Uniqueness of Regular and Singular Equilibria for Spherically Symmetric Problems of Nonlinear Elasticity

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Communicated by S. ANTMAN

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Introduction

In this paper we prove the uniqueness of certain regular and singular radial solutions to the equilibrium equations of nonlinear elasticity for an isotropic material.

Consider a homogeneous isotropic elastic body which in its reference state occupies the open subset $\Omega \subset \mathbb{R}^3$. We study the two cases when Ω is a ball

$$B = \{\mathbf{X} \in \mathbb{R}^3; |\mathbf{X}| < 1\}$$

or a spherical shell

$$B^\varepsilon = \{\mathbf{X} \in \mathbb{R}^3; \varepsilon < |\mathbf{X}| < 1\}$$

of any inner radius $\varepsilon \in (0, 1)$. A *deformation* of the body is a function $\mathbf{x} : \Omega \to \mathbb{R}^3$. In this paper we are concerned with *radial* deformations, for which $\mathbf{x}$ has the form

$$\mathbf{x}(\mathbf{X}) = \frac{r(R)}{R}\mathbf{X}, \tag{1}$$

where $R = |\mathbf{X}|$. We specify the displacements on the outer surface of the ball and shell by requiring that

$$r(1) = \lambda > 0. \tag{2}$$

BALL (1982) showed that for an isotropic material the study of weak solutions of the form (1) to the equilibrium equations of nonlinear elasticity under zero body force is equivalent to the study of solutions to the radial equilibrium equation

$$\frac{d}{dR}\left(R^2\Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right)\right) = 2R\Phi_{,2}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right), \tag{3}$$

where $\Phi_{,i}$ denotes differentiation of Φ with respect to its i^{th} argument, Φ being the stored energy function of the material. BALL exhibited a class of solutions to (3) for B that satisfy

$$r(0) > 0. \tag{4}$$

Such solutions describe the formation of a hole at the centre of the deformed ball B. BALL called this phenomenon cavitation. We follow his terminology and say that a solution of r of (3) for B is a *cavitating equilibrium solution* if it satisfies (4) together with the natural boundary condition that the cavity surface is stress free.

Our main results are proofs of uniqueness of regular and cavitating equilibria of the form (1) that satisfy (2) (Theorems 2.4, 2.5 and 3.8). The regular deformations correspond to equilibrium configurations for shells B^ε of any cavity size under either the displacement or mixed displacement-traction boundary value problems. To our knowledge these are the first such results for shells. Our proof of uniqueness of cavitating equilibria is a natural one under mild hypotheses on the stored energy function; it generalises a result of BALL (1982), who used an *ad hoc* Gronwall inequality together with very restrictive conditions. Under somewhat less restrictive hypotheses than BALL'S, STUART (1984) used an elegant shooting argument showing that the stress on the cavity surface is a monotone function of the derivative on the boundary. Thus in STUART'S approach uniqueness is an immediate consequence of his proof of existence.

As an application of our uniqueness results in section 5 we study the asymptotic behaviour of solutions to the mixed problem for B^ε in the limit $\varepsilon \to 0$ and we determine the sense in which equilibrium solutions for B^ε approximate those for B when ε is small. A necessary prerequisite in this analysis is to prove the existence of the relevant equilibria as is done in section 4.

Equilibrium solutions to the mixed displacement-traction boundary value problem for a shell B^ε correspond to solutions r_ε of (3) that satisfy $r_\varepsilon(1) = \lambda$ and generate zero traction on the inner surface. A change of variables gives (3) an autonomous form. Thus any such solution r_ε generates an orbit in phase space with the property that it intersects two given curves. The idea of our proof is to parametrise the set of all orbits with this property and to show that an appropriate 'time map' is a strictly monotone function of the parameter. This enables us to prove uniqueness of r_ε in Theorem 2.4. A different choice of time map yields a proof of uniqueness of solutions to the pure displacement boundary value problem for B^ε (see Theorem 2.5).

It is readily observed that the uniqueness proofs of Section 2 for shells rely on the fact that two distinct orbits in phase space cannot cross and thus one solution curve either lies wholly below or above any other. Motivated by this consideration we make a change of variables in the energy functional to an integral over phase plane variables of a *convex* integrand. This procedure leads to a simple proof of uniqueness of cavitating equilibria in Theorem 3.8.

In Section 4 we examine the existence of equilibria for B and B^ε by using the direct method of the calculus of variations to show that an absolute minimiser of the energy exists in the class of radial deformations. This method of proof enables us to show that the deformed cavity size of a shell B^ε is a monotone function of the boundary displacement (Proposition 4.10).

In Proposition 4.7 we demonstrate the existence of cavitating minimisers of the energy for B under assumptions on the growth of $\lambda^{-3}\Phi(\lambda, \lambda, \lambda)$ for large λ. Theorem 1.11 gives a characterisation of the phenomenon of cavitation. The combination of Theorem 1.11 with Proposition 4.7 yields the existence of a critical boundary displacement λ_c with the property that if the boundary displacement $\lambda \leqq \lambda_c$, then a homogeneous deformation is the unique minimiser of the energy, and if $\lambda > \lambda_c$, then a deformation with a cavity is the unique energy minimiser. These results extend and generalise work of BALL (1982) and enable us in Section 5 to examine the asymptotic behaviour of the $\{r_\varepsilon\}$ as $\varepsilon \to 0$.

Intuitively we should expect $\{r_\varepsilon\}$ to approximate equilibrium solutions to the displacement boundary value problem for the solid ball B as $\varepsilon \to 0$. In Proposition 5.3 we show that r_ε is a solution to the mixed problem for B^ε if and only if it is the global minimiser of the energy. We combine this proposition with energy arguments to prove the following convergence results:

$$\text{If } \lambda \leqq \lambda_c, \text{ then } \sup_{R\in[\varepsilon,1]} |r_\varepsilon(R) - \lambda R| \to 0 \quad \text{as } \varepsilon \to 0. \tag{5}$$

$$\text{If } \lambda > \lambda_c, \text{ then } \sup_{R\in[\varepsilon,1]} |r_\varepsilon(R) - r_c(R)| \to 0 \quad \text{as } \varepsilon \to 0, \tag{6}$$

where $r_c(R)$ is a cavitating equilibrium solution for B.

We remark finally that exactly analogous results hold for two-dimensional problems in elasticity. However for ease of presentation we only consider the case of dimension three.

Constitutive Assumptions

Throughout this paper unless otherwise stated we assume that Φ is in C^3 on its domain of definition and that $\Phi_{,i}(1, 1, 1,) = 0$ so that the undeformed configuration is a natural state. (For results relating differentiability properties of Φ to those of W (where W is the corresponding stored energy function that satisfies (0.6)) see BALL (1984)). In the course of this paper we refer to a number of constitutive hypotheses on the stored energy function Φ; for ease of reference these are listed together below.

(H1) $$\Phi_{,11}(v_1, v_2, v_3) > 0.$$

This is known as the tension-extension inequality. For an interpretation of (H1) see TRUESDELL & NOLL (1965):

(H2) $$\left(\frac{v_i\Phi_{,i}(v_1, v_2, v_3) - v_j\Phi_{,j}(v_1, v_2, v_3)}{v_i - v_j}\right) \geqq 0, \quad i \neq j,\ v_i \neq v_j.$$

We say that Φ satisfies H2$^+$ if strict inequality holds. The set of inequalities H2$^+$ are known as the Baker-Ericksen inequalities (see TRUESDELL & NOLL for an interpretation).

(H3) Either

$$\operatorname*{Lim}_{\substack{v_1, v_2 \to \infty \\ v_1 < v_2}} \left(\frac{\Phi_{,1}(v_1, v_2, v_2)}{v_2^2}\right) = \infty$$

or

$$\operatorname*{Lim}_{\substack{v_1, v_2 \to \infty \\ v_1 < v_2}} (\Phi(v_1, v_2, v_2) - v_1\Phi_{,1}(v_1, v_2, v_2)) = -\infty.$$

(H4) Either

$$\operatorname*{Lim}_{\substack{v_1, v_2 \to 0 \\ v_1 > v_2}} \left(\frac{\Phi_{,1}(v_1, v_2, v_2)}{v_2^2}\right) = -\infty$$

or

$$\operatorname*{Lim}_{\substack{v_1, v_2 \to 0 \\ v_1 > v_2}} (\Phi(v_1, v_2, v_2) - v_1\Phi_{,1}(v_1, v_2, v_2)) = +\infty.$$

(H5) $$\Phi_{,1}(v, a, a) \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \text{ as } v \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \text{ for fixed } a \in (0, \infty).$$

(H6) $$\det(\text{Hess}\,\Phi)|_{v_i=1} \overset{\text{def}}{=} \det(\Phi_{,ij}(1, 1, 1)) > 0.$$

(H7) $$\frac{\Phi_{,i} - \Phi_{,j}}{v_i - v_j} + \Phi_{,ij} \geqq 0 \quad \text{for } v_i \neq v_j.$$

(H8) $$\Phi_{,i}(v, v, v) = 0 \text{ for all } i \text{ if and only if } v = 1.$$

This is the assumption that Φ has only one natural state.

(H9) $$\frac{v^2}{(v^3 - 1)^2}\hat{\Phi}(v) \in L^1(\delta, \infty) \quad \text{for } \delta \in (1, \infty) \quad \text{where}$$

$$\hat{\Phi}(v) \overset{\text{def}}{=} \Phi\left(\frac{1}{v^2}, v, v\right).$$

(H10) There exist constants k, $M > 0$ such that

$$M \leqq \frac{\Phi(\lambda, \lambda, \lambda)}{\lambda^3} \quad \text{for } \lambda \geqq k.$$

(H11) $\Phi(v_1, v_2, v_3) > \Phi(1, 1, 1)$ if $v_i \neq 1$ for some i.

(E1) $$\Phi(v_1, v_2, v_3) \geqq \sum_{i=1}^{3} \psi(v_i) \text{ where}$$

$\psi: (0, \infty) \to (0, \infty)$ satisfies

(i) $\psi \in C((0, \infty))$,

(ii) $\dfrac{\psi(v)}{v} \to \infty$ as $v \to \infty$,

(iii) $\psi(v) \to \infty$ as $v \to 0$.

(E2) There exist constants M, $\varepsilon_0 \in (0, \infty)$ such that

$$|\Phi_{,i}(v_1, \alpha_2 v_2, \alpha_3 v_3)\, v_i| < M(\Phi(v_1, v_2, v_3) + 1) \quad \text{if } |\alpha_i - 1| < \varepsilon_0,\ i = 2, 3.$$

Notation. We write $M^{3\times 3}$ for the space of all 3×3 matrices over $\mathbb{R}$. We set

$$M_+^{3\times 3} = \{F \in M^{3\times 3};\ \det F > 0\}$$

and denote by $SO(3)$ the special orthogonal group on $\mathbb{R}^3$.

If $E \subset \mathbb{R}^3$ is measurable, $n \geqq 1$, $1 \leqq p \leqq +\infty$, then we denote by $L^p(E; \mathbb{R}^n)$ the Banach space of equivalence classes of Lebesgue measurable functions $u: E \to \mathbb{R}^n$ with norm $\|\cdot\|_p$ defined by

$$\|u\|_p = \begin{cases} \left(\int_E |u(\mathbf{x})|^p\, dx\right)^{\frac{1}{p}} & 1 \leqq p < +\infty \\ \operatorname{Ess\,Sup}_{\mathbf{x}\in E} |u(\mathbf{x})| & p = \infty \end{cases}$$

(see ADAMS (1975)). We write $L^p(E) = L^p(E; \mathbb{R})$.

The Sobolev space $W^{1,p}(E; \mathbb{R}^n)$ is the space of equivalence classes of Lebesgue measurable functions u satisfying $u, \nabla u \in L^p(E; \mathbb{R}^n)$ where ∇u is the distributional derivative of u. $W^{1,p}(E; \mathbb{R}^n)$ then becomes a Banach space under the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p.$$

(see ADAMS (1975)). We write $W^{1,p}(E) = W^{1,p}(E; \mathbb{R})$.

0. The Stored Energy Function and Weak Equilibrium Solutions

Consider a homogeneous hyperelastic body which in a reference configuration occupies the bounded, open, connected set $\Omega \subset \mathbb{R}^3$. In a typical deformation $\mathbf{x}: \Omega \to \mathbb{R}^3$ a particle with position vector $\mathbf{X}$ in Ω moves to a point having position vector $\mathbf{x}(\mathbf{X})$.

If $W: M_+^{3\times 3} \to \mathbb{R}^+$ is the stored energy function of the material, then the total stored energy E associated with the deformation $\mathbf{x}$ is given by

$$E(\mathbf{x}) = \int_\Omega W(\nabla \mathbf{x}(\mathbf{X})\, dX. \tag{0.1}$$

We say that $\mathbf{x}$ is an *admissible deformation* if it satisfies the local invertibility condition

$$\det(\nabla \mathbf{x}(\mathbf{X})) > 0 \quad \text{for all } \mathbf{X} \in \Omega. \tag{0.2}$$

To account for the idea that large energies must accompany large extensions or compressions we require that

$$W(F) \to \infty \quad \text{as } \det F \to \infty \text{ or } 0. \tag{0.3}$$

We assume that W is frame indifferent, *i.e.*, that

$$W(QF) = W(F) \quad \text{for all } F \in M_+^{3\times 3}, \quad Q \in \mathrm{SO}(3), \tag{0.4}$$

and we assume that W is *isotropic i.e.*, that W in addition satisfies

$$W(FQ) = W(F) \quad \text{for all } F \in M_+^{3\times 3}, \quad Q \in \mathrm{SO}(3). \tag{0.5}$$

It can be shown that (0.4) and (0.5) hold if and only if there exists a symmetric function $\Phi : \mathbb{R}^3_{++} \to \mathbb{R}$ satisfying

$$W(F) = \Phi(v_1, v_2, v_3) \quad \text{for all } F \in M_+^{3\times 3}, \tag{0.6}$$

where

$$\mathbb{R}^3_{++} = \{(c_1, c_2, c_3) \in \mathbb{R}^3;\ c_i > 0,\ i = 1, 2, 3\} \tag{0.7}$$

and where the v_i, known as the *principal stretches*, are the eigenvalues of $(F^T F)^{\frac{1}{2}}$ (for a proof see TRUESDELL & NOLL (1965)).

By the definition of hyperelasticity the *Piola-Kirchhoff stress tensor* $T_R : M_+^{3\times 3} \to M^{3\times 3}$ is given by

$$T_R(F) = \frac{\partial W}{\partial F}(F) \stackrel{\text{def}}{=} \left(\frac{\partial W(F)}{\partial F^i_j}\right). \tag{0.8}$$

If W is isotropic and $F = \operatorname{diag}(v_1, v_2, v_3)$ with $v_i > 0$ for all i, then

$$T_R(F) = \operatorname{diag}(\Phi_{,1}, \Phi_{,2}, \Phi_{,3}) \tag{0.9}$$

where $\Phi_{,i} = \Phi_{,i}(v_1, v_2, v_3)$.

