DISCONTINUOUS LIMIT SEMIGROUPS FOR THE SINGULAR ZHANG EQUATION AND ITS HYDRODYNAMIC VERSION

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Abstract. We study properties of the solutions of two semilinear parabolic equations, the singular Zhang equation and its hydrodynamic version:

$$v_t = v_{xx} + v_x/v \quad \text{and} \quad u_t = u_{xx} + \ln|u_x|,$$

which exhibit strong singularities on the sets $\{v = 0\}$ and $\{u_x = 0\}$ respectively.

Using the concept of maximal solutions constructed by a regular monotone approximation, we prove existence and boundedness of such solutions. We show that for some classes of initial data, the solutions become instantly entirely singular: $v(t) \equiv 0$ and $u(t) \equiv -\infty$ and for any arbitrarily small $t > 0$. In general, we establish that for both equations the solutions cannot satisfy the initial condition in any weak sense, i.e., the corresponding semigroups are not continuous at $t = 0$. Such discontinuous semigroups are exhibited by a wide class of singular nonlinear parabolic equations.

Dedicated to the memory of Anatolii S. Kalashnikov

1. Introduction: singular models and main results

1.1. **Nonlinear models.** The singular limit as $q \to 0$ of the generalised deterministic Kardar-Parisi-Zhang equation [8] or Krug-Spohn equation [9]

$$u_t = u_{xx} + \frac{1}{q}\,|u_x|^q, \quad q > 0, \tag{1.1}$$

is the singular Zhang equation [11]

$$u_t = u_{xx} + \ln|u_x|. \tag{1.2}$$

It occurs as a by-product in the study of complex directed polymers and is considered as a "prototype of much more complicated systems" [6]. Neither (1.2) nor its "hydrodynamic version"

$$v_t = v_{xx} + v_x/v, \tag{1.3}$$

are well understood mathematically in spite of their simple form. These are examples of semilinear parabolic equations with strong singularities of special type. Observe that (1.3)

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can be formally obtained from (1.2) by differentiating and setting $u_x = v$. Nevertheless, we will show that the solutions of these two equations exhibit completely different properties. Observe that only the solutions of (1.2) with non-empty set of extremal singular points, where $u_x = 0$, are of physical interest. They correspond to white noise initial conditions which are supposed to create eventually special asymptotic patterns.

Equation (1.3) is the limit case of the diffusion-(singular) convection equation

$$v_t = v_{xx} + v^{\alpha-1} v_x \tag{1.4}$$

as $\alpha \to +0$. For $\alpha > 0$ the mathematical theory is rather well-developed for this model (see the references in [7]) and exhibits several interesting properties such as infinite one-sided propagation, existence of back front, instantaneous shrinking of the support etc. No results in the limit case $\alpha = 0$ were available in the mathematical literature.

These examples belong to a class of singular uniformly parabolic equations. Though singularities are presented by lower order operators, they are so essential that, as we will show, the corresponding nonlinear semigroups are not of the class C_0 and in any (weak) sense are not continuous at the initial instant $t = 0$.

The first fundamental results on the existence, uniqueness, comparison, regularity for the heat equation and the porous medium equation with strong (singular) absorption and the localization criterion for such equations are one of the brilliant contributions of A.S. Kalashnikov to the PDEs theory. His paper [7] is one of the most important and complete surveys on this subject.

1.2. **Main results.** We prove existence and regularity results for the above equations (1.3), (1.2). The study of (1.3) is essentially based on the recent geometric theory [1] (the G-theory) of nonlinear parabolic equations with singular propagation. We adapt its main concepts to the above singular semilinear parabolic equations. We use the concept of the *maximal (proper) solutions* defined by a monotone regular approximation [1], [5]. In general, for very singular equations, this is a physically correct concept of solutions. Moreover, we will show that for both equations any concept of weak solutions does not apply. More precisely, the maximal solution, in fact, is not a solution of the Cauchy or Dirichlet problem with prescribed initial data since the corresponding limit semigroups (see the definition below) are not continuous at $t = 0$. This is the main important difference between these two equations and a wide class of others with less singular operators like (1) with $q > 0$ or (1.4) with $\alpha > 0$.

The main idea of the G-theory consists in using the families of the travelling wave (TW) solutions $V(x,t) = f(\xi), \quad \xi = x - \lambda t$, where the speed $\lambda \in \mathbf{R}$ is a parameter of the family. These particular solutions describe finite or infinite propagation on the singular level $\{v = 0\}$. The functions $f(\xi)$ then solve a nonlinear ODE with a parameter λ. In general, the intersection comparison technique based on the Sturmian argument establishes that existence-nonexistence, regularity and finite propagation properties of these TW solutions guarantee a similar behaviour of more general maximal solutions of the PDE.

The analysis of equation (1.2) also uses some comparison ideas and we first study special self-similar solutions describing effects of singularity formation for this PDE.

In Sections 2-5 we present a general existence-regularity theory for the above nonlinear singular equations (some results of Section 2 were first announced in [2]). Finally, we study the relationships between these two equations and show that the Dirichlet problem for (1.2) gives a unique solution of (1.3) subject a non-local constraint, which is equivalent to restricting the equation to a linear subspace of codimension 1.

2. Existence and "nonexistence" results for the hydrodynamic version

We consider the Cauchy problem for the hydrodynamic version of the Zhang equation

$$v_t = v_{xx} + v_x/v, \quad (x,t) \in Q = \mathbf{R} \times \mathbf{R}_+, \tag{2.1}$$

with non-negative bounded continuous initial data

$$v(0,x) = v_0(x), \quad x \in \mathbf{R}. \tag{2.2}$$

2.1. **Maximal solution and extended semigroup.** Equation (2.1) exhibits the obvious divergent structure:

$$v_t = (v_x + \ln v)_x. \tag{2.3}$$

Nevertheless, it contains a strong singularity at $v = 0$, and a standard concept of weak solutions does not apply directly and in fact is not available in general, see comments below. Existence and uniqueness results for weak solutions are unknown.

In what follows we consider *maximal* solutions which are constructed by regular monotone approximations of the problem, [5], Section 2. It is easy to see that for (2.1) this can be done by a suitable uniformly positive approximation of the initial data only. Then no regular approximation of the equation considered on a subset of uniformly positive solutions is necessary. Let $\{v_\varepsilon\}$, $\varepsilon > 0$, be a family of global classical solutions of equation (2.1) with strictly positive bounded initial data

$$v_{0\varepsilon}(x) = v_0(x) + \varepsilon, \quad x \in \mathbf{R}.$$

By the Maximum Principle, $v_\varepsilon \geq \varepsilon > 0$ everywhere, so that the family consists of nonsingular solutions uniformly bounded away from 0. Therefore the classical parabolic theory applies to guarantee global existence of smooth bounded v_ε.

