forall F \in M_+^{3\times 3}.$$

Henceforth we will assume that all incompressible stored energy functions are defined on all of $M_+^{3\times 3}$. Thus, we may also assume that the corresponding symmetric function $\Phi^{\text{inc}}(v_1, v_2, v_3)$ is also defined on all of $\mathbf{R}^3_{++}$.

The equilibrium equations for *incompressible* materials are the Euler-Lagrange equations for the functional

$$\int_\Omega W^{\text{inc}}(\nabla \mathbf{u}(\mathbf{X})) - p(\mathbf{X})[\det \nabla \mathbf{u}(\mathbf{X}) - 1]\, dX, \tag{1.6}$$

where $p(\mathbf{X})$, the pressure, is a Lagrange multiplier corresponding to the constraint of incompressibility (1.5).

Radial deformations. We will consider only radial deformations $\mathbf{u}$ of B in which case

$$\mathbf{u}(\mathbf{X}) = \frac{r(R)}{R}\mathbf{X} \quad \forall \mathbf{X} \in B, \tag{1.7}$$

for some function $r(R)$, where $R = |\mathbf{X}|$. In this case the corresponding principal stretches are

$$v_1 = r'(R), \qquad v_2 = v_3 = r(R)/R. \tag{1.8}$$

For the *displacement* boundary value problem we specify the boundary condition

$$\mathbf{u}|_{\partial B} = \lambda \mathbf{X} \tag{1.9}$$

for some $\lambda > 0$.

In the *Cauchy traction problem* we specify, for some $P \in \mathbf{R}$, that

$$T\mathbf{X} = P\mathbf{X} \quad \forall \mathbf{X} \in \partial B, \tag{1.9a}$$

where T is the Cauchy stress tensor.

For *compressible* materials

$$T(F) = \frac{1}{\det F}\frac{\partial W}{\partial F}(F)F^T, \tag{1.10}$$

and for *incompressible* materials

$$T(F) = -p1 + \frac{\partial W^{\text{inc}}}{\partial F}(F)F^T, \tag{1.10a}$$

where p is the pressure. By (1.2), (1.4), and (1.8), on radial maps the energy functional takes the form

$$E(\mathbf{u}) = 4\pi I(r) \overset{\text{def}}{=} 4\pi \int_0^1 R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) dR. \tag{1.11}$$

The *radial equilibrium equation* is the Euler-Lagrange equation for I given by (1.11),

$$\frac{d}{dR}\left[R^2 \Phi_{,1}\left(r', \frac{r}{R}, \frac{r}{R}\right)\right] = 2R\Phi_{,2}\left(r', \frac{r}{R}, \frac{r}{R}\right), \tag{1.12}$$

where $\Phi_{,i}$ denotes the differential of Φ with respect to its ith argument.

REMARK. Equation (1.12) is invariant under the scaling $(r, R) \to (\alpha r, \alpha R)$ for $\alpha > 0$. Notice also that the *homogeneous deformations*, corresponding to $r(R) \equiv \lambda R$, are always solutions of (1.9) and (1.12).

Cavitating equilibria. Following [2], for *compressible* materials we say that $r \in C^2((0, 1])$ is a cavitating equilibrium solution if it is a solution of (1.12) on $(0, 1]$ satisfying

(i) $r(0) = \lim_{r \to 0} r(R) > 0$,

(ii) $r'(R) > 0 \ \forall R \in (0, 1]$,

together with the natural boundary condition

(iii) $\lim_{R \to 0} T(r(R)) = 0$, (1.13)

where

$$T(r(R)) = \left(\frac{R}{r}\right)^2 \Phi_{,1}\left(r', \frac{r}{R}, \frac{r}{R}\right) \tag{1.14}$$

is the radial component of the Cauchy stress. Thus, under the corresponding deformation $\mathbf{u}$ (given by (1.7)) a cavity of radius $r(0)$ forms at the centre of the ball and (iii) is the natural boundary condition that the cavity surface is stress free. The existence of cavitating solutions is studied in [2, 4, 10, 12, 16].

For *incompressible* materials the constraint (1.5) together with (1.7), (1.8) imply that any radial deformation must satisfy

$$r'\left(\frac{r}{R}\right)^2 = 1 \quad \text{a.e. } R \in [0, 1],$$

hence

$$r(R) = (R^3 + A^3)^{1/3} \quad \forall R \in [0, 1] \text{ for some } A \geq 0, \tag{1.15}$$

are the only kinematically admissible maps. If $\mathbf{u}$ satisfies (1.9) and $\lambda > 1$ then $A^3 = \lambda^3 - 1$. Notice that by (1.7) this represents a deformation $\mathbf{u}$ of the ball which opens a hole of size A. For maps of the form (1.15)

$$v_1 = \left(\frac{R}{r}\right)^2, \qquad v_2 = v_3 = \frac{r}{R}. \tag{1.16}$$

Ball [2] gives necessary and sufficient conditions for the map (1.7) corresponding to (1.15) to generate a weak solution of the Euler-Lagrange equations for (1.6). The corresponding radial Cauchy stress is then given by

$$\begin{aligned}\hat{T}(R) &= -p(R) + \left(\frac{R}{r}\right)^2 \Phi_{,1}^{\text{inc}}\left(\left(\frac{R}{r}\right)^2, \frac{r}{R}, \frac{r}{R}\right) \\ &= C + \int_R^1 2\frac{s^2}{r^3}\left[\frac{r}{s}\Phi_{,2}^{\text{inc}}\left(\left(\frac{s}{r}\right)^2, \frac{r}{s}, \frac{r}{s}\right) - \left(\frac{s}{r}\right)^2 \Phi_{,1}^{\text{inc}}\left(\left(\frac{s}{r}\right)^2, \frac{r}{s}, \frac{r}{s}\right)\right] ds,\end{aligned}$$

where $r(R)$ is given by (1.15) and $p(R)$ is the pressure. This follows from [2, Theorem 4.3] on changing variables or can be obtained formally by taking the Euler-Lagrange equation for the functional

$$\int_0^1 R^2 \left[\Phi^{\text{inc}}\left(r', \frac{r}{R}, \frac{r}{R}\right) - p(R)\left(r'\frac{r}{R}\frac{r}{R} - 1\right)\right] dR$$

and noting (1.16) and (1.8).

For later use we note that, on changing variables from R to R/A we obtain

$$\hat{T}(R) = C + \int_{R/A}^{1/A} \frac{2s^2}{\tilde{r}^3}\left[\frac{\tilde{r}}{s}\Phi,_2^{\text{inc}}\left(\left(\frac{s}{\tilde{r}}\right)^2, \frac{\tilde{r}}{s}, \frac{\tilde{r}}{s}\right) - \left(\frac{s}{\tilde{r}}\right)^2 \Phi,_1^{\text{inc}}\left(\left(\frac{s}{\tilde{r}}\right)^2, \frac{\tilde{r}}{s}, \frac{\tilde{r}}{s}\right)\right] ds, \tag{1.17}$$

where $\tilde{r}(R) = (R^3 + 1)^{1/3}$.

2. Convergence of critical loads and displacements. In this section we consider a family of compressible stored energy functions

$$\Phi^k(v_1, v_2, v_3) = \Phi^{\text{inc}}(v_1, v_2, v_3) + h(k, v_1v_2v_3 - 1) \tag{2.1}$$

parametrised by $k \in (0, k_0)$, where Φ^{inc} represents the stored energy function of an incompressible material and h is a compressibility term. We will denote by I^k the corresponding energy functional (1.11). We will assume throughout this section that Φ^{inc} is C^3 on $\mathbf{R}^3_{++} = \{(v_1, v_2, v_3) : v_1, v_2, v_3 > 0\}$; we will also refer to a number of hypotheses on Φ^{inc} and h which for convenience we list together below.