The *Cauchy stress tensor* $T(F)$ is related to $T_R(F)$ through the formula

$$T(F) = (\det F)^{-1}\, T_R(F)\, F^T. \tag{0.10}$$

The tensors T_R and T measure the force on the body per unit area in the undeformed and deformed configurations respectively.

For an elastic body with stored energy function W the *equilibrium equations* under zero body force are given by

$$\frac{\partial}{\partial X^\alpha}\left(\frac{\partial W}{\partial x^i_{,\alpha}}(\nabla \mathbf{x}(\mathbf{X}))\right) = 0 \quad \text{for } i = 1, 2, 3, \tag{0.11}$$

for all $\mathbf{X} = (X^1, X^2, X^3) \in \Omega$. These are the *Euler-Lagrange equations* for the functional E (as given by (0.1)).

The *displacement boundary value problem* in elasticity consists in finding a solution $\mathbf{x}$ to (0.11) taking prescribed values on the boundary $\partial\Omega$. We now restrict attention to the case where

$$\Omega = B \stackrel{\text{def}}{=} \{\mathbf{X} \in \mathbb{R}^3;\ |\mathbf{X}| < 1\} \tag{0.12}$$

is the open unit ball. We consider only radial deformations; *i.e.*, deformations $\mathbf{x}$ of the form

$$\mathbf{x}(\mathbf{X}) = \frac{r(R)}{R}\mathbf{X}, \tag{0.13}$$

satisfying

$$\mathbf{x}(\mathbf{X}) = \lambda\mathbf{X} \text{ for } \mathbf{X} \in \partial\Omega \text{ and for some } \lambda \in (0, \infty), \tag{0.14}$$

where $R = |\mathbf{X}|$.

The following proposition, taken from BALL (1982, Lemma 4.1), relates the properties of $\mathbf{x}$ and r as defined by (0.13).

Proposition 0.1. *Let* $1 \leqq p < +\infty$ *and let* $\mathbf{x}$ *be given by* (0.13). *Then* $\mathbf{x} \in W^{1,p}(B; \mathbb{R}^3)$ *if and only if* $r(\cdot)$ *is absolutely continuous on compact subintervals of* (0, 1) *and*

$$\int_0^1 R^2 \left(|r'(R)|^p + \left|\frac{r(R)}{R}\right|^p\right) dR < +\infty. \tag{0.15}$$

The weak derivatives of $\mathbf{x}$ *are then given by*

$$\nabla\mathbf{x}(\mathbf{X}) = \frac{r(R)}{R}\mathbf{1} + \frac{\mathbf{X} \otimes \mathbf{X}}{R^2}\left(r'(R) - \frac{r(R)}{R}\right), \tag{0.16}$$

where $\mathbf{1}$ *denotes the identity tensor.*

Following BALL (1982) we say that $\mathbf{x} \in W^{1,1}(B; \mathbb{R}^3)$ is a weak equilibrium solution of the displacement boundary value problem if

$$\det(\nabla\mathbf{x}(\mathbf{X})) > 0 \quad \text{for a.e. } \mathbf{X} \in B, \frac{\partial W}{\partial F}(\nabla\mathbf{x}(\cdot)) \in L^1(B; \mathbb{R}^9)$$

and

$$\int_B \frac{\partial W(\nabla\mathbf{x})}{\partial x^i_{,\alpha}} \psi^i_{,\alpha}\, dX = 0 \quad \text{for all } \psi \in C_0^\infty(B; \mathbb{R}^3). \tag{0.17}$$

BALL reduced the analysis of weak equilibrium solutions of the form (0.13) to the study of solutions of a particular ordinary differential equation by means of the following result. (*Cf.* BALL (1982, Theorem 4.2 and Proposition 6.1).)

Theorem 0.2. *Let* $\Phi \in C^m(\mathbb{R}^3_{++})$, $m \geqq 1$. *Then* $\mathbf{x}$ *defined by* (0.13) *is a weak equilibrium solution if and only if*

$$r'(R) > 0 \quad \text{for a.e. } R \in (0, 1),$$

$$R^2\Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right), R^2\Phi_{,2}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \in L^1(0, 1),$$

$$R^2\Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) = 2\int_1^R \varrho\Phi_{,2}\left(r'(\varrho), \frac{r(\varrho)}{\varrho}, \frac{r(\varrho)}{\varrho}\right) d\varrho + \text{const.}$$

for a.e.

$$R \in (0, 1). \tag{0.18}$$

The v_i are given almost everywhere by

$$v_1 = r'(R), \quad v_2 = v_3 = \frac{r(R)}{R}. \tag{0.19}$$

Moreover, if Φ satisfies (H1) *and* (H5), *then* $r \in C^m((0, 1])$, $r'(R) > 0$ *for every* $R \in (0, 1]$ *and r satisfies the* radial equilibrium equation

$$\frac{d}{dR}\left(R^2\Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right)\right) = 2R\Phi_{,2}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \tag{0.20}$$

for every $R \in (0, 1]$.

Notice that the *homogeneous deformation*

$$r(R) \equiv \lambda R \tag{0.21}$$

is always a solution of (0.20) and (0.14).

Equations (0.6), (0.1) and (0.19) reduce the energy corresponding to the radial deformation (0.13) to the form

$$E(\mathbf{x}) = 4\pi I(r) \stackrel{\text{def}}{=} 4\pi \int_0^1 R^2\Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) dR. \tag{0.22}$$

Notice that (0.20) is the Euler-Lagrange equation for (0.22).

To demonstrate the existence of non-trivial solutions of (0.20) corresponding to cavitation, BALL used a variational technique, showing that the functional I attains its infimum on a set of admissible functions A_λ where

$$A_\lambda = \{r \in W^{1,1}(0, 1);\, r(1) = \lambda,\, r'(R) > 0 \text{ a.e } R \in (0, 1),\, r(0) \geqq 0\}. \tag{0.23}$$

Our next proposition is a modified version of Theorem 7.1 of BALL (1982); for convenience a proof is given in the Appendix.

Proposition 0.3. *Let* $\Phi \in C^m(\mathbb{R}^3_{++})$ *for* $m \geqq 1$ *and let Φ satisfy* (H1), (H5) *and* (E1). *If r is an absolute minimiser of I on A_λ then*

(i) $r'(R) > 0$ for $R \in (0, 1]$, (0.24)

(ii) $r \in C^m((0, 1])$ and satisfies (0.20) for every $R \in (0, 1]$. (0.25)

Moreover if $r(0) = \operatorname{Lim}_{R \to 0} r(R) > 0$ *then*

$$\operatorname{Lim}_{R \to 0} T(r(R)) = 0 \tag{0.26}$$

where

$$T(r(R)) \stackrel{\text{def}}{=} \left(\frac{R}{r(R)}\right)^2 \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) \tag{0.27}$$

is the radial component of the Cauchy stress.

From (0.13) we see that there is a cavity at the centre of the deformed ball if and only if $r(0) > 0$, and that (0.26) is the natural boundary condition that the cavity surface be stress-free.

BALL showed that for sufficiently large values of the boundary displacement λ the minimiser r of I on A_λ satisfies $r(0) > 0$. By Proposition 0.3 r is a solution of the radial equilibrium equation (0.20) and by theorem 0.2 $\mathbf{x}$ defined by (0.13) corresponds to a weak solution of the three dimensional equilibrium equations.

In the remainder of this paper when referring to a *cavitating equilibrium solution r* we mean a function $r \in C^2((0, 1])$ with $r(0) > 0$ that satisfies (0.20) on (0, 1] and satisfies

$$\text{(i)} \quad r'(R) > 0 \quad \text{for} \quad R \in (0, 1], \tag{0.28}$$

$$\text{(ii)} \quad \lim_{R \to 0} T(r(R)) = 0. \tag{0.29}$$

1. Properties of Radial Equilibrium Solutions

In this section we gather properties of solutions $r(R)$ to the radial equilibrium equation (0.20). These results will be central to the arguments in the rest of this paper.

Proposition 1.1. *Let Φ satisfy* (H1) *and let $r \in C^2((0, 1])$ be a solution of* (0.20) *satisfying* (0.28). *If there is an $R_0 \in (0, 1]$ and a $\lambda_0 \in (0, \infty)$ such that*

$$\frac{r(R_0)}{R_0} = r'(R_0) \overset{\text{def}}{=} \lambda_0,$$

then $r(R) \equiv \lambda_0 R$ for $R \in (0, 1]$.

Proof. Equation (0.20) is of the form $r'' = f(R, r, r')$ *where f is in C^1. Standard results for ordinary differential equations then imply that the solution $r(R)$ to the initial value problem with data $r(R_0) = \lambda_0 R_0$, $r'(R_0) = \lambda_0$ is unique. Hence $r(R) \equiv \lambda_0 R$.*

Corollary 1.2. *If $r \in C^2((0, 1])$ is a solution of* (0.20) *for which* (0.28) *holds and for which the function $r(R)/R$ is not constant* (*on any nonempty open interval*), *then $r(R)/R$ is a strictly monotone function on* (0, 1]. *In particular, if $r(0) = \lim_{R \to 0} r(R) > 0$, then $r'(R) < \dfrac{r(R)}{R}$ for $R \in (0, 1]$.*

Proof. The first part of the corollary is an easy consequence of Proposition 1.1 and the formula

$$\frac{d}{dR}\left[\frac{r(R)}{R}\right] = \frac{1}{R}\left[r'(R) - \frac{r(R)}{R}\right].$$

The second part then follows immediately from the observation that if $r(0) > 0$, then $r(R)/R \to \infty$ as $R \to 0$.

We now give conditions under which the radial Cauchy stress $T(r(R))$ is monotone on any interval where $r'(R) \neq \dfrac{r(R)}{R}$ for any solution $r \in C^2((0, 1])$ of (0.20).

Proposition 1.3. *If* Φ *satisfies* (H1), (H2) *and if* $r \in C^2((0, 1])$ *satisfies* (0.20) *and* (0.28), *then*

$$\frac{dT(r(R))}{dR}\left(r'(R) - \frac{r(R)}{R}\right) \leqq 0 \quad \text{for any } R \in (0, 1]. \tag{1.1}$$

Proof. By (0.20) and (0.27),

$$\frac{dT(r(R))}{dR} = \frac{2R^2}{r^3(R)}\left(\frac{r(R)}{R}\Phi_{,2}(R) - r'(R)\,\Phi_{,1}(R)\right) \tag{1.2}$$

where

$$\Phi_{,i}(R) = \Phi_{,i}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right). \tag{1.3}$$

The result then follows by (H2) and (0.19). (We use the notation (1.3) and the analogously defined expression $\Phi(R)$ when the arguments of Φ and its derivatives are clear.)

We define the inverse Cauchy stress $\tilde{T}(r(R))$ by

$$\tilde{T}(r(R)) = \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) - r'(R)\,\Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right). \tag{1.4}$$

We refer to BALL (1982) for an interpretation of $\tilde{T}$ and the proof of the following analogue of proposition 1.3.

Proposition 1.4. *If* Φ *satisfies* (H1), (H2) *and if* $r \in C^2((0, 1])$ *is a solution of* (0.20) *satisfying* (0.28), *then*

$$\frac{d\tilde{T}(r(R))}{dR}\left(r'(R) - \frac{r(R)}{R}\right) \geqq 0 \quad \text{for any } R \in (0, 1].$$

Notice that Propositions 1.1, 1.3 and 1.4 show that T and $\tilde{T}$ are Lyapunov functions for (0.20). A third related Lyapunov function is given by the following identity

$$\frac{d}{dR}\left\{R^3\left[\Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) + \left(\frac{r(R)}{R} - r'(R)\right)\Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right)\right]\right\}$$
$$= 3R^2\Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right). \tag{1.6}$$

For future reference we introduce the related notation

$$H(X, Y) \stackrel{\text{def}}{=} \Phi(X, Y, Y) + (Y - X)\,\Phi_{,1}(X, Y, Y). \tag{1.7}$$

It was noted by BALL (1982) that (1.6) is the radial version of the following three-dimensional conservation law

$$\frac{\partial}{\partial X^\alpha}\left(X^\alpha W - \frac{\partial W}{\partial x^j_{,\alpha}}(X^\beta x^j_{,\beta} - x^j)\right) = 3W, \tag{1.8}$$

(see GREEN (1973)). Equation (1.8) was recently used by KNOPS & STUART (1984) to prove the uniqueness of smooth equilibrium solutions to the displacement boundary value problem of elasticity for star-shaped domains under assumptions of quasiconvexity.

Proposition 1.5. *If Φ satisfies* (H1), (H7) *and if* $r \in C^2((0, 1])$ *is a solution of* (0.20) *satisfying* $r'(R) \neq \frac{r(R)}{R}$ *for* $R \in (0, 1]$, *then*

$$r''(R)\,[r'(R) - r(R)/R] \leqq 0 \quad \text{for } R \in (0, 1].$$

The proof is an immediate consequence of (H7) and (0.20).

Proposition 1.6. *Let Φ satisfy* (H1), (H2) *and* (H5). *If* $r \in C^2((0, 1])$ *is a cavitating equilibrium solution, then r is extendable to* $r \in C^2((0, \infty))$ *as a solution of* (0.20) *and satisfies*

$$\text{(a)} \quad \frac{r(R)}{R} > r'(R) > 0 \tag{1.9}$$

for $R \in (0, \infty)$,

$$\text{(b)} \quad \lim_{R\to\infty} \frac{r(R)}{R} = \lim_{R\to\infty} r'(R) = \lambda_c \quad \text{for some } \lambda_c \in [1, \infty). \tag{1.10}$$

Proof. By the continuation principle (see HARTMAN (1973), *e.g.*) r may be extended to a maximal interval of existence $(0, \delta)$, $\delta > 1$, as a solution of (0.20) satisfying (1.9) on $(0, \delta)$. We suppose for a contradiction that δ is finite; then one of the following cases must occur:

(i) $\lim_{R\to\delta} \frac{r(R)}{R} = \infty$,

(ii) $\lim_{R\to\delta} \frac{r(R)}{R} = 0$,

(iii) $\lim_{R\to\delta} r'(R) = \infty$,

(iv) $\lim_{R\to\delta} r'(R) = 0$.