The family $\{v_\varepsilon\}$ is monotone decreasing in ε by the Maximum Principle and is uniformly bounded above: for small $\varepsilon > 0$, $v_\varepsilon \leq v_1 \leq \sup v_0 + 1$. Hence, there exists a finite limit

$$v = \lim_{\varepsilon \to +0} v_\varepsilon \geq 0.$$

This gives us the unique *proper* solution which is maximal in a natural sense. It does not depend on the type of monotone decreasing approximation [5]. This defines the extended limit semigroup of maximal solutions

$$v(x,t) = T_t v_0(x) \geq 0.$$

This definition gives no regularity properties of the maximal solutions. We even do not know if, for arbitrarily smooth initial data, $v(x,t)$ is a continuous function (in fact, it is not). As usual in the study of singular nonlinear evolution equations with blow-up or extinction [4], [5], [10], the first existence question consists in the following preliminary classification of singularities. We distinguish two cases:

(i) *complete singularity*: for arbitrarily small $t > 0$ the maximal solution is entirely singular, i.e., $v(x,t) \equiv 0$, and

(ii) *incomplete singularity*: $v(x,t) \not\equiv 0$ for all small $t > 0$.

Actually, such a classification concerns first continuity-like properties of maximal solutions at $t = 0$. Usually, in the classical parabolic theory, the case (i) is treated as *nonexistence* (and (ii) as *existence*) of a solution of the Cauchy problem, since the maximal solution $v(x,t) \equiv 0$ cannot satisfy any nontrivial initial condition. On the other hand, as we have mentioned in the introduction, the nonexistence treatment makes no sense for such singular equations with essentially *discontinuous* semigroups. For convenience, sometimes, we keep using the term nonexistence where the context dictates.

2.2. **Set of travelling waves.** Substituting the TW solution

$$V(x,t) = f(\xi) \ge 0, \quad \xi = x - \lambda t, \quad \lambda \in \mathbf{R}, \tag{2.4}$$

into equation (2.1), we obtain the ODE for the continuous profile $f = f_\lambda$

$$f'' + \lambda f' + f'/f = 0. \tag{2.5}$$

It is integrated once and gives that a non-trivial $f' = P(f) \not\equiv 0$ satisfies

$$P = C - \ln f - \lambda f, \quad f \ge 0, \tag{2.6}$$

where $C \in \mathbf{R}$ is a free constant of integration.

We study singular solutions $f \ge 0$ of (2.5) in the sense that

$$0 \in \overline{\mathrm{Im}}\{f(\xi)\}, \tag{2.7}$$

i.e., $f(\xi)$ attains the singular 0-level at a finite or infinite point. We next define the set

$$\Delta = \{\lambda \in \mathbf{R} \ : \ (2.5),\ (2.7) \text{ has a maximal solution}\}. \tag{2.8}$$

By the maximal solution of the ODE we mean that one which can be constructed by a regular strictly positive approximation (similar to the definition for the PDE). Fixing an $n \gg 1$, we consider the orbit $P = P_n(f)$ satisfying

$$P_n(1/n) = 0. \tag{2.9}$$

It follows from (2.6) that such an orbit exists with the constant

$$C = C_n = (\lambda/n) - \ln n \to -\infty, \quad n \to \infty. \tag{2.10}$$

Then the minimal orbit on the phase plane $\{f, P\}$ (which gives the maximal TW) does not exist since

$$f' \equiv P_{\min}(f) = \lim_{n \to \infty} P_n(f) = -\infty \quad \text{for any } f > 0. \tag{2.11}$$

Therefore, $\Delta = \emptyset$.

Let us classify the limit $n \to \infty$. We describe the properties of $\{f_n(\xi)\}$ for large n. For any $\lambda \in \mathbf{R}$, we obtain from (2.6), (2.7) a small monotone increasing ("flat") profile f_n^+ such that

$$f_n^+(0) = 0, \;\; f_n^+(\xi) > 0, \;\; (f_n^+)'(\xi) > 0 \text{ for } \xi > 0,$$

and

$$f_n^+(\xi) \to 1/n - 0 \;\; \text{as } \xi \to \infty. \tag{2.12}$$

Therefore, uniformly in ξ

$$f_n^+(\xi) \to 0 \quad \text{as } n \to \infty. \tag{2.13}$$

This means that for increasing TW profiles with $P > 0$ we observe the sa-called *flat limit* as $n \to \infty$, [1], Section 2.1.

On the other hand, for any $\lambda \in \mathbf{R}$ there exists a "steep" profile f_n^- satisfying

$$f_n^-(\xi) > 1/n, \;\; (f_n^-)'(\xi) < 0 \quad \text{for } \xi \in \mathbf{R}, \;\; f_n^-(\xi) \to 1/n + 0, \; \xi \to \infty. \tag{2.14}$$

Due to (2.11), we have that on any fixed level $\delta > 0$ there holds

$$f'(\xi)|_{f(\xi)=\delta} \to -\infty \quad \text{as } \; n \to \infty. \tag{2.15}$$

This is called the *steep limit* of nonexistence. One can choose a sequence $\{f_n^-\}$ such that

$$\lim_{n \to \infty} f_n^-(\xi) = H_\infty(\xi) = \{\infty, \; \xi \le 0; \;\; 0, \; \xi > 0\}, \tag{2.16}$$

where H_∞ is the infinite-step Heaviside function.

2.3. **Discontinuity: first example of complete singularity.** The steep limit as $n \to \infty$ of negative orbits $P_n(f) < 0$ implies nonexistence for the initial data exhibiting the corresponding negative slope behaviour. Below, we present a simple geometric proof of this singular property of nonexistence. For nonnegative functions we denote

$$\operatorname{supp} v_0 = \{x \in \mathbf{R} : \;\; v_0(x) > 0\}.$$

Theorem 2.1. *Let*

$$\sup \{\operatorname{supp} v_0\} < \infty. \tag{2.17}$$

Then

$$v(x,t) \equiv 0 \quad \textit{for any } t > 0. \tag{2.18}$$

Proof. Fix $\varepsilon = 1/n \ll 1$ and choose the corresponding steep TW profile f_n^- satisfying properties (2.14)-(2.16). By (2.17) there exists a constant a independent of n such that the regularised initial data satisfy $v_{0\varepsilon}(x) \equiv v_0(x) + 1/n \le f_n^-(x-a)$. Then by the standard comparison of bounded classical solutions of a uniformly parabolic equation we get that $v_\varepsilon(x,t) \le f_n^-(x - \lambda t - a)$ *in* Q.

Passing to the limit $\varepsilon = 1/n \to 0$, we arrive at the inequality

$$v(x,t) \le H^\infty(x - \lambda t - a).$$

Now passing to the limit $\lambda \to -\infty$ (then $x - \lambda t - a \to +\infty$ for any fixed x) we get that $v(x,t) \le 0$ so that (2.18) holds. $\square$

2.4. **Nonexistence for solutions changing sign.** We show that the above comparison technique applies to solutions with bounded initial data v_0 changing sign. Assume that we can construct a proper solution by a regular approximation, so that these proper solutions satisfy the Maximum Principle. Then we get the following result.

