Solution by quadratures of the problem of a cylindrical crack by the method of matrix factorization

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Abstract

In this paper the axi-symmetric problem on a semi-infinite cylindrical crack is considered. On the surfaces of the crack, the normal and tangential components of the traction are prescribed whereas the displacement vector components are unknown and supposed to be discontinuous. The problem is reduced to a 2×2 matrix Wiener-Hopf factorization. The solution is found by quadratures. Thus, this is the first example of successful closed-form matrix factorization arisen in the theory of mixed boundary value problems for elastic bodies with curvilinear spatial defects. In addition, the weight functions for the stress-intensity factors are constructed. Numerical results for the stress-intensity factors for two types of loading: (i) the exponential functions and (ii) a point force acting along the axis of symmetry, are reported.

1 Introduction

The problem on a cylindrical crack that occupies a semi-infinite cylindrical surface can be interpreted as a model describing a debonding of an elastic fibre from an elastic space when the materials of the elastic matrix and the cylindrical fibre are the same. Mathematical models for a cylindrical surfaces in elasticity and diffraction theory have been widely discussed in the scientific literature. Mostly, the authors use one of the following three techniques. The first one is the method of integral equations which is efficient for finite cylindrical surfaces. Some of the papers of this direction are listed by Antipov et al (2000). Additionally, we point out the work by Martynenko and Ulitko (1982) where an axi-symmetric elastic problem for a finite cylindrical crack was reduced to a system of Fredholm equations. Popov and Cablis (1997) considered a 3-d-problem for a cylindrical defect. The authors derived integral equations for two particular (scalar) cases when either the tangential (angular) displacement or the normal (radial) displacement were discontinuous and other components were continuous. A model problem for an infinite fibre in an elastic space when there is a finite zone of dry friction along the surface of the fibre, was analysed by Antipov et al (2000). The problem was reduced to a singular integral equation and solved approximately by the method of orthogonal polynomials.

The second approach is asymptotic one for thin fibres which has been developed by Movchan and Willis (1997) (see also Antipov et al (2000)).

Finally, the third method is based on the Wiener-Hopf technique and is applied to semi-infinite cylindrical surfaces. We mention solution of a problem of vibration of a semi-infinite cylindrical shell by Lawrie (1986) where a mixed boundary value problem for the Helmholtz equation was reduced to a scalar Wiener-Hopf functional equation and was solved exactly. Antipov et al (2000) considered an axi-symmetric boundary value problem problem for the Lamé equation when along a semi-infinite cylindrical surface the tangential displacements were discontinuous and the normal displacements were continuous whereas the tangential and normal traction components satisfied the Coulomb dry friction law. This problem was solved in closed form by the Wiener-Hopf method. The stress-intensity factors have been worked out. The behaviour of the displacement field at infinity was studied.

The main aim of the present paper is to construct a solution of an elastic axi-symmetric problem for a semi-infinite cylindrical crack in the general formulation, namely when the both displacement components are discontinuous. We show that such a simple idea as to represent a given matrix kernel $\mathbf{G}_0(\alpha)$ in the form

$\mathbf{R}_1(\alpha)\mathbf{G}(\alpha)\mathbf{R}_2(\alpha)$ works in this situation. Here $\mathbf{R}_1(\alpha), \mathbf{R}_2(\alpha)$ are rational matrices and $\mathbf{G}(\alpha)$ is a matrix which admits a constructive factorization. This idea allowed to solve the 3-d-problem of an interface semi-infinite plane crack (Antipov, 1999). In addition, we construct the weight functions for a semi-infinite cylindrical crack. The weight functions for the 3-d-problem of an interface semi-infinite plane crack were found by Bueckner (1987) and Movchan et al (1998) and for an interface crack by Antipov (1999).