(Φ1) $\Phi,_{11}(v_1, v_2, v_3) > 0$,

(Φ2) $\dfrac{v^2}{(v^3-1)^2}\hat{\Phi}(v) \in L^1(\delta, \infty)$ for any $\delta > 1$, where $\hat{\Phi}(v) \overset{\text{def}}{=} \Phi(1/v^2, v, v)$,

(Φ3) $\left(\dfrac{v_i\Phi,_i(v_1, v_2, v_3) - v_j\Phi,_j(v_1, v_2, v_3)}{v_i - v_j}\right) > 0, \qquad i \neq j,\ v_i \neq v_j$,

(Φ4) $\dfrac{\Phi,_i - \Phi,_j}{v_i - v_j} + \Phi,_{ij} \geq 0$ for $v_i \neq v_j$,

(Φ5) There exist constants $M, \varepsilon_0 \in (0, \infty)$ such that

$$|\Phi,_i(v_1, \alpha_2 v_2, \alpha_3 v_3)v_i| < M(\Phi(v_1, v_2, v_3) + 1) \quad \text{(No summation)}$$

if $|\alpha_i - 1| < \varepsilon_0$, $i = 2, 3$.

(Φ6) $\dfrac{1}{(v^3-1)}\dfrac{d}{dv}\hat{\Phi}(v) \in L^1(1, \infty)$, where $\hat{\Phi}(v) = \Phi(1/v^2, v, v)$.

(Φ7) There exist constants $A, B > 0$ and $\beta \in (0, 2)$ such that

$$\left(\frac{v_1\Phi,_1(v_1, v_2, v_2) - v_2\Phi,_2(v_1, v_2, v_2)}{v_1 - v_2}\right) \leq A + B(v_2)^\beta$$

for $0 < v_1 \leq v_2$.

We will say that the compressibility term h satisfies (H) If $h: (0, k_0) \times (-1, \infty) \to \mathbf{R}^+ \cup \{0\}$ is C^3, strictly convex in its second argument for each $k \in (0, k_0)$, and satisfies

(i) $h(k, v) \to \infty$ as $v \to -1$ for each $k \in (0, k_0)$,
(ii) $h(k, 0) \to c$ as $k \to 0$ for some constant c,
(iii) $h(k, v)/v \to \infty$ as $v \to \infty$ for each $k \in (0, k_0)$,
(iv) $h(k, v) \geq c_1|v|/k + c_2$ $\forall v > -1$, $\forall k \in (0, k_0)$, for some constants $c_1 > 0$, c_2.

(v) $|\frac{\partial h}{\partial \mathrm{v}}(k, \mathrm{v})| \leq M[h(k, \mathrm{v})/(\mathrm{v}+1)+1] \quad \forall \mathrm{v} > -1, \ \forall k \in (0, k_0)$, for some constant $M \in (0, \infty)$.

Finally we shall suppose that $\Phi, {}_1^{\mathrm{inc}}(1, 1, 1) + \frac{\partial h}{\partial \mathrm{v}}(k, 1) = 0 \ \ \forall k \in (0, k_0)$ so that the undeformed configuration $\mathrm{v}_1 = \mathrm{v}_2 = \mathrm{v}_3 = 1$ is a natural state for each Φ^k.

REMARKS ON HYPOTHESES. To illustrate the implications of hypotheses (Φ1)–(Φ7) it is useful to focus on the class of stored energies of the form

$$\Phi(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) = \sum_{i=1}^{3} \varphi(\mathrm{v}_i) + \sum_{\substack{i,j=1 \\ i<j}}^{3} \psi(\mathrm{v}_i \mathrm{v}_j) + \pi(\mathrm{v}_1 \mathrm{v}_2 \mathrm{v}_3). \qquad (*)$$

(Of course the term in π is redundant if Φ is an incompressible stored energy function, however it is of interest to include this term in discussing the restrictions imposed by the hypotheses.)

(Φ1) is known as the tension-extension inequality and is satisfied by energy functions of the form (*) if φ, ψ, and π are convex, with at least one of them having strictly positive second derivative. (Φ2) guarantees that the incompressible map (1.15) has finite energy. (Φ3) is often referred to as the Baker-Ericksen inequalities and is satisfied, for example, by all energy functions of the form (*) provided $t\varphi'(t)$ and $t\psi'(t)$ are increasing functions of t. (Φ4) is satisfied by all smooth energy functions of the form (*) provided that φ, ψ, and π are convex. (Φ5) is often satisfied by polynomial or simple rational functions Φ. (Φ6) is the requirement that the critical load for bifurcation to a deformation with a cavity given by (2.2.7) (or equivalently (2.2.6)) is finite. (Φ7) is a growth assumption consistent with polynomial growth of Φ: notice that for functions of the form (*) this condition is a restriction on the growth of φ and ψ only.

(H)(i) is the requirement that compression to zero volume requires infinite energy. (H)(ii) guarantees that the incompressible map (1.15) has bounded energy as $k \to 0$. (H)(iii) ensures that expansion to infinite volume requires infinite energy. (H)(v) is often satisfied by polynomial or simple rational functions.

Roughly speaking, the hypotheses in (H) state that the graph of the convex function h has a well close to $\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3 = 1$ for small k and that the sides of the well become steeper as $k \to 0$ (with the bottom of the well converging to the value c at $\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3 = 1$).

EXAMPLE. Examples of simple stored energy functions satisfying (Φ1)–(Φ7) and (H) are

$$\Phi^{\mathrm{inc}}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) = c_1 \left(\sum_{i=1}^{3} \mathrm{v}_i^{\alpha} - 3 \right) + c_2 \left(\sum_{\substack{i,j=1 \\ i<j}}^{3} (\mathrm{v}_i \mathrm{v}_j)^{\beta} - 3 \right),$$

$$1 < \alpha < 3, \ \ 1 \leq \beta < \frac{3}{2},$$

$$h(k, v_1v_2v_3 - 1) = \left[\frac{c_3}{k}|v_1v_2v_3 - 1|^{\gamma} + c_4(v_1v_2v_3)^{\delta} + \frac{c_5}{(v_1v_2v_3)^{\varepsilon}}\right], \quad \gamma > 1,\ \delta \geq 1,\ \varepsilon > 0,$$

where $c_1, c_3, c_5 > 0$, $c_2, c_4 \geq 0$ are constants. (Choosing $\alpha c_1 + 2\beta c_2 + \delta c_4 - c_5\varepsilon = 0$ ensures that the undeformed configuration is a natural state for all k.) See also Ball [2, Example 7.6] for an illustrative example which motivated this current work.

We next state a central result on the existence of cavitating minimisers for Φ^k given by (2.1). Recall that I^k is given by (1.11) with $\Phi = \Phi^k$.

THEOREM 1. Let Φ^{inc} satisfy $(\Phi 1)$–$(\Phi 5)$ and let h satisfy (H). Then for each $k \in (0, k_0)$ there exist $\lambda^k_{\text{crit}} \in (1, \infty)$ and a cavitating equilibrium solution $r^k_c \in C^2((0, 1])$ for Φ^k with the following properties:

(i) r^k_c is extendable to a solution of (1.12) on $(0, \infty)$,

(ii) $r^k_c(R)/R \searrow \lambda^k_{\text{crit}}$ as $R \to \infty$,

(iii) if $\lambda \leq \lambda^k_{\text{crit}}$, then $r(R) \equiv \lambda R$ is the global minimiser of I^k on

$$\mathscr{A}_\lambda = \{r \in W^{1,1}((0, 1)) : r(0) \geq 0,\ r(1) = \lambda,\ r' > 0 \text{ a.e. } R \in (0, 1)\}. \tag{2.2}$$

If $\lambda > \lambda^k_{\text{crit}}$, then the global minimiser of I^k on $\mathscr{A}_k$ is given by $r(R) \equiv \alpha r^k_c(R/\alpha)$, where α is the unique root of $\lambda = \alpha r^k_c(1/\alpha)$ (i.e., r represents a deformation with cavity size proportional to α).