Proposition 2.1. *If v_0 is compactly supported, then $v(x,t) \equiv 0$ for any $t > 0$.*

Proof. It is quite similar. Indeed by the same comparison from the right-hand side with the steep decreasing TWs $f_n^-(x - \lambda t)$, $\lambda \ll -1$, we first establish the the positive parts or the solutions vanish identically:

$$[v(x,t)]_+ \equiv 0, \quad t > 0.$$

Since the equation (2.1) is invariant under the symmetry transformation $v \mapsto -v,\ x \mapsto -x$, we perform the same comparison from the left-hand side with the steep increasing TWs $-f_n(x + \lambda t)$, $\lambda \gg 1$ which in a similar way gives that

$$[v(x,t)]_- \equiv 0, \quad t > 0,$$

completing the proof. □

2.5. **Singular finite-time limit of self-similar solutions.** We now show that the singular behaviour of the solutions crucially depends on the behaviour of the initial data $v_0(x)$ for $x \gg 1$. Consider the self-similar solutions depending on the standard blow-up similarity variables

$$v_*(x,t) = (T-t)^{1/2} g(\eta), \quad \eta = x/(T-t)^{1/2}, \quad t \in (0,T), \tag{2.19}$$

where $T > 0$ is the blow-up time. Then $g \geq 0$ solves the ODE

$$g'' - g'\eta/2 + g/2 + g'/g = 0, \quad \eta \in \mathbf{R}. \tag{2.20}$$

By a standard local analysis of (2.20) for $\eta \gg 1$ one can see that it admits a family of solutions $g(\eta)$ satisfying

$$g(\eta) = \eta^{-1}(1 + o(1)),\ \eta \to \infty. \tag{2.21}$$

Therefore, near the blow-up time, $t \to T^-$, we have for any fixed $x > 0$ that $\eta = x/(T-t)^{1/2} \to \infty$ so that uniformly on any subset $[d, \infty)$, $d > 0$, the self-similar solution has the asymptotic behaviour

$$v_*(x,t) = (T-t)x^{-1}\,(1 + o(1)) \to 0, \quad t \to T^-. \tag{2.22}$$

We thus observe the effect of disappearance of the solution in a finite time. This depends on the special spatial structure of the corresponding initial data:

$$v_*(x,0) = Tx^{-1}\,(1 + o(1)), \quad x \to \infty. \tag{2.23}$$

We claim that if the condition (2.23) is violated and $v_0(x) = o(x^{-1})$, then the maximal solution disappears instantaneously. As in the above nonexistence theorem, the proof is based on a careful comparison of the maximal solution with the steep TW solutions $f_n^-(x - \lambda_n t - a_n)$ and passage to the limit $n \to \infty$, where both parameters $\lambda_n \to -\infty$ and $a_n \to \infty$ become unbounded eventually.

2.6. **Positivity.** The flat limit $n \to \infty$ of nonexistence in the ODE implies existence and positivity of the solutions for another class of initial data.

Theorem 2.2. *Let there exist a constant $\delta > 0$ such that*

$$v_0(x) \ge \delta \quad for \;\; x \gg 1. \tag{2.24}$$

Then

$$v(x,t) > 0 \quad in \;\; Q. \tag{2.25}$$

Proof. We again set $\varepsilon = 1/n \ll 1$, choose $a \gg 1$ and and $\lambda \ll -1$ so that by (2.24) and (2.13) we guarantee that $v_{0\varepsilon}(x) \ge f_n^+(x-a)$, $x \ge a$. Then by comparison since $v_\varepsilon(x,t) > 0$

$$v_\varepsilon(x,t) \ge \; f_n^+(x - \lambda t - a) \quad \text{in } Q. \tag{2.26}$$

We first fix $n = N \gg 1$ on the right-hand side and pass to the limit $\varepsilon \to 0$ to get that $v(x,t) \ge f_N^+(x-\lambda t-a) \not\equiv 0$. The positivity follows from (2.26) by setting $\lambda = \lambda(N) \to -\infty$ as $N \to \infty$. In the limit the perturbations propagate to the left with infinite speed. This completes the proof. $\square$

2.7. **Nonexistence and positivity for fully nonlinear equations.** The proof of Theorems 2.1 (nonexistence) and 2.2 (positivity and existence) are entirely geometrical in the sense that they are based on comparison with a family of the TW solutions $\{f_n^-\}$ and $\{f_n^+\}$ satisfying certain properties. Therefore, as in the G-theory of singular propagation [1], such results hold for general fully nonlinear parabolic equations

$$v_t = F(v, v_x, v_{xx}), \quad F_r'(p,q,r) \ge 0 \;\; \text{(the parabolicity condition)},$$

provided that the orbits are smooth beyond a singular subset (level), unique maximal solutions can be constructed by monotone regularization and the maximal TWs $\{f_n^\pm(\xi)\}$ obtained from the family of the ODEs

$$F(f, f', f'') + \lambda f = 0 \quad (\lambda \in \mathbf{R})$$

satisfy the required properties.

2.8. **Continuous self-similar collapse of singularity.** We now construct a global in time self-similar solution of the form

$$\tilde v(x,t) = t^{1/2} h(\zeta), \quad \zeta = x/t^{1/2}, \tag{2.27}$$

which describes a continuous collapse of singularity for monotone increasing initial data. The function $h > 0$ solves the ODE (cf. (2.20) with the opposite sign in the linear first-order operator)

$$h'' + h'\zeta/2 - h/2 + h'/h = 0,$$

which admits the solutions satisfying (cf. (2.21))

$$h(\zeta) = -\zeta^{-1}(1 + o(1)), \quad \zeta \to -\infty.$$

Such monotone increasing profiles have the following asymptotic behaviour:

$$h(\zeta) = C\zeta(1 + o(1)), \quad \zeta \to +\infty,$$

where $C > 0$ is a constant. Passing to the limit $t \to +0$ in (2.27) and using the above expansions, we obtain the corresponding initial data

$$\tilde{v}(x, +0) = \{0, \;\; x \le 0; \quad Cx, \;\; x > 0\}. \tag{2.28}$$

The self-similar solutions exhibit the continuous behaviour at $t = +0$:

$$\tilde{v}(x,t) = -tx^{-1} + o(1) \to 0 = \tilde{v}(x,+0), \quad x < 0;$$

$$\tilde{v}(0,t) = h(0)t^{1/2} \to 0 = \tilde{v}(0,+0);$$

$$\tilde{v}(x,t) = Cx + o(1) \to Cx = \tilde{v}(x,+0), \quad x > 0.$$

In particular, the uniform estimate for $x < 0$ takes the form: as $t \to +0$

$$\tilde{v}(x,t) \le \tilde{v}(0,t) = h(0)t^{1/2} \to 0.$$

This limit analysis as $t \to +0$ proves that the self-similar solutions are the maximal ones. Each one $\tilde{v}$ can be obtained as the limit of the regularised sequence $v_\varepsilon(x,t) = \tilde{v}(x, t+\epsilon) > 0$ uniformly on any compact subset from Q.

2.9. **Continuity: local comparison with similarity solutions.** We now show that a standard local comparison applies to ensure a continuous collapse of singularity for more general maximal solutions. This approach will be used later on in other problems and now we discuss it in some detail.