The structure of the paper is following. We formulate the problem as a discontinuous boundary value problem for Love's function in Section 2. Section 3 reduces this problem to a Riemann-Hilbert problem for a vector of the second order. The matrix coefficient differs from Chebotarev-Khrapkov matrices (Chebotarev, 1956; Khrapkov, 1971). In Section 4 we introduce a new unknown vector that admits a pole on the real axis and that satisfies a new boundary value problem with the coefficient which may be factorized by Khrapkov's method (1971). We analyze the characteristic functions of the matrix coefficient and factorize the matrix. Then we study the behaviour of the factors at zero and at infinity. At the end of the Section we write down the solution of the original Riemann-Hilbert problem in closed form. Section 5 shows that for a particular case of loading, namely when the loads are linear combinations of the exponential functions, the solution can be simplified. The stress-intensity factors K_I, K_{II} are found and a numerical example is considered. The dependence of the factors K_I, K_{II} upon Poisson's ratio is studied. In Section 6 the weight functions for a semi-infinite cylindrical crack are constructed in terms of the Fourier integrals. These formulae are transformed into an improper integral which converges exponentially at infinity and an infinite sum of residues at the zeros of the determinant of the matrix coefficient of the original Riemann-Hilbert problem. We mention a rapid (exponential) convergence of the last series. An asymptotic approach for definition of the zeros of the determinant is working out. Numerical computations are implemented for the case of a point force along the axis of symmetry of the crack (Kelvin's problem). The zones where the crack is open and where it is closed are found out.

2 Formulation

We consider an elastic isotropic space $\mathbb{R}_3 = \{0 < r < \infty,\ 0 \le \varphi \le 2\pi,\ -\infty < z < \infty\}$ with Poisson's ratio ν and the shear modulus G. Assume that there is a semi-infinite cylindrical crack $\{r = a \pm 0,\ 0 \le \varphi \le 2\pi,\ 0 < z < \infty\}$ (Fig.1) that is acted on by the tangential and normal loads

$$\tau_{rz} = p_1(z), \quad \sigma_r = p_2(z), \quad r = a \pm 0,\ 0 \le \varphi \le 2\pi,\ 0 < z < \infty \tag{2.1}$$

which are assumed to be independent of φ.

The displacement components u_r, u_z are discontinuous across the crack. We introduce the following functions

$$\chi_1(z) = 2G\frac{\partial}{\partial z}[u_r], \quad \chi_2(z) = 2G\frac{\partial}{\partial z}[u_z], \quad \operatorname{supp}\chi_j \subset [0,\infty), \tag{2.2}$$

where $[u]$ determines a discontinuity of a function u while crossing the crack

$$[u] = u|_{r=a-0} - u|_{r=a+0}. \tag{2.3}$$

To define the stress field and the displacement vector, one needs to know the functions $\chi_1(z), \chi_2(z)$. The displacement and stress components are represented through the biharmonic Love function $\Psi(r,z)$ as follows

$$2Gu_r = -\frac{\partial^2\Psi}{\partial r\partial z}, \quad 2Gu_z = \left(\varkappa_+\Delta - \frac{\partial^2}{\partial z^2}\right)\Psi + const,$$

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\Delta - \frac{\partial^2}{\partial r^2}\right)\Psi, \quad \tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\Psi,$$

$$\varkappa_\pm = \frac{1}{2}(\varkappa \pm 1), \quad \varkappa = 3 - 4\nu, \tag{2.4}$$

where Δ is the Laplace operator, that for the axially-symmetric case is

$$\Delta = \Delta_r + \frac{\partial^2}{\partial z^2}, \quad \Delta_r = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{2.5}$$

Allowing for the discontinuity of the displacements (2.2) and continuity of stress components across the crack, we arrive at the discontinuous boundary value problem for the biharmonic operator

$$\Delta^2\Psi(r,z) = 0, \quad (r,z) \in (0,\infty)\times(-\infty,\infty)\setminus\mathcal{C},$$

