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On cavitation and degenerate cavitation under internal hydrostatic pressure

By J. Sivaloganathan

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

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In this paper we present some interesting variants on the mathematical phenomenon of cavitation in nonlinear elasticity. The paper is motivated by experimental work of Gent & Tompkins on pressurized elastomers, the fundamental mathematical work of Ball on cavitation and an example of degenerate cavitation due to Sivaloganathan.

Keywords: nonlinear elasticity; cavitation; degenerate cavitation; calculus of variations

1. Introduction

The setting for many† of the previous studies on cavitation in finite elasticity (see, for example, Horgan & Polignone (1995) and references therein) is one in which a ball of homogeneous compressible hyperelastic material with stored energy function W is held in tension under prescribed radial boundary displacements or loads. It is known that (under appropriate hypotheses on W) the radial deformation that minimizes the total stored energy of the body is one which produces a hole at the centre of the deformed ball, provided the boundary displacements or loads are sufficiently severe (see, for example, Ball 1982; Sivaloganathan 1986*a*). This is the mathematical phenomenon of cavitation. These cavitating deformations have the further property that they are invertible mappings on the punctured ball (i.e. on the domain of the ball minus its centre—which is where the singularity forms).

We first demonstrate that there are classes of materials which may not exhibit the above type of cavitation but which are capable of undergoing cavitation when subjected to an *internal* hydrostatic pressure. One way in which such a situation arises is when, as in the experiments of Gent & Tompkins (1969), samples of polymers are left in a gas, at pressure P_0 say, over a period of hours (Gent & Tompkins (1969) used a number of different gases including argon and nitrogen). The pressurizing gas then dissolves into the minute pores of the polymer and, on release of the external pressure, we can view the polymer as being internally subject to a negative hydrostatic pressure. Gent & Tompkins (1969) allowed the samples to come into equilibrium first and then removed the confining pressure. They found that small holes would then appear in the sample if this pressure P_0 was sufficiently large.

Mathematically, let us assume that our material is homogeneous, compressible, hyperelastic and occupies the domain $\Omega \subset \mathbb{R}^3$ in its reference state. Let $M_+^{3\times 3}$ denote the set of real 3×3 matrices with positive determinant and let $W: M_+^{3\times 3} \to \mathbb{R}^+$ be

† A notable exception is the interesting paper by Müller & Spector (1995).

the stored energy function of the material. A deformation of the body is a map $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ which satisfies the local invertibility condition

$$\det \nabla \boldsymbol{u}(\boldsymbol{x}) > 0 \text{ almost everywhere.} \tag{1.1}$$

Then the total energy stored in the deformation is given by

$$E(\boldsymbol{u}) = \int_\Omega W(\nabla \boldsymbol{u}(\boldsymbol{x}))\, \mathrm{d}x. \tag{1.2}$$

In the *displacement* boundary-value problem we specify the boundary values $\boldsymbol{u}|_{\partial\Omega}$. Under the conditions of internal hydrostatic loading described above we may attempt to model the initial internal material response by the new stored-energy function†

$$W^{(P_0)}(F) = W(F) + P_0 \det F \quad \text{for } F \in M^{3\times 3}. \tag{1.3}$$

The extra term is chosen so that it contributes a term $P_0 I$ (corresponding to the internal hydrostatic pressure) to the Cauchy stress tensor for $W^{(P_0)}$. Of course this will only be valid on a short time-scale since, over a period of time, the gas will come out of solution and the response of the sample will return to that described by the original stored-energy function W. (This appears reasonable provided that the pressure P_0 has not been sufficient to cause permanent damage to the sample—as occurs for large pressures in the experiments of Gent & Tompkins (1969).)

The new stored energy function $W^{(P_0)}$ may now satisfy some of the many hypotheses under which it is known that cavitating equilibria exist (see, for example, Ball 1982; Stuart 1985; Sivaloganathan 1986*a*, *b*; Meynard 1992) even though the original stored-energy function W *may not allow cavitation.* In §§ 2 and 3 we study the displacement and dead-load traction problems for the stored-energy function $W^{(P_0)}$. We give conditions on $W^{(P_0)}$ under which cavitation does not occur when $P_0 = 0$ but does occur for $P_0 > 0$ for sufficiently large boundary displacements or loads (see theorems 2.3, 2.4 and 3.2). We also note that for any fixed boundary displacements, energy minimizers exhibit cavitation if the internal pressure P_0 is sufficiently large (see theorem 2.4 and the subsequent remark). An example of a stored-energy function to which this analysis applies is given in example 2.6.

The experiments of Gent & Tompkins (1969) described earlier correspond to a *traction* boundary-value problem in which the boundary of the specimen is left free (i.e. subject to zero tractions). In this case there is a difficulty in trying to use the modified stored-energy function $W^{(P_0)}$ and the corresponding energy,

$$E^{(P_0)}(\boldsymbol{u}) = \int_\Omega W(\nabla \boldsymbol{u}(\boldsymbol{x})) + P_0 \det \nabla \boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}x, \tag{1.4}$$

to model the situation throughout the polymer sample, since at the boundary of the specimen we expect the dissolved gas to come out of solution almost immediately on release of the confining pressure. To overcome this we first consider extremals of the modified energy functional:

$$\hat{E}^{(P_0)}(\boldsymbol{u}) = \int_\Omega W(\nabla \boldsymbol{u}(\boldsymbol{x})) + P_0 \det \nabla \boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}x - \int_{\partial\Omega} \tfrac{1}{3} P_0 \boldsymbol{u} \cdot (\mathrm{Adj}\, \nabla \boldsymbol{u})^{\mathrm{T}} \boldsymbol{N}\, \mathrm{d}S, \tag{1.5}$$

† It would be interesting to try and derive a form of $W^{(P_0)}$ from a homogenization argument (e.g. by considering the material as an elastic body 'reinforced' through the superposition of gas pockets) but we do not pursue this possibility in this paper.

where $\operatorname{Adj} \nabla \boldsymbol{u}$ is the adjugate matrix of $\nabla \boldsymbol{u}$. The boundary term is chosen so that the natural boundary condition for the variational problem on the outer boundary is that the deformed surface is traction free (for the unmodified stored-energy function W). We also note that the equilibrium equations (Euler–Lagrange equations) corresponding to the two integral functionals (1.4), (1.5) are the same since $\det \nabla \boldsymbol{u}$ is a null Lagrangian and the surface integral in (1.5) only depends on derivatives of $\boldsymbol{u}$ tangential to $\partial \Omega$. The last integral term in (1.5) is closely linked to the distributional Jacobian (see, for example, Müller & Spector 1995, § 8). There also appear to be interesting connections between the functional (1.5) and representations for the relaxed energy functional obtained in Marcellini (1986), but we do not pursue these issues in this paper.

To make further analytic progress in the two types of problem described above (the displacement and traction problems) we restrict attention to the case when $\Omega = B$ is the unit ball in $\mathbb{R}^3$ and the deformations $\boldsymbol{u}$ are radial, i.e. of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{r(R)}{R}\boldsymbol{x}, \quad \text{where } R = |\boldsymbol{x}|, \quad \boldsymbol{x} \in B. \tag{1.6}$$

We say that the deformation (1.6) exhibits cavitation if $r(0) > 0$ so that a hole forms at the centre of B. In the case of radial deformations the functional (1.5) takes the form

$$\hat{E}^{(P_0)}(\boldsymbol{u}) = 4\pi \hat{I}^{(P_0)}(r), \tag{1.7}$$

where

$$\begin{aligned} \hat{I}^{(P_0)}(r) &= \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) + P_0 r'(R) r^2(R)\,\mathrm{d}R - \tfrac{1}{3} P_0 r^3(1) \\ &= \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \mathrm{d}R - \tfrac{1}{3} P_0 r^3(0) \end{aligned} \tag{1.8}$$

and Φ is the representation of the stored-energy function W in terms of the principal stretches of the deformation $\boldsymbol{u}$ (see § 2).