First, given arbitrary increasing initial data

$$v_0(x) \equiv 0 \text{ for } x \le 0, \;\; v_0(x) > 0 \text{ for } x > 0, \tag{2.29}$$

and assuming first that $v_0(x) = Cx(1+o(1))$, $x \to +0$, by a standard local comparison with self-similar solutions $\tilde{v}(x \pm \delta, t)$ shifted in x with slightly perturbed expansion coefficients $C \mp \delta$, it is not difficult to prove that for any data like that the singularity at $x = 0$ collapses continuously in $C_{\rm loc}(\mathbf{R})$.

Second, one can see that the same is true for any such monotone continuous initial data not necessarily satisfying the linear behaviour near the singularity $x = +0$. In fact, such local comparison shows that once the problem admits a particular continuous (self-similar) solution, any solution from the corresponding class is continuous at the singularity. Indeed, if a solution would be discontinuous at $x = 0$, $t = +0$, this would contradict the local comparison from above with the self-similar solution $\tilde{v}(x + \delta, t)$ with $C = C(\delta) \gg 1$. Indeed, using the asymptotic properties of $\tilde{v}$, we conclude that $v(0,t) \le \tilde{v}(\delta, t) = C(\delta)\delta + o(1) \le 2C(\delta)\delta$ for small $t > 0$, and since $C(\delta)\delta \to 0$ as $\delta \to 0$ for continuous initial data v_0, the continuity at $x = 0$, $t = +0$ follows.

In such an analysis it is important to bear in mind that the parabolic flow is regular and smooth on any compact subset where the solution is uniformly bounded away from zero, so that the only discontinuity would occur first from the singularity at the origin $x = 0$, $t = 0$ (and next this discontinuity can give an essential change of the solution beyond the singularity in the smooth part of the flow), the point which can be controlled by a standard local comparison with continuous similarity solutions. Let us state this simple continuity result.

Theorem 2.3. *For any continuous monotone increasing initial data* (2.29), *the maximal solution $v(x,t)$ is continuous at $t=+0$, so that it satisfies the initial condition in the usual sense: $v(x,0)=v_0(x)$.*

It is easy to see that in this case, even for discontinuous initial data, say,

$$v_0(x) = \{1,\ x>0;\quad 0,\ x \le 0\},$$

the singularity collapses in a continuous manner though we cannot prove this by comparison with similarity solutions (2.27) with continuous piece-wise linear initial data (2.28) (indeed, here $C(\delta)=1/\delta$ so that $C(\delta)\delta = 1 \not\to 0$ as $\delta \to 0$). Nevertheless, one can see that for such discontinuous initial data, the collapse is described by the standard similarity solution

$$\bar v(x,t) = \theta(\zeta), \quad \zeta = x/t^{1/2}, \tag{2.30}$$

of the heat equation $v_t = v_{xx}$. Then $\theta>0$ solves the ODE

$$\theta'' + \theta'\zeta/2 = 0, \zeta \in \mathbf{R};\quad \theta(-\infty)=0,\ \theta(\infty)=1.$$

Indeed, we have that on such scaling structures, as $t\to 0$,

$$|\bar v_{xx}| = \frac{1}{t}\,|\theta''(\zeta)| \gg \left|\frac{\bar v_x}{\bar v}\right| = \frac{1}{\sqrt t}\left|\frac{\theta'(\zeta)}{\theta(\zeta)}\right|,$$

so that the lower-order singular term is negligible in a small neighbourhood of initial singularity in comparison with the diffusion operator. As usual, this makes it possible to construct a supersolution for $t\approx +0$ with a slightly perturbed asymptotic scaling structure given in (2.30), see examples in Chapt. 2 and 4 in [10].

2.10. **Discontinuous semigroup.** Theorem 2.1 implies that the maximal solution, in fact, is not a solution of the Cauchy problem: it cannot satisfy the initial condition in any (weak) sense. The limit semigroup T_t of maximal solutions is not continuous at $t=0$. To emphasise that once more, consider the initial data with a zero gap on an interval of non-zero measure, say, $[-a,a]$, $a>0$:

$$v_0(x) = 0 \quad \text{for } |x| \le a. \tag{2.31}$$

Consider two continuous non-negative bounded functions $\phi(x)\not\equiv 0$ and $\psi(x)\not\equiv 0$ where $\phi(x)>0$ is monotone increasing for $x>a$, $\phi(x)=0$ for $x\le a$, and vice versa $\psi(x)\equiv 0$ for $x \ge -a$.

The initial data

$$v_0(x) = \phi(x) + \psi(x), \quad x\in \mathbf{R}, \tag{2.32}$$

satisfy the gap condition (2.31). With a slight modification of the proofs of Theorem 2.1 and 2.2, we obtain the following result.

Theorem 2.4. *The maximal solution $v(x,t)>0$ of the Cauchy problem* (2.1), (2.32) *satisfies*

$$v(x,+0) \equiv \lim_{t\to+0} v(x,t) = \phi(x), \quad x\in\mathbf{R}, \tag{2.33}$$

i.e., $v(x,t)$ does not depend on $\psi(x)$.

Proof. In order to prove that the left-hand structure of the initial data given by $\psi(x)$ for $x \le -a$ disappears in no time, we apply the proof of Theorem 2.1. Fix $\varepsilon \ll 1$ and consider the classical solution $v_\varepsilon \ge \varepsilon$. By continuity, there exists $t_\varepsilon > 0$ such that $v_\varepsilon(0,t) \le 2\varepsilon$ on $(0, t_\varepsilon]$. Choosing $n \gg 1$ such that $1/n \ge 2\varepsilon$, we now can compare on $P_\varepsilon = \mathbf{R}_- \times (0, t_\varepsilon]$ the solution v_ε and the steep TW $f_n^-(x - \lambda t)$ with arbitrary $\lambda \ll -1$ which gives $v_\varepsilon(x,t) \le f_n^-(x - \lambda t)$ in P_ε. Passing to the limit $\lambda \to -\infty$ we obtain that $v_\varepsilon(x, t_\varepsilon) \le 1/n$. Setting $\varepsilon \to 0$ yields that $v(x, +0) \equiv 0$ for $x < 0$. The rest of (2.33), the continuity for $x > 0$ is proved by the local comparison with the continuous maximal solutions $\tilde{v}(x + \delta, t)$ as in the previous subsection. □

3. A generalised model

Let us show that the above results apply to more general equations. For instance, consider the generalized equation

$$v_t = v_{xx} + v_x/\varphi(v), \tag{3.1}$$

where $\varphi(v) > 0$ for $v > 0$ is a smooth function satisfying the singularity condition

$$\varphi(v) \to 0, \quad v \to +0.$$

We then define the unique maximal solution by the positive approximation of the initial data, and the further theory of equation (3.1) depends on the properties of the TW solutions $V(x,t) = f(\xi)$, $\xi = x - \lambda t$, where $f \ge 0$ solves the ODE

$$f'' + \lambda f + f'/\varphi(f) = 0. \tag{3.2}$$

Integrating it, we obtain (cf. (2.6))

$$f' \equiv P = C - \Psi(f) - \lambda f, \quad \Psi(f) = \int_1^f ds/\varphi(s). \tag{3.3}$$

We consider two cases.