Proof. The existence or r^k_c satisfying the above would follow from [12, Propositions 4.7 and 0.3 and Theorem 1.11] were it not for the fact that our Φ^k do not satisfy hypothesis (E1)(ii) of [12]. However, this condition is only used in [12] in proving the existence of a minimiser of the energy on $\mathscr{A}_\lambda$. Our hypothesis (H) guarantees that a minimiser still exists by adapting the arguments of Ball [2, Theorem 7.1].

2.1. *Convergence of critical displacement.* In this subsection we prove that the critical displacements λ^k_{crit} for Φ^k, whose existence is guaranteed by the previous theorem, satisfy $\lambda^k_{\text{crit}} \to 1$ as $k \to 0$. (By (1.15) $\lambda^{\text{inc}}_{\text{crit}} = 1$.) To do this we need the following preparatory results.

The first proposition examines the behaviour of minimisers of I^k as $k \to 0$.

PROPOSITION 2. Suppose that Φ^{inc} satisfies $(\Phi 1)$, $(\Phi 2)$ and h satisfies (H) and let $\{k_n\}$ be a sequence with $k_n \in (0, k_0)$, $k_n \to 0$ as $n \to \infty$. If y_{k_n} is a minimiser of I^{k_n} on $\mathscr{A}_\lambda$, then for each $\lambda > 1$

$$y_{k_n} \xrightarrow{W^{1,1}(0,1)} \hat{r} \quad \text{as } n \to \infty, \tag{2.1.1}$$

where

$$\hat{r}(R) = (R^3 + (\lambda^3 - 1))^{1/3} \tag{2.1.2}$$

is an *incompressible* deformation.

Proof. The existence of y_{k_n} is a consequence of Theorem 1. It then follows that

$$I^{k_n}(y_{k_n}) = \inf_{y \in A_\lambda} I^{k_n}(y) \leq I^{k_n}(\tilde{r}) = \int_0^1 R^2 \Phi^{\text{inc}}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right) dR + \int_0^1 R^2 h(k_n, 0)\, dR \tag{2.1.3}$$

for all n. Setting $\mathrm{v} = \tilde{r}(R)/R$ in the right-hand side expression in (2.1.3) and using $(\Phi 2)$, we obtain

$$\int_0^1 R^2 \Phi^{\text{inc}}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right) dR = A^3 \int_\lambda^\infty \frac{\mathrm{v}^2}{(\mathrm{v}^3 - 1)^2} \hat{\Phi}^{\text{inc}}(\mathrm{v})\, d\mathrm{v} < +\infty,$$

where $A = (\lambda^3 - 1)^{1/3}$, and hence (H)(ii) and (2.1.3) imply that

$$I^{k_n}(y_{k_n}) \leq \text{constant}, \tag{2.1.4}$$

uniformly in n. We now make the change of variables $u_{k_n} = y_{k_n}^3$ and $\rho = R^3$ in (2.1.4). Then writing $\dot{u}_{k_n}$ for $du_{k_n}/d\rho$ we obtain

$$\dot{u}_{k_n} = \left(\frac{y_{k_n}}{R}\right)^2 y'_{k_n}. \tag{2.1.5}$$

From (2.1.4), (2.1.5), and (H)(iv) we obtain

$$\int_0^1 |\dot{u}_{k_n} - 1|\, d\rho \leq c \cdot k_n \quad \text{for all } n, \tag{2.1.6}$$

where c is a constant. Then since $u_{k_n}(1) = \lambda^3$

$$u_{k_n} \xrightarrow{W^{1,1}(0,1)} \tilde{u} \quad \text{as } n \to \infty, \tag{2.1.7}$$

where $\tilde{u}(\rho) = (\rho + (\lambda^3 - 1))$. As $\lambda > 1$, it follows from (2.1.7) that $u_{k_n}(0)$ is uniformly bounded away from zero. Hence by (2.1.2) and (2.1.5) we obtain

$$\int_0^1 R^2 \left| \left(\frac{y_{k_n}}{R}\right)^2 y'_{k_n} - \left(\frac{\tilde{r}}{R}\right)^2 \tilde{r}' \right| dR = \int_0^1 |\dot{u}_{k_n} - \dot{\tilde{u}}|\, d\rho \tag{2.1.8}$$

and

$$\int_0^1 |y'_{k_n} - \tilde{r}'|\, dR \leq \text{const} \cdot \left(\int_0^1 |y_{k_n}^2 y'_{k_n} - y_{k_n}^2 \tilde{r}'|\, dR \right) \tag{2.1.9}$$

for all n. Since

$$|y_{k_n}^2 y'_{k_n} - y_{k_n}^2 \tilde{r}'| \leq |y_{k_n}^2 y'_{k_n} - \tilde{r}^2 \tilde{r}'| + |\tilde{r}^2 \tilde{r}' - y_{k_n}^2 \tilde{r}'| \tag{2.1.10}$$

for all n, by the triangle inequality and as the second term on the right-hand side of (2.1.10) is bounded by $2\lambda^2 \tilde{r}' \in L^1(0, 1)$, it follows from (2.1.7)–(2.1.10) and the dominated convergence theorem that

$$y'_{k_n} \xrightarrow{L^1(0,1)} \tilde{r}' \quad \text{as } n \to \infty.$$

Finally, as $y_{k_n}(1) = \lambda$ for all n (2.1.1.) holds. The next result proves convergence of the critical displacements.

THEOREM 3. Let Φ^{inc} satisfy $(\Phi 1)$–$(\Phi 5)$ and let h satisfy (H). If λ^k_{crit} is the critical displacement for cavitation corresponding to the stored energy function Φ^k (given by (2.1)), then

$$\lambda^k_{\text{crit}} \to 1 \quad \text{as } k \to 0. \tag{2.1.11}$$

Proof. The existence of λ^k_{crit} is a consequence of Proposition 2. Suppose for a contradiction that (2.1.11) does not hold. Then by Theorem 1 there exist a sequence $k_n \to 0$ as $n \to \infty$ and $\delta_0 > 1$ which satisfy

$$\lambda^{k_n}_{\text{crit}} \geq \delta_0 > 1 \quad \forall n.$$

Now fix $\lambda \in (\delta_0, 1)$. By Theorem 1 the global minimiser of I^{k_n} on $\mathscr{A}_\lambda$ is λR for each n, but Proposition 2 then gives a contradiction.

2.2. *Convergence of critical loads.*

Critical load: compressible case. The Cauchy traction problem for radial maps differs from the displacement boundary value problem in that the displacement condition (1.9) (which corresponds to $r(1) = \lambda$) on the outer boundary of the ball is replaced by the condition (1.9a), that the radial component of Cauchy stress be specified. Using (1.14), this takes the form

$$T^k(r(1)) = \frac{1}{(r(1))^2}\Phi,^k_1(r'(1), r(1), r(1)) = P, \tag{2.2.1}$$

where $P \in \mathbf{R}$ is given.

For the purposes of the discussion which follows, first fix any $k \in (0, k_0)$. Given any cavitating equilibrium solution $r_k(R)$ for the stored energy function Φ^k, Theorem 1 shows that $r_k(R)$ may be extended to $(0, \infty)$ as a solution of (1.12) with $\lim_{R\to\infty} r_k(R)/R = \lambda^k_{\text{crit}}$. We will show that the critical load at which bifurcation to a deformation with a cavity occurs, P^k_{crit}, is the "stress at infinity" produced by this extended radial deformation of $\mathbf{R}^3$, i.e., that

$$P^k_{\text{crit}} = \lim_{R\to\infty} T^k(r_k(R)). \tag{2.2.2}$$

To see this notice first, by Theorem 1, any other cavitating equilibrium $\tilde{r}(R)$ for Φ^k may also be extended to $(0, \infty)$ as a solution of (1.12) and that

$$\tilde{r}(R) \equiv d r_k\left(\frac{R}{d}\right), \tag{2.2.3}$$

where $d > 0$ is uniquely determined by the cavity size

$$\tilde{r}(0) = d r_k(0). \tag{2.2.4}$$

Conversely, by the Remark following (1.12), for any $d > 0$, $\tilde{r}$ defined by (2.2.3) is a cavitating equilibrium with cavity given by (2.2.4). The critical Cauchy load at which bifurcation occurs, P^k_{crit}, is given by the limiting value of the radial Cauchy stress on the boundary of the ball, produced by a cavitating equilibrium, as the size

of the cavity produced by the deformation tends to zero. Thus by (2.2.3) and (2.2.1)

$$P^k_{\text{crit}} = \lim_{d\to 0} T^k(r_k(1/d)) = \lim_{d\to 0}\left(\frac{1/d}{r(1/d)}\right)^2 \Phi,\,{}^k_1(r'_k(1/d),\, dr_k(1/d),\, dr_k(1/d))$$
$$= \left(\frac{1}{\lambda^k_{\text{crit}}}\right)^2 \Phi,\,{}^k_1(\lambda^k_{\text{crit}},\, \lambda^k_{\text{crit}},\, \lambda^k_{\text{crit}}),$$

where we have used Theorem 1(iii). It now follows that

$$P^k_{\text{crit}} = \lim_{R\to\infty} T^k(r_k(R)),$$

as required.