It turns out that the above functional is unbounded below for analogous reasons to those noted in Ball (1982) (for the Cauchy traction problem). To see this unboundedness let r be given by $r(R) = [R^3 + A^3]^{1/3}$ (so that the corresponding deformation (1.6) produces a hole of radius A). Next, change the independent variable to $v = r(R)/R$ in (1.8) to obtain

$$\hat{I}^{(P_0)}(r) = A^3 \int_{(1+A^3)^{1/3}}^{\infty} \frac{v^2}{(v^3-1)^2} \Phi\left(\frac{1}{v^2}, v, v\right) \mathrm{d}v - \tfrac{1}{3} P_0 A^3 \to -\infty \quad \text{as } A \to \infty, \tag{1.9}$$

provided that the energy function Φ satisfies a suitable growth hypothesis so that the integral term is finite (see § 2, condition (H3)). Despite this mathematical observation of unboundedness, in terms of the physical situation being modelled, it appears unreasonable to expect the internal inflating pressure in the cavity to remain constant at the magnitude P_0 for arbitrarily large cavity sizes A. For small cavity sizes A it is reasonable to suppose that the inflating pressure will be maintained at P_0 by diffusion of gas into the cavity from the surrounding material. For larger cavity sizes it is more reasonable to suppose that the corresponding pressure $P(A)$ exerted by the gas in the cavity should approach that predicted by the general gas law (i.e. $P(A)A^3 = \text{const.}$). Hence, noting that the last term in (1.8) represents the work done by the gas at pressure P_0 in opening the cavity from zero radius to radius A, we are led to replace the functional in (1.8) by the functional

$$\tilde{I}(r) = \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \mathrm{d}R - H(r(0)), \tag{1.10}$$

where

$$H(r(0)) = \int_0^{r(0)} P(t)t^2 \, \mathrm{d}t \tag{1.11}$$

and the function $P : [0,\infty) \to (0,\infty)$ is continuous satisfying $P(A) = P_0$ for all $|A| \leqslant \epsilon$ and $A^3 P(A) \to c$ as $A \to \infty$, where $\epsilon, c > 0$ are constants.

With this modification the previous argument showing unboundedness below the energy functional fails and theorem 4.2 gives conditions under which $\tilde{I}$ attains a minimum in the class of radial deformations. Moreover, we prove that minimizers must exhibit cavitation if P_0 is sufficiently large (in accord with the experimental observations of Gent & Tompkins (1969)).

For general existence theorems allowing cavitation without the assumption of radial symmetry we refer to Müller & Spector (1995) (and to Sivaloganathan & Spector (1999) for the case in which the possible points of discontinuity of the deformations are prescribed and the energy functional includes terms penalizing cavity initiation and growth).

A final purpose of this paper is to describe the mathematical phenomenon of degenerate cavitation, which can occur for stored-energy functions $W^{(P_0)}$ of the form (1.3). This phenomenon suggests links between the nonlinear elasticity problems described herein and aspects of the continuum theory of defects and linear fracture mechanics. To describe the degenerate nature of this type of cavitation, recall that in order that a deformation $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be physically admissible we require that it satisfies the local invertibility condition (1.1). Mathematically, the constraint (1.1) is accommodated by requiring that the stored-energy function W satisfies (see, for example, Ball 1977; Ciarlet 1988)

$$W(F) \to \infty \quad \text{as } \det F \to 0. \tag{1.12}$$

The phenomenon of degenerate cavitation can occur when (1.1) fails. In cases when it occurs, for sufficiently severe boundary displacements or loads, any minimizing sequence of the total stored energy in a class of mappings satisfying (1.1) has a subsequence converging weakly to a minimizing deformation $\boldsymbol{u}$ which satisfies

$$\det \nabla \boldsymbol{u}(\boldsymbol{x}) \geqslant 0 \quad \text{for } \boldsymbol{x} \in \Omega, \tag{1.13}$$

where in particular $\det \nabla \boldsymbol{u}(\boldsymbol{x}) = 0$ on a set of non-zero measure. Specifically, in the radial-displacement boundary-value problem (in which $\Omega = B$, the unit ball in $\mathbb{R}^3$) this limit deformation still produces a hole at the centre of the deformed ball for large boundary displacements as in the non-degenerate case. However, in degenerate cavitation, there is also a 'core' region† around the centre in which the

† Our terminology comes from the striking similarity between some of these solutions (such as example 5.9), and solutions used to model defects by a centre of dilation in the continuum theory of defects in crystals (see, for example, Teodosiu 1982).

limit deformation satisfies $\det \nabla \boldsymbol{u}(\boldsymbol{x}) = 0$. The size of the core region varies with the boundary conditions and increases, for example, as the boundary displacement is increased (see example 5.9). In theorem 5.1 we give conditions (for the radial-displacement boundary-value problem) on the stored-energy function under which any minimizing sequence for the total stored energy in a class of maps satisfying (1.1) converges weakly to a minimizer in the class of deformations satisfying (1.13). Theorems 5.7 and 5.8 together give sufficient conditions under which degenerate cavitation occurs.

Example 5.9 demonstrates that the mathematical phenomenon of degenerate cavitation under internal pressure loading can occur even when the underlying constitutive law for the material is *linear*, the nonlinear behaviour arising from the invertibility constraint (1.13). The minimizer given in the example bears a striking similarity to singular solutions of *linear elasticity* which have been classically used in the continuum theory of defects to model point defects as centres of dilation (see Teodosiu 1982). In this classical approach one postulates the existence of a 'core region' around the defect in which, typically, an atomistic model is used to calculate the energy stored by the deformation. The main obstacle that the use of a core region overcomes is that the singular linear elastic solutions employed have infinite energy. Typically, the size and shape of the core region is determined by ad hoc arguments. Interestingly, in our example a 'core region', B_{R_0}, emerges as part of the minimization and its radius is uniquely determined by the boundary data. Examples such as this may indicate a way to unify classical approaches to modelling defects and fracture with more recent work in nonlinear elasticity and the calculus of variations on the existence of singular discontinuous equilibria (such as Ball 1982; Stuart 1985; Sivaloganathan 1986*a*; James & Spector 1991, 1992; Müller & Spector 1995).

Assumptions

We assume that our material is compressible, homogeneous, hyperelastic and occupies the region $B = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < 1\}$ in its reference state.

We suppose further that the stored energy function W is both *frame indifferent* and *isotropic* so that

$$W(F) = W(QF) = W(FQ) \quad \forall Q \in SO(3), \quad F \in M_+^{3\times 3}. \tag{1.14}$$

Hence there is a symmetric function $\Phi : \mathbb{R}^3_{++} \to [0, \infty)$ such that $\Phi(v_1, v_2, v_3) = W(F)$ for all $F \in M_+^{3\times 3}$, where the v_i are the eigenvalues of $\sqrt{F^{\mathrm{T}}F}$ (known as the principal stretches) and $\mathbb{R}^3_{++} = \{\boldsymbol{v} \in \mathbb{R}^3 : v_i > 0, i = 1, 2, 3\}$, denotes the positive octant of $\mathbb{R}^3$. We denote the partial derivatives $\partial\Phi/\partial v_i$ by $\Phi_{,i}$, $i = 1, 2, 3$. We restrict attention to radial deformations of the ball, i.e. to deformations of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{r(R)}{R}\boldsymbol{x}, \quad \text{where } R = |\boldsymbol{x}|, \quad \boldsymbol{x} \in B. \tag{1.15}$$

For such deformations

$$\nabla \boldsymbol{u}(\boldsymbol{x}) = r'(R)\frac{\boldsymbol{x} \otimes \boldsymbol{x}}{R^2} + \frac{r(R)}{R}\left[I - \frac{\boldsymbol{x} \otimes \boldsymbol{x}}{R^2}\right] \tag{1.16}$$

and hence the corresponding principal stretches are

$$v_1 = r'(R), \qquad v_2 = v_3 = \frac{r(R)}{R}. \tag{1.17}$$

(See Ciarlet (1988) and Truesdell & Noll (1965) for further details concerning hyperelasticity.)