3.1. **Complete singularity (nonexistence).** If

$$|\Psi(0)| = \int_0^1 ds/\varphi(s) = \infty, \tag{3.4}$$

then as in Section 2.2 we can construct the steep TW profiles f_n^- exhibiting similar fast propagation properties. This gives Theorem 1 on complete singularity (in fact, nonexistence) of a nontrivial maximal solution for any initial data with finite right-hand interface.

3.2. **Positivity and finite propagation.** We always can prove Theorem 2.2 using the appropriate flat profiles f_n^+. If

$$\int_0^1 ds/\varphi(s) < \infty, \tag{3.5}$$

then we can observe finite propagation of the right-hand interface of the maximal solution. Given arbitrary $\lambda \in \mathbf{R}$, we define the orbit $P_{\min}(f)$ with $C = \Psi(0)$ such that $P_{\min}(0) = 0$. This minimal orbit is a maximal solution of the PDE and can be constructed via a regular limit $P_{\min}(f) = \lim P_n(f)$, $P_n(1/n) = 0$. Then solving the corresponding first-order equation

$$f' = P_{\min}(f) \equiv \int_0^f ds/\varphi(s) - \lambda f = \int_0^f (1 + o(1))ds/\varphi(s), \;\; f \to 0,$$

one can derive the criterion of finite propagation of the TWs

$$\int_0^1 \left(\int_0^z ds/\varphi(s) \right)^{-1} dz < \infty. \tag{3.6}$$

By comparison of both the maximal solutions $v(x,t)$ and the TW $f(\xi)$ we obtain the result.

Theorem 3.1. *Let* (3.5) *and* (3.6) *hold. Then if* $\sup\{\mathrm{supp} v_0\} < \infty$*, the maximal solution has the right-hand interface*

$$s(t) = \sup\{\mathrm{supp}\, v(\cdot, t)\} < \infty, \quad t > 0.$$

In the latter case the regularity and the interface propagation can be studied by the complete set $B = \{V(x,t), \; \lambda \in \mathbf{R}\}$ of the TWs. See [1] and [3] where the techniques of the G-theory applied to the PME with strong absorption

$$u_t = (u^m)_{xx} - u^p, \quad m > 1, \; p \in (-m, 2 - m).$$

4. Singular Zhang equation: the Cauchy problem

4.1. **Nonexistence: bounded data and single-well singularity.** We now consider the Cauchy problem for the semilinear equation

$$u_t = u_{xx} + \ln |u_x|, \quad u(x,0) = u_0(x), \tag{4.1}$$

where u_0 is a bounded continuous function. We first study the solutions of the physical version of the equation which does not admit spatially flat solution $U(t)$. This corresponds to the following definition of the singular logarithmic term via continuity:

$$\ln |u_x| = -\infty \;\text{ if }\; u_x = 0. \tag{4.2}$$

The singularity has a typical single-well shape. With this definition, in order to determine the maximal solution, we introduce a monotone regular approximation of the equation

$$u_t = u_{xx} + F_\varepsilon(u_x), \quad \varepsilon > 0, \tag{4.3}$$

where for any $p = u_x \in \mathbf{R}$ there is a monotone convergence from above

$$F_\varepsilon(p) = \ln[\varepsilon^2 + p^2]^{1/2} \to \ln|p| + 0, \quad \varepsilon \to +0. \tag{4.4}$$

Let $\{u_\varepsilon\}$ be the global classical solutions of the regularised equation with the same initial data u_0. Then $\{u_\varepsilon\}$ is monotone decreasing in ε so that there exists the limit

$$u = T_t u_0 \equiv \lim_{\varepsilon \to 0} u_\varepsilon \quad \text{in } \; Q,$$

which is the maximal solution independent of the type of monotone decreasing approximation and T_t is the corresponding limit semigroup. The maximal solutions is unique and satisfies the Maximum Principle [5].

Once we start our study of the actual properties of the limit semigroup, we arrive at a typical complete singularity (nonexistence-like) result.

Theorem 4.1. *For any bounded initial data u_0, the maximal solution of* (4.1) *is*

$$u(x,t) \equiv -\infty \quad \text{in } \; Q. \tag{4.5}$$

Proof. Let $u_0 \le C_0$. We compare $u_\varepsilon(x,t)$ with the flat solution $U_\varepsilon(t)$ satisfying

$$U_\varepsilon' = F_\varepsilon(0), \quad t > 0; \;\; U_\varepsilon(0) = C_0.$$

Then $U_\varepsilon(t) = F_\varepsilon(0)t + C_0$. By comparison of classical solutions $u_\varepsilon(x,t) \le F_\varepsilon(0)t + C_0$ for $t > 0$. Passing to the limit $\varepsilon \to 0$ and using the definition of the logarithmic singularity and of the approximation (see (4.2), (4.4)), we arrive at (4.5). □

Remark: nonexistence for the homogeneous Neumann problem. The same nonexistence result is true for the homogeneous Neumann problem in $Q_1 = \{x \in (-1,1), t > 0\}$ with bounded initial data u_0 and boundary conditions

$$u_x = 0 \quad \text{for } x = \pm 1, \;\; t > 0.$$

Since the flat solutions $U_\varepsilon(t)$ satisfy this Neumann condition, by comparison we arrive at the same conclusion that $u(x,t) \equiv -\infty$ for $t > 0$.

4.2. **Existence for unbounded data.** In the case of the single-well singularity, the maximal solutions can be nontrivial for unbounded data. The analysis is relied on the fact that equation (4.1) admits an obvious stationary solution $u_s(x) = x$.

Theorem 4.2. *Let $u_0(x) \ge x$ in $\mathbf{R}$. Then the maximal solution to* (4.1) *with the single-well singularity* (4.2) *satisfies*

$$u(x,t) \ge x \quad \text{in } \; Q.$$

Proof. With the approximation (4.4), $\underline{u}_\varepsilon = x$ is a subsolution of (4.3) and the result follows. □

4.3. **Existence: bounded data and double-well singularity.** In order to get non-trivial maximal solutions for bounded data in $\mathbf{R}$, we need a "slight" modification of the singularity. Unlike the single-well structure (4.2), we consider a singular double-well structure described by the vanishing condition

$$\ln|u_x| = 0, \quad u_x = 0.$$

Formally, this corresponds to the problem

$$u_t = u_{xx} + \ln|u_x|\chi_{\{u_x \neq 0\}}, \quad u(x,0) = u_0(x). \tag{4.6}$$

Then a suitable regular approximation also exhibits a similar double-well structure and can be given by

$$F_\varepsilon(p) = \ln[(1+\varepsilon)|p|]\,\frac{p^2}{\varepsilon^2 + p^2} \to \ln|p|\chi_{\{p\neq 0\}}, \;\; \varepsilon \to 0.$$

The convergence is pointwise for any $u_x \in \mathbf{R}$ and uniform on any subset bounded away from $p = u_x = 0$. We next construct the maximal solution $u = \lim u_\varepsilon$ which is bounded by comparison with bounded flat stationary solutions.