It follow from Corollary 1.2 that (i) cannot occur and that (iii) cannot occur because (iii) implies (i). If (ii) holds, then there is an $R_0 \in (0, \delta)$ satisfying $\frac{r(R_0)}{R_0} = 1$ as $\lim_{R\to 0} \frac{r(R)}{R} = \infty$. Since r satisfies (1.9) on $(0, \delta)$ we can apply Proposition 1.3 (which is valid for $R \in (0, \delta)$) to conclude that $T(r(R))$ is non-decreasing and hence that

$$0 = T(r(0)) = \lim_{R\to 0} T(r(R)) \leqq T(r(R_0)) = \Phi_{,1}(r'(R_0), 1, 1) < \Phi_{,1}(1, 1, 1) = 0, \tag{1.11}$$

a contradiction. (Equality holds in the last term in (1.11) since the reference configuration is a natural state.) Now suppose that (iv) holds. Since (ii) is false it then follows that there is a $b \in (0, \infty)$ such that $r(R)/R \searrow b$ as $R \to \delta$. Assumption (H5) gives the existence of $a \in (0, \infty)$ satisfying

$$\Phi_{,1}(a, b, b) < 0. \tag{1.12}$$

Then (H1) implies that for R sufficiently close to δ

$$T(r(R)) = \left(\frac{R}{r(R)}\right)^2 \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) < \left(\frac{R}{r(R)}\right)^2 \Phi_{,1}\left(a, \frac{r(R)}{R}, \frac{r(R)}{R}\right). \tag{1.13}$$

Since $\dfrac{r(R)}{R} \to b$ as $R \to \delta$, (1.12) implies that $T(r(R))$ is negative for R sufficiently close to δ, which contradicts proposition 1.3 as $T(r(0)) = 0$.

Hence (iv) cannot hold and $\delta = \infty$.

We next prove that (b) of the proposition.

By (1.9) and Corollary 1.2, $\dfrac{r(R)}{R}$ is decreasing, so there is a $\lambda_c \in [0, \infty)$ such that

$$\frac{r(R)}{R} \searrow \lambda_c \text{ as } R \to \infty. \tag{1.14}$$

An argument analogous to that used in the negation of case (ii) then implies that $\lambda_c \in [1, \infty)$. Finally, the monotonicity of $T(r(R))$ together with the inequality

$$T(r(R)) < \left(\frac{R}{r}\right)^2 \Phi_{,1}\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) < \text{const.}$$

(which is a consequence of (H1) and (1.14)) implies that $\operatorname{Lim}_{R\to\infty} T(r(R)) = d$ for some $d \in [0, \infty)$.

By (H5) there exists an $a \in (0, \infty)$ such that

$$\frac{1}{\lambda_c^2} \Phi_{,1}(a, \lambda_c, \lambda_c) = d.$$

We assert that

$$\operatorname*{Lim}_{R\to\infty} r'(R) = a. \tag{1.15}$$

Suppose this were not true. Then there would exist an $\varepsilon_0 > 0$ and a sequence $R_n \to \infty$ as $n \to \infty$ with the property that $|r'(R_n) - a| \geqq \varepsilon_0$ for all n. We assume without loss of generality that $r'(R_n) \geqq a + \varepsilon_0$ for all n. (An exactly analogous argument holds in the case when $r'(R_n) \leqq a - \varepsilon_0$ for all n.) It then follows from (H1) that

$$T(r(R_n)) \geqq \left(\frac{R_n}{r(R_n)}\right)^2 \Phi_{,1}\left(a + \varepsilon_0, \frac{r(R_n)}{R_n}, \frac{r(R_n)}{R_n}\right).$$

Letting $n \to \infty$ and using (1.14) and (H1) we then obtain

$$d \geqq \frac{1}{\lambda_c^2} \Phi_{,1}(a + \varepsilon_0, \lambda_c, \lambda_c) > d,$$

which is a contradiction, proving (1.15). We now suppose for a contradiction that (b) does not hold, so that $|a - \lambda_c| = 2\varepsilon_1 > 0$. Without loss of generality we assume that $a = \lambda_c + 2\varepsilon_1$. It follows from (1.15) that there exists an N such that $|r'(R) - a| < \varepsilon_1$ if $R \in (N, \infty)$ and hence by the mean value theorem that

$$\frac{r(R)}{R} - \frac{r(N)}{R} = \left(1 - \frac{N}{R}\right) r'(\theta(R)) > \left(1 - \frac{N}{R}\right)(a - \varepsilon_1) = \left(1 - \frac{N}{R}\right)(\lambda_c + \varepsilon_1)$$

for $R \in (N, \infty)$ where $\theta(R) \in (N, R)$. This contradicts (1.14) for large R.

Remark 1.7. If H2+ holds, then $\lambda_c \in (1, \infty)$ because $T(r(R))$ is then strictly increasing.

Corollary 1.8. *The results of proposition 1.6 hold if* (H5) *is replaced by* (H7).
Proof. The proof follows from proposition 1.5, the continuation principle and arguments analogous to those used in proposition 1.6 on noting that

$$0 < r'(R) \leq r'(S) < \frac{r(S)}{S} < \frac{r(R)}{R} \quad \text{for } S > R.$$

(See also Ball (1982), p. 601.)

Proposition 1.9. *Let Φ satisfy* (H1), (H2) *and* (H3). *If $r \in C^2((0, 1])$ is a solution of* (0.20) *satisfying* (0.28) *with $r'(R) < \frac{r(R)}{R}$ for $R \in (0, 1]$, then there is an $M > 0$ such that*

$$0 < |r'(R)| = r'(R) \leq M \quad \text{for } R \in (0, 1]. \tag{1.16}$$

Proof. We assume without loss of generality that Φ satisfies the first condition of (H3); otherwise exactly analogous arguments hold on using the inverse Cauchy stress $\tilde{T}$ and Proposition 1.4 instead of the radial Cauchy stress T. It follows from Proposition 1.3 that $T(r(R))$ is nowhere decreasing.

Let $\alpha = T(r(1))$. We assume for a contradiction that (1.16) does not hold for any M. This implies the existence of a sequence $\{R_n\} \in (0, 1]$, $R_n \to 0$ as $n \to \infty$, satisfying $n < r'(R_n) < \frac{r(R_n)}{R_n}$ for all n. It then follows from (H3) that $T(r(R_n)) \to \infty$ as $n \to \infty$ and so $T(r(R_N)) > \alpha$ for some N, contradicting the fact that T is nowhere decreasing.

Remark 1.10. If Φ satisfies (H1) and (H7), then the above result follows trivially from Proposition 1.5.

The following fundamental theorem embodies some of the central ideas associated with the phenomenon of cavitation.

Theorem 1.11. *Suppose that Φ satisfies* (E1), (E2), (H1), (H2+), (H3), (H4), (H5) *and that there is a cavitating equilibrium solution $r_c \in C^2((0, 1])$ with $r_c(1) = \lambda$.*

Then

(i) r_c *is unique and extendable to* $r_c \in C^2((0,\infty))$ *as a solution of* (0.20),

(ii) $\lim_{R\to\infty} \frac{r_c(R)}{R} = \lambda_c$ *for some* $\lambda_c \in (1,\infty)$,

(iii) *if* $\mu \leqq \lambda_c$ *then* $r_\mu(R) \equiv \mu R$ *is the unique global minimiser of* I *on* A_μ, (*where* A_μ *is defined by* (0.23)),

(iv) *if* $\mu > \lambda_c$ *then the global minimiser* r_μ *of* I *on* A_μ *exists, is unique and satisfies* $r_\mu(0) > 0$. *Moreover*

$$r_\mu(R) \equiv \delta r_c\left(\frac{R}{\delta}\right) \quad \textit{for } R \in (0,1],$$

where δ *is the unique root of* $\delta r_c\left(\frac{1}{\delta}\right) = \mu$.

($\lambda_c = \lambda_{cr}$ was defined by BALL (1982) p. 601.)

Corollary 1.12. *The conclusion of Theorem 1.11 holds with* (H3) *and* (H4) *replaced by* (H7).

The proof of this theorem and corollary is given at the end of Section 4. The theorem also holds with (H2) in place of (H2+) with the exception that in this case $\lambda_c \in [1,\infty)$. (See Remark 1.7).

We refer to Section 4 for results concerning the existence of cavitating equilibria. The next proposition uses the conservation law (1.6) and will play a central role in our analysis.

Proposition 1.13. *Suppose that* Φ *satisfies* (H1) *and that* $r_c \in C^2((0,1])$ *is a cavitating equilibrium solution. Then*

$$\text{(i)} \quad \lim_{R\to 0} R^3\left\{\Phi\left(r_c', \frac{r_c}{R}, \frac{r_c}{R}\right) + \left(\frac{r_c}{R} - r_c'\right)\Phi_{,1}\left(r_c', \frac{r_c}{R}, \frac{r_c}{R}\right)\right\} = 0, \tag{1.17}$$

$$\text{(ii)} \quad I(r_c) = \tfrac{1}{3}\{\Phi(r_c'(1), r_c(1), r_c(1)) + (r_c(1) - r_c'(1))\,\Phi_{,1}(r_c'(1), r_c(1), r_c(1))\}. \tag{1.18}$$

In particular, any cavitating equilibrium solution has finite energy.

Proof. Equation (1.6) implies that for $\tau \in (0,1)$

$$\tau^3\Phi(\tau) + 3\int_\tau^1 R^2\Phi(R)\,dR$$
$$= \Phi(1) + (r_c(1) - r_c'(1))\,\Phi_{,1}(1) + \tau^3\left(r_c'(\tau) - \frac{r_c(\tau)}{\tau}\right)\Phi_{,1}(\tau), \tag{1.19}$$

where the expression $\Phi_{,i}(R)$ is given by (1.3) with $r_c(R)$ in place of $r(R)$, and $\Phi(R)$ is analogously defined. The last term on the right-hand side of (1.19) may be written as

$$(\tau r_c'(\tau) - r_c(\tau))\, r_c^2(\tau)\left(\frac{\tau}{r_c(\tau)}\right)^2 \Phi_{,1}(\tau). \tag{1.20}$$

It follows from (0.29) that the limit as $\tau \to 0$ of (1.20) is zero as $r_c'(R) < \frac{r_c(R)}{R}$ by Corollary 1.2. Hence the limit as $\tau \to 0$ of the right-hand side of (1.19) exists. But the left-hand side of (1.19) is the sum of two positive terms; so by the monotone convergence theorem applied to the sequence $R^2\Phi(R)\,\chi_{(1/n,1)}$ $n \in \mathbb{N}$ we obtain

$$R^2\Phi\left(r_c', \frac{r_c}{R}, \frac{r_c}{R}\right) \in L^1(0, 1). \tag{1.21}$$

Therefore $\operatorname{Lim}_{\tau\to 0} \tau^3\Phi(\tau)$ exists and by (1.21) this limit is equal to zero. Thus (i) follows. Statement (ii) is then a consequence of (i) and (1.19).

2. Uniqueness of Solutions to Boundary Value Problems for Shells

In this section we use phase plane techniques to prove the uniqueness of solutions to the displacement and mixed displacement/traction boundary value problems for shells of internal radius ε. The proofs consist in showing that an appropriate 'time map' is monotone. They rely on the change of variables

$$v = \frac{r}{R}, \qquad e^s = R, \tag{2.1}$$

which gives (0.20) the autonomous form

$$\frac{d}{ds}(\Phi_{,1}(\dot{v} + v, v, v)) = 2(\Phi_{,2}(\dot{v} + v, v, v) - \Phi_{,1}(\dot{v} + v, v, v, v)), \tag{2.2}$$

where $\dot{v}$ denotes $\frac{dv}{ds}$.

The results of this section motivate a change of variables in the energy functional, which is used in Section 3 to prove the uniqueness of cavitating equilibrium solutions.

Recall that the shell B^ε is defined by

$$B^\varepsilon = \{\mathbf{X} \in \mathbb{R}^3; \varepsilon < |\mathbf{X}| < 1\}. \tag{2.3}$$

A. The Mixed Problem. We define a radial equilibrium solution to the mixed displacement/traction problem for B^ε to be any solution $r_\varepsilon \in C^2([\varepsilon, 1])$ of (0.20) satisfying

(i) $r_\varepsilon'(R) > 0$ for $R \in [\varepsilon, 1]$, (2.4)

(ii) $r_\varepsilon(1) = \lambda$, (2.5)

(iii) $r_\varepsilon(\varepsilon) > 0$ and (2.6)

(iv) $T(r_\varepsilon(\varepsilon)) = 0$. (2.7)

In condition (ii) $\lambda > 0$ is the boundary displacement, and (iv) is the natural boundary condition that the cavity is stress free.

We first give conditions on the stored energy function that guarantee that the points of zero radial stress form a well defined curve in phase space crossing the $\dot{v} = 0$ axis with negative slope.

Proposition 2.1. *If* Φ *satisfies* (H1) *and* (H5), *then there exists a unique function* $\sigma \in C^1((0, \infty))$ *satisfying*

$$\Phi_{,1}(\sigma(v) + v, v, v) = 0 \quad \text{for all } v \in (0, \infty). \tag{2.8}$$

If, in addition, Φ *satisfies* (H6), *then*

$$\sigma'(1) < 0. \tag{2.9}$$

Proof. It follows from (H1) and (H5) that for each $v_0 \in (0, \infty)$ there exists a unique $\alpha(v_0) \in (-v_0, \infty)$ such that $\Phi_{,1}(\alpha(v_0) + v_0, v_0, v_0) = 0$. The existence of $\sigma \in C^1((0, \infty))$ satisfying (2.8) is then a consequence of (H1) and the implicit function theorem. Implicit differentiation of (2.8) with respect to v gives

$$(\sigma'(v) + 1)\, \Phi_{,11}(\sigma(v) + v, v, v) + 2\Phi_{,12}(\sigma(v) + v, v, v) = 0 \tag{2.10}$$

and hence

$$\sigma'(v) = \frac{-2\Phi_{,12}(\sigma(v) + v, v, v)}{\Phi_{,11}(\sigma(v) + v, v, v)} - 1. \tag{2.11}$$

Hypothesis (H6) is the condition that

$$\det\, (\Phi_{,ij}(v_1, v_2, v_3))|_{v_i=1} > 0. \tag{2.12}$$

On setting $X = \dfrac{\Phi_{,12}(1, 1, 1)}{\Phi_{,11}(1, 1, 1)}$, we can write (2.12) as

$$1 - 3X^2 + 2X^3 = (X - 1)^2\, (2X + 1) > 0$$

and hence $X > \dfrac{-1}{2}$. Condition (2.9) then follows from (2.11) and the definition of X.

Corollary 2.2. *The functions* $\Phi_{,1}(\lambda, \lambda, \lambda)$ *and* $\dfrac{1}{\lambda^2}\Phi_{,1}(\lambda, \lambda, \lambda)$ *are monotone in a neighbourhood of* $\lambda = 1$.