2. Displacement boundary-value problem

We first consider the radial-displacement boundary-value problem in which we require that

$$\boldsymbol{u}(\boldsymbol{x}) = \lambda \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \partial B, \quad \text{for some } \lambda > 0. \tag{2.1}$$

The total energy stored in a radial deformation $\boldsymbol{u}$ of the ball is given by

$$E(\boldsymbol{u}) = \int_{\Omega} W(\nabla \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}x = 4\pi I(r) = 4\pi \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right)\mathrm{d}R, \tag{2.2}$$

and this is defined on

$$\mathcal{A}_\lambda = \{r \in W^{1,1}((0,1)) : r(1) = \lambda,\ r' > 0 \text{ almost everywhere},\ r(0) \geqslant 0\}. \tag{2.3}$$

The condition $r' > 0$ corresponds to the invertibility condition (1.1) for maps of the form (1.15) and the condition $r(1) = \lambda$ corresponds to the boundary condition (2.1).

Constitutive hypotheses

We assume that $\Phi \in \mathrm{C}^2(\mathbb{R}^3_{++})$ and will refer to the following hypotheses on Φ. (We refer to Ogden (1984) for examples of stored-energy functions and their correlation with experimental data.)

(H1) $\Phi_{,ii}(v_1, v_2, v_3) > 0$, $i = 1, 2, 3$ on $\mathbb{R}^3_{++}$ (the tension-extension inequality).

(H2) $\Phi(v_1, v_2, v_3) \geqslant \psi(v_1)$ where $\psi : (0, \infty) \to \mathbb{R}^+$ is continuous and satisfies

$$\text{(i)}\ \frac{\psi(t)}{t} \to \infty, \quad \text{as } t \to \infty,$$
$$\text{(ii)}\ \psi(t) \to \infty, \quad \text{as } t \to 0.$$

(H3)

$$\frac{v^2}{(v^3-1)^2}\Phi\left(\frac{1}{v^2}, v, v\right) \in L^1((\delta, \infty)) \quad \text{for } \delta > 1.$$

(H4)

$$\left(\frac{v_i\Phi_{,i} - v_j\Phi_{,j}}{v_i - v_j}\right) \geqslant 0$$

for $i \neq j$ and $v_i \neq v_j$.

(H5) (i) Either

$$\lim_{v_1, v_2 \to \infty, v_1 < v_2} \left(\frac{\Phi_{,1}(v_1, v_2, v_2)}{v_2^2}\right) = \infty$$

or

$$\lim_{v_1,v_2\to\infty, v_1<v_2} [\Phi(v_1,v_2,v_2) - v_1\Phi_{,1}(v_1,v_2,v_2)] = -\infty.$$

(ii) Either

$$\lim_{v_1,v_2\to 0, v_1>v_2} \left(\frac{\Phi_{,1}(v_1,v_2,v_2)}{v_2^2}\right) = -\infty$$

or

$$\lim_{v_1,v_2\to 0, v_1>v_2} [\Phi(v_1,v_2,v_2) - v_1\Phi_{,1}(v_1,v_2,v_2)] = \infty.$$

(H6) For any fixed $a \in (0,\infty)$,

$$\Phi_{,1}(v_1,a,a) \to +\infty, -\infty$$

as $v_1 \to \infty, 0$, respectively.

(H7) There exist constants $M, \varepsilon_0 > 0$ such that

$$|\Phi_{,i}(v_1,\alpha_2 v_2,\alpha_3 v_3)v_i| < M[\Phi(v_1,v_2,v_3)+1] \quad \text{(no summation)} \tag{2.4}$$

if $|\alpha_i - 1| < \varepsilon_0$, $i = 2,3$.

Remark 2.1. Hypothesis (H3) guarantees that if $\lambda > 1$, then the energy (2.2) associated with the discontinuous incompressible map $\boldsymbol{u}$ (given by (2.1)) with $r(R) = (R^3 + (\lambda^3 - 1))^{1/3}$ is finite. The inequalities (H4) are a weakened form of the Baker–Ericksen inequalities (see Ball 1982, expression (3.10)). Hypothesis (H7) is often satisfied by polynomial or simple rational functions of v_1, v_2, v_3.

Theorem 2.2. *Let Φ satisfy (H1), (H2) and let $\lambda > 0$, $P \geqslant 0$ be given. Let $I^{(P)}$ be defined by*

$$I^{(P)}(r) = I(r) + \frac{1}{4\pi}P\int_B \det\nabla\boldsymbol{u} = I(r) + P\int_0^1 r^2 r'\,\mathrm{d}R. \tag{2.5}$$

Then any minimizing sequence for $I^{(P)}$ on $\mathcal{A}_\lambda$ has a subsequence converging weakly in $W^{1,1}((1,\delta))$, any $\delta \in (0,1)$, to some $\tilde{r} \in \mathcal{A}_\lambda$, where

$$I^{(P)}(\tilde{r}) = \inf_{\mathcal{A}_\lambda} I^{(P)}$$

(i.e. $\tilde{r}$ is a minimizer of $I^{(P)}$ on $\mathcal{A}_\lambda$).

Proof. This follows by the statement and method of proof of proposition 4.1 in Sivaloganathan (1986*a*) applied to $\Phi^{(P)}(v_1,v_2,v_3) = \Phi(v_1,v_2,v_3) + Pv_1v_2v_3$. ■

Theorem 2.3. *Let Φ satisfy (H1)–(H7) and let $P > 0$ be given. Then for λ sufficiently large, any minimizer $r(R)$ of $I^{(P)}$ on $\mathcal{A}_\lambda$ satisfies $r(0) > 0$.*

Proof. This follows directly from proposition 4.7 in Sivaloganathan (1986*a*) applied to $\Phi^{(P)}(v_1,v_2,v_3) = \Phi(v_1,v_2,v_3) + Pv_1v_2v_3$. ■

The next result notes that if the boundary condition λ is held fixed and the internal pressure P is increased, then again cavitation occurs. It also notes hypotheses under which cavitation does not occur in the case $P = 0$.

Theorem 2.4. *Let Φ satisfy (H1)–(H7) and let $\lambda > 0$ be given. If $P = 0$ and Φ satisfies*

$$\frac{\Phi(v,v,v)}{v^3} \to 0 \quad \text{as } v \to \infty, \tag{2.6}$$

then the unique minimizer of $I^{(0)} = I$ is the homogeneous map $r_\lambda^{\text{hom}}(R) \equiv \lambda R$. Given $\lambda > 1$, any minimizer r of $I^{(P)}$ on $\mathcal{A}_\lambda$ satisfies $r(0) > 0$ if P is sufficiently large.

Proof. If $P = 0$ and condition (2.6) holds, then by theorem 2.2 a minimizer $r(R)$ exists. By (H1),

$$R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \geqslant R^2\left[\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) + \left(r' - \frac{r}{R}\right)\Phi_{,1}\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right)\right] \tag{2.7}$$

for almost every $R \in [0, 1]$.

If $r(0) = 0$, then, by propositions 0.3, 4.2 of Sivaloganathan (1986*a*), $r(R)$ satisfies $r \in \mathbb{C}^2((0,1]) \cap \mathbb{C}^1([0,1])$ and $r(R)/R \to l$ as $R \to 0$ for some $l \in (0, \infty)$.