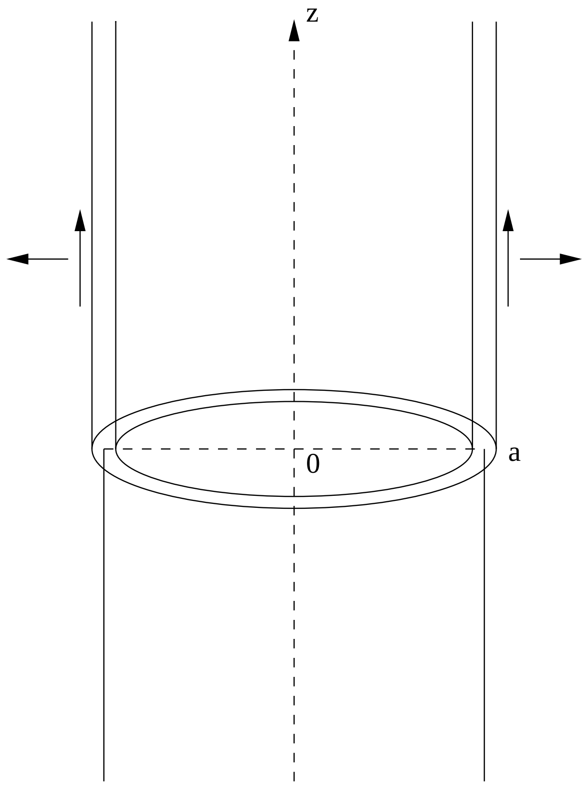

Figure 1: A semi-infinite cylindrical crack.

$$
\begin{gathered}
-\frac{\partial^2}{\partial r \partial z}\Psi|_{r=a-0}+\frac{\partial^2}{\partial r \partial z}\Psi|_{r=a+0}=\chi_1(z),\\
\frac{\partial}{\partial z}\left(\varkappa_+\Delta_r+\varkappa_-\frac{\partial^2}{\partial z^2}\right)\Psi|_{r=a-0}-\frac{\partial}{\partial z}\left(\varkappa_+\Delta_r+\varkappa_-\frac{\partial^2}{\partial z^2}\right)\Psi|_{r=a+0}=\chi_2(z),\\
\frac{\partial}{\partial z}\left(\nu\Delta_r-\frac{\partial^2}{\partial r^2}+\nu\frac{\partial^2}{\partial z^2}\right)\Psi|_{r=a-0}-\frac{\partial}{\partial z}\left(\nu\Delta_r-\frac{\partial^2}{\partial r^2}+\nu\frac{\partial^2}{\partial z^2}\right)\Psi|_{r=a+0}=0,\\
\frac{\partial}{\partial r}\left[(1-\nu)\Delta_r-\nu\frac{\partial^2}{\partial z^2}\right]\Psi|_{r=a-0}-\frac{\partial}{\partial r}\left[(1-\nu)\Delta_r-\nu\frac{\partial^2}{\partial z^2}\right]\Psi|_{r=a+0}=0,\\
-\infty<z<\infty,
\end{gathered}
\tag{2.6}
$$

where $\mathcal{C}=\{(r,z)|\ r=a,0<z<\infty\}$.

3 Reduction to a Riemann-Hilbert boundary value problem for a vector

The first step of the solution is to reduce the boundary value problem for the biharmonic equation to a boundary value problem of the theory of analytic functions. Let us apply the Fourier transform

$$
\Psi_\alpha(r)=\int_{-\infty}^{\infty}\Psi(r,z)e^{i\alpha z}dz, \quad \Psi(r,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_\alpha(r)e^{-i\alpha z}d\alpha \tag{3.1}
$$

to the boundary value problem (2.6). Assuming the notations

$$
\chi_{j\alpha}=\int_0^\infty \chi_j(z)e^{i\alpha z}dz \quad (j=1,2) \tag{3.2}
$$

we obtain the discontinuous boundary value problem for an ordinary differential equation, namely

$$(\Delta_r^2 - 2\alpha^2\Delta_r + \alpha^4)\Psi_\alpha(r) = 0, \quad r \in (0,\infty)\setminus\{a\},$$

$$\Psi'_\alpha(a-0) - \Psi'_\alpha(a+0) = \frac{1}{\alpha^2}\chi_{1\alpha},$$

$$(\varkappa_+\Delta_r - \varkappa_-\alpha^2)\Psi_\alpha(a-0) - (\varkappa_+\Delta_r - \varkappa_-\alpha^2)\Psi_\alpha(a+0) = -\frac{\chi_{2\alpha}}{i\alpha},$$

$$\left(\nu\Delta_r - \frac{d^2}{dr^2} - \nu\alpha^2\right)\Psi_\alpha(a-0) - \left(\nu\Delta_r - \frac{d^2}{dr^2} - \nu\alpha^2\right)\Psi_\alpha(a+0) = 0,$$

$$\frac{d}{dr}[(1-\nu)\Delta_r + \nu\alpha^2]\Psi_\alpha(a-0) - \frac{d}{dr}[(1-\nu)\Delta_r + \nu\alpha^2]\Psi_\alpha(a+0) = 0. \tag{3.3}$$