Having this characterisation, the key to proving the convergence of the critical loads (P^k_{crit}) as $k \to 0$ lies in the observation that

$$P^k_{\text{crit}} = \int_0^\infty \frac{d}{dR} T^k(r_k(R))\, dR = \int_0^\infty \frac{2R^2}{(r_k(R))^3}\left[\frac{r_k(R)}{R}\Phi,\,{}^k_2 - r'_k(R)\Phi,\,{}^k_1\right] dR$$
$$= \int_0^\infty \frac{2R^2}{(r_k(R))^3}\left[\frac{r_k(R)}{R}\Phi,\,{}^{\text{inc}}_2 - r'_k(R)\Phi,\,{}^{\text{inc}}_1\right] dR, \tag{2.2.5}$$

where we have used (1.14), (1.12), (1.13), and the definition of Φ^k (2.1). Notice in particular that the compressibility term h does not enter explicitly in (2.2.5).

Critical load: incompressible case. By (1.17) if the Cauchy load on the boundary of the ball is $P \in \mathbf{R}$, then $C = P$. If we now specify that the cavity surface be stress free (i.e., that $\hat{T}(0) = 0$), then we obtain

$$P = \int_0^{1/A} \frac{2R^2}{\tilde{r}^3}\left[\frac{\tilde{r}}{R}\Phi,\,{}^{\text{inc}}_2\left(\left(\frac{R}{\tilde{r}}\right)^2, \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right) - \left(\frac{R}{\tilde{r}}\right)^2 \Phi,\,{}^{\text{inc}}_1\left(\left(\frac{R}{\tilde{r}}\right)^2, \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right)\right] dR.$$

Hence, by previous arguments, taking the limit $A \to 0$, we obtain for the bifurcation point

$$P^{\text{inc}}_{\text{crit}} = \int_0^\infty \frac{2R^2}{\tilde{r}^3}\left[\frac{\tilde{r}}{R}\Phi,\,{}^{\text{inc}}_2 - \left(\frac{R}{\tilde{r}}\right)^2 \Phi,\,{}^{\text{inc}}_1\right] dR$$
$$= \int_0^\infty \frac{2R^2}{r^3}\left[\frac{r}{R}\Phi,\,{}^{\text{inc}}_2\left(\left(\frac{R}{r}\right)^2, \frac{r}{R}, \frac{r}{R}\right) - \left(\frac{R}{r}\right)^2 \Phi,\,{}^{\text{inc}}_1\left(\left(\frac{R}{r}\right)^2, \frac{r}{R}, \frac{r}{R}\right)\right] dR \tag{2.2.6}$$

for any r of the form (1.15).

Clearly the convergence of the critical Cauchy loads P^k_{crit} to $P^{\text{inc}}_{\text{crit}}$ is proved if we can pass to the limit $k \to 0$ from (2.2.5) to (2.2.6). However this presents some technical difficulties, the main one being that the convergence result established in Proposition 2 is only on $(0, 1)$ and we require convergence of the extended functions on $(0, \infty)$. Notice that if we set $\mathrm{v} = r/R$ we obtain the equivalent form

$$P^{\text{inc}}_{\text{crit}} = \int_1^\infty \frac{1}{(\mathrm{v}^3 - 1)} \frac{d}{d\mathrm{v}} \Phi^{\text{inc}}\left(\frac{1}{\mathrm{v}^2}, \mathrm{v}, \mathrm{v}\right) d\mathrm{v} \tag{2.2.7}$$

as given by Ball [2].

The idea now is to use $v = r/R$ as the independent variable in the compressible case also, but to do this we require the following two preparatory results.

PROPOSITION 4. Let Φ^{inc} satisfy $(\Phi 1)$–$(\Phi 5)$ and let h satisfy (H). Then for each $k \in (0, k_0)$ there exists $g_k : (\lambda^k_{crit}, \infty) \to (0, \infty)$, where $g_k \in C^2((\lambda^k_{crit}, \infty))$ and satisfies

$$(g_k(v)-v)\frac{d}{dv}[\Phi, {}^k_1(g^k(v), v, v)] = 2[\Phi, {}^k_2(g^k(v), v, v)-\Phi, {}^k_1(g^k(v), v, v)] \quad (2.2.8)$$

for $v \in (\lambda^k_{crit}, \infty)$.

Moreover, if $\lambda \in (\lambda^k_{crit}, \infty)$ and r_k is the minimiser of I^k on $\mathscr{A}_\lambda$, then

$$g_k\left(\frac{r_k(R)}{R}\right) = r_k'(R), \qquad R \in (0, 1]. \quad (2.2.9)$$

Proof. The existence of λ^k_{crit} is a consequence of Theorem 1. It also follows from this theorem that if $\lambda \in (\lambda^k_{crit}, \infty)$, then the global minimiser r_k on I^k on $\mathscr{A}_\lambda$ satisfies $r_k(0) > 0$, moreover, r_k can be extended to $(0, \infty)$ as a solution of (1.12) and $\lim_{R\to\infty} r_k(R)/R = \lambda^k_{crit}$. By a well-known property of solutions of (1.12) (see, e.g., [12, Corollary 1.2]) $r_k(R)/R$ is strictly monotone decreasing on $(0, \infty)$. Thus, we can invert the relation

$$v = \frac{r_k(R)}{R} \quad (2.2.10)$$

to give R as a function of $v \in (\lambda^k_{crit}, \infty)$. Now define

$$g_k(v) = r_k'(R(v)) \quad \text{for } v \in (\lambda^k_{crit}, \infty).$$

Clearly (2.2.9) holds by construction and Theorem 1(iii) implies that the validity of (2.2.9) is independent of the choice of $\lambda \in (\lambda_{crit}, \infty)$.

Expression (2.2.8) follows from rewriting (1.12) using v as the independent variable and noting that

$$R\frac{d}{dR} = R\frac{dv}{dR}\frac{d}{dv} = (g_k(v) - v)\frac{d}{dv}.$$

The idea will be to use v as the independent variable in proving convergence of (2.2.5) to (2.2.6) (or (2.2.7)) as $k \to 0$ and $g_k(v)$ as the dependent variable instead of R and r. This has the advantage that all cavitating solutions for a given Φ^k give rise to only one corresponding function g_k. The next proposition concerns the behaviour of g_k as $k \to 0$.

PROPOSITION 5. Let Φ^{inc} satisfy $(\Phi 1)$–$(\Phi 7)$ and let h satisfy (H). Then

$$g_k(v) \to 1/v^2 \quad \text{as } k \to 0 \quad (2.2.11)$$

for each $v \in (1, \infty)$, where g_k is as given in Proposition 4.

Notice that by Theorem 3, given any $v \in (1, \infty)$, $g_k(v)$ is defined for k sufficiently small. The proof of Proposition 5 is of a technical nature and is given in the Appendix.