Now, given $\varepsilon \in (0,1)$, using the symmetry of Φ and integrating (2.7) gives

$$\begin{aligned}\int_\varepsilon^1 R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \mathrm{d}R &\geqslant \left[\tfrac{1}{3}R^3\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right)\right]_\varepsilon^1 \\ &= \tfrac{1}{3}\Phi(\lambda,\lambda,\lambda) - \tfrac{1}{3}\varepsilon^3\Phi\left(\frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}\right).\end{aligned}$$

Passing to the limit $\epsilon \to 0$ gives

$$I(r) \geqslant \tfrac{1}{3}\Phi(\lambda,\lambda,\lambda) = I(r_\lambda^{\text{hom}}). \tag{2.8}$$

Next suppose that $r(0) > 0$ so that $\lim_{R\to 0} r(R)/R = \infty$. Then integrating (2.7) and proceeding analogously to the above argument gives

$$\begin{aligned}\int_\varepsilon^1 R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \mathrm{d}R &\geqslant \tfrac{1}{3}\Phi(\lambda,\lambda,\lambda) - \tfrac{1}{3}\varepsilon^3\Phi\left(\frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}\right) \\ &= I(r_\lambda^{\text{hom}}) - \tfrac{1}{3}r(\varepsilon)^3\Phi\left(\frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}\right)\Big/\left(\frac{r(\varepsilon)}{\varepsilon}\right)^3.\end{aligned}$$

Passing to the limit $\varepsilon \to 0$ using (2.6) gives again

$$I(r) \geqslant I(r_\lambda^{\text{hom}}). \tag{2.9}$$

Hence, by (2.8) and (2.9), cavitation does not occur for $P = 0$ since the homogeneous map r_λ^{hom} is the global minimizer in this case for any $\lambda > 0$ (note that the inequality (2.9) is strict if $r \neq r_\lambda^{\text{hom}}$ on a set of non-zero measure).

The proof of the second part of this theorem that for large P any minimizer $r(R)$ satisfies $r(0) > 0$ follows from a straightforward modification of the proof of proposition 4.7 in Sivaloganathan (1986*a*) (by showing that the incompressible map

$$r(R) = (R^3 + (\lambda^3 - 1))^{1/3} \tag{2.10}$$

has less energy than the homogeneous map r_λ^{hom} for large P). ∎

Remark 2.5. The restriction $\lambda > 1$ in the second part of theorem 2.4 stems from the use of the test map (2.10) in the above proof. Notice that this map is not physically admissible if $\lambda < 1$. This restriction can be relaxed to allow any $\lambda > 0$ if we replace hypothesis (H3) by the assumption that for any $\lambda > 0$ there exists some map $r \in \mathcal{A}_\lambda$, satisfying $r(0) > 0$ that has finite energy (this assumption is often satisfied by stored-energy functions that satisfy (H3)).

Example 2.6. Consider the stored-energy function

$$\Phi(v_1, v_2, v_3) = \sum_{i=1}^{3} \mu\left[\frac{2}{v_i} + v_i^2\right],$$

where $\mu > 0$ is a constant.

By theorems 2.3, 2.4, cavitation does not occur in the above example for $P = 0$ but does occur for any given $P > 0$ for sufficiently large boundary displacements λ, or for fixed λ for sufficiently large P.

3. The traction boundary-value problem

In the radial dead-load traction problem we replace the condition (2.1) by specifying that the radial Piola stress $\boldsymbol{T}^{\mathrm{R}}(\boldsymbol{x})$ corresponding to the deformation (1.15) satisfies

$$\boldsymbol{T}^{\mathrm{R}}(\boldsymbol{x}) = T\boldsymbol{x} \quad \forall \boldsymbol{x} \in \partial B,$$

where $T \in \mathbb{R}$ is a given constant.

Corresponding to this boundary condition and under internal hydrostatic pressure of magnitude $P \geqslant 0$, in the class of radial deformations, we seek to minimize

$$E_P(r) = \int_0^1 R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) + Pr'r^2 \,\mathrm{d}R - Tr(1) \tag{3.1}$$

on

$$\mathcal{A} = \{r \in W^{1,1}((0,1)) : r(0) \geqslant 0,\ r' > 0 \text{ a.e.}\}. \tag{3.2}$$

Our first result is that E_P attains a minimum on $\mathcal{A}$.

Proposition 3.1. *Let Φ satisfy (H1), (H2). Then E_P attains a minimum on*

$$\mathcal{A} = \{r \in W^{1,1}((0,1)) : r(0) \geqslant 0,\ r' > 0 \text{ a.e}\}.$$

Proof. By (H2)(i) it follows that

$$E_P(r) \geqslant \frac{1}{2}\int_0^1 R^2 \psi(r') \,\mathrm{d}R + k_1 \int_0^1 R^2 r' + k_2 \int_0^1 R^2 \frac{r}{R} + k_3 - Tr(1)$$

for any $r \in \mathcal{A}$, where $k_1, k_2 \geqslant 0$ may be chosen arbitrarily large by suitable choice of k_3. Thus, integrating the second integral by parts yields

$$E_P(r) \geqslant \frac{1}{2}\int_0^1 R^2 \psi(r') \,\mathrm{d}R + (k_1 - T)r(1) + k_3 + (k_2 - 2k_1)\int_0^1 Rr \,\mathrm{d}R\,;$$

choosing $k_1 > T$, $k_2 - 2k_1 \geqslant 0$, yields

$$E_P(r) \geqslant \frac{1}{2}\int_0^1 R^2\psi(r')\,\mathrm{d}R + (k_1 - T)r(1) + k_3. \tag{3.3}$$

By (3.3) if $(r_n) \subset \mathcal{A}$ is a minimizing sequence for E_P, then $(r_n(1))$ is a bounded sequence and by the arguments of proposition 4.1 of Sivaloganathan (1986*a*) it now follows that there exists $\tilde{r} \in \mathcal{A}$ such that $r_n \rightharpoonup \tilde{r}$ in $W^{1,1}((\delta, 1))$ as $n \to \infty$ for any $\delta \in (0, 1)$. Thus, by standard lower semicontinuity arguments (see, for example, Ball *et al.* 1981), it now follows that

$$E_P(\tilde{r}) \leqslant \liminf_{n\to\infty} E_P(r_n) = \inf_{\mathcal{A}} E_P$$

so that $\tilde{r}$ is a minimizer. ■

The next theorem notes conditions under which cavitation does not occur when $P = 0$, but does occur for large boundary tractions T for any fixed $P > 0$.

Theorem 3.2. *If Φ satisfies (H1)–(H7) and $P > 0$, then for all T sufficiently large any minimizer of E_P on $\mathcal{A}$ satisfies $r(0) > 0$. If $P = 0$ and Φ satisfies*

$$\frac{\Phi(v, v, v)}{v^3} \to 0$$

as $v \to \infty$, then any minimizer r of E_P on $\mathcal{A}$ satisfies $r(R) \equiv \lambda R$ for some $\lambda > 0$.