The solution of the problem (3.3) bounded at $r = 0$ and decaying at infinity is given by

$$\Psi_\alpha(r) = \begin{cases} A_0 I_0(|\alpha|r) + A_1|\alpha|rI_1(|\alpha|r), & r < a, \\ B_0K_0(|\alpha|r) + B_1|\alpha|rK_1(|\alpha|r), & r > a, \end{cases} \tag{3.4}$$

with the coefficients

$$A_0 = \frac{(\varkappa_+ + 1)|\alpha|aK_0(|\alpha|a) + (\alpha^2a^2 + 2\varkappa_+)K_1(|\alpha|a)}{\varkappa_+|\alpha|^3 d(|\alpha|a)}\chi_{1\alpha} + \frac{\varkappa_+K_1(|\alpha|a) + |\alpha|aK_0(|\alpha|a)}{i\varkappa_+\alpha^2 d(|\alpha|a)}a\,\mathrm{sgn}\,\alpha\chi_{2\alpha},$$

$$A_1 = -\frac{K_1(|\alpha|a) + |\alpha|aK_0(|\alpha|a)}{\varkappa_+|\alpha|^3 d(|\alpha|a)}\chi_{1\alpha} - \frac{K_1(|\alpha|a)}{i\varkappa_+\alpha^2 d(|\alpha|a)}a\,\mathrm{sgn}\,\alpha\chi_{2\alpha},$$

$$d(\alpha) = \alpha[I_0(\alpha)K_1(\alpha) + I_1(\alpha)K_0(\alpha)], \tag{3.5}$$

and

$$B_0 = \frac{-A_0[2\varkappa_+I_1(|\alpha|a) - |\alpha|aI_0(|\alpha|a)] + |\alpha|^{-3}(2\varkappa_+ - \alpha^2a^2\varkappa_+^{-1})\chi_{1\alpha} - i|\alpha|^{-2}a\,\mathrm{sgn}\,\alpha\chi_{2\alpha}}{2\varkappa_+K_1(|\alpha|a) + |\alpha|aK_0(|\alpha|a)},$$

$$B_1 = A_1\frac{I_1(|\alpha|a)}{K_1(|\alpha|a)} + \frac{\chi_{1\alpha}\,\mathrm{sgn}\,\alpha}{\varkappa_+\alpha^3K_1(|\alpha|a)}, \tag{3.6}$$

where $I_j(x), K_j(x)$ $(j = 0,1)$ are the Bessel functions. In order to satisfy the boundary conditions (2.1), we need the expressions for the Fourier transforms of the stresses σ_r, τ_{rz}

$$(\sigma_r^{(\alpha)},\ \tau_{rz}^{(\alpha)}) = \int_{-\infty}^{\infty}(\sigma_r,\ \tau_{rz})e^{i\alpha z}d\alpha \tag{3.7}$$

as $r = a \pm 0$. The functions $\sigma_r^{(\alpha)}(r)$, $\tau_{rz}^{(\alpha)}(r)$ are continuous at the point $r = a$, and from (2.4) and (3.4)- (3.6) we get

$$\sigma_r^{(\alpha)}(a) = \frac{i\chi_{1\alpha}\,\mathrm{sgn}\,\alpha}{\varkappa_+d(|\alpha|a)}\Sigma_1(|\alpha|a) + \frac{\chi_{2\alpha}}{\varkappa_+d(|\alpha|a)}\Sigma_0(|\alpha|a),$$

$$\tau_{rz}^{(\alpha)}(a) = -\frac{\chi_{1\alpha}}{\varkappa_+d(|\alpha|a)}\Sigma_0(|\alpha|a) + \frac{i\chi_{2\alpha}\,\mathrm{sgn}\,\alpha}{\varkappa_+d(|\alpha|a)}\Sigma_2(|\alpha|a), \tag{3.8}$$

where

$$\Sigma_0(\alpha) = \alpha[A(\alpha) + B(\alpha)], \quad \Sigma_1(\alpha) = (\alpha^2 + \varkappa_+ + 1)A(\alpha) + 2B(\alpha), \quad \Sigma_2(\alpha) = \alpha^2A(\alpha),$$

$$A(\alpha) = I_0(\alpha)K_1(\alpha) - I_1(\alpha)K_0(\alpha),$$

$$B(\alpha) = \alpha I_0(\alpha)K_0(\alpha) - \left(\frac{\varkappa_+}{\alpha} + \alpha\right)I_1(\alpha)K_1(\alpha). \tag{3.9}$$