Theorem 4.3. *For any bounded u_0, the problem* (4.6) *admits a global bounded non-trivial maximal solution.*

On the basis of the geometric theory, using the complete families of the maximal TWs, we can study optimal regularity properties and finite propagation of singularities.

5. Singular Zhang equation: the Dirichlet problem

5.1. **Existence for bounded data and single-well singularity.** Consider equation (4.1) with a single-well monotone approximation (4.4) in $Q_1 = (-1,1) \times \mathbf{R}_+$ with the Dirichlet boundary conditions

$$u(\pm 1, t) = 0, \quad t \geq 0. \tag{5.1}$$

We establish global existence of non-trivial maximal solutions. At $t = +0$, instantaneously, the solutions take the form of a minimal inverse bell-shaped function.

Such a procedure of the instant *two-sided monotonisation* of the solution is described as follows. Given a function $w \in C([-1,1])$, $w(\pm 1) = 0$, we first define the left-hand and the right-hand monotone prolongation

$$w_-(x) = \inf_{s\in(-1,x)} w(s) \quad \text{and} \quad w_+(x) = \inf_{s\in(x,1)} w(s).$$

Finally we introduce the *operator of minimal shape optimisation*:

$$\hat{w}(x) = \max\{w_-(x),\, w_+(x)\}, \quad x \in (-1,1). \tag{5.2}$$

The main existence result is as follows.

Theorem 5.1. *The Dirichlet problem* (4.1), (5.1) *has a bounded maximal solution* $u(x,t)$ *which for* $t>0$ *has an inverse bell-shaped form, i.e., the derivative* $u_x(x,t)$ *for* $t>0$ *changes sign on* $(-1,1)$ *exactly once and from minus to plus (no nontrivial maximum points of* $u(x,t)$ *exist). It satisfies the initial condition*

$$u(x,+0)=\hat{u}_0(x), \quad x\in(-1,1). \tag{5.3}$$

The proof consists of a few steps.

5.2. **A priori bound.** In order to construct a uniform lower bound on $\{u_\varepsilon\}$, we consider a family of symmetric stationary solutions $U_\mu=U_\mu(|x|)$ of (4.1) satisfying the ODE

$$U''+\ln|U'|=0, \quad x>0; \;\; U(0)=\mu, \;\; U'(0)=0, \tag{5.4}$$

where $\mu\in\mathbf{R}$ is a parameter. By translation, we have that $U_\mu=U_0+\mu$, where U_0 is given by the quadrature

$$\int_0^{U_0'(x)} dz/(-\ln z)=x.$$

It follows that as $x\to+\infty$ there holds $U'(x)\to 1-0$. Hence, $U_0(x)$ is monotone increasing with the behaviour

$$U_0(x)=x+O(1), \quad x\to+\infty. \tag{5.5}$$

Observe that $U_0\in C^1$ for $x\geq 0$ and $U_0\in C^\infty$ at any non-singular point $x>0$. At the singularity point it exhibits the following behaviour

$$U_0(x)=\frac{1}{2}x^2|\ln|x||-\frac{1}{2}x^2\ln|\ln|x||(1+o(1)), \quad x\to 0. \tag{5.6}$$

This means that $U_0\in C^{2-\varepsilon}$ for any $\varepsilon>0$, and more precisely

$$\left(\frac{U_0}{-\ln U_0}\right)^{1/2} \quad \text{and} \quad \frac{1}{U_0''}\,e^{-U_0''} \quad \text{are Lipschitz continuous at } x=0. \tag{5.7}$$

We now prove a lower bound.

Proposition 5.1. *There exists* $\mu<0$ *such that for any* $\varepsilon>0$ *small*

$$u_\varepsilon\geq\mu \quad \text{in } Q_1.$$

Proof. Since $U_\mu(x)$ is a convex function, it is a subsolution of the regularised equation (4.3), (4.4). Therefore, choosing $\mu<0$ such that $u_{0\varepsilon}\equiv u_0\geq U_\mu$ in $(-1,1)$, we deduce by comparison that $u_\varepsilon\geq U_\mu$ in Q_1. □

Passing to the limit $\varepsilon\to 0$, we conclude that the maximal solution, which is bounded above by approximation: $u\leq u_1$, is also uniformly bounded from below

$$u\geq\mu \quad \text{in } Q_1. \tag{5.8}$$

5.3. **Self-similar solution: local singularity formation.** As a natural extension of the above study of differential properties of stationary profiles, we now consider the general question of formation of a differential singularity from a smooth profile. It has a typical self-similar structure. Consider the following self-similar solution of equation (4.1):

$$u_*(x,t) = 2^{-1} t \ln t + t\theta(\eta), \quad \eta = x/t^{1/2}, \tag{5.9}$$

where a symmetric profile $\theta = \theta(|\eta|)$ satisfies the ODE

$$\mathbf{Z}(\theta) \equiv \theta'' + \theta'\eta/2 - \theta + \ln|\theta'| = 1/2, \quad \eta > 0; \;\; \theta'(0) = 0. \tag{5.10}$$

By a standard application of the Banach Contraction Principle to the equivalent integral equation one can show that the solution with $\theta(0) = \mu$ exists locally for small $\eta > 0$. The solutions obey the same regularity property (5.7) as the stationary ones. The global properties of the self-similar profiles are listed below.

Proposition 5.2. *For any μ the solution $\theta(\eta)$ is strictly monotone increasing and has the asymptotic behaviour*

$$\theta(\eta) = C\eta^2 + \ln\eta + 2C + \ln(2C) + o(1), \quad \eta \to \infty, \tag{5.11}$$

where $C = C(\mu) > 0$ is strictly monotone increasing.

Proof. Obviously, due to the weak nonlinearity, θ is a global solution. Monotonicity of θ follows from the Maximum Principle. The expansion follows from a standard local analysis of the singular point $\eta = \infty$. Monotonicity of $C(\mu)$ is guaranteed by comparison of the solutions. □