Proof. By proposition 3.1 a minimizer $r(R)$ of E_P exists. By propositions 0.3 and 4.5 of Sivaloganathan (1986*a*) it follows that if $r(0) = 0$, then $r(R) \equiv \mu R$ for some $\mu > 0$. The proof of the first part of theorem 3.2 follows the method of Ball (1982) by showing that the homogeneous map $r_\mu^{\mathrm{hom}}(R) \equiv \mu R$ is not a minimizer of E_p for any $\mu > 0$. To do this we use two sets of test functions, the first given by

$$r_\mu(R) = [R^3 + \mu^3]^{1/3}.$$

In this case the energy difference between $r_\mu(R)$ and r_μ^{hom} is given by

$$\begin{aligned}\Delta E &= E_P(r_\mu) - E_P(r_\mu^{\mathrm{hom}}) \\ &= \mu^3 \int_{\hat{\mu}}^{\infty} \frac{v^2}{(v^3 - 1)^2}\hat{\Phi}^{(P)}(v)\,\mathrm{d}v - \tfrac{1}{3}\Phi^{(P)}(\mu, \mu, \mu) - T(\hat{\mu} - \mu),\end{aligned}$$

where $\Phi^{(P)} = \Phi + Pv_1v_2v_3$ and $\hat{\Phi}^{(P)}(v) = \Phi^{(P)}(1/v^2, v, v)$ and we have made the change of variable $v = r_\mu/R$ and set $\hat{\mu} = r_\mu(1) = (1 + \mu^3)^{1/3}$. Hence if T is positive,

$$\Delta E \leqslant \mu^3\left[\int_{\hat{\mu}}^{\infty} \frac{v^2}{(v^3 - 1)^2}\Phi(v)\,\mathrm{d}v + \frac{P}{3\mu^3} - \frac{\Phi^{(P)}(\mu, \mu, \mu)}{3\mu^3}\right].$$

Since

$$\frac{\Phi^{(P)}(\mu, \mu, \mu)}{\mu^3} \geqslant P > 0$$

for all μ, it follows from (H3) that ΔE is negative provided μ is sufficiently large, $\mu > k$ say, where $k \in (0, \infty)$.

For $\mu \leqslant k$ we use the test functions

$$r_\mu(R) = [R^3 + \delta^3]^{1/3},$$

where $\delta^3 = (2+\mu)^3 - 1$. The energy difference then takes the form

$$\begin{aligned}\Delta E &= E_P(r_\mu) - E_P(r_\mu^{\text{hom}}) \\ &= \delta^3 \int_{(2+\mu)}^{\infty} \frac{v^2}{(v^3-1)^2} \hat{\Phi}^{(P)}(v)\,\mathrm{d}v - \tfrac{1}{3}\Phi^{(P)}(\mu,\mu,\mu) - 2T. \end{aligned} \tag{3.4}$$

By (H3), the integral term is clearly bounded for $\mu \leqslant k$ and so the right-hand side of (3.4) is negative for T sufficiently large. Thus r_μ^{hom} cannot be a minimizer of E_P for any μ and the result follows.

The proof of the second part of theorem 3.2 follows from theorem 2.4: suppose that $P = 0$ and let $r(R)$ be a minimizer of E_0. Then r minimizes $I^{(0)} = I$ on $\mathcal{A}_\lambda$ with $\lambda = r(1)$ and hence, by theorem 2.4, $r(R) \equiv \lambda R$. ∎

4. Relationship of analysis and experiment

In this section we relate the analytical approaches to cavitation to the experimental results of Gent & Tompkins (1969) outlined earlier. For reasons outlined in the introduction, we model the situation in these experiments by introducing the modified energy functional:

$$\tilde{I}(r) = \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \mathrm{d}R - H(r(0)), \tag{4.1}$$

where

$$H(r(0)) = \int_0^{r(0)} P(t)t^2\,\mathrm{d}t, \tag{4.2}$$

and the function $P : [0,\infty) \to (0,\infty)$ is continuous satisfying $P(A) = P_0$ for all $|A| \leqslant \epsilon$ and $A^3 P(A) \to c$ as $A \to \infty$, where $\epsilon, c > 0$ are constants.

The next lemma notes, in particular, that under the above hypotheses the function $H(t)$ cannot grow quicker than logarithmically in t.

Lemma 4.1. *There exist constants $\alpha > 0, \beta$ such that*

$$\tfrac{1}{3} P_0 \min\{t^3, \epsilon^3\} \leqslant H(t) \leqslant \alpha \log t + \beta \quad \forall t \geqslant 0. \tag{4.3}$$

Proof. We first prove the upper bound on $H(t)$. By the assumption $A^3 P(A) \to c$ as $A \to \infty$ there exists d such that

$$t^2 P(t) < \frac{c+1}{t} \quad \forall t \geqslant d$$

and hence

$$\int_d^t s^2 P(s)\,\mathrm{d}s < (c+1)\log(t/d),$$

from which it follows that

$$H(t) \leqslant \alpha \log t + \beta \quad \forall t \geqslant 0,$$

where

$$\alpha = c + 1 \quad \text{and} \quad \beta = \int_0^d s^2 P(s)\,\mathrm{d}s - (c+1)\log d.$$

The lower bound on $H(t)$ follows easily from the non-negativity of $P(t)$ and the assumption that $P(t) = P_0$ for $0 \leqslant t \leqslant \epsilon$. ■

The next result shows that if the internal pressure P_0 is sufficiently large, then any minimizer must exhibit cavitation.

Theorem 4.2. *If Φ satisfies (H1)–(H7), then $\tilde{I}$ attains a minimum on $\mathcal{A}$ (given by (3.2)) and any minimizer r exhibits cavitation for P_0 sufficiently large.*

Proof. By lemma 4.1 it follows that for any $r \in \mathcal{A}$ we have

$$\tilde{I}(r) \geqslant \int_0^1 R^2 \psi(r'(R))\,\mathrm{d}R - \alpha \log(r(0)) - \beta.$$

By setting $T = 0$ in the first steps the proof of proposition 3.1 that lead to expression (3.3), we obtain

$$\tilde{I}(r) \geqslant \frac{1}{2}\int_0^1 R^2 \psi(r'(R))\,\mathrm{d}R + k_1 r(1) + k_3 - \alpha \log(r(0)) - \beta$$

and (since $\log t \leqslant t$ and $r(1) > r(0)$ for $r \in \mathcal{A}$) we obtain, on choosing $k_1 > 2\alpha$, that

$$\tilde{I}(r) \geqslant \frac{1}{2}\int_0^1 R^2 \psi(r'(R))\,\mathrm{d}R + \alpha r(1) - \beta + k_3.$$

The remainder of the proof of the existence of a minimizer $r(R)$ is exactly analogous to that of proposition 3.1.

We next prove that a minimizer $r(R)$ of $\tilde{I}(r)$ must exhibit cavitation (i.e. $r(0) > 0$) if P_0 is sufficiently large. By the remarks in the proof of theorem 3.2 if $r(0) = 0$, then $r(R) \equiv \mu R$ for some $\mu > 0$ (note that $H(0) = 0$). We show that $r_\mu^{\mathrm{hom}} \equiv \mu R$ is not a minimizer for any μ by using the test function

$$r(R) = (R^3 + \epsilon^3)^{1/3}$$

and observing that, by lemma 4.1,

$$\Delta E = \tilde{I}(r) - \tilde{I}(r_\mu^{\mathrm{hom}}) = I(r) - \tfrac{1}{3}\Phi(\mu,\mu,\mu) - H(\epsilon) + H(0) \leqslant I(r) - \tfrac{1}{3}P_0\epsilon^3$$

uniformly in μ (where we have used the non-negativity of Φ). The integral $I(r)$ is finite by assumption (H3) and hence ΔE is clearly negative if P_0 is sufficiently large. Hence r_μ^{hom} is not a minimizer for any μ and thus any minimizer r must satisfy $r(0) > 0$ for large P_0 as claimed. ■

5. Degenerate cavitation

In this section we introduce the mathematical phenomenon of degenerate cavitation. This phenomenon typically occurs when the stored-energy function W does not satisfy the condition $W(F) \to \infty$ as $\det F \to 0$. To outline the main features of degenerate cavitation we introduce the following alternative to hypothesis (H2).

(H2′) $\Phi(v_1, v_2, v_3) \geqslant \psi(v_1)$, where $\psi : [0, \infty) \to \mathbb{R}^+$ is continuous and satisfies:

(i) $\psi(t)/t \to \infty$ as $t \to \infty$;

(ii) Φ can be extended to the *closed* octant $\bar{\mathbb{R}}^3_{++}$ as a $\mathbb{C}^2$ function.

Under the above hypothesis we have the following analogue of theorem 2.2.