THEOREM 6. Let Φ^{inc} satisfy $(\Phi 1)$–$(\Phi 7)$ and let h satisfy (H). Then

$$P^k_{\text{crit}} \to P^{\text{inc}}_{\text{crit}} \quad \text{as } k \to 0, \tag{2.2.12}$$

where P^k_{crit} is the critical Cauchy load for cavitation for Φ^k and $P^{\text{inc}}_{\text{crit}}$ (given by (2.2.6)) is the critical Cauchy load for an incompressible material with stored energy function Φ^{inc}.

Proof. Setting $v = r_k(R)/R$ in (2.2.5) and using Proposition 4 we obtain

$$P^k_{\text{crit}} = \int_{\infty}^{\lambda^k_{\text{crit}}} \frac{d\tilde{T}_k(v)}{dv}\, dv = 2\int_{\lambda^k_{\text{crit}}}^{\infty} \frac{v\Phi,_2^{\text{inc}}(g_k(v), v, v) - g_k(v)\Phi,_1^{\text{inc}}(g_k(v), v, v))}{v^3(v - g_k(v))}\, dv, \tag{2.2.13}$$

where $\tilde{T}_k(v) \overset{\text{def}}{=} \Phi,_1^k(g_k(v), v, v)/v^2$ and g_k is as defined in Proposition 4. As Φ^{inc} satisfies $(\Phi 7)$ by assumption, the integrand in (2.2.13) is bounded by $(A+Bv^{\beta})/v^3 \in L^1((1, \infty))$. Expression (2.2.12) now follows from Proposition 5, Theorem 3, (2.2.7), and the dominated convergence theorem.

3. Cavitation in inhomogeneous materials. In this section we examine some consequences of the arguments of Sec. 2.2 for cavitation in inhomogeneous materials. We will only consider stored energy functions which are radially inhomogeneous, i.e., of the form $\Phi(Ry_1, v_2, v_3)$. We will show that the critical values at which bifurcation to a deformation with a cavity occurs depends crucially on the nature of the material found at the origin. In particular, in the inhomogeneous incompressible case, we show that the critical load at which cavitation occurs is the same as the critical load at which cavitation occurs for a *homogeneous* ball composed entirely of the material found at the origin. In the inhomogeneous compressible case, we show that cavitation occurs for sufficiently severe boundary displacements or loads. We also show that any radial deformation r which keeps the origin fixed (i.e., with $r(0) = 0$) is unstable if $\lim_{R\to 0} r'(R) = l > \lambda^{\text{hom}}_{\text{crit}}$, where $\lambda^{\text{hom}}_{\text{crit}}$ is the critical displacement for cavitation for a homogeneous ball composed entirely of the material found at the origin of the inhomogeneous ball (i.e., a ball with stored energy function $\Phi(0, v_1, v_2, v_3)$). This result for the compressible case should be compared with James and Spector [7, Theorem 4.2].

We remark finally that the assumption of radial inhomogeneity is rather a severe one and that for general inhomogeneities one would expect that singularities such as cavitation could occur at points of the ball other than the centre.

3.1. *The incompressible case.* As in the homogeneous case the constraint of incompressibility forces all kinematically admissible radial maps to be of the form

$$r(R) = (R^3 + A^3)^{1/3}, \qquad R \in [0, 1] \text{ for some } A \geq 0. \tag{3.1.1}$$

By the arguments of Ball [2] (see also Sec. 2), if we specify the radial component of the Cauchy stress on the boundary of B to be P, then the radial component of Cauchy stress corresponding to the map (3.1.1) is given by

$$\hat{T}(R) = P + \int_1^R \frac{2s^2}{r^3}\left[\frac{r}{s}\Phi,_2\left(s, \left(\frac{s}{r}\right)^2, \frac{r}{s}, \frac{r}{s}\right) - \left(\frac{s}{r}\right)^2 \Phi,_1\left(s, \left(\frac{s}{r}\right)^2, \frac{r}{s}, \frac{r}{s}\right)\right] ds,$$

where $\Phi,_i$ denotes $\partial\Phi/\partial v_i$.

If we require that the cavity surface is stress free, then $\hat{T}(0) = 0$ and so P is a root of

$$P = \int_0^{1/A} \frac{2s^2}{\tilde{r}^3}\left[\frac{\tilde{r}}{s}\Phi_{,2}\left(As, \left(\frac{s}{\tilde{r}}\right)^2, \frac{\tilde{r}}{s}, \frac{\tilde{r}}{s}\right) - \left(\frac{s}{\tilde{r}}\right)^2 \Phi_{,1}\left(As, \left(\frac{s}{\tilde{r}}\right)^2, \frac{\tilde{r}}{s}, \frac{\tilde{r}}{s}\right)\right] ds, \tag{3.1.2}$$

where $\tilde{r}(R) \equiv (R^3 + 1)^{1/3}$ (this follows from the change of independent variable $s \to s/A$).

REMARK. In the following theorem we will require that our inhomogeneous, incompressible stored energy function $\Phi(R, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$ satisfy inhomogeneous versions of the constitutive hypotheses $(\Phi 2)$, $(\Phi 6)$ of Sec. 2; by this we will mean that those hypotheses hold with $\hat{\Phi}$ replaced by $\Phi(A(\mathrm{v}^3 - 1)^{1/3}, 1/\mathrm{v}^2, \mathrm{v}, \mathrm{v})$ for any $A > 0$. Thus, for example, $(\Phi 2)$ then becomes the assumption that the incompressible deformation (3.1.1) has finite energy for this inhomogeneous energy function.

THEOREM 7. If the incompressible inhomogeneous stored energy function Φ satisfies the inhomogeneous versions of $(\Phi 2)$, $(\Phi 6)$ and $(\Phi 3)$, $(\Phi 7)$ uniformly in R, then the critical load $P_{\text{crit}}^{\text{inhom}}$ for cavitation is the same as the critical load for a homogeneous ball composed *entirely* of the material found at the origin.

Proof. The critical load $P_{\text{crit}}^{\text{inhom}}$ is given by the limit of (3.1.2) as $A \to 0$. On setting $\mathrm{v} = \tilde{r}/s$ the integral in (3.1.2) takes the form

$$\int_{(1+A^3)^{1/3}}^{\infty} \frac{2}{(\mathrm{v}^2 - 1)\mathrm{v}} \times \left[\mathrm{v}\Phi_{,2}\left(A(\mathrm{v}^3 - 1)^{-1/3}, \frac{1}{\mathrm{v}^2}, \mathrm{v}, \mathrm{v}\right) - \frac{1}{\mathrm{v}^2}\Phi_{,1}\left(A(\mathrm{v}^3 - 1)^{-1/3}, \frac{1}{\mathrm{v}^2}, \mathrm{v}, \mathrm{v}\right)\right] d\mathrm{v}. \tag{3.1.3}$$

By $(\Phi 3)$ this integrand is positive and by $(\Phi 7)$ it is dominated by

$$\frac{2}{(\mathrm{v}^3 - 1)\mathrm{v}}[A + B\mathrm{v}^{\beta}]\left[\mathrm{v} - \frac{1}{\mathrm{v}^2}\right] \in L^1(1, \infty).$$

Thus, by the dominated convergence theorem, we pass to the limit $A \to 0$ in (3.1.3) to obtain

$$P_{\text{crit}}^{\text{inhom}} = \int_1^{\infty} \frac{1}{(\mathrm{v}^3 - 1)} \frac{d}{d\mathrm{v}}\left[\Phi\left(0, \frac{1}{\mathrm{v}^2}, \mathrm{v}, \mathrm{v}\right)\right] d\mathrm{v}. \tag{3.1.4}$$

Comparison with (2.2.7) then yields the result.

3.2. *The compressible case.* In this subsection we study cavitation for radially inhomogeneous materials. In particular, for simplicity, we will focus on the class of stored energy functions of the form

$$\tilde{\Phi}(R, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) = \sum_{i=1}^{3} \alpha(R)\varphi(\mathrm{v}_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \beta(R)\psi(\mathrm{v}_i\mathrm{v}_j) + \gamma(R)h(\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3), \tag{3.2.1}$$

where α, β, γ are smooth positive functions in $C^2([0, 1])$. (However, many of the results and techniques will apply to more general energy functions.) We will assume

throughout that the following hypotheses hold:

(I) φ, ψ, h are nonnegative and C^3 on $(0, \infty)$.