Proof. It is easily seen that

$$\frac{d}{d\lambda}\left(\frac{1}{\lambda^2}\, \Phi_{,1}(\lambda, \lambda, \lambda)\right)\Bigg|_{\lambda=1} = \frac{\Phi_{,11}(\lambda)}{\lambda^2}\left(1 + \frac{2\Phi_{,12}(\lambda)}{\Phi_{,11}(\lambda)}\right)\Bigg|_{\lambda=1}, \tag{2.13}$$

when we use the first that the undeformed configuration is a natural state. The result follows on noting that the right-hand side of (2.13) is strictly positive by (2.9) and (2.11). A similar argument applies in the case of $\Phi_{,1}(\lambda, \lambda, \lambda)$.

Remark 2.3. Notice that $v \equiv$ constant is always a solution of (2.2). Hence the v-axis is a line of rest points and consideration of the phase portrait then shows that any non-constant C^2-solution $v(s)$ of (2.2) satisfies one of the two following conditions

(i) $\dot{v}(s) > 0$ for all s in the interval of existence or

(ii) $\dot{v}(s) < 0$ for all s in the interval of existence.

Hence

$$\left(\frac{d}{ds}\dot{v}(s)\right)\frac{1}{\dot{v}(s)} = \frac{d\dot{v}}{dv} = 2\left(\frac{\Phi_{,2}(\dot{v}+v,v,v) - \Phi_{,1}(\dot{v}+v,v,v)}{\dot{v}\Phi_{,11}(\dot{v}+v,v,v)} - \frac{\Phi_{,12}(\dot{v}+v,v,v)}{\Phi_{,11}(\dot{v}+v,v,v)}\right) - 1 \tag{2.14}$$

$$= 2\left(\frac{\int_0^1 (\Phi_{,21}(t\dot{v}+v,v,v) - \Phi_{,11}(t\dot{v}+v,v,v))\,dt - \Phi_{,12}(\dot{v}+v,v,v)}{\Phi_{,11}(\dot{v}+v,v,v)}\right) - 1 \tag{2.15}$$

$$\stackrel{\text{def}}{=} G(v,\dot{v}) \tag{2.16}$$

and

$$G \in C^1(H) \quad \text{where } H = \{(v,\dot{v}) \in \mathbb{R}^2;\ v > 0,\ \dot{v} + v > 0\}. \tag{2.17}$$

It then follows that solutions $v(s)$ of (2.2) generate solutions of

$$\frac{d\dot{v}}{dv} = G(v,\dot{v}) \tag{2.18}$$

and conversely solution curves of (2.18) are invariant manifolds for the flow generated by (2.2).

If Φ satisfies (H1) and (H5), then the points of zero radial stress lie on a curve $\dot{v} = \sigma(v)$ in phase space by Proposition 2.1. Moreover if $\dot{v} = f_\delta(v)$ is a C^1 solution of (2.18) on an interval containing the points $\lambda > 0$, $\delta > 0$ satisfying

(i) $f_\delta(\delta) = \sigma(\delta)$ and

(ii) $f_\delta(v) \neq 0$ for $v \in [\delta, \lambda]$ (or $v \in [\lambda, \delta]$)

where δ and λ are positive constants, then we define the time map $\mathcal{I}$ by

$$\mathcal{I}(\delta) = \int_\delta^\lambda \frac{1}{f_\delta(v)}\,dv. \tag{2.19}$$

Our next theorem concerns the uniqueness of equilibrium solutions to the mixed problem for shells of internal radius ε and is one of the main results of this section.

Theorem 2.4. *Let Φ satisfy* (H1), (H2), (H5). *Then for each* $\varepsilon \in (0, 1)$ *and* $\lambda \in (0, \infty)$ *there exists at most one solution* $r_\varepsilon \in C^2([\varepsilon, 1])$ *of* (0.20) *satisfying* (2.4)–(2.7).

Proof. The proof proceeds in 3 stages; first we characterise the phase portraits corresponding to r_ε, secondly we prove a monotonicity property associated with the time map $\mathcal{I}$ and finally we show that this monotonicity implies the uniqueness of r_ε.

Step 1. Fix $\varepsilon \in (0, 1)$ and let $r \in C^2([\varepsilon, 1])$, $r(R) \not\equiv \lambda R$ be a solution of (0.20) which satisfies (2.4)–(2.7). Then, under the change of variables given by (2.1), $r(R)$ gives rise to a non constant solution $v(s)$ of (2.2), where $v \in C^2([\log \varepsilon, 0])$ and satisfies

$$\text{(i)} \quad v(0) = \lambda, \tag{2.20}$$

$$\text{(ii)} \quad \dot{v}(s) + v(s) > 0 \quad \text{for } s \in [\log \varepsilon, 0], \tag{2.21}$$

$$\text{(iii)} \quad \Phi_{,1}(\dot{v} + v, v, v)|_{s=\log \varepsilon} = 0. \tag{2.22}$$

We assert that $v(s)$ satisfies one of the two following conditions: either

$$\text{(a)} \quad \sigma(v(s)) \leqq \dot{v}(s) < 0 \quad \text{for } s \in [\log \varepsilon, 0], \text{ or} \tag{2.23}$$

$$\text{(b)} \quad \sigma(v(s)) \geqq \dot{v}(s) > 0 \quad \text{for } s \in [\log \varepsilon, 0]. \tag{2.24}$$

The arguments contained in Remark 2.3 imply that $\dot{v}(s)$ has only one sign for $s \in [\log \varepsilon, 0]$. We suppose that $\dot{v}(s) < 0$ for $s \in [\log \varepsilon, 0]$; then Proposition 1.3 together with (2.1) and (0.27) implies that $\frac{1}{v^2(s)} \Phi_{,1}(\dot{v}(s) + v(s), v(s), v(s))$ is nowhere decreasing on $[\log \varepsilon, 0]$. Consequently, by Proposition 2.1 and (2.22), we obtain

$$\frac{1}{v^2(s)} \Phi_{,1}(\dot{v}(s) + v(s), v(s), v(s)) \geqq 0 = \frac{1}{v^2(s)} \Phi_{,1}(\sigma(v(s)) + v(s), v(s), v(s)) \tag{2.25}$$

for $s \in [\log \varepsilon, 0]$. Hence (2.23) follows from (H1). A similar proof holds for (2.24) in the case $\dot{v}(s) > 0$ for $s \in [\log \varepsilon, 0]$. To justify our consideration of non constant solutions $v(s)$ we make the following remark; if $v(s) \equiv \lambda$ satisfies (2.22), then $\sigma(\lambda) = 0$. Condition (2.23) or (2.24) evaluated at $s = 0$ together with (H1) then imply that this constant solution is unique amongst *all* solutions of (2.2) satisfying (2.20)–(2.22) and the theorem holds.

Step 2. Let f_{δ_i}, $i = 1, 2$ be two distinct non-trivial C^1 solutions of (2.18) on $[\lambda, \delta_i]$ (or $[\delta_i, \lambda]$) satisfying

$$\sigma(\delta_i) = f_{\delta_i}(\delta_i), \quad i = 1, 2. \tag{2.26}$$

We claim that if

$$\text{(i)} \quad \sigma(v) \leqq f_{\delta_i}(v) < 0 \quad \text{for } v \in [\lambda, \delta_i], i = 1, 2, \tag{2.27}$$

where δ_i are positive constants with $\lambda < \delta_1 < \delta_2$, or if

$$\text{(ii)} \quad 0 < f_{\delta_i} \leqq \sigma(v) \quad \text{for } v \in [\delta_i, \lambda], i = 1, 2, \tag{2.28}$$

where δ_i are positive constants with $\delta_2 < \delta_1 < \lambda$, then

$$\mathcal{I}(\delta_1) < \mathcal{I}(\delta_2). \tag{2.29}$$

We prove (i) (the proof of (ii) is identical in nature and will be omitted). Uniqueness of solutions to the initial value problem for (2.18) implies that

$$f_{\delta_1}(v) < f_{\delta_2}(v) \quad \text{for } v \in [\lambda, \delta_1]. \tag{2.30}$$

Using the definition of the time map (2.19) we obtain

$$\mathscr{I}(\delta_1) = -\int_{\lambda}^{\delta_1} \frac{1}{f_{\delta_1}(v)}\,dv < -\int_{\lambda}^{\delta_2} \frac{1}{f_{\delta_2}(v)}\,dv = \mathscr{I}(\delta_2) \tag{2.31}$$

from (2.27) and (2.30) and hence (2.29) holds.

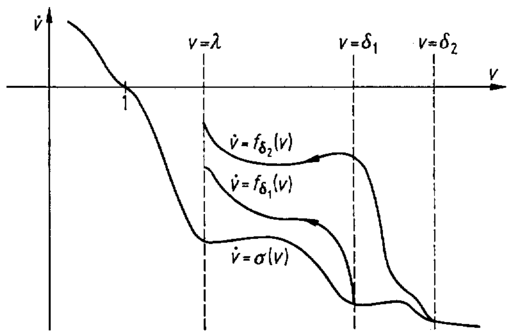

Fig. 1. *A possible phase portrait*

Step 3. Now let $v \in C^2([\log \varepsilon, 0])$ be a solution of (2.2) that satisfies (2.20)–(2.22) and (2.23). (An exactly analogous argument that holds in the case of (2.24)). The arguments contained in remark 2.3 imply that $v(s)$ generates a solution $f_\delta \in C^1([\lambda, \delta])$ of (2.18) satisfying $f_\delta(\delta) = \sigma(\delta)$, where $\delta = v(\log \varepsilon)$. It then follows that

$$\frac{1}{f_\delta(v(s))}\frac{dv(s)}{ds} = 1 \quad \text{for } s \in [\log \varepsilon, 0] \tag{2.32}$$

and so

$$\mathscr{I}(\delta) = \int_{\delta}^{\lambda} \frac{1}{f_\delta(v)}\,dv = \int_{\log\varepsilon}^{0} \frac{1}{f_\delta(v(s))}\frac{dv(s)}{ds}\,ds = \int_{\log\varepsilon}^{0} 1\,ds = \log\frac{1}{\varepsilon}. \tag{2.33}$$

The proof of the theorem is completed on noting that by (2.23) and (2.24) any two distinct solutions $v_i(s)$, $i = 1, 2$ of (2.2) that satisfy (2.20)–(2.22) will generate two distinct functions f_{δ_i}, $i = 1, 2$, satisfying the conditions (2.26) and (2.27) (or (2.28)) of Step 2; (2.29) and (2.33) then yield a contradiction.

B. The Displacement Boundary Value Problem. Our next result concerns the uniqueness of solutions to the displacement boundary value problem for a shell of internal radius $\varepsilon \in (0, 1)$; equilibrium configurations for this problem correspond to solutions $r_\varepsilon \in C^2([\varepsilon, 1])$ of (0.20) that satisfy

(i) $r_\varepsilon(1) = \lambda$, (2.34)

(ii) $r'_\varepsilon(R) > 0$ for $R \in [\varepsilon, 1]$ and (2.35)

(iii) $r_\varepsilon(\varepsilon) = \mu$, (2.36)

where λ and μ are given constants with $0 < \mu < \lambda$.

Theorem 2.5. *Suppose that Φ satisfies* (H1). *Then for each* $\varepsilon \in (0, 1)$ *there exists at most one solution* $r_\varepsilon \in C^2([\varepsilon, 1])$ *of* (0.20) *satisfying* (2.34)–(2.36).

Proof. We proceed in a manner analogous to that for the proof of Theorem 2.4.

Step 1. Fix $\varepsilon \in (0, 1)$; then any solution $r \in C^2([\varepsilon, 1])$ that satisfies (2.34)–(2.36) generates a solution $v \in C^2([\log \varepsilon, 0])$ of (2.2) satisfying

$$\text{(i)} \quad v(0) = \lambda, \tag{2.37}$$

$$\text{(ii)} \quad \dot{v}(s) + v(s) > 0 \quad \text{for } s \in [\log \varepsilon, 0], \tag{2.38}$$

$$\text{(iii)} \quad v(\log \varepsilon) = \frac{\mu}{\varepsilon}. \tag{2.39}$$

We assume without loss of generality that $\frac{\mu}{\varepsilon} < \lambda$. (An analogous argument holds in the case $\frac{\mu}{\varepsilon} > \lambda$; clearly if $\frac{\mu}{\varepsilon} = \lambda$ then by Corollary 1.2 $r(R) \equiv \lambda R$ is the only solution of (0.20) that satisfies (2.34)–(2.36)).

Step 2. If $f_\delta \in C^1\left(\left[\frac{\mu}{\varepsilon}, \lambda\right]\right)$ is a solution of (2.18) satisfying

$$\text{(i)} \quad f_\delta(v) > 0 \quad \text{for } v \in \left[\frac{\mu}{\varepsilon}, \lambda\right], \tag{2.40}$$

$$\text{(ii)} \quad f_\delta\left(\frac{\mu}{\varepsilon}\right) = \delta, \tag{2.41}$$

where $\delta > 0$ is a constant, then we define the time map $\mathscr{I}^*$ by

$$\mathscr{I}^*(\delta) = \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_\delta(v)}\, dv. \tag{2.42}$$

Now let $f_{\delta_i} \in C^1\left(\left[\frac{\mu}{\varepsilon}, \lambda\right]\right)$ $i = 1, 2$ be any two distinct solutions of (2.18) satisfying (2.40) and (2.41) where δ_1 and δ_2 are constants with $0 < \delta_1 < \delta_2$.

It then follows from (2.41) and the uniqueness of solutions to the initial value problem for (2.18) that

$$f_{\delta_1}(v) < f_{\delta_2}(v) \quad \text{for } v \in \left[\frac{\mu}{\varepsilon}, \lambda\right].$$

Hence

$$\mathscr{I}^*(\delta_2) = \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_{\delta_2}(v)}\, dv < \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_{\delta_1}(v)}\, dv = \mathscr{I}^*(\delta_1)$$

and so

$$\mathscr{I}^*(\delta_1) > \mathscr{I}^*(\delta_2). \tag{2.43}$$

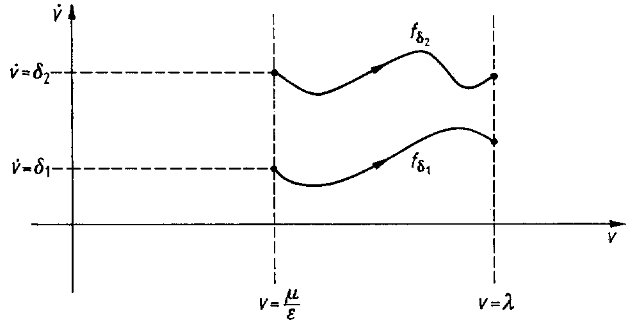

Fig. 2

Step 3. Now let $v_0 \in C^2([\log \varepsilon, 0])$ be a non-constant solution of (2.2) that satisfies (2.37)–(2.39). The arguments contained in Remark 2.3 then imply that $v_0(s)$ generates a solution $f_{\delta_0} \in C^1\left(\left[\frac{\mu}{\varepsilon}, \lambda\right]\right)$ of (2.18) satisfying (2.40) and (2.41) with $\delta_0 = \dot v_0(\log \varepsilon)$. It then follows that

$$\mathcal{I}^*(\delta_0) = \int_{\frac{\mu}{\varepsilon}}^{\lambda} \frac{1}{f_{\delta_0}(v)}\, dv = \int_{\log\varepsilon}^{0} \frac{1}{f_{\delta_0}(v_0(s))} \frac{dv_0(s)}{ds}\, ds = \int_{\log\varepsilon}^{0} 1\, ds = \log \frac{1}{\varepsilon}. \tag{2.44}$$

The proof of the theorem is completed on noting that any two distinct solutions $v_i(s)$, $i = 1, 2$ of (2.2) that satisfy (2.37)–(2.39) generate two distinct functions f_{δ_i} $i = 1, 2$ satisfying the conditions (2.40) and (2.41) of Stèp 2; (2.43) and (2.44) then yield a contradiction.