Next, we introduce the functions

$$\Phi_k^+(\alpha) = \int_0^\infty \chi_k(z)e^{i\alpha z/a}dz, \quad F_k^+(\alpha) = \int_0^\infty p_k(z)e^{i\alpha z/a}dz \quad (k = 1,2) \tag{3.10}$$

and

$$\Phi_1^-(\alpha)=\int\limits_{-\infty}^{0}\tau_{rz}(a,z)e^{i\alpha z/a}dz,\quad \Phi_2^-(\alpha)=\int\limits_{-\infty}^{0}\sigma_r(a,z)e^{i\alpha z/a}dz. \tag{3.11}$$

The functions $\Phi_k^+(\alpha), F_k^+(\alpha)$ $(k=1,2)$ are analytic in the upper half-plane $\mathbb{C}^+=\{\alpha\in\mathbb{C}|\ \Im\alpha>0\}$ and the functions $\Phi_k^-(\alpha)$ $(k=1,2)$ are analytic in the lower half-plane $\mathbb{C}^-=\{\alpha\in\mathbb{C}|\ \Im\alpha<0\}$. The functions $F_k^+(\alpha)$ are known and the other functions $\Phi_k^+(\alpha),\Phi_k^-(\alpha)$ are to be determined. Using definitions (3.2), (3.7) and (3.10), (3.11) we get

$$\tau_{rz}^{(\alpha)}(a)=F_1^+(\alpha a)+\Phi_1^-(\alpha a),\quad \sigma_r^{(\alpha)}(a)=F_2^+(\alpha a)+\Phi_2^-(\alpha a),$$
$$\chi_{1\alpha}=\Phi_1^+(\alpha a),\quad \chi_{2\alpha}=\Phi_2^+(\alpha a). \tag{3.12}$$

Let $\mathbf{\Phi}(\alpha)=\mathbf{\Phi}^\pm(\alpha),\ \mathbf{F}(\alpha)=\mathbf{F}^\pm(\alpha),\ \alpha\in\mathbb{C}^\pm$ denote the vectors

$$\mathbf{\Phi}^\pm(\alpha)=\begin{pmatrix}\Phi_1^\pm(\alpha)\\ \Phi_2^\pm(\alpha)\end{pmatrix},\quad \mathbf{F}^\pm(\alpha)=\begin{pmatrix}F_1^\pm(\alpha)\\ F_2^\pm(\alpha)\end{pmatrix}. \tag{3.13}$$

Substituting relations (3.12) into (3.8) provides the boundary condition of the following vector Riemann-Hilbert problem:

It is required to determine the vector $\mathbf{\Phi}(\alpha)$ which is sectionally-analytic in the α-plane and vanishes at infinity

$$\mathbf{\Phi}(\alpha)=O(\alpha^{-1/2}),\quad \alpha\to\infty,\alpha\in\mathbb{C}^\pm. \tag{3.14}$$

Its boundary values $\mathbf{\Phi}^\pm(t)$ on $\mathbb{R}_1$ are Hölder's vector-functions and satisfy the boundary condition

$$\mathbf{\Phi}^-(t)=\mathbf{G}_0(t)\mathbf{\Phi}^+(t)-\mathbf{F}^+(t),\quad t\in\mathbb{R}_1, \tag{3.15}$$

where

$$\mathbf{G}_0(\alpha)=\frac{1}{\varkappa_+ d(|\alpha|)}\begin{pmatrix}-\Sigma_0(|\alpha|) & i\,\mathrm{sgn}\,\alpha\Sigma_2(|\alpha|)\\ i\,\mathrm{sgn}\,\alpha\Sigma_1(|\alpha|) & \Sigma_0(|\alpha|)\end{pmatrix}. \tag{3.16}$$

Remark. The class of solutions (3.14) is due to the behaviour of the stresses

$$\sigma_r(a,z)=O(z^{-1/2}),\quad \tau_{rz}(a,z)=O(z^{-1/2}),\quad z\to 0 \tag{3.17}$$

on the edge of the crack and the Abelian theorem for the Fourier transform (Noble, 1988).