(II) φ, h are strictly convex and ψ is convex.

(III) $h(\delta) \to \infty$ as $\delta \to 0$ and $h(\delta)/\delta \to \infty$ as $\delta \to \infty$.

(IV) $(v^2/(v^3-1)^2)\tilde{\Phi}(1/(v^3-1)^{1/3}, 1/v^2, v, v) \in L^1(\delta, \infty)$ for $\delta \in (1, \infty)$.

(V) There exist constants $M, \varepsilon_0 \in (0, \infty)$ such that

$$\left|\frac{\partial \tilde{\Phi}}{\partial v_i}(R, v_1, \alpha_2 v_2, \alpha_3 v_3)v_i\right| \leq M[\tilde{\Phi}(R, v_1, v_2, v_3) + 1]$$

if $|\alpha_i - 1| < \varepsilon_0$, $i = 2, 3$, $\forall R \in [0, 1]$.

In this section we treat the displacement boundary value problem. We define the corresponding inhomogeneous energy functional I^{inhom} on $\mathscr{A}_\lambda$ (given by (2.2)) by

$$I^{\text{inhom}}(r) = \int_0^1 R^2 \tilde{\Phi}\left(R, r', \frac{r}{R}, \frac{r}{R}\right) dR \tag{3.2.2}$$

and the corresponding equilibrium equation is

$$\frac{d}{dR}\left[R^2 \frac{\partial \tilde{\Phi}}{\partial v_1}\left(R, r', \frac{r}{R}, \frac{r}{R}\right)\right] = 2R \frac{\partial \tilde{\Phi}}{\partial v_2}\left(R, r', \frac{r}{R}, \frac{r}{R}\right). \tag{3.2.3}$$

PROPOSITION 8. For each $\lambda \in (0, \infty)$ there exists a minimiser $r(R)$ of I^{inhom} on $\mathscr{A}_\lambda$, moreover $r \in C^2((0, 1])$ and satisfies (3.2.3).

Proof. The existence of a minimiser r of I^{inhom} on $\mathscr{A}_\lambda$ follows, for example, by the techniques of [2, Theorem 7.1]. The smoothness of r follows by the method employed in [12, Proposition 0.3].

REMARK 9. Notice that our class of stored energy functions satisfies

$$\begin{aligned} k_1 \Phi(v_1, v_2, v_3) &\leq \tilde{\Phi}(R, v_1, v_2, v_3) \\ &\leq k_2 \Phi(v_1, v_2, v_3) \quad \forall v_i > 0, \ i = 1, 2, 3 \ \forall R \in [0, 1], \end{aligned}$$

where $k_1, k_2 > 0$ and Φ is the stored energy function of a homogeneous material which exhibits cavitation: simply choose

$$\begin{aligned} k_2 &= \max\{\sup \alpha, \sup \beta, \sup \gamma\}, \\ k_1 &= \min\{\inf \alpha, \inf \beta, \inf \gamma\}, \end{aligned}$$

and

$$\Phi(v_1, v_2, v_3) = \sum \varphi(v_i) + \sum_{i \neq j} \psi(v_i v_j) + h(v_1 v_2 v_3). \tag{3.2.4}$$

We will demonstrate that, for sufficiently large boundary displacements λ, any minimiser of I^{inhom} on $\mathscr{A}_\lambda$ corresponds to a deformation with a cavity. To do this we need the following preparatory result which is a consequence of Ball [2, Proposition 6.7].

PROPOSITION 10. If $r \in \mathscr{A}_\lambda$ and $r(0) = 0$ then $I(r) \geq I(r_{\text{hom}})$, where $r_{\text{hom}}(R) \equiv \lambda R$ and I is the energy functional (1.11) corresponding to the homogeneous stored energy function Φ given by (3.2.4).

We next demonstrate that cavitation occurs for sufficiently large λ.

PROPOSITION 11. If λ is sufficiently large, then any minimiser r of I^{inhom} on $\mathscr{A}_\lambda$ satisfies $r(0) > 0$.

Proof. The result follows by the following elementary argument based on [2] (see also [12]). Let $r_{\text{inc}}(R) = (R^3 + (\lambda^3 - 1))^{1/3}$. Then it is easily verified that $r_{\text{inc}} \in \mathscr{A}_\lambda$. Now let r_λ denote any minimiser of I^{inhom} on $\mathscr{A}_\lambda$ (at least one exists by Proposition 8). Then

$$\Delta E \stackrel{\text{def}}{=} I^{\text{inhom}}(r_{\text{inc}}) - I^{\text{inhom}}(r_\lambda) \leq k_2 I(r_{\text{inc}}) - k_1 I(r_\lambda),$$

where I is the energy functional corresponding to the homogeneous stored energy Φ given by (3.2.4). We now demonstrate that $r_\lambda(0) > 0$ for sufficiently large λ: suppose that $r_\lambda(0) = 0$ for an unbounded set of λ. Then by Proposition 10

$$\Delta E \leq k_2 I(r_{\text{inc}}) - k_1 I(r_{\text{hom}}).$$

Now, setting $\mathrm{v} = r_{\text{inc}}(R)/R$ in the first integral and evaluating the second, we obtain

$$\Delta E \leq k_2(\lambda^3 - 1)\left[\int_\lambda^\infty \frac{\mathrm{v}^2}{(\mathrm{v}^3 - 1)^2}\Phi\left(\frac{1}{\mathrm{v}^2}, \mathrm{v}, \mathrm{v}\right) d\mathrm{v} - \frac{k_1}{k_2}\frac{\Phi(\lambda, \lambda, \lambda)}{(\lambda^3 - 1)}\right].$$

The first term in square brackets is monotone decreasing in λ and the second tends to $-\infty$ as $\lambda \to \infty$, hence ΔE is negative for large λ contradicting the assumption that r_λ is a minimiser of I^{inhom} on $\mathscr{A}_\lambda$. Thus, $r_\lambda(0) > 0$ for all λ sufficiently large.

REMARK. One of the main problems with treating the inhomogeneous case is that we have no explicit representation of a "trivial" solution of (3.2.3), i.e., a solution with $r(0) = 0$. (In the homogeneous case it is the homogeneous deformation $r(R) \equiv \lambda R$.) However, in the next result, we give conditions under which any sufficiently regular deformation $r \in \mathscr{A}_\lambda$ with $r(0) = 0$ is unstable to a variation with a cavity and we relate this to the nature of the material found at the origin of the ball. In particular, if the radial stretch at the origin corresponding to such a deformation exceeds $\lambda_{\text{crit}}^{\text{hom}}$, where $\lambda_{\text{crit}}^{\text{hom}}$ is the critical displacement for a homogeneous ball composed entirely of the material found at the origin, then it is energetically more favourable for the ball to cavitate.

PROPOSITION 12. Let $r \in \mathscr{A}_\lambda$ satisfy $r(0) = 0$ and suppose that $\lim_{R \to 0} r'(R) = l \in (0, \infty]$.

If $l > \lambda_{\text{crit}}^{\text{hom}}$, then r is not a minimiser of I^{inhom} on $\mathscr{A}_\lambda$, where $\lambda_{\text{crit}}^{\text{hom}}$ denotes the critical displacement after which cavitation occurs for a homogeneous ball with stored energy function $\tilde{\Phi}(0, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$.

Proof. We assume without loss of generality that $I^{\text{inhom}}(r) < +\infty$, otherwise the result is trivial. We first consider the case $0 < l < +\infty$.