Theorem 5.1. *If Φ satisfies (H1), (H2′), then $I^{(P)}$ (given by (2.5)) attains a minimum on*

$$\bar{\mathcal{A}}_\lambda = \{r \in W^{1,1}((0,1)) : r(1) = \lambda,\ r' \geqslant 0 \text{ a.e.}, r(0) \geqslant 0\} . \tag{5.1}$$

Proof. This is a straightforward modification of theorem 2.2, where the condition (H2)(ii) is used to show that any minimizer r satisfies $r'(R) > 0$ a.e. (note that it is a consequence of Mazur's theorem that any weak limit r of a sequence in $\mathcal{A}_\lambda$ satisfies $r' \geqslant 0$ a.e.). Hence under the hypothesis (H2′)(ii) we can only conclude that a minimizer $r(R)$ satisfies $r'(R) \geqslant 0$ almost everywhere. It is straightforward to prove that all the results proved in §§ 2–4 hold if (H2) is replaced by (H2′) and $\mathcal{A}_\lambda$ and $\mathcal{A}$ are replaced by $\bar{\mathcal{A}}_\lambda$, $\bar{\mathcal{A}}$, respectively (where $\bar{\mathcal{A}}$ is similarly obtained from the definition (see (3.2)) of $\mathcal{A}$ on replacing the condition $r' > 0$ a.e. by $r' \geqslant 0$ a.e.). For simplicity, however, we restrict attention throughout this section to the displacement boundary-value problem. ■

The remainder of this section studies the displacement problem and hypotheses under which degenerate cavitation will occur: i.e. hypotheses under which a minimizer (whose existence is guaranteed by theorem 5.1) satisfies $r \in \bar{\mathcal{A}}_\lambda \backslash \mathcal{A}_\lambda$.

The next proposition gives conditions under which the energy of a map $r \in \bar{\mathcal{A}}_\lambda$ can be approximated by the energy of a sequence of (invertible) maps $(r_n) \subset \mathcal{A}_\lambda$ which converge to r as $n \to \infty$. This will enable us to prove that $\inf_{\bar{\mathcal{A}}_\lambda} I^{(P)} = \inf_{\mathcal{A}_\lambda} I^{(P)}$. In order to prove this result we introduce the following growth hypothesis on Φ.

(H8) There exist constants $C_1, C_3 > 0, C_2, C_4$ such that, for some $\alpha \in (1, 3)$,

$$C_1 \sum_{i=1}^{3} {v_i}^\alpha + C_2 \leqslant \Phi(v_1, v_2, v_3) \leqslant C_3 \sum_{i=1}^{3} {v_i}^\alpha + C_4 \tag{5.2}$$

for all $v_i \geqslant 0$.

Notice that this assumption precludes singular behaviour of Φ as $v_i \to 0$ and is consistent with (H2′)(ii) (but not with (H2)(ii)). Notice also that, in particular, (H8) implies (H3).

Example 5.2. The hypothesis (H8) is satisfied, for example, if we let

$$\Phi(v_1, v_2, v_3) = \sum_{i=1}^{3} \phi(v_i) + \sum_{i,j=1}^{3} \psi(v_i v_j),$$

where $\phi, \psi : [0, \infty) \to [0, \infty)$ are in $\mathbb{C}^2([0, \infty))$ and for some $\alpha \in (1, 3)$, $\beta \in (0, \frac{3}{2})$ satisfying $2\beta \leqslant \alpha$, we have

$$\frac{\phi(t)}{t^\alpha} \to \text{const.} \neq 0 \quad \text{as } t \to \infty, \qquad \frac{\psi(t)}{t^\beta} \to \text{const.} \quad \text{as } t \to \infty.$$

Proposition 5.3. *Let Φ satisfy (H8) then for any $r \in \bar{\mathcal{A}}_\lambda$, with $I(r) < +\infty$, there exists a sequence $(r_n) \subset \mathcal{A}_\lambda$ such that $(r_n - r) \in W^{1,\infty}((0,1))$ for all $n \in \mathbb{N}$, $\|r_n - r\|_{1,\infty} \to 0$ as $n \to \infty$ and*

$$I(r_n) \to I(r) \quad \text{as } n \to \infty.$$

Proof. Let $r \in \bar{\mathcal{A}}_\lambda$ and suppose that for some $\delta > 0$

$$T = \{R \in [0,1] : r'(R) > \delta\}$$

has non-zero measure. (If not, then $r \equiv \lambda$ and we choose $r_n(R) \equiv \lambda + (R-1)/n$ for all n.) Now define

$$S = \{R \in [0,1] : r'(R) = 0\}.$$

If $\operatorname{meas} S = 0$, then $r \in \mathcal{A}_\lambda$ and there is nothing to prove (choose $r_n = r$ for all $n \in \mathbb{N}$).

Otherwise, define

$$\pi = \tfrac{1}{2}\delta\left[\frac{\chi_S}{\operatorname{meas} S} - \frac{\chi_T}{\operatorname{meas} T}\right] \tag{5.3}$$

(where χ_A denotes the characteristic function of the set A). Then $\pi \in L^\infty$ and $\int_0^1 \pi = 0$. Now, given $\varepsilon \in (0, 2\operatorname{meas} T)$, define

$$r_\varepsilon(R) = \lambda + \int_1^R r'(s) + \varepsilon\pi(s)\,\mathrm{d}s. \tag{5.4}$$

Then

$$r_\varepsilon \in \mathcal{A}_\lambda, \quad r_\varepsilon(0) = r(0), \quad r_\varepsilon - r \in W^{1,\infty}((0,1)) \tag{5.5}$$

and $\|r_\varepsilon - r\|_{1,\infty} \to 0$ as $\varepsilon \to 0$.

Let $(\varepsilon_n) \subset (0, 2\operatorname{meas} T)$ be a sequence converging to zero and let (r_n) denote the corresponding sequence (r_{ε_n}).

Since $I(r) < \infty$, it follows from (H8) that $R^2(r')^\alpha$, $R^2(r/R)^\alpha \in L^1((0,1))$. Thus by (5.5), (5.2) and the dominated convergence theorem it follows that $I(r_n) \to I(r)$ as $n \to \infty$. ■

Corollary 5.4. *For the sequence (r_n) given in proposition 5.3, for any $P \geqslant 0$, $I^{(P)}(r_n) \to I^{(P)}(r)$ as $n \to \infty$.*

Proof. This follows immediately from proposition 5.3, (5.5) and the definition of $I^{(P)}$ (see (2.5)). ■

Theorem 5.5. *Let Φ satisfy hypotheses (H1), (H2′), (H3), (H8) and let $P \geqslant 0$. Then any minimizing sequence for $I^{(P)}$ on $\mathcal{A}_\lambda$ has a subsequence which converges weakly in $W^{1,1}((\delta,1))$, any $\delta \in (0,1)$, to some $\tilde{r} \in \bar{\mathcal{A}}_\lambda$ which satisfies*

$$I^{(P)}(\tilde{r}) = \inf_{\mathcal{A}_\lambda} I^{(P)} = \inf_{\bar{\mathcal{A}}_\lambda} I^{(P)}.$$

Proof. By corollary 5.4,

$$\inf_{\mathcal{A}_\lambda} I(P) = \inf_{\bar{\mathcal{A}}_\lambda} I^{(P)}.$$

It now follows from theorem 5.1 that

$$I^{(P)}(\tilde{r}) = \inf_{r\in\mathcal{A}_\lambda} I^{(P)}(r) = \inf_{r\in\bar{\mathcal{A}}_\lambda} I^{(P)}(r)$$

as required. ■

Remark 5.6. By theorem 5.5, since (H2′) holds, there is the possibility that the minimizer $\tilde{r}$ lies in $\bar{\mathcal{A}}_\lambda\backslash\mathcal{A}_\lambda$. If, in addition, $\tilde{r}(0) > 0$, then this would be an example of degenerate cavitation occurring. Our next theorem gives sufficient conditions that cavitation does not occur for $P = 0$ for any $\lambda > 0$ and that (possibly degenerate) cavitation occurs for any $P > 0$ for λ sufficiently large (depending on P).