3. Uniqueness of Cavitating Solutions

In this section we make a change of variables to construct a new energy function that is convex. (see Proposition 3.7). This new function supports a proof of uniqueness of cavitating equilibrium solutions in Theorem 3.8.

First we state a proposition concerning the invertibility of the relation $v = \frac{r(R)}{R}$ when r is a cavitating equilibrium solution.

Proposition 3.1. *Let* $r \in C^2((0, 1])$ *be a cavitating equilibrium solution with* $r(1) = \lambda > 0$. *Then there exists a function* $g: [\lambda, \infty) \to (0, 1]$, $g \in C^2([\lambda, \infty))$ *satisfying*

(i) $g\left(\frac{r(R)}{R}\right) = R^3$ for $R \in (0, 1]$,

(ii) $g(\lambda) = 1$,

(iii) $\operatorname{Lim}_{v \to \infty} g(v) = 0$,

(iv) $$\frac{3g\left(\frac{r(R)}{R}\right)}{g'\left(\frac{r(R)}{R}\right)}+\frac{r(R)}{R}=r'(R) \quad \text{for } R\in(0,1].$$

Proof. The existence of g satisfying (i) is a consequence of corollary 1.2 and the inverse function theorem. Conditions (ii) and (iii) then follow from (i) as does (iv) on implicit differentiation.

Proposition 3.2. *Let* Φ *satisfy* (H1). *If* $r\in C^2((0,1])$ *is a cavitating equilibrium solution with* $r(1)=\lambda>0$, *then*

(i) $$\operatorname*{Lim}_{v\to\infty}\frac{1}{v^3}\left(\Phi\left(\frac{3g(v)}{g'(v)}+v,v,v\right)-\frac{3g(v)}{g'(v)}\Phi_{,1}\left(\frac{3g(v)}{g'(v)}+v,v,v\right)\right)=0,$$

(ii) $$I(r)=\Phi(r'(1),\lambda,\lambda)+(\lambda-r'(1))\,\Phi_{,1}(r'(1),\lambda,\lambda)=H(r'(1),\lambda),$$

where g is defined as in Proposition 3.1 and H *is given by* (1.7).

Proof. Condition (ii) is a direct consequence of Proposition 1.13. From the proof of Proposition 1.13 it also follows that Part (i) follows from Proposition 3.1 and (1.17) on setting $v=\frac{r(R)}{R}$, since $r(0)>0$.

Remark 3.3. If $r\in C^2((0,1])$ is a cavitating equilibrium solution with $r(1)=\lambda$ and if g is defined as in Proposition 3.1, then

$$g(v)=r^3\,(g^{\frac{1}{3}}(v))\frac{1}{v^3} \quad \text{for } v\in[\lambda,\infty) \tag{3.1}$$

and hence

$$\frac{r^3(0)}{v^3}\leqq g(v)\leqq\frac{\lambda^3}{v^3} \quad \text{for } v\in[\lambda,\infty). \tag{3.2}$$

Remark 3.4. The function $H(X,Y)$ as defined by (1.7) satisfies

$$\frac{\partial}{\partial X}H(X,Y)>0 \quad \text{for } X\in(0,Y)$$

whenever Φ satisfies (H1).

Proposition 3.5. *Suppose that* Φ *satisfies* (H1) *and that* $r\in C^2((0,1])$ *is a cavitating equilibrium solution with* $r(1)=\lambda>0$. *Then the energy of the deformation is finite and given by*

$$I(r)=\tilde{I}(g)\stackrel{\text{def}}{=}-\int_{\lambda}^{\infty}\frac{g'(v)}{3}\,\Phi\left(\frac{3g(v)}{g'(v)}+v,v,v\right)dv, \tag{3.3}$$

where g is defined as in Proposition 3.1.

Proof. The energy $I(r)$ is finite by Proposition 1.13; (3.3) then follows immediately from Proposition 3.1 on noting that

$$g'(v)\frac{dv}{dR} = 3R^2.$$

We next show that the corresponding function g of Proposition 3.1 satisfies the Euler-Lagrange equations corresponding to the functional $\tilde{I}$ as defined by (3.3).

Proposition 3.6. *Let* $r \in C^2((0, 1])$ *be a cavitating equilibrium solution with* $r(1) = \lambda > 0$. *Then*

$$\begin{aligned}\frac{d}{dv}\left(\frac{-1}{3}\Phi\left(\frac{3g(v)}{g'(v)} + v, v, v\right) + \frac{g(v)}{g'(v)}\Phi_{,1}\left(\frac{3g(v)}{g'(v)} + v, v, v\right)\right)& \\ = -\Phi_{,1}\left(\frac{3g(v)}{g'(v)} + v, v, v\right) \quad \text{for } v \in [\lambda, \infty),&\end{aligned} \tag{3.4}$$

where g *is defined as in Proposition 3.1.*

Proof. As $g \in C^2([\lambda, \infty))$, (3.4) is equivalent to

$$\frac{-1}{3}\left(3 - \frac{3gg''}{(g')^2} + 1\right)\Phi_{,1} - \tfrac{2}{3}\Phi_{,2} + \left(1 - \frac{gg''}{(g')^2}\right)\Phi_{,1} + \frac{g}{g'}\frac{d}{dv}\Phi_{,1} = -\Phi_{,1} \tag{3.5}$$

for $v \in [\lambda, \infty)$, where $\Phi_{,i} = \Phi_{,i}\left(\frac{3g(v)}{g'(v)} + v, v, v\right)$. Equation (3.5) may be rewritten as

$$\frac{g}{g'}\frac{d}{dv}\left(\Phi_{,1}\left(\frac{3g}{g'} + v, v, v\right)\right) = \tfrac{2}{3}\left(\Phi_{,2}\left(\frac{3g}{g'} + v, v, v\right) - \Phi_{,1}\left(\frac{3g}{g'} + v, v, v\right)\right). \tag{3.6}$$

The function r is a solution of (0.20) and hence

$$R\frac{d}{dR}\left(\Phi_{,1}\left(r', \frac{r}{R}, \frac{r}{R}\right)\right) = 2\left(\Phi_{,2}\left(r', \frac{r}{R}, \frac{r}{R}\right) - \Phi_{,1}\left(r', \frac{r}{R}, \frac{r}{R}\right)\right) \tag{3.7}$$

for $R \in (0, 1]$.

We set $v = r/R$ and use Proposition 3.1 to convert (3.7) into (3.6), thereby completing the proof.

The last proposition is an example of the general invariant nature of the Euler-Lagrange equations (see CESARI (1983) p. 48). We next examine a property of the integrand of $\tilde{I}$ as defined by (3.3).

Proposition 3.7. *If* Φ *satisfies* (H1), *then for each* $v \in (0, \infty)$ *the function* $G: S \to \mathbb{R}$ *defined by*

$$G\begin{pmatrix}X_1\\X_2\end{pmatrix} = G(\mathbf{X}) = \frac{-X_2}{3}\Phi\left(\frac{3X_1}{X_2} + v, v, v\right) \tag{3.8}$$

is a convex function on

$$S = \left\{ \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathbb{R}^2;\ X_1 \in (0, \infty),\ X_2 \in (-\infty, 0),\ \frac{3X_1}{v} + X_2 \leqq 0 \right\}. \tag{3.9}$$

Proof. Since S is a convex subset of $\mathbb{R}^2$ for each $v \in (0, \infty)$ it is sufficient to show that the hessian of G is positive semidefinite on S. An easy calculation gives

$$\operatorname{Hess} G(\mathbf{X}) \stackrel{\text{def}}{=} \left(\frac{\partial^2 G(\mathbf{X})}{\partial X_i\, \partial X_j} \right) = 3 \begin{pmatrix} \dfrac{-1}{X_2} & \dfrac{X_1}{X_2^2} \\ \dfrac{X_1}{X_2} & \dfrac{-X_1^2}{X_2^3} \end{pmatrix} \Phi_{,11} \left(\frac{3X_1}{X_2} + v, v, v \right). \tag{3.10}$$

It follows from (H1), (3.10) and (3.9) that the trace of Hess $G(\mathbf{X})$ and the determinant of Hess $G(\mathbf{X})$ satisfy.

$$\det(\operatorname{Hess} G(\mathbf{X})) = 0 \quad \text{and} \quad \operatorname{tr}(\operatorname{Hess} G(\mathbf{X})) > 0 \quad \text{for } \mathbf{X} \in S,$$

for each $v \in (0, \infty)$. Hence Hess $G(\mathbf{X})$ is positive semidefinite, completing the proof.

The main result of this section is the following theorem:

Theorem 3.8. *If Φ satisfies* (H1), *then for each $\lambda \in [1, \infty)$ there exists at most one cavitating equilibrium $r \in C^2((0, 1])$ satisfying $r(1) = \lambda$.*

Proof. We suppose for a contradiction that there is a $\lambda \in [1, \infty)$ for which there are two distinct cavitating equilibrium solutions $r_i(R) \in C^2((0, 1])$ with $r_i(1) = \lambda$ for $i = 1, 2$. Let g_i, $i = 1, 2$ be the corresponding functions as defined in Proposition 3.1. Then by Proposition 3.5

$$I(r_i) = \tilde{I}(g_i) = -\int_\lambda^\infty \frac{g_i'(v)}{3} \Phi \left(\frac{3g_i(v)}{g_i'(v)} + v, v, v \right) dv, \quad i = 1, 2. \tag{3.11}$$

It follows from Proposition 3.7 that

$$\int_\lambda^M G \begin{pmatrix} g_1(v) \\ g_1'(v) \end{pmatrix} dv \leqq \int_\lambda^M G \begin{pmatrix} g_2(v) \\ g_2'(v) \end{pmatrix} dv + \int_\lambda^M \left\{ \frac{\partial G}{\partial X_1} \begin{pmatrix} g_1(v) \\ g_1'(v) \end{pmatrix} (g_1(v) - g_2(v)) \right.$$
$$\left. + \frac{\partial G}{\partial X_2} \begin{pmatrix} g_1(v) \\ g_1'(v) \end{pmatrix} (g_1'(v) - g_2'(v)) \right\} dv \quad \text{for each } M \in (\lambda, \infty), \tag{3.12}$$

where G is defined by (3.8). (This is an elementary consequence of the convexity of G.) We integrate the second integral on the right hand side of (3.12) by parts to convert it to the form

$$\int_\lambda^M \left\{ \frac{\partial G}{\partial X_1} \begin{pmatrix} g_1 \\ g_1' \end{pmatrix} - \frac{d}{dv} \left(\frac{\partial G}{\partial X_2} \begin{pmatrix} g_1 \\ g_1' \end{pmatrix} \right) \right\} (g_1 - g_2)\, dv + \left((g_1 - g_2) \frac{\partial G}{\partial X_2} \begin{pmatrix} g_1 \\ g_1' \end{pmatrix} \right) \Bigg|_\lambda^M. \tag{3.13}$$

Proposition 3.6 and (3.8) then imply that the integrand in (3.13) is identically equal to zero. We thus conclude from (3.12), (3.13) that

$$\int_{\lambda}^{M} \frac{\partial G}{\partial X_1}\binom{g_1}{g_1'}(g_1 - g_2) + \frac{\partial G}{\partial X_2}\binom{g_1}{g_1'}(g_1' - g_2')\,dv = [g_2(M) - g_1(M)]\frac{\partial G}{\partial X_2}\binom{g_1(M)}{g_1'(M)} \tag{3.14}$$

for each $M \in (\lambda, \infty)$, where we have used the fact that $g_1(\lambda) = g_2(\lambda) = 1$. Remark 3.3, Proposition 3.2(i) and (3.8) then imply that the right-hand side of (3.14) tends to zero as M tends to infinity. Thus (3.14) and (3.12) imply that

$$I(r_1) = \tilde{I}(g_1) \leqq \tilde{I}(g_2) = I(r_2).$$

Interchanging the roles of r_1 and r_2 in the above arguments we obtain

$$H(r_1'(1), \lambda) = I(r_1) = I(r_2) = H(r_2'(1), \lambda),$$

where we have used Proposition 3.2(ii). Corollary 1.2 implies that $r_i'(1) < \lambda$, $i = 1, 2$, and thus it follows from Remark 3.4 that $r_1'(1) = r_2'(1)$. Hence $r_1(R) \equiv r_2(R)$, a contradiction.

4. Existence of Radial Equilibria

In subsection A we prove the existence of energy minimisers for the displacement boundary value problem for a ball and in subsection B the existence of cavitating minimisers for sufficiently large boundary data. Proposition 0.3 then gives conditions under which these minimisers are solutions of the radial equilibrium equation (0.20). Finally in subsection C we show existence of radial equilibria for shells.

A. Existence of minimisers for a ball. Our first proposition concerns the existence of energy minimisers to the displacement boundary value problem for a ball.

Proposition 4.1. *Let Φ satisfy* (E1) *and* (H1) *and let I be defined by*

$$I(r) = \int_0^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) dR. \tag{4.1}$$

Then I attains its infimum on A_λ (where A_λ is defined by (0.23).

Proof. Let $\{y_n\}$ be a minimising sequence for I on A_λ and let $\beta = \operatorname{Inf}_{A_\lambda} I$.

Assumption (E1) implies that for each positive integer m

$$\int_{\frac{1}{m}}^{1} \left(\frac{1}{m}\right)^2 \cdot \psi(y_n')\,dR \leqq I(y_n) \leqq \text{constant} \quad \text{for all } n. \tag{4.2}$$

Using Theorem 10.3 from CESARI (1983), we choose the following sequences inductively:

$$\{y_{m,n}\}_{n=1}^{\infty} \text{ is a subsequence of } \{y_{m-1,n}\}_{n=1}^{\infty} \text{ satisfying} \tag{4.3}$$

$$y'_{m,n} \xrightarrow{L^1\left(\frac{1}{m},1\right)} Z_m \qquad \text{as } n \to \infty \tag{4.4}$$

for some $Z_m \in L^1\left(\frac{1}{m}, 1\right)$ and we define $\{y_{1,n}\}$ by $y_{1,n} = y_n$ for all n.