For $\varepsilon \in (0, 1)$ define

$$\lambda(\varepsilon) = \frac{r(\varepsilon)}{\varepsilon}. \tag{3.2.5}$$

By the hypotheses of the proposition

$$\lambda(\varepsilon) \to l \quad \text{as } \varepsilon \to 0. \tag{3.2.6}$$

By Theorem 1 and our assumption on the structure of $\tilde{\Phi}$, the homogeneous stored energy function $\tilde{\Phi}(0, v_1, v_2, v_3)$ exhibits cavitation. Let $r_c(R)$ be a cavitating equilibrium for $\tilde{\Phi}(0, v_1, v_2, v_3)$. Then by Theorem 1, since $l > \lambda_{\text{crit}}^{\text{hom}}$ for each ε sufficiently small, there exists $\alpha(\varepsilon)$ such that

$$\frac{\alpha}{\varepsilon} r_c\left(\frac{\varepsilon}{\alpha}\right) = \lambda(\varepsilon). \tag{3.2.7}$$

Now define

$$r_\varepsilon(R) = \begin{cases} \alpha r_c(R/\alpha) & \text{for } R \in [0, \varepsilon], \\ r(R) & \text{for } R \in [\varepsilon, 1]. \end{cases}$$

Then

$$\begin{aligned}
\Delta E &= I^{\text{inhom}}(r_\varepsilon) - I^{\text{inhom}}(r) \\
&= \int_0^\varepsilon R^2 \left[\tilde{\Phi}\left(R, r_c'(R/\alpha), \frac{\alpha r_c(R/\alpha)}{R}, \frac{\alpha r_c(R/\alpha)}{R}\right)\right. \\
&\qquad \left. - \tilde{\Phi}\left(R, r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right)\right] dR \\
&= \varepsilon^3 \int_0^1 R^2 \left\{ \left[\tilde{\Phi}\left(\varepsilon R, r_c'(R/\alpha), \alpha\frac{r_c(R/\alpha)}{R}, \frac{\alpha r_c(R/\alpha)}{R}\right)\right.\right. \\
&\qquad \left. - \tilde{\Phi}\left(0, r_c'(R/\alpha), \frac{\alpha r_c(R/\alpha)}{R}, \frac{\alpha r_c(R/\alpha)}{R}\right)\right] \\
&\qquad + \left[\tilde{\Phi}\left(0, r_c'(R/\alpha), \frac{\alpha r_c(R/\alpha)}{R}, \frac{\alpha r_c(R/\alpha)}{R}\right)\right. \\
&\qquad\qquad \left. - \tilde{\Phi}(0, \lambda(\varepsilon), \lambda(\varepsilon), \lambda(\varepsilon))\right] \\
&\qquad + \left[\tilde{\Phi}(0, \lambda(\varepsilon), \lambda(\varepsilon), \lambda(\varepsilon))\right. \\
&\qquad\qquad \left.\left. - \tilde{\Phi}\left(\varepsilon R, r'(\varepsilon R), \frac{r(\varepsilon R)}{\varepsilon R}, \frac{r(\varepsilon R)}{\varepsilon R}\right)\right]\right\} dR.
\end{aligned}$$

In the above integral the first and third terms in square brackets tend to zero as $\varepsilon \to 0$, however the second term converges to a negative constant as $\varepsilon \to 0$, thus ΔE is negative for sufficiently small ε and r is not a minimiser of I^{inhom}.

We next consider the case $l = +\infty$. In this case let

$$r_\varepsilon(R) = \begin{cases} (R^3 + (\lambda^3 - 1)\varepsilon^3)^{1/3} & \text{on } [0, \varepsilon], \\ r(R) & \text{on } (\varepsilon, 1], \end{cases} \tag{3.2.8}$$

where $\lambda = \lambda(\varepsilon)$ as given by (3.2.5). Then

$$\Delta E = I^{\text{inhom}}(r_\varepsilon) - I^{\text{inhom}}(r) \leq k_2 I(r_\varepsilon) - k_1 I(r),$$

where I is the energy functional corresponding to the homogeneous stored energy Φ as given by (3.2.4). By Proposition 10

$$\Delta E \leq k_2 I(r_\varepsilon) - k_1 I(r_{\text{hom}}),$$

where $r_{\text{hom}}(R) = \lambda(\varepsilon)R$. On setting $\mathrm{v} = r_\varepsilon/R$ in the first of these integrals we obtain

$$\Delta E \leq k_2\varepsilon^3(\lambda^3 - 1)\left[\int_{\lambda(\varepsilon)}^{\infty} \frac{\mathrm{v}^2}{(\mathrm{v}^3 - 1)^2}\Phi\left(\frac{1}{\mathrm{v}^2}, \mathrm{v}, \mathrm{v}\right) d\mathrm{v} - \frac{k_1}{k_2}\frac{\Phi(\lambda, \lambda, \lambda)}{(\lambda^3 - 1)}\right].$$

The first term in square brackets is finite by hypothesis (IV) and monotone decreasing since $\lambda(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Hypothesis (III) implies that the second term tends to $-\infty$ as $\varepsilon \to 0$, hence ΔE is negative for sufficiently small ε and r is not a minimiser of I^{inhom}.

REMARK. For general coefficient functions α, β, γ in (3.2.1) one would expect nonuniqueness of "trivial" solutions of (3.2.3) (i.e., those satisfying $r(1) = \lambda$, $r(0) = 0$). These could be obtained, for example, by taking two appropriate maps r_1, r_2 satisfying $r_1(1) = r_2(1) = \lambda$, $r_1(0) = r_2(0) = 0$, substituting them into the differential equation (3.2.3), using (3.2.1), and treating the two equations thus obtained as a pair of differential equations to be solved for the coefficient functions α, β, γ.

In cases where there is uniqueness of solutions to (3.2.3) satisfying $r(1) = \lambda$, $r(0) = 0$, one would expect, by Proposition 12, that bifurcation to a deformation with a cavity occurs when the radial stretch at the origin of this "trivial" deformation which keeps the ball intact reaches $\lambda_{\text{crit}}^{\text{hom}}$ (the critical boundary displacement for a homogeneous ball composed entirely of the material found at the origin). However, a justification of this conjecture would require a detailed study of solutions of (3.2.3).

4. Appendix.

Proof of Proposition 5. The proof proceeds in two stages: we first show that

$$\Phi, {}_1^k(g_k(\mathrm{v}_0), \mathrm{v}_0, \mathrm{v}_0) \to \text{constant} \quad \text{as } k \to 0$$

for each $\mathrm{v}_0 \in (1, \infty)$ and then that this together with our assumptions on Φ^k imply (2.1.11).

Step 1. Fix $\mathrm{v}_0 \in (1, \infty)$. Then by Theorem 3 there exists a constant $c_0 \in (0, \infty)$ such that

$$\lambda_{\text{crit}}^k < \mathrm{v}_0 \quad \text{for } k \in (0, c_0),$$

and so $g_k(\mathrm{v}_0)$ is well defined. Let (k_n), $k_n \in (0, c_0)$, be a sequence with $k_n \to 0$ as $n \to \infty$. Now applying Proposition 2 we may assume without loss of generality that the minimisers r_{k_n} of I^{k_n} on $\mathscr{A}_{\mathrm{v}_0}$ converge to an incompressible deformation and in particular that they satisfy

$$r'_{k_n}(R) \to \tilde{r}'(R) = \left(\frac{R}{\tilde{r}(R)}\right)^2 \quad \text{as } n \to \infty \text{ pointwise for a.e. } R \in (0, 1),$$

and

$$\frac{r_{k_n}(R)}{R} \to \frac{\tilde{r}(R)}{R} \quad \text{as } n \to \infty \text{ pointwise for a.e. } R \in (0, 1), \tag{4.1}$$

where $\tilde{r}(R) = (R^3 + \mathrm{v}_0^3 - 1)^{1/3}$.