Theorem 5.7. *Suppose that Φ satisfies (H1), (H2′), (H8) (so that (H3) holds in particular). If $P = 0$, then, for any $\lambda > 0$, the minimizer of I on $\bar{\mathcal{A}}_\lambda$ (and hence on $\mathcal{A}_\lambda$) is the homogeneous map*

$$r_{\text{hom}}(R) \equiv \lambda R.$$

Given any $P, \lambda > 0$ let $(r_n) \subset \mathcal{A}_\lambda$ be a minimizing sequence for $I^{(P)}$ on $\mathcal{A}_\lambda$. Then there exists $\tilde{r} \in \bar{\mathcal{A}}_\lambda$ and a subsequence of (r_n) converging weakly in $W^{1,1}((\delta, 1))$ to $\tilde{r}$ for any $\delta \in (0, 1)$ and satisfying

$$I^{(P)}(\tilde{r}) = \inf_{\mathcal{A}_\lambda} I^{(P)} = \lim_{n\to\infty} I^{(P)}(r_n).$$

For sufficiently large λ (depending on P), $\tilde{r}$ satisfies

$$\tilde{r}(0) > 0.$$

Proof. We first treat the case $P = 0$ and demonstrate that if $r \in \bar{\mathcal{A}}_\lambda$, then $I(r) \geqslant I(r_{\text{hom}})$. This proof is exactly analogous to that of theorem 2.4. Let $r \in \bar{\mathcal{A}}_\lambda$, then by (H1)

$$R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \geqslant R^2\left[\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) + \left(r' - \frac{r}{R}\right)\Phi_{,1}\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right)\right] \tag{5.6}$$

for almost every $R \in [0, 1]$. We consider separately the two possibilities that

$$\liminf_{R\to 0} r(R)/R$$

is finite or infinite. If $\liminf_{R\to 0} r(R)/R = \eta \geqslant 0$, then choose a sequence $(\varepsilon_n) \subset (0, 1)$ with $\varepsilon_n \to 0$ as $n \to \infty$ for which $r(\epsilon_n)/\epsilon_n \to \eta$ as $n \to \infty$. Then integrating (5.6) gives

$$\int_{\varepsilon_n}^1 R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \mathrm{d}R \geqslant \Big[\tfrac{1}{3}R^3\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right)\Big]_{\varepsilon_n}^{1}$$

$$= \tfrac{1}{3}\Phi(\lambda, \lambda, \lambda) - \tfrac{1}{3}\varepsilon_n^3\Phi\left(\frac{r(\varepsilon_n)}{\varepsilon_n}, \frac{r(\varepsilon_n)}{\varepsilon_n}, \frac{r(\varepsilon_n)}{\varepsilon_n}\right).$$

Passing to the limit $n \to \infty$ and using (H2′)(ii) gives

$$I(r) \geqslant \tfrac{1}{3}\Phi(\lambda, \lambda, \lambda) = I(r_{\text{hom}}). \tag{5.7}$$

Next suppose that $\liminf_{R \to 0} r(R)/R = \infty$. Then integrating (5.6) and proceeding analogously to the above argument gives

$$\begin{aligned}\int_\varepsilon^1 R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \mathrm{d}R &\geqslant \tfrac{1}{3}\Phi(\lambda, \lambda, \lambda) - \tfrac{1}{3}\varepsilon^3\Phi\left(\frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}\right) \\ &= I(r_{\text{hom}}) - \tfrac{1}{3}[r(\varepsilon)]^3\Phi\left(\frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}, \frac{r(\varepsilon)}{\varepsilon}\right) \Big/ \left(\frac{r(\varepsilon)}{\varepsilon}\right)^3.\end{aligned}$$

Passing to the limit $\varepsilon \to 0$ using (H8) again gives

$$I(r) \geqslant I(r_{\text{hom}}). \tag{5.8}$$

Hence, by (5.7) and (5.8), cavitation does not occur for $P = 0$ since the homogeneous map r_{hom} is the global minimizer in this case for any $\lambda > 0$.

Now let $P > 0$. By theorem 5.5 we have that there exists $\tilde{r} \in \bar{\mathcal{A}}_\lambda$ for which $I^{(P)}(\tilde{r}) = \inf_{\mathcal{A}_\lambda} I^{(P)}$. We next demonstrate that $\tilde{r}$ must satisfy

$$\tilde{r}(0) > 0 \tag{5.9}$$

for sufficiently large λ. To see this first observe that a straightforward test-function argument from the proof of proposition 4.7 in Sivaloganathan (1986*a*), using assumption (H3), shows that

$$\inf_{\mathcal{A}_\lambda} I^{(P)} < I^{(P)}(r_{\text{hom}}) \tag{5.10}$$

for sufficiently large λ. (The argument in proposition 4.7 of Sivaloganathan (1986*a*) shows that, if λ is large, then $I^{(P)}(r_{\text{hom}}) > I^{(P)}(\hat{r})$ for the test map $\hat{r} \in \mathcal{A}_\lambda$, $\hat{r}(R) = (R^3 + (\lambda^3 - 1))^{1/3}$, which satisfies $\hat{r}(0) > 0$, $\hat{r}$ being chosen so that the corresponding deformation (1.15) is incompressible. The hypothesis (H3) guarantees that $\hat{r}$ has finite energy.) Let $(r_n) \subset \mathcal{A}_\lambda$ be a minimizing sequence for $I^{(P)}$ for a value of λ sufficiently large that (5.10) holds. Then define

$$\varepsilon_0 = I^{(P)}(r_{\text{hom}}) - \inf_{\mathcal{A}_\lambda} I^{(P)} > 0. \tag{5.11}$$

By the first part of this proof we have that

$$I(r) = \int_0^1 R^2\Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) \mathrm{d}R \geqslant I(r_{\text{hom}}) \quad \forall r \in \mathcal{A}_\lambda.$$

Hence by definition of $I^{(P)}$ and (5.11) it follows that

$$\int_0^1 Pr_n^2 r_n' \,\mathrm{d}R + \tfrac{1}{2}\varepsilon_0 < \int_0^1 R^2 P\lambda^3 \,\mathrm{d}R$$

for large n. Hence

$$\tfrac{1}{3}P[\lambda^3 - r_n^3(0)] + \tfrac{1}{2}\varepsilon_0 < \tfrac{1}{3}P\lambda^3 3 \Rightarrow r_n^3(0) > \frac{3\varepsilon_0}{2P} \quad \text{for large } n. \tag{5.12}$$

Finally (passing to a subsequence if necessary), since the minimizing sequence (r_n) converges weakly to $\tilde{r} \in \bar{\mathcal{A}}_\lambda$ in $W^{1,1}((\delta, 1))$ for any $\delta > 0$, it follows that (r_n) converges to $\tilde{r}$ in $C([\delta, 1])$ for any $\delta > 0$, which together with (5.12) and the definition of $\bar{\mathcal{A}}_\lambda$ implies $\tilde{r}(0) > 0$. ∎

The next result gives sufficient conditions for degenerate cavitation to occur: i.e. we prove that if the minimizer $\tilde{r}$ of theorem 5.7 satisfies $\tilde{r}(0) > 0$, then $\tilde{r} \in \bar{\mathcal{A}}_\lambda \backslash \mathcal{A}_\lambda$. To state the result we require the following hypothesis on the stored energy function Φ.