We define the function Z by

$$Z(R) = Z_k(R) \quad \text{where } k \text{ is so chosen that } R \in \left(\frac{1}{k}, 1\right). \tag{4.5}$$

The function Z is then well defined for a.e. R since if $m_1 > m_2$ then

$$y'_{m_1,n} \xrightarrow{L^1\left(\frac{1}{m_2},1\right)} Z_{m_i} \qquad \text{as } n \to \infty, \qquad i = 1, 2, \tag{4.6}$$

so by the uniqueness of weak limits $Z_{m_1}(R) = Z_{m_2}(R)$ for a.e. $R \in \left(\frac{1}{m_2}, 1\right)$.

We now set

$$y(R) = \lambda - \int_R^1 Z(s)\, ds \tag{4.7}$$

and

$$r_m = y_{m,m} \qquad \text{for all } m. \tag{4.8}$$

The sequence $\{r_m\}$ defined by (4.8) then satisfies

$$r_m \xrightarrow{W^{1,1}(\delta,1)} y \qquad \text{as } m \to \infty \tag{4.9}$$

for each $\delta \in (0, 1)$.

We extend the definition of Φ by setting $\Phi(v_1, v_2, v_3) = \infty$ if $v_i \leqq 0$ for any i so that for each $R \in (0, 1)$, $g(R, \cdot, \cdot)$ defined by

$$g(R, r, r') = R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right)$$

becomes a continuous function from $\mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}$. Then using (E1), (H1) and a standard lower semicontinuity theorem (*cf.* BALL, CURRIE & OLVER 1980, Theorem 5.4) we conclude that

$$\int_\delta^1 R^2 \Phi\left(y'(R), \frac{y(R)}{R}, \frac{y(R)}{R}\right) dR \leqq \varliminf_{m \to \infty} \int_\delta^1 R^2 \Phi\left(r'_m(R), \frac{r_m(R)}{R}, \frac{r_m(R)}{R}\right) \tag{4.10}$$

for each $\delta \in (0, 1)$. Since $\{r_m\}$ is a subsequence of a minimising sequence for I on A_λ

$$\int_\delta^1 R^2 \Phi\left(y'(R), \frac{y(R)}{R}, \frac{y(R)}{R}\right) dR \leqq \beta = \inf_{A_\lambda} I \quad \text{for each } \delta \in (0, 1). \tag{4.11}$$

Using the fact that Φ is positive we obtain by the monotone convergence theorem that

$$\int_0^1 R^2\Phi\left(y'(R), \frac{y(R)}{R}, \frac{y(R)}{R}\right) dR \leqq \beta. \tag{4.12}$$

To complete the proof we show that $y \in A_\lambda$ so that equality holds in (4.12). It follows from (E1), (4.7) and (4.12) that $y'(R) > 0$ for a.e. $R \in (0, 1)$. Clearly $y(1) = \lambda$ and as

$$\int_\delta^1 |y'|\, ds = \int_\delta^1 y'\, ds \leqq 2\lambda \quad \text{for each } \delta \in (0, 1),$$

the monotone convergence theorem implies that $y' \in L^1(0, 1)$, whence $y \in W^{1,1}(0, 1)$. Finally (4.9) implies that

$$r_m \xrightarrow{C([\delta,1])} y \quad \text{as } m \to \infty \quad \text{for each } \delta \in (0, 1);$$

hence $y(R) \geqq 0$ for $R \in (0, 1)$ and so $y(0) \geqq 0$. This establishes that $y \in A_\lambda$.

B. Existence of cavitating minimisers. With a view to proving the existence of cavitating minimisers we establish conditions under which any solution $r \in C^2((0, 1])$ of (0.20) satisfying (0.24), $r(1) = \lambda$ and $r(0) = \operatorname{Lim}_{R\to 0} r(R) = 0$ must be identically equal to λR. We first state a preparatory result, the proof of which is given by Ball (1982), Theorem 6.5.

Proposition 4.2. *Let* Φ *satisfy* (H1)–(H4) *and let* $r \in C^2((0, 1])$ *be a solution of* (0.20) *satisfying* (0.28) *with* $\operatorname{Lim}_{R\to 0} r(R) = r(0) = 0$. *Then*

$$r \in C^1([0, 1]) \cap C^2((0, 1]) \tag{4.13}$$

and there is an $l \in (0, \infty)$ *such that*

$$r'(0) = \operatorname{Lim}_{R\to 0} r'(R) = \operatorname{Lim}_{R\to 0} \frac{r(R)}{R} = l. \tag{4.14}$$

Proposition 4.3. *Let* Φ *satisfy* (H1)–(H4) *and let* $r \in C^2((0, 1])$, $r(R) \not\equiv \lambda R$ *be a solution of* (0.20) *satisfying* (0.28) *with* $\operatorname{Lim}_{R\to 0} r(R) = r(0) = 0$, $r(1) = \lambda$. *Then*

$$I(r) < I(\lambda R). \tag{4.15}$$

Proof. The proof is analogous to that of Proposition 1.13. It follows from (1.6) that

$$\tau^3\Phi(\tau) + 3\int_\tau^1 R^2\Phi(R)\, dR = \Phi(1) + (\lambda - r'(1))\, \Phi_{,1}(1) + \tau^3\left(r'(\tau) - \frac{r(\tau)}{\tau}\right)\Phi_{,1}(\tau). \tag{4.16}$$

The last term on the right-hand side of (4.16) may be written as

$$r^2(\tau)\,(\tau r'(\tau) - r(\tau))\left(\frac{\tau^2}{r^2(\tau)}\Phi_{,1}(\tau)\right). \tag{4.17}$$

Using Proposition 4.2 and the fact that $r(0) = 0$, we conclude that (4.17) tends to zero as $\tau \to 0$. Since the left-hand side of (4.16) is the sum of two positive terms one of which is monotone, the limit as $\tau \to 0$ of each of them exists. By the monotone convergence theorem $I(r) < +\infty$ and so $\operatorname{Lim}_{\tau \to 0} \tau^3 \Phi(\tau) = 0$. Equation (4.16) then takes the form

$$I(r) = \tfrac{1}{3}\left(\Phi(r'(1), \lambda, \lambda) + (\lambda - r'(1))\, \Phi_{,1}(r'(1), \lambda, \lambda)\right). \tag{4.18}$$

Using (H1), we obtain

$$I(r) < \frac{\Phi(\lambda, \lambda, \lambda)}{3} = \int_0^1 R^2 \Phi(\lambda, \lambda, \lambda)\, dR = I(\lambda R) \tag{4.19}$$

as required ($r'(1) \neq \lambda$ by Proposition 1.1).

Proposition 4.4. *Let* Φ *satisfy* (H1)–(H4) *and let* $r \in C^2((0, 1])$, $r(R) \not\equiv \lambda R$ *be a solution of* (0.20) *satisfying* (0.28) *with* $r(1) = \lambda$ *and* $\operatorname{Lim}_{R \to 0} r(R) = r(0) = 0$. *Then*

$$I(\lambda R) < I(r). \tag{4.20}$$

Proof. It follows from (H1) and Proposition 1.1 that

$$R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) > R^2 \left(\Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right) + \left(r' - \frac{r}{R}\right) \Phi_{,1}\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right)\right) \tag{4.21}$$

for $R \in (0, 1]$. Then for $\tau \in (0, 1)$

$$\int_\tau^1 R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) dR > \left(\frac{R^3}{3} \Phi\left(\frac{r}{R}, \frac{r}{R}, \frac{r}{R}\right)\right)_\tau^1. \tag{4.22}$$

Letting $\tau \to 0$, we obtain from Proposition 4.2, part (4.14) that

$$I(r) = \int_0^1 R^2 \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right) dR > \tfrac{1}{3}\, \Phi(\lambda, \lambda, \lambda) = I(\lambda R), \tag{4.23}$$

as required.

The observation that (H1) implies (4.22) was made by BALL (in a private communication). On combining the last two propositions we obtain the following result.

Proposition 4.5. *Let* Φ *satisfy* (H1)–(H4) *and let* $r \in C^2((0, 1])$ *be a solution of* (0.20) *satisfying* (0.28) *with* $r(1) = \lambda$ *and* $\operatorname{Lim}_{R \to 0} r(R) = r(0) = 0$. *Then* $r(R) \equiv \lambda R$.

Proof. We suppose for a contradiction that $r(R) \not\equiv \lambda R$; then applying Propositions 4.3 and 4.4 we conclude that $I(r) > I(\lambda R)$ and $I(r) < I(\lambda R)$.

Remark 4.6. We refer to BALL (1982) for an alternative proof of Proposition 4.5 and for the analogous result when the hypotheses (H3) and (H4) are replaced by (H7).

Proposition 4.5 is in the spirit of a recent result by KNOPS & STUART (1984) concerning the uniqueness of smooth solutions to the equilibrium equations of elasticity.

Our next result concerns the existence of cavitating minimisers for the displacement boundary value problem.

Proposition 4.7. *Let Φ satisfy* (H1)–(H5), (H9), (H10), (E1) *and* (E2). *Then any minimiser r of I on A_λ satisfies $r(0) > 0$ for λ sufficiently large.*

Proof. A minimiser r exists by Proposition 4.1 and is a smooth solution of the radial equilibrium equation by Proposition 0.3. It follows from Proposition 4.5 that if $r(0) = 0$ then $r(R) \equiv \lambda R$. To prove the proposition it therefore suffices to exhibit a function $\tilde{r} \in A_\lambda$ satisfying $\tilde{r}(0) > 0$ and having less energy than the homogeneous deformation for sufficiently large λ. To this end we choose the following test function

$$\tilde{r}(R) = \begin{cases} [R^3 + \varepsilon^3]^{\frac{1}{3}} & \text{if } R \in [0, \delta], \quad (4.24) \\ \lambda R & \text{if } R \in [\delta, 1] \quad (4.25) \end{cases}$$

where $\delta = \varepsilon/(\lambda^3 - 1)^{\frac{1}{3}}$. It is easily checked that $\tilde{r} \in A_\lambda$. The difference in energies ΔE is then given by

$$\Delta E = I(\tilde{r}) - I(\lambda R) = \int_0^\delta R^2 \left(\Phi\left(\tilde{r}', \frac{\tilde{r}}{R}, \frac{\tilde{r}}{R}\right) - \Phi(\lambda, \lambda, \lambda)\right) dR. \quad (4.26)$$

Setting $v = \dfrac{\tilde{r}}{R}$ and using the definition of δ we can write (4.26) the form

$$\Delta E = \varepsilon^3 \int_\lambda^\infty \frac{v^2}{(v^3 - 1)^2} \hat{\Phi}(v)\, dv - \frac{\varepsilon^3 \Phi(\lambda, \lambda, \lambda)}{3(\lambda^3 - 1)} \quad (4.27)$$

$$\leqq \varepsilon^3 \left(\int_\lambda^\infty \frac{v^2}{(v^3 - 1)^2} \hat{\Phi}(v)\, dv - \frac{\Phi(\lambda, \lambda, \lambda)}{3\lambda^3}\right). \quad (4.28)$$

Hence by (H9) and (H10), ΔE is negative for λ sufficiently large, as the first term in (4.28) is decreasing and the second is bounded away from zero.

Remark 4.8. The above proposition holds with (H3) and (H4) replaced by (H7) or by any conditions under which Proposition 4.5 holds (for a variety of such results see BALL (1982) chapter 6).

C. Existence of radial equilibria for shells.

Proposition 4.9. *Suppose that Φ satisfies* (H1), (E1). *For each $\lambda > 0$, $\varepsilon \in (0, 1)$ let*

$$I_\varepsilon(r) \stackrel{\text{def}}{=} \int_\varepsilon^1 R^2 \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}\right) dR \quad (4.29)$$

whenever $r \in A_\lambda^\varepsilon$ *where*

$$A_\lambda^\varepsilon \stackrel{\text{def}}{=} \{r \in W^{1,1}(\varepsilon, 1); r(1) = \lambda, r' > 0 \text{ a.e.}, r(\varepsilon) \geqq 0\}. \tag{4.30}$$

Then there exists an absolute minimiser r_ε *of* I_ε *on* A_λ^ε. *Moreover, if* Φ *also satisfies* (H5), (H11), (E2), *then* $r_\varepsilon \in C^2((\varepsilon, 1])$ *is a solution of* (0.20) *and there exists a* $\delta(\varepsilon) > 0$ *such that if* $\lambda \in (1 - \delta(\varepsilon), \infty)$, *then* $r_\varepsilon \in C^2([\varepsilon, 1])$ *and satisfies* (2.4)–(2.7).

Proof. Applying the techniques of Proposition 4.1 we obtain the existence of a minimiser r_ε of I_ε on A_λ^ε for each $\varepsilon \in (0, 1)$. Arguments identical to those contained in the appendix then imply that $r_\varepsilon \in ((\varepsilon, 1])$ and is a solution of (0.20) satisfying (2.4) and (2.5). It therefore suffices to show that r_ε satisfies $r_\varepsilon(\varepsilon) > 0$ since this implies that r_ε satisfies (2.7) (by arguments analogous to those in the appendix). We consider the two cases $\lambda > 1$ and $\lambda \leqq 1$.

Case (i). We suppose for a contradiction that $r_{\varepsilon_0}(\varepsilon_0) = 0$ for some $\varepsilon_0 \in (0, 1)$. Since $\lambda > 1$ by assumption, $r_{\varepsilon_0}(R_0) = R_0$ for some $R_0 \in (\varepsilon_0, 1)$. By the optimality of r_{ε_0} it then follows that $\tilde{r}$ defined by

$$\tilde{r}(R) = \begin{cases} R & \text{if } R \in [R_0, 1] \\ r_{\varepsilon_0}(R) & \text{if } R \in [\varepsilon_0, R_0) \end{cases}$$

satisfies

$$I_\varepsilon(\tilde{r}(R)) \leqq I_\varepsilon(R),$$

in contradiction to (H11).