We will need self-similar subsolutions of the same structure

$$\underline{u}_*(x,t) = \frac{1}{2} t \ln t + t\underline{\theta}(\eta), \tag{5.12}$$

corresponding to $\mu \ll -1$ and $C(\mu) \approx +0$. Then $\underline{\theta}$ solves the differential inequality $\mathbf{Z}(\underline{\theta}) \geq 1/2$ and we can choose (cf. (5.11))

$$\underline{\theta}(\eta) = C\eta^2 + \ln\eta + B \quad \text{for } \eta \gg 1,$$

where the constant B satisfies

$$B < 2C + \ln 2C. \tag{5.13}$$

For sufficiently small $\eta > 0$, the transitional structure of the subsolution $\underline{\theta}$ is chosen in accordance to the stationary profiles. It is important that by (5.13), $C \to +0$ implies that $B \to -\infty$, so that for fixed $\eta \gg 1$, $\underline{\theta}(\eta) \to -\infty$. This means that, after a suitable extension up to the origin, we have that $\underline{\theta}(0) \to -\infty$. In fact, this proves that the solutions with the behaviour (5.11) satisfy

$$C(\mu) \to +0 \;\text{ as } \mu \to -\infty. \tag{5.14}$$

Using a similar supersolution $\overline{u}_*$ with $C \gg 1$, one can show that the self-similar profiles satisfy

$$C(\mu) \to +\infty \;\text{ as } \mu \to +\infty. \tag{5.15}$$

Fixing μ and hence $C = C(\mu)$, we see that the self-similar solution is continuous at $t = 0$ in $C_{\rm loc}(\mathbf{R})$ and satisfies the following C^∞-smooth initial condition: as $t \to +0$ uniformly on any compact subset in x bounded away from the origin there holds

$$u_*(x,t) = Cx^2 + t\ln(xB) + o(1), \tag{5.16}$$

so that the initial condition is

$$u_*(x,+0) = Cx^2, \quad x \in \mathbf{R}.$$

The same is true for the corresponding subsolutions $\underline{u}_*$ and supersolutions $\overline{u}_*$.

It can be shown by comparison below with the stationary solutions that the self-similar solution is a maximal solution, i.e., it is constructed by regular monotone approximations. It describes the generic local singularity formation for equation (4.1) from C^∞ smooth initial profile (5.16). The local singularity formation on compact subsets in $\eta = x/\sqrt{t}$ is then given by

$$u_*(x,t) \approx \frac{1}{2} t\ln t + \mu t + \frac{x^2}{2} |\ln(|x|/\sqrt{t})| + ...,$$

where $\mu = \theta(0)$ is the only constant depending on the parameter $C(\mu) > 0$ in the initial condition (5.16). By comparison, it also shows the instability of a uniform pattern $u \equiv$ const, which is destroyed faster than the above singular patterns. This means that the piece-wise constant in x parts of the maximal solution profile $u(x,t)$ do not exist for $t > 0$, cf. the statement of Theorem 5.1.

This singularity formation means that the singularity operator $\ln|u_x|$ produces bounded C^1-perturbations of initial profiles in a neighbourhood of smooth minima like for the parabola (5.16). In C^2 these perturbations are unbounded. This conclusion is important for the further construction of continuous limit semigroups.

5.4. **Instant formation of the minimal shape.** We begin with the following simple initial configuration which explains the first principle of the initial formation of minimal shape.

Proposition 5.3. *Let $u_0(x) \ge 0$ (so that $\hat{u}_0 \equiv 0$). Then the maximal solution satisfies*

$$u(x,+0) \equiv 0. \tag{5.17}$$

Proof. As in the proof of Theorem 4.1, by comparison from above with the flat solution, we deduce that

$$u_\varepsilon(x,t) \le F_\varepsilon(0)t + C_0, \quad t \in (0, C_0/|F_\varepsilon(0)|]; \quad C_0 = \sup\, u_0 > 0.$$

Passing to the limit $\varepsilon \to 0$, we get that $u(x,+0) \le 0$. On the other hand, using the self-similar subsolution (5.12), $\underline{u}_*(x,t) - C$ with positive $C(\mu) \ll 1$ such that

$$u_0(x) \ge \underline{u}_*(x,0) - C \equiv -C(1-x^2), \quad x \in (-1,1),$$

we conclude that $u(x,t) \ge \underline{u}_*(x,t) - C$ in Q_1 by comparison and passing $\varepsilon \to 0$. Thus, setting $t = +0$, we get

$$u(x,+0) \ge -C(1-x^2), \quad x \in (-1,1).$$

Passing to the limit $C \to +0$, we obtain the converse inequality $u(x, +0) \geq 0$ completing the proof. $\square$

We now consider general continuous initial data u_0 and prove the main result.

Proof of Theorem 5.1. Observe that given arbitrary bounded continuous initial data, exactly as in the proof of Proposition 5.3, we deduce that

$$u(x, +0) \leq 0. \tag{5.18}$$

We now prove that

$$u(x, +0) \leq \hat{u}_0(x). \tag{5.19}$$

We need a slight modification of the construction of the maximal solution. Consider the regularized solution $u_{\varepsilon\rho}(x, t)$ with smooth regularized initial data $u_{0\rho}$ having a finite number of isolated minima. Here $\rho > 0$ is the new parameter, $\{u_{0\rho}\}$ is monotone decreasing in ρ and $u_{0\rho} \to u_0$ as $\rho \to +0$ uniformly. Since the maximal solutions does not depend on the type monotone decreasing regularization [5], [1], we have that

$$u = \lim_{\rho \to +0} \lim_{\varepsilon \to +0} u_{\varepsilon\rho}.$$

Denote

$$u_\rho(x, t) = \lim_{\varepsilon \to +0} u_{\varepsilon\rho}(x, t).$$

Fix a minimum point $x_1 \in (-1, 1)$ of $u_{0\rho}(x)$ so that $u_{0\rho}(x) \geq u_{0\rho}(x_1)$ in a neighbourhood $(x_1 - \nu, x_1 + \nu)$, $\nu > 0$ small. By the local comparison from below directly with the self-similar subsolution $\underline{u}_*(x - x_1, t) - \delta$ with positive arbitrarily small δ and C, as in the proof of Proposition 5.3, we deduce that $u_\rho(x_1, t)$ is lower semicontinuous at $t = +0$. By a similar comparison from above with the supersolution $\overline{u}_{*\varepsilon}(x - x_1, t) + \delta$, $C \gg 1$, $\gamma > 0$ small and $\varepsilon \to +0$, we prove that $u_\rho(x_1, t)$ is upper semicontinuous at $t = +0$. Consider now two nontrivial neighbour minimum points $-1 < x_1 < x_2 < 1$, $u_{0\rho}(x) \not\equiv \text{const}$ on (x_1, x_2). Since both $u_{\varepsilon\rho}(x_1, t)$ and $u_{\varepsilon\rho}(x_2, t)$ are continuous at $t = +0$, by the local comparison of $u_{\varepsilon\rho}$ in $(x_1, x_2) \times (0, \delta)$ with the flat solution $U_\varepsilon(t) = F_\varepsilon(0)t + C_0$ and passing to the limit $\varepsilon \to +0$, as in the proof of Theorem 4.1, we conclude that $u_\rho(x, +0) \leq \max\{u_{0\rho}(x_1), u_{0\rho}(x_2)\}$ on (x_1, x_2). Performing successively this comparison with all the nontrivial pairs $\{x_1, x_2\}$ of minimum points, we obtain that

$$u_\rho(x, +0) \leq \hat{u}_{0\rho}(x).$$

Now passing to the limit $\rho \to +0$ and using the fact that $\hat{u}_{0\rho} \to \hat{u}_0$, we arrive at (5.19).