(H9)

$$\liminf_{w \to \infty} \frac{1}{w^2} \Phi_{,1}(v, w, w) \geqslant 0 \quad \text{uniformly in } v \geqslant 0. \tag{5.13}$$

(This hypothesis is satisfied, for example, if

$$\Phi(v_1, v_2, v_3) = \sum_{i=1}^{3} \phi(v_i) + \sum_{i,j=1}^{3} \psi(v_i v_j) + h(v_1 v_2 v_3),$$

where $\phi, \psi, h \in \mathrm{C}^1([0, \infty))$ are all convex functions and h is increasing on $[0, \infty)$. Again, this condition (H9) is often compatible with (H2′) but not (H2).)

Theorem 5.8. *Suppose that Φ satisfies (H1), (H7), (H9) and that $P > 0$. If $\tilde{r}$ is a minimizer of $I^{(P)}$ on $\bar{\mathcal{A}}_\lambda$ and $\tilde{r}(0) > 0$, then $\tilde{r} \in \bar{\mathcal{A}}_\lambda \backslash \mathcal{A}_\lambda$.*

Suppose for a contradiction that $\tilde{r} \in \mathcal{A}_\lambda$. Define

$$\tilde{\Phi}(v_1, v_2, v_3) = \Phi(v_1, v_2, v_3) + P v_1 v_2 v_3 \tag{5.14}$$

and observe that $\tilde{\Phi}$ also satisfies (H7). By (H7),

$$R\tilde{\Phi}_{,2}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right) \in L^1((0,1))$$

and, since $\tilde{r}$ is a minimizer, it follows by the arguments in the appendix of Sivaloganathan (1986*a*) that

$$R^2\tilde{\Phi}_{,1}\left(\tilde{r}'(R), \frac{\tilde{r}(R)}{R}, \frac{\tilde{r}(R)}{R}\right) + 2\int_R^1 s\tilde{\Phi}_{,2}\left(\tilde{r}'(s), \frac{\tilde{r}(s)}{s}, \frac{\tilde{r}(s)}{s}\right) \mathrm{d}s = \text{const.} \tag{5.15}$$

for a.e. $R \in (0, 1)$.

Hence

$$R^2\tilde{\Phi}_{,1}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right) \in W^{1,1}((0,1)).$$

Without loss of generality, there exist $R_0, \delta > 0$ such that

$$T = \{R \geqslant R_0 : \tilde{r}'(R) > \delta\} \tag{5.16}$$

satisfies $\mathrm{meas}(T) \neq 0$. (Otherwise, $\tilde{r}(R) \equiv \lambda$ and so $\tilde{r} \notin \mathcal{A}_\lambda$, a contradiction.)

Let $\pi = \chi_T$ and let $v(R) = \int_R^1 \pi(s)\,\mathrm{d}s$. Then $r_\varepsilon(R) = \tilde{r}(R) + \varepsilon v(R)$ satisfies $r_\varepsilon(1) = \lambda$, $r_\varepsilon' > 0$, a.e. $R \in (0, 1)$ for all $|\epsilon|$ sufficiently small and hence $r_\varepsilon \in \mathcal{A}_\lambda$ for $|\epsilon|$

sufficiently small. Thus, as $\tilde{r}$ is a minimizer, it follows using the arguments in the appendix of Sivaloganathan (1986*a*) that

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[I^{(P)}(\tilde{r}+\varepsilon v)]|_{\varepsilon=0} \\
&= \int_0^1 R^2\left[\tilde{\Phi}_{,1}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right)v'(R) + \frac{2}{R}\tilde{\Phi}_{,2}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right)v(R)\right]\mathrm{d}R \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}R}\left[R^2\tilde{\Phi}_{,1}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right)v(R)\right]\mathrm{d}R
\end{aligned}
$$

and, since the product of absolutely continuous functions is absolutely continuous,

$$
= -\lim_{R\to 0}\left[\frac{R^2}{\tilde{r}^2}\tilde{\Phi}_{,1}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right)v\tilde{r}^2\right].
$$

Since $v(R)\tilde{r}^2(R) \to v(0)\tilde{r}^2(0) > 0$ it follows that

$$
\lim_{R\to 0} T(R) = 0 \tag{5.17}
$$

as required, where

$$
T(R) = \left(\frac{R}{\tilde{r}}\right)^2 \tilde{\Phi}_{,1}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right)
$$

is the radial component of the Cauchy stress. However, by (5.13) and (5.14) it follows that

$$
\lim_{R\to 0}\left[\left(\frac{R}{\tilde{r}}\right)^2 \Phi_{,1}\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right) + P\right] \geqslant P > 0,
$$

which is a contradiction of (5.17). Hence $\tilde{r} \notin \mathcal{A}_\lambda$ and so $\tilde{r} \in \bar{\mathcal{A}}_\lambda \backslash \mathcal{A}_\lambda$ as required.

Example 5.9. Consider the radial-displacement boundary-value problem with the stored-energy function

$$
\Phi(v_1, v_2, v_3) = \sum_{i=1}^{3} \tfrac{1}{2}\mu(v_i - 1)^2,
$$

where $\mu > 0$.

In this case, by theorems 5.7 and 5.8, degenerate cavitation occurs for $P > 0$ and large λ (and if $P = 0$, then the homogeneous deformation is the energy minimizer). We next give more detailed information on the minimizers of $I^{(P)}$ using the results in Sivaloganathan (1992). Notice first that

$$
\begin{aligned}
4\pi I(r) &= \tfrac{1}{2}\mu\int_0^1 4\pi R^2\left[(r'-1)^2 + 2\left(\frac{r}{R}-1\right)^2\right]\mathrm{d}R \\
&= \tfrac{1}{2}\mu\int_0^1 4\pi R^2\left[(r')^2 + 2\left(\frac{r}{R}\right)^2\right]\mathrm{d}R + 4\pi\tfrac{1}{2}\mu\int_0^1 -2\frac{\mathrm{d}}{\mathrm{d}R}[R^2 r]\,\mathrm{d}R + \text{const.} \\
&= \tfrac{1}{2}\mu\int_0^1 4\pi R^2\left[(r')^2 + 2\left(\frac{r}{R}\right)^2\right]\mathrm{d}R + \text{const.} = \tfrac{1}{2}\mu\int_B |\nabla \boldsymbol{u}|^2 + \text{const.},
\end{aligned}
$$

where $\boldsymbol{u}$ is given by (1.15).

Hence if $P > 0$, then

$$4\pi I^{(P)}(r) = \int_B \tfrac{1}{2}\mu|\nabla \boldsymbol{u}|^2 + P \det \nabla \boldsymbol{u} \, dx + \text{const}.$$

Now, by the results of theorem 1.3 of Sivaloganathan (1992), it follows that for $P > 0$:
(i) if $\lambda \leqslant \lambda_{\text{crit}}$, then $\tilde{r} \equiv \lambda R$ is the global minimizer of $I^{(P)}$ on $\bar{\mathcal{A}}_\lambda$;
(ii) if $\lambda > \lambda_{\text{crit}}$, then

$$\tilde{r}(R) = \begin{cases} AR + \dfrac{\lambda - A}{R^2}, & \text{on } [R_0, 1], \\ AR_0 + \dfrac{\lambda - A}{R_0^2}, & \text{on } [0, R_0] \end{cases} \tag{5.18}$$

is the global minimizer of $I^{(P)}$ on $\bar{\mathcal{A}}_\lambda$, where

$$\lambda_{\text{crit}} = A = \frac{4\mu}{3P} \quad \text{and} \quad R_0 = \left[2\left(\frac{\lambda}{A} - 1\right)\right]^{1/3},$$

i.e. for $\lambda > \lambda_{\text{crit}}$, $\tilde{r}$ is constant on $[0, R_0]$ and so degenerate cavitation occurs (since, for the corresponding radial deformation (1.15), $\det \nabla \boldsymbol{u} = \tilde{r}'(\tilde{r}/R)^2 = 0$ on $[0, R_0]$ by (1.15)–(1.17) and (ii) above).

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