Case (ii). We now suppose for a contradiction that for some $\tilde{\varepsilon} \in (0, 1)$ there does not exist δ with the stated properties. Then there exists a sequence $\lambda_n \nearrow 1$ as $n \to \infty$ with corresponding minimisers $r^{(n)}$ of $I_{\tilde{\varepsilon}}$ on $A_{\lambda_n}^{\tilde{\varepsilon}}$ with the property that

$$r^{(n)}(\tilde{\varepsilon}) = 0 \quad \text{for all } n. \tag{4.31}$$

This property of $r^{(n)}$ then implies the existence of a subsequence which we also denote by $\{r^{(n)}\}$, and a function $r \in A_1^{\tilde{\varepsilon}}$ with the property that

$$r^{(n)} \xrightarrow{W^{1,1}(\tilde{\varepsilon},1)} r \quad \text{as } n \to \infty. \tag{4.32}$$

(This follows by arguments analogous to those in Proposition 4.1). But

$$I_{\tilde{\varepsilon}}(r^{(n)}) \leqq I_{\tilde{\varepsilon}}(\lambda_n R) \quad \text{for all } n, \text{ since } \lambda_n R \in A_{\lambda_n}^{\tilde{\varepsilon}}.$$

Hence by (4.32) and the weak lower semicontinuity of $I_{\tilde{\varepsilon}}$ we obtain

$$I_{\tilde{\varepsilon}}(r) \leqq \varliminf_{n \to \infty} I_{\tilde{\varepsilon}}(r^{(n)}) \leqq \varliminf_{n \to \infty} I_{\tilde{\varepsilon}}(\lambda_n R) = I(R). \tag{4.33}$$

Clearly (H11) implies that $r(R) \equiv R$ in contradiction (4.31) and (4.32).

Our next proposition shows that the deformed cavity size is a monotone function of the boundary displacement for shells.

Proposition 4.10. *Suppose that Φ satisfies* (H1), (H5), (E1), (E2) *and that* $\varepsilon \in (0, 1)$. *If r_ε^λ is a minimiser of I_ε on A_λ^ε, then $r_\varepsilon^\lambda(\varepsilon)$ is a nowhere decreasing function of λ.*

Proof. The existence of r_ε^λ is a consequence of Proposition 4.9. We suppose for a contradiction that there exist an $\varepsilon \in (0, 1)$, displacements λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda_2 < +\infty$ and corresponding minimisers $r_\varepsilon^{\lambda_i}$ of I_ε on $A_{\lambda_i}^\varepsilon$ such that

$$r_\varepsilon^{\lambda_1}(\varepsilon) > r_\varepsilon^{\lambda_2}(\varepsilon).$$

Then there exists an $R_0 \in (\varepsilon, 1)$ such that

$$r_\varepsilon^{\lambda_1}(R_0) = r_\varepsilon^{\lambda_2}(R_0).$$

Consequently $\tilde{r}$ defined by

$$\tilde{r}(R) = \begin{cases} r_\varepsilon^{\lambda_1}(R) & \text{if } R \in [R_0, 1] \\ r_\varepsilon^{\lambda_2}(R) & \text{if } R \in [\varepsilon, R_0) \end{cases}$$

satisfies

(i) $\tilde{r} \in A_{\lambda_1}^\varepsilon$,

(ii) $I_\varepsilon(\tilde{r}) = I_\varepsilon(r_\varepsilon^{\lambda_1}) = \operatorname*{Inf}_{A_{\lambda_1}^\varepsilon} I_\varepsilon$.

The arguments contained in Proposition 4.9 then imply that $\tilde{r}, r_\varepsilon^{\lambda_1}$ belong to $C^2((\varepsilon, 1])$ and satisfy (0.20) on $(\varepsilon, 1]$. Hence by the definition of $\tilde{r}$ and the uniqueness of solutions to the initial value problem for (0.20) it follows that $r_\varepsilon^{\lambda_1}(R) \equiv r_\varepsilon^{\lambda_2}(R)$ which is a contradiction.

Finally we indicate the proof of Theorem 1.11.

Proof Theorem 1.11. The uniqueness of r_c follows from Theorem 3.8. Proposition 1.6 then implies that r_c is uniquely extendable to $r_c \in C^2((0, \infty))$ as a solution of (0.20) with $\dfrac{r_c(R)}{R} \searrow \lambda_c$ as $R \to \infty$, where $\lambda_c \in (1, \infty)$ by Remark 1.7. This proves (i) and (ii).

It follows from Proposition 4.1 that for each $\mu \in (0, \infty)$ there exists a global minimiser r_μ of I on A_μ. Propositions 0.3 and 4.5 together imply that

(a) r_μ is a solution of (0.20),

(b) $r_\mu(0) = 0$ if and only if $r_\mu(R) \equiv \mu R$.

We first treat the case in which $\mu > \lambda_c$. By the monotonicity of $r(R)/R$ ensured by Corollary 1.2 there is a unique solution δ of $\delta r_c\left(\dfrac{1}{\delta}\right) = \mu$ if and only if $\mu > \lambda_c$. We now define

$$\tilde{r}(R) \equiv \delta r_c\left(\frac{R}{\delta}\right).$$

As the equilibrium equation (0.20) is invariant under this rescaling, $\tilde{r}$ is a cavitating equilibrium solution that satisfies $\tilde{r}(1) = \mu$. We then obtain

$$I(\tilde{r}) = \tfrac{1}{3}\,[\Phi(\tilde{r}'(1), \mu, \mu) + (\mu - \tilde{r}'(1))\,\Phi_{,1}(\tilde{r}'(1), \mu, \mu)] < \frac{\Phi(\mu, \mu, \mu)}{3} = I(\mu R)$$

from Proposition 1.13 and (H1). Thus μR is not the global minimiser of I on A_μ and so $r_\mu(0) > 0$ by (b). It then follows from Proposition 0.3 that r_μ is a cavitating equilibrium solution, which must be unique by Theorem 3.8. Hence $r_\mu(R) \equiv \tilde{r}(R)$ proving (iv).

We now consider the case in which $\mu \leqq \lambda_c$. Suppose for a contradiction that there exists $\mu \leqq \lambda_c$ with $r_\mu(R) \not\equiv \mu R$. It follows from (b) that $r_\mu(0) > 0$ and so Proposition 0.3 implies that r_μ is a cavitating equilibrium solution. Thus by Proposition 1.6 r_μ is extendable to $r_\mu \in C^2((0, \infty))$ as a solution of (0.20) with

$$\frac{r_\mu(R)}{R} \to \lambda_{\tilde{c}} \quad \text{as } R \to \infty, \quad \text{where } \lambda_{\tilde{c}} \in [1, \infty).$$

If we then choose $\bar{\lambda} > \max(\lambda_c, \lambda_{\tilde{c}})$ we can find appropriate rescalings r_1, r_2 of r_c and r_μ satisfying $r_1(1) = r_2(1) = \bar{\lambda}$, *i.e.*, we can exhibit two distinct cavitating equilibria contradicting Theorem 3.8. Thus (iii) holds.

Corollary 1.12 follows by an identical argument with the exception that we use Remark 4.6 (in place of Proposition 4.5) to conclude that (b) holds.

5. Asymptotic Behaviour of Equilibria for Shells

In this section we present results on the asymptotic behavior of solutions to the mixed problem for shells studied in Section 2.

Proposition 5.1. *Suppose Φ satisfies* (H1), (E1). *Let $\lambda > 0$. If $\{\varepsilon_n\}$ is a sequence of positive numbers with $\varepsilon_n \to 0$ as $n \to \infty$ and if r_{ε_n} is a minimiser of I_{ε_n} on $A_\lambda^{\varepsilon_n}$, then there exist $r \in A_\lambda$ and a subsequence $\{\varepsilon_{n(j)}\}$ such that*

$$r_{\varepsilon_{n(j)}} \xrightarrow{W^{1,1}(\delta,1)} r \quad \text{as } j \to \infty \tag{5.1}$$

for each $\delta \in (0, 1)$. Moreover

$$I(r) = \operatorname*{Inf}_{A_\lambda} I. \tag{5.2}$$

Proof. The existence of r_{ε_n} follows from Proposition 4.9. For fixed $\delta \in (0, 1)$ there exists $N(\delta)$ such that $0 < \varepsilon_n < \delta$ whenever $n > N(\delta)$. It then follows from (E1) that

$$\int_0^1 \delta^2 \psi(r'_{\varepsilon_n})\, dR \leqq I_\delta(r_{\varepsilon_n}) \leqq I_{\varepsilon_n}(r_{\varepsilon_n}) = \operatorname*{Inf}_{A_\lambda^{\varepsilon_n}} I_{\varepsilon_n} \leqq I_{\varepsilon_n}(\tilde{r}) \leqq I(\tilde{r}) \leqq \text{const.} \tag{5.3}$$

for $n > N(\delta)$, where $\tilde{r}$ is any global minimiser of I on A_λ (by Proposition 4.1 at least one exists). Theorem 10.3 of CESARI (1983) then implies the existence of a subsequence $\{r_{\varepsilon_n}^\delta\}$ of $\{r_{\varepsilon_n}\}$ which is weakly convergent in $W^{1,1}(\delta, 1)$. Using the techniques of Proposition 4.1 and choosing inductive subsequences $\{r_n^{\delta_k}\}$ of $\{r_n^{\delta_{k-1}}\}$ for some positive sequence $\{\delta_k\} \to 0$ as $k \to \infty$, we can show that the diagonal sequence then satisfies (5.1) for some $r \in A_\lambda$.

Finally, to prove (5.2) we note that for each $\delta \in (0, 1)$, $r_{\varepsilon_{n(j)}} \in A_\lambda^\delta$ for j sufficiently large. Hence (5.3) implies that $I_\delta(r_{\varepsilon_{n(j)}}) \leqq I(\tilde{r})$ for j sufficiently large. The weak lower semicontinuity of I_δ then implies that

$$I_\delta(r) \leqq I(\tilde{r}).$$

But this inequality holds for each $\delta \in (0, 1)$ and so by the monotone convergence theorem

$$I(r) \leqq I(\tilde{r}). \tag{5.4}$$

Since $r \in A_\lambda$, equality holds in (5.4).

The idea that minimisers of I^ε on A_λ^ε converge to minimisers of I on A_λ as $\varepsilon \to 0$ was first noticed by BALL (1982).

Proposition 5.2. *Suppose that Φ satisfies* (H1)–(H5), (H11), (E1), (E2) *and that for each $\varepsilon > 0$, r_ε is a global minimiser of I_ε on A_λ^ε.*

(i) *If $\lambda \leqq \lambda_c$, then* $\operatorname{Sup}_{[\varepsilon,1]} |r_\varepsilon(R) - \lambda R| \to 0$ as $\varepsilon \to 0$, (5.5)

(ii) *If $\lambda > \lambda_c$, then* $\operatorname{Sup}_{[\varepsilon,1]} |r_\varepsilon(R) - r_c(R)| \to 0$ as $\varepsilon \to 0$, (5.6)

where r_c is the cavitating equilibrium solution satisfying $r_c(1) = \lambda$ (if there is no cavitation we set $\lambda_c = \infty$).

Proof. It follows from Proposition 5.1 and Theorem 1.11 that for each $\delta \in (0, 1)$

$$r_\varepsilon \xrightarrow{W^{1,1}(\delta,1)} r_t \quad \text{as } \varepsilon \to 0 \quad \text{if} \quad \lambda \leqq \lambda_c, \tag{5.7}$$

where

$$r_t(R) \equiv \lambda R$$

and

$$r_\varepsilon \xrightarrow{W^{1,1}(\delta,1)} r_c \quad \text{as } \varepsilon \to 0 \quad \text{if} \quad \lambda > \lambda_c. \tag{5.8}$$

We first treat the case in which $\lambda \leqq \lambda_c$. We suppose for a contradiction that (5.5) does not hold. Then there are an $\varepsilon_0 > 0$ and positive sequences $\{\varepsilon_n\}$, $\{x_n\}$ such that

(a) $\varepsilon_n \to 0$ as $n \to \infty$,

(b) $x_n \in [\varepsilon_n, 1]$ for each n,

(c) $|r_{\varepsilon_n}(x_n) - \lambda x_n| \geqq \varepsilon_0$ for all n.

Condition (5.7) implies that for each $\delta \in (0, 1)$

$$\operatorname{Sup}_{[\delta,1]} |r_\varepsilon(R) - \lambda R| \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.9}$$

We may therefore assume without loss of generality that $x_n \to 0$ as $n \to \infty$. On choosing $\delta = \varepsilon_0/(2\lambda)$ we obtain a contradiction of the fact that $r_\varepsilon'(R) > 0$ for $R \in (\varepsilon, 1]$. (r_ε would necessarily have the form indicated in Figure 3).

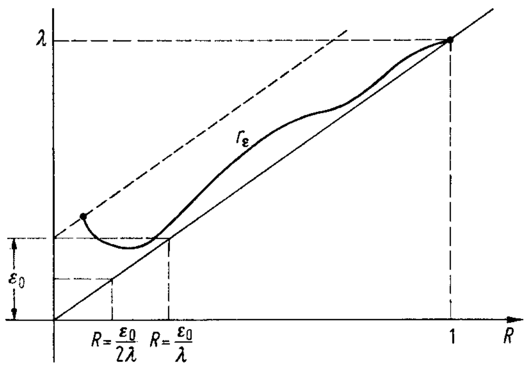

Fig. 3

We next consider the case in which $\lambda > \lambda_c$. We again suppose for a contradiction that (5.6) does not hold. Then there exist an $\varepsilon_0 > 0$ and positive sequences $\{\varepsilon_n\}$, $\{x_n\}$ with the properties

(a) $\varepsilon_n \to 0$ as $n \to \infty$, (5.10)

(b) $x_n \in [\varepsilon_n, 1]$ for each n, (5.11)

(c) $|r_{\varepsilon_n}(x_n) - r_c(x_n)| \geqq \varepsilon_0$ for all n. (5.12)

Again (5.8) implies that for each $\delta \in (0, 1)$

$$\sup_{[\delta,1]} |r_\varepsilon(R) - r_c(R)| \to 0 \quad \text{as } \varepsilon \to 0 \tag{5.13}$$

and we therefore assume that

$$x_n \to 0 \quad \text{as } n \to \infty. \tag{5.14}$$

Note that $r_{\varepsilon_n} \in C^2([\varepsilon_n, 1])$ for all n by (1.10) and Proposition 4.9. We assert that

$$r'_{\varepsilon_n}(1) < r'_c(1) \quad \text{for all } n, \tag{5.15}$$

since if for some N, $r'_c(1) < r'_{\varepsilon_N}(1)$, it then follows from (H1) that

$$0 \leqq T(r_c(1)) \leqq T(r_{\varepsilon_N}(1)).$$

As $T(r_{\varepsilon_N}(\varepsilon_N)) = 0$ (by Proposition 4.9) we conclude from Proposition 1.3 that $r'_{\varepsilon_N}(R) < \dfrac{r_{\varepsilon_N}(R)}{R}$ for $R \in [\varepsilon_N, 1]$. Consideration of the phase portrait together with (H1) then implies that

$$0 \leqq \frac{1}{v^2} \Phi_{,1}(r'_c(g^{\frac{1}{3}}(v)), v, v))\Big|_{v=\frac{r_{\varepsilon_N}(R)}{R}} < T(r_{\varepsilon_N}(R)) \tag{5.16}$$

for $R \in [\varepsilon_N, 1]$, where g is defined in Proposition 3.1. Condition (5.16) evaluated at $R = \varepsilon_N$ contradicts the fact that r_{ε_N} satisfies (2.7); thus (5.15) holds. The continuity of r_c implies the existence of a δ_0 such that

$$r_c(0) \leqq r_c(R) \leqq r_c(0) + \frac{\varepsilon_0}{3} \quad \text{for } R \in (0, \delta_0]. \tag{5.17}$$

Setting $\delta = \delta_0$ in (5.13) we obtain

$$|r_{\varepsilon_n}(R) - r_c(R)| < \frac{\varepsilon_0}{3} \quad \text{for } R \in [\delta_0, 1], \tag{5.18}$$

for sufficiently large n. Hence (5.17) implies that

$$r_{\varepsilon_n}(\delta_0) \leqq r_c(\delta_0) + \frac{\varepsilon_0}{3} \leqq r_c(0) + \frac{2\varepsilon_0}{3} \tag{5.19}$$

if n is sufficiently large. The arguments of Theorem 2.5 imply that

$$r_c(R) \neq r_{\varepsilon_n}(R) \quad \text{for } R \in [\varepsilon_n, 1),$$

and so by (5.12) and (5.15) we conclude that

$$r_{\varepsilon_n}(x_n) \geqq r_c(x_n) + \varepsilon_0 \geqq r_c(0) + \varepsilon_0 \tag{5.20}$$

for all n. The limit (5.14) shows that conditions (5.20) and (5.19) together contradict (2.4) for large n.