By Theorem 1 the r_{k_n} are cavitating equilibrium solutions, thus

$$\frac{1}{v_0^2}\Phi,_1^{k_n}(g_{k_n}(v_0), v_0, v_0) = \frac{1}{(r_{k_n}(1))^2}\Phi,_1^{k_n}(r'_{k_n}(1), r_{k_n}(1), r_{k_n}(1)) = T^k(r_k(1))$$
$$= \int_0^1 \frac{d}{dR}\left[\left(\frac{R}{r_{k_n}}\right)^2 \Phi,_1^k\left(r'_{k_n}, \frac{r_{k_n}}{R}, \frac{r_{k_n}}{R}\right)\right] dR \qquad (4.2)$$

for each n, where we have incorporated the zero stress boundary condition and T^k is given by (1.14) with $\Phi = \Phi^k$. By (1.12) and the definition of Φ^k (2.1), the above expression takes the form

$$\int_0^1 \frac{2R^2}{(r_{k_n})^3}\left[\frac{r_{k_n}}{R}\Phi,_2^{\text{inc}} - r'_{k_n}\Phi,_1^{\text{inc}}\right] dR. \qquad (4.3)$$

The integrand is positive by $(\Phi 3)$, hence using $(\Phi 7)$ we obtain the bounds

$$0 \le \frac{2R^2}{(r_{k_n})^3}\left[\frac{r_{k_n}}{R}\Phi,_2^{\text{inc}} - r'_{k_n}\Phi,_1^{\text{inc}}\right] \le \frac{2R^2}{r_{k_n}^3}\left[\frac{r_{k_n}}{R} - r'_{k_n}\right]\left[A + B\left(\frac{r_{k_n}}{R}\right)^\beta\right]$$
$$\le \frac{4R}{r_{k_n}^2}\left[A + B\left(\frac{r_{k_n}}{R}\right)^\beta\right] \le c_1 + c_2\frac{R^{1-\beta}}{(r_{k_n})^{2-\beta}} \le c_1 + c_3R^{1-\beta} \qquad (4.4)$$

for all n, where c_1, c_2, and c_3 are constants. To obtain (4.4) we have used the fact that $r'_{k_n} < r_{k_n}/R \ \forall R$ and the fact that $r_{k_n}(0)$ is uniformly bounded away from zero by Proposition 2. Since $\beta < 2$ the right-hand side of (4.3) is in $L^1(0, 1)$, hence by (4.1) and the Dominated Convergence Theorem we can pass to the limit in (4.2) and (4.3) to obtain

$$\frac{1}{v_0^2}\Phi,_1^{k_n}(g_{k_n}(v_0), v_0, v_0) \to \int_0^1 \frac{2R^2}{(\tilde{r})^3}\left[\frac{\tilde{r}}{R}\Phi,_2^{\text{inc}} - \tilde{r}'\Phi,_1^{\text{inc}}\right] dR \qquad (4.5)$$

as $n \to \infty$, where $\tilde{r}$ is defined as in (4.1). Setting $v = \tilde{r}(R)/R$ now gives the integral in (4.5) the form

$$\int_{v_0}^\infty \frac{1}{(v^3 - 1)}\frac{d\Phi^{\text{inc}}}{dv}\left(\frac{1}{v^2}, v, v\right) dv \qquad (4.6)$$

and this expression is finite by $(\Phi 6)$. Hence

$$\frac{1}{v_0^2}\Phi,_1^{k_n}(g_{k_n}(v_0), v_0, v_0) \to \text{constant} \quad \text{as } n \to \infty. \qquad (4.7)$$

Step 2. From (4.7) and the definition of Φ^k it follows that

$$\frac{1}{v_0^2}\Phi,_1^{\text{inc}}(g_{k_n}(v_0), v_0, v_0) + h'(k_n, g_{k_n}(v_0)v_0^2 - 1) \to \text{constant} \quad \text{as } n \to \infty. \qquad (4.8)$$

We now suppose for a contradiction that (2.1.11) does not hold so that, without loss

of generality, there exist $\bar{v} \in (1, \infty)$, $\delta_0 > 0$, and a sequence (k_j) converging to zero satisfying either

(i) $g_{k_j}(\bar{v})\bar{v}^2 < 1 - \delta_0 \quad \forall j$ or

(ii) $g_{k_j}(\bar{v})\bar{v}^2 > 1 + \delta_0 \quad \forall j$.

If (i) holds then by the convexity of h for each j

$$h'(k_j, g_{k_j}(\bar{v})\bar{v}^2 - 1) \leq h'(k_j, -\delta_0). \tag{4.9}$$

However,

$$h(k_j, -\delta_0) - h(k_j, 0) = -\delta_0 h'(k_j, \theta_j) \quad \text{for each } j \tag{4.10}$$

for some $\theta_j \in (-\delta_0, 0)$. It then follows from (4.10) and parts (ii), (iv) of hypothesis (H) that

$$h'(k_j, \theta_j) \to -\infty \quad \text{as } j \to \infty. \tag{4.11}$$

Thus, by the convexity of $h(k, \cdot)$,

$$h'(k_j, -\delta_0) \to -\infty \quad \text{as } j \to \infty, \tag{4.12}$$

(4.12), (4.9), (Φ1) and (i) now together contradict (4.8). A similar contradiction is obtained if case (ii) holds.

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References

[1] S. S. Antman and P. V. Negron-Marrero, *The remarkable nature of radially symmetric equilibrium states of aelotropic nonlinearly elastic bodies*, J. Elasticity **18**, 131–164 (1987)

[2] J. M. Ball, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Philos. Trans. Roy. Soc. London Ser. A **306**, 557–611 (1982)

[3] A. N. Gent and P. B. Lindley, *Internal rupture of bonded rubber cylinders in tension*, Proc. Roy. Soc. London Ser. A **249**, 195–205 (1958)

[4] C. O. Horgan and R. Abeyaratne, *A bifurcation problem for a compressible nonlinearly elastic medium: growth of a microvoid*, J. Elasticity **16**, 189–200 (1986)

[5] C. O. Horgan and T. J. Pence, *Cavity formation at the center of a composite incompressible nonlinearly elastic sphere*, J. Appl. Mech. **56**, 302–308 (1989)

[6] C. O. Horgan and T. J. Pence, *Void nucleation in tensile loading of a composite incompressible nonlinearly elastic sphere*, J. Elasticity **21**, 61–82 (1989)

[7] R. D. James and S. J. Spector, *The formation of filamentary voids in solids*, IMA preprint, no. 572, University of Minnesota, 1989

[8] R. J. Knops and C. A. Stuart, *Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity*, Arch. Rational Mech. Anal. **86**, 233–249 (1984)

[9] K. A. Pericak-Spector and S. J. Spector, *Nonuniqueness for a hyperbolic system: cavitation in nonlinear elastodynamics*, Arch. Rational Mech. Anal. **101**, 293–317 (1988)

[10] P. Podio-Guidugli, G. Vergara Caffarelli, and E. G. Virga, *Discontinuous energy minimisers in nonlinear elastostatics: an example of J. Ball revisited*, J. Elasticity **16**, 75–96 (1986)

[11] J. Sivaloganathan, *Cavitation in nonlinear elasticity and related problems*, Ph.D. thesis, Heriot-Watt University, 1984

[12] J. Sivaloganathan, *Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity*, Arch. Rational Mech. Anal. **96**, 97–136 (1986)

[13] J. Sivaloganathan, *A field theory approach to stability of equilibria in radial elasticity*, Math. Proc. Cambridge Philos. Soc. **99**, 589–604 (1986)

[14] J. Sivaloganathan, *The generalised Hamilton-Jacobi inequality and the stability of equilibria in nonlinear elasticity*, Arch. Rational Mech. Anal. **107**, 347–369 (1989)
[15] J. Sivaloganathan, *Implications of rank-one convexity*, Ann. Inst. H. Poincaré Anal. non linéaire **5**, no. 4, 99–118 (1988)
[16] C. A. Stuart, *Radially symmetric cavitation for hyperelastic materials*, Ann. Inst. H. Poincaré Anal. non linéaire **2**, 33–66 (1985)
[17] K. Zhang, *Polyconvexity and stability of equilibria in nonlinear elasticity*, Quart. Mech. Appl. Math. **43**, 215–221 (1990)