As the last step, we establish the opposite inequality

$$u(x, +0) \geq \hat{u}_0(x). \tag{5.20}$$

Consider the left-hand and the right-hand monotone prolongation $u_{0-}(x)$ and $u_{0+}(x)$ which are decreasing and increasing functions respectively. Given an arbitrarily small constant $\delta > 0$, one can construct a strictly monotone smooth regularisations of these functions, $u_{0-\delta}(x)$ and $u_{0+\delta}(x)$ respectively, such that on $[-1, 1]$

$$u_{0-}(x) - \delta \leq u_{0-\delta}(x) \leq u_{0-}(x), \quad u'_{0-\delta}(x) \leq -\delta/2,$$

$$u_{0+}(x) - \delta \le u_{0+\delta}(x) \le u_{0+}(x), \quad u'_{0-\delta}(x) \ge \delta/2,$$

and uniformly on $[-1,1]$

$$u_{0\pm\delta}(x) \to u_{0\pm}(x), \quad \delta \to +0.$$

We now compare the maximal solution $u(x,t)$ with the solutions $u_{\pm\delta}(x,t)$ corresponding to the initial data $u_{0\pm\delta}(x)$ and fixed boundary conditions at $x=1$ and $x=-1$: $u_{\pm\delta}(\pm1,t) \equiv u_{0\pm\delta}(\pm1)$. Observe that due to the strictly monotone properties of such regularised initial data, the local in time maximal solutions are not singular, so that they are continuous and can be compared directly with the corresponding approximating sequences $\{u_\varepsilon\}$. Indeed, differentiating the equation, we have that $v = u_x$ solves the hydrodynamic version $v_t = v_{xx} + v_x/v$ with data v_0 of constant sign, so that singularities cannot occur in finite time by the Maximum Principle. This gives that $u_\varepsilon(x,t) \ge u_{\pm\delta}(x,t)$ for all small $t>0$, so that passing to the limit $\varepsilon \to 0$, we obtain that

$$u(x,t) \ge \max\{u_{-\delta}(x,t),\, u_{+\delta}(x,t)\},$$

where the right-hand side is convex and, indeed, is a sufficiently regular subsolution of the singular equation. Setting here $t \to +0$ and passing to the limit $\delta \to +0$, we arrive at inequality (5.20).

The same comparison guarantees that $u(x,t)$ has a single minimum and no maxima for any $t>0$. This completes the proof.

In similar lines, the proof applies to a wider class of initial data. Then a smooth approximation $u_{0\varepsilon}$ satisfying the above properties is necessary. We leave details to the reader.

5.5. **On correspondence with the hydrodynamic version.** The solutions of the singular Zhang equation (1.2) and its hydrodynamic version (1.3) are formally related by the first-order differential (contact) transformation

$$u_x = v. \tag{5.21}$$

For the Dirichlet problem with the homogeneous conditions (5.1), this means that

$$u(x,t) = \int_{-1}^{x} v(y,t)dy. \tag{5.22}$$

Then $u=0$ for $x=-1$ and $u(1,t)=0$ provided that $v(x,t)$ satisfies the *nonlocal condition* (constraint)

$$\int_{-1}^{1} v(y,t)dy = 0, \quad t>0. \tag{5.23}$$

Thus, the unique maximal solution u of the singular Zhang equation generates the unique solution $v = u_x$ of its hydrodynamic version with nonlocal constraint (5.23) and initial data $v_0 = u_0'$. By Theorem 5.1, $v(x,t)$ exists, is bounded and for any $t>0$ (and for $t=+0$) changes sign exactly once on $[-1,1]$.

It is worth mentioning that problem (1.3), (5.23) belongs to the class of nonlinear parabolic equation on a subspace. Indeed, the nonlocal condition means that the solution belongs to the subspace F_{-1} of codimension 1: $F_{-1} = F \setminus \mathrm{Span}\{1\}$ of the Fourier space

$$F = \mathrm{Span}\{\sin(\pi n x), \cos(\pi n x),\ n \ge 0\}.$$

Obviously, though the equation is parabolic, the subspace F_{-1} is not ordered in the usual sense (no functions $v_1 \le v_2$ are available on F_{-1} unless $v_1 \equiv v_2$) so that both Maximum Principle and comparison are not effective in there.

5.6. **Generalised models.** The properties of the maximal solutions described above for the singular Zhang equation are available for other equations with a single-well singularity like

$$u_t = u_{xx} + f(|u_x|) \tag{5.24}$$

where $f(q)$ is a continuous monotone increasing function satisfying

$$f(q) \to -\infty, \quad q \to 0 \quad (f(0) = -\infty).$$

Then the maximal solutions are constructed by monotone approximations via smooth solutions of the equation

$$u_t = u_{xx} + f(\sqrt{\varepsilon^2 + (u_x)^2}).$$

For instance, consider the equation

$$u_t = u_{xx} - |u_x|^{-\alpha}, \quad \alpha > 0, \tag{5.25}$$

with a single-well singularity $f(|q|) = |q|^{-\alpha}$. Then given initial data in the Dirichlet problem, the limit semigroup T_t is not continuous at $t = +0$ and performs the instant minimal shape optimisation

$$u_0 \mapsto \hat{u}_0.$$

The symmetric stationary solutions $U_\mu = U_0 + \mu$,

$$U_0'' - \frac{1}{|U_0'|^\alpha} = 0 \implies U_0(x) = C_0|x|^\gamma, \ \gamma = \frac{\alpha+2}{\alpha+1}, \ C_0 = \frac{1}{\alpha+2}(\alpha+1)^\gamma,$$

give the actual regularity of the solutions, $U_0 \in C^{1+\nu}$, $\nu = 1/(\alpha+1)$. By comparison, these stationary solutions prove a lower bound on $\{u_\varepsilon\}$ similar to Proposition 5.1, which establishes that the maximal solution is non-trivial.

A generic behaviour close to the stationary profiles is described by the symmetric self-similar solutions

$$u_*(x,t) = t^{\gamma/2}\theta(\eta), \quad \eta = x/t^{1/2},$$

where θ solves a nonlinear ODE

$$\theta'' + \theta'\eta/2 - \gamma\theta/2 - |\theta'|^{-\alpha} = 0, \ \eta > 0; \quad \theta'(0) = 0,$$

with well-understood local and global properties. In particular, $\theta(\eta) = C\eta^\gamma(1 + o(1))$ as $\eta \to \infty$, and $C = C_0$ corresponds to the stationary solution: $\theta(\eta) \equiv U_0(\eta)$. If $C > C_0$ then $\theta(0) > 0$ and $C < C_0$ implies $\theta(0) < 0$. Therefore, fixing $-\mu = \theta(0) < 0$, so that

$$u_*(x, +0) \equiv Cx^\gamma < C_0 x^\gamma, \ x > 0,$$

we obtain that for $t > 0$

$$\inf_x u_*(x,t) \equiv u_*(0,t) = -\mu t^\gamma.$$

Hence, $O(t^\gamma)$ is the actual rate of such a slow ordered evolution of the maximal solutions from the stationary ones. The analysis of the instant minimal shape optimisation is done similarly.

For a double-well singularity approximation, equation (5.25) describes finite interface (free-boundary) propagation on singular levels. The corresponding Bernstein estimates and the second-order interface equation can be studied by the G-theory [1], [3].

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