Proposition 5.3. *Let* Φ *satisfy* (H1)–(H5), (H11), (E1), (E2). *If* $\varepsilon \in (0, 1)$, $\lambda \in (1 - \delta(\varepsilon), \infty)$ *with* $\delta(\varepsilon)$ *defined as in Proposition 4.9, then* $r_\varepsilon \in C^2([\varepsilon, 1])$ *and is a solution of* (0.20) *satisfying* (2.4)–(2.7) *if and only if it is the global minimiser of* I_ε *on* A_λ^ε. *Moreover,*

(i) if $1 - \delta(\varepsilon) < \lambda \leqq \lambda_c$, then $\sup_{[\varepsilon,1]} |r_\varepsilon(R) - \lambda R| \to 0$ as $\varepsilon \to 0$

and

(ii) if $\lambda > \lambda_c$, then $\sup_{[\varepsilon,1]} |r_\varepsilon(R) - r_c(R)| \to 0$ as $\varepsilon \to 0$.

Proof. It follows from the arguments of Proposition 4.9 that a global minimiser r_ε always exists and satisfies (0.20) and (2.4)–(2.7). Theorem 2.4 implies that r_ε is unique. Thus the first half of the proposition is true. Statements (i) and (ii) follow from Proposition 5.2.

Remark 5.4. As a consequence of Corollary 1.12, Proposition 5.3 holds with (H3) and (H4) replaced by (H7).

Remark 5.5. It is clear from Theorem 1.11(iv) and Corollary 1.2 that for cavitating equilibria the deformed cavity size is a continuous monotone function of the boundary displacement λ. Combining this observation with Propositions 4.2 and 5.3 yields the following rigorous picture of the bifurcation that has occured. In Figure 4 we have plotted the deformed cavity size against the corresponding boundary displacement λ. The solid curve represents the values of $r_\lambda(0)$ where r_λ

is the minimiser of I on A_λ and the broken curve represents the values of $r_\varepsilon^\lambda(\varepsilon)$ where r_ε^λ is the minimiser of I_ε on A_λ^ε. ε is fixed and chosen to be small. $r_\varepsilon^\lambda(\varepsilon)$ forms a continuous curve by Proposition 2.1 and by the continuous dependence of solutions to (3.2) on initial data. It is at present unclear whether the broken curve rejoins the λ axis for sufficiently small values of λ; *i.e.* whether the opposite phenomenon to cavitation occurs with the cavity of a shell disappearing for sufficiently small boundary displacements.

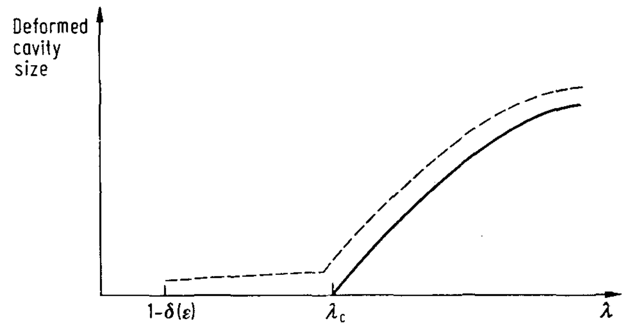

Fig. 4*
($\delta(\varepsilon)$ is defined as in Proposition 4.9)

6. Concluding Remarks

The solutions r_ε exhibit boundary layer behaviour with significant changes in strain in a neighbourhood of the cavity. If suitable conditions on the stored energy function Φ are imposed and if $\lambda < \lambda_c$ it is possible to prove the following *uniform* first order expansion for r_ε:

$$r_\varepsilon(R) = \varepsilon r_0\left(\frac{R}{\varepsilon}\right) + o(\varepsilon), \tag{6.1}$$

where r_0 is the unique solution of (0.20) on the exterior domain $[1, \infty)$ that satisfies (i) $\lim_{R\to\infty} \frac{r(R)}{R} = \lambda$, (ii) $r_0'(R) > 0$ for $R \in [1, \infty)$, (iii) $T(r_0(1)) = 0$. Thus (6.1) provides a uniform approximation of strains within the boundary layer. Expansions of this type together with the other results presented are of interest in studying the interaction between voids in an elastic material.

Work on metals (*e.g.* COX & LOW (1974), HANCOCK & COWLING (1977)) indicates that void nucleation and coalescence is a possible mechanism for the initiation of fracture. This type of ductile fracture is often considered to be a phenomenon of plasticity. However there is evidence that suggests that this type of phenomenon may be treated within the framework of nonlinear elasticity provided

* (In a recent paper HORGAN & ABEYARATNE (1985) obtain a similar picture using the two-dimensional stored energy function

$$\Phi(v_1, v_2) = v_1^{-2} + v_2^{-2} + 2v_1v_2).$$

unloading does not take place. I conjecture that in weak materials the presence holes leads to high stresses giving rise to cavitation. The cumulative effect of this state of stress across a body could be a mechanism for the initiation of fracture, with the creation of a series of holes leading to the formation of a crack.

7. Appendix

Proof of Proposition 0.3. The proof uses a technique from BALL (1982) Theorem 7.3. Let $k \in (1, \infty)$ and define s_k by

$$s_k = \left\{\varrho \in \left(\frac{1}{k}, 1\right);\ \frac{1}{k} < r'(\varrho) < k\right\}. \tag{7.1}$$

Let $v \in L^\infty(0, 1)$ satisfy

$$\int_{s_k} v\, d\varrho = 0. \tag{7.2}$$

Then setting

$$r_\varepsilon(\varrho) = r(\varrho) + \varepsilon \int_0^\varrho v(\tau)\, x_k(\tau)\, d\tau, \tag{7.3}$$

where x_k is the characteristic function of s_k, we find from (7.2) and (7.3) that r_ε satisfies

(i) $r_\varepsilon(1) = \lambda$,

(ii) $r_\varepsilon(0) = r(0)$,

(iii) $r'_\varepsilon(\varrho) = r'(\varrho)$ if $\varrho \leqq \frac{1}{k}$ or if $r'(\varrho) \notin \left(\frac{1}{k}, k\right)$.

Since $r \in C\left(\left[\frac{1}{k}, 1\right]\right)$ and $r' > 0$ a.e., it follows that $r\left(\frac{1}{k}\right) > 0$ and so $r_\varepsilon(\varrho) > 0$ for $\varrho \in (0, 1)$ provided ε is sufficiently small. It follows from (iii) that $r'_\varepsilon(\varrho) > 0$ for a.e. $\varrho \in (0, 1)$ provided $\varepsilon < \frac{1}{2k\, \|v\|_\infty}$. Thus $r_\varepsilon \in A_\lambda$ for sufficiently small ε. The triangle inequality implies that

$$\left|\frac{\Phi\left(r'_\varepsilon, \frac{r_\varepsilon}{\varrho}, \frac{r_\varepsilon}{\varrho}\right) - \Phi\left(r', \frac{r}{\varrho}, \frac{r}{\varrho}\right)}{\varepsilon}\right| \leqq \left|\frac{\Phi\left(r'_\varepsilon, \frac{r_\varepsilon}{\varrho}, \frac{r_\varepsilon}{\varrho}\right) - \Phi\left(r', \frac{r_\varepsilon}{\varrho}, \frac{r_\varepsilon}{\varrho}\right)}{\varepsilon}\right| + \left|\frac{\Phi\left(r', \frac{r_\varepsilon}{\varrho}, \frac{r_\varepsilon}{\varrho}\right) - \Phi\left(r', \frac{r}{\varrho}, \frac{r}{\varrho}\right)}{\varepsilon}\right| \tag{7.4}$$

for $\varrho \in (0, 1)$.

Notice that each side of (7.4) is identically zero for $\varrho \in \left(0, \frac{1}{k}\right)$. If $\varrho \in \left(\frac{1}{k}, 1\right)$ and $r'(\varrho) \in [1/k, k]$, then the two terms on the right hand side of (7.4) are bounded by a constant independent of ε. If $\varrho \in \left(\frac{1}{k}, 1\right)$ and $r'(\varrho) \notin \left(\frac{1}{k}, k\right)$ then we multiply the right-hand side of (7.4) by ϱ^2 and use (iii) and the mean value theorem to obtain

$$2\varrho^2 \,|\Phi_{,2}(r'(\varrho), g(\varrho, \Theta(\varrho), \varepsilon), g(\varrho, \Theta(\varrho), \varepsilon))|\, \frac{1}{\varrho} \int_0^\varrho v(\tau)\, x_k(\tau)\, d\tau \tag{7.5}$$

where

$$g(\varrho, \Theta(\varrho), \varepsilon) = \frac{r(\varrho)}{\varrho} + \frac{\varepsilon\Theta(\varrho)}{\varrho} \int_0^\varrho \varrho\, v(\tau)\, x_k(\tau)\, d\tau \tag{7.6}$$

and $\Theta(\varrho) \in (0, 1)$. We now write

$$g(\varrho, \Theta, \varepsilon) = \frac{r}{\varrho} \left(\frac{r(\varrho) + \varepsilon\Theta(\varrho) \int_0^\varrho x_k v\, d\tau}{r(\varrho)} \right). \tag{7.7}$$

Since

$$\left| \frac{r(\varrho) + \varepsilon\Theta(\varrho) \int_0^\varrho x_k v\, d\tau}{r(\varrho)} - 1 \right| < \varepsilon_0 \tag{7.8}$$

for ε sufficiently small, we conclude from (E2) that the right-hand side of (7.4) is bounded by

$$2\varrho^2 M \left(\Phi\left(r', \frac{r}{\varrho}, \frac{r}{\varrho}\right) + 1 \right) \frac{1}{\varrho} \int_0^\varrho x_k v\, d\tau \left(\frac{\varrho}{r(\varrho)}\right). \tag{7.9}$$

Since $r(\varrho) \geqq r\left(\frac{1}{k}\right)$ for $\varrho \in \left(\frac{1}{k}, 1\right)$ and since $I(r) < +\infty$ by assumption, it follows that (7.9) lies in $L^1\left(\frac{1}{k}, 1\right)$. As r is a global minimiser of I, on using the dominated convergence theorem we obtain

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \int_0^1 R^2 \left(\frac{\Phi\left(r'_\varepsilon, \frac{r_\varepsilon}{R}, \frac{r_\varepsilon}{R}\right) - \Phi\left(r', \frac{r}{R}, \frac{r}{R}\right)}{\varepsilon} \right) dR \\ &= \int_{\frac{1}{k}}^1 R^2 \left(\Phi_{,1}(R)\, x_k(R)\, v(R) + \frac{2\Phi_{,2}(R)}{R} \int_0^R x_k(\tau)\, v(\tau)\, d\tau \right) dR. \end{aligned} \tag{7.10}$$

Since $I(r) < +\infty$, $r\left(\frac{1}{k}\right) > 0$, it follows from (E2) that $R\Phi_2\left(r', \frac{r}{R}, \frac{r}{R}\right) \in L^1\left(\frac{1}{k}, 1\right)$. Integrating (7.10) by parts then gives

$$\int_{s_k} \left(R^2 \Phi_{,1}\left(r', \frac{r}{R}, \frac{r}{R}\right) + 2 \int_R^1 \varrho \Phi_{,2}\left(r', \frac{r}{\varrho}, \frac{r}{\varrho}\right) d\varrho \right) v(R)\, dR = 0. \tag{7.11}$$

As (7.11) holds for all $v \in L^\infty(0, 1)$ with $\int_{s_k} v\, d\varrho = 0$ it follows that

$$R^2\Phi_{,1}(R) + 2\int_R^1 \varrho\Phi_{,2}(\varrho)\, d\varrho = c_k \quad \text{for a.e. } \varrho \in s_k,$$

where c_k is a constant. Finally since meas $\left((0, 1) \setminus \bigcup_1^\infty s_k\right) = 0$, the c_k are all equal. An application of Theorem 4.2 of BALL (1982) implies $r \in C^m((0, 1])$ and satisfies (0.20) and (0.28).

If $r(0) > 0$ then let $w \in C^\infty((0, 1))$ satisfy $w(\varrho) = 1$ for $\varrho \in (0, \frac{1}{3})$ and $w(\varrho) = 0$ for $\varrho \in (\frac{2}{3}, 1)$. It is a consequence of (E2) that $R\Phi_{,2}(R) \in L^1(0, 1)$. On setting

$$u_\varepsilon(\varrho) = r(\varrho) + \varepsilon w(\varrho)$$

we obtain

$$0 = \frac{d}{d\varepsilon}[I(u_\varepsilon)]|_{\varepsilon=0} = \int_0^1 R^2\Phi_{,1}(R)\, w'(R) + 2R\Phi_{,2}(R)\, w(R)\, dR = -\operatorname*{Lim}_{R\to 0} R^2\Phi_{,1}(R),$$

proving the proposition.

Acknowledgements. The work contained in this paper was carried out at Heriot-Watt University under a Science and Engineering Research Council Studentship and grant. It is a pleasure to thank Professor J. M. BALL for many useful discussions during the course of this work.

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(Received January 7, 1986)