4 Analysis of the Riemann-Hilbert problem

4.1 Transformation of the matrix coefficient

First, we mention that matrix (3.16) is not a Chebotarev-Khrapkov matrix (Khrapkov, 1971). Nevertheless, it is possible to find a rational matrix $\mathbf{M}(\alpha)$ such that the product $\mathbf{M}(\alpha)\mathbf{G}_0(\alpha)$ possesses the structure which allows factorization in terms of two matrices analytic in the upper and lower half-planes except for at most a finite number of poles or points where the matrices are singular. Let us multiply the boundary condition by the rational matrix

$$\mathbf{M}(\alpha)=\begin{pmatrix}-1 & 0\\ \frac{2i}{\alpha} & 1\end{pmatrix}. \tag{4.1}$$

Then we get a new Riemann-Hilbert problem for the vector $\boldsymbol{\varphi}(\alpha)=\boldsymbol{\varphi}^\pm(\alpha),\ \alpha\in\mathbb{C}^\pm$:

$$\boldsymbol{\varphi}^+(\alpha)=\frac{1}{\varkappa_+}\mathbf{\Phi}^+(\alpha),$$

$$\boldsymbol{\varphi}^-(\alpha)=\mathbf{M}(\alpha)\mathbf{\Phi}^-(\alpha)=\begin{pmatrix}-\Phi_1^-(\alpha)\\ \frac{2i}{\alpha}\Phi_1^-(\alpha)+\Phi_2^-(\alpha)\end{pmatrix}. \tag{4.2}$$

Although the second component of the vector $\boldsymbol{\varphi}(\alpha)$ admits a pole at the point of the contour $\mathbb{R}_1$ $\alpha=0$, the structure of the new coefficient of the Riemann-Hilbert problem is better and the boundary condition (3.15) turns into the following one

$$\boldsymbol{\varphi}^-(t)=\frac{i}{2}\mathbf{G}(t)\boldsymbol{\varphi}^+(t)+\mathbf{g}(t),\quad t\in\mathbb{R}_1, \tag{4.3}$$

where

$$\mathbf{G}(\alpha)=\begin{pmatrix} b(\alpha)+c(\alpha)l(\alpha) & c(\alpha)m(\alpha)\\ c(\alpha)n(\alpha) & b(\alpha)-c(\alpha)l(\alpha)\end{pmatrix}, \tag{4.4}$$

$$b(\alpha)=\frac{2|\alpha|B(|\alpha|)}{id(|\alpha|)},\quad c(\alpha)=2\operatorname{sgn}\alpha\frac{A(|\alpha|)}{d(|\alpha|)}, \tag{4.5}$$

$$l(\alpha)=-i\alpha,\quad m(\alpha)=-\alpha^2,\quad n(\alpha)=\alpha^2+\varkappa_-,$$

$$\mathbf{g}(\alpha)=\begin{pmatrix} F_1^+(\alpha)\\ -\frac{2i}{\alpha}F_1^+(\alpha)-F_2^+(\alpha)\end{pmatrix}. \tag{4.6}$$

The matrix coefficient $\mathbf{G}(\alpha)$ of the new problem (4.3) possesses the Chebotarev-Khrapkov structure. The behaviour of the vector-function $\boldsymbol{\varphi}^\pm(\alpha)=(\varphi_1^\pm(\alpha),\varphi_2^\pm(\alpha))$ as $\alpha\to\infty$ and $\alpha\in\mathbb{C}^\pm$ is still the same as that for the original functions $\mathbf{\Phi}^\pm(\alpha)$, i.e.

$$\varphi_j^\pm(\alpha)=O(\alpha^{-1/2}),\quad \alpha\to\infty,\quad \alpha\in\mathbb{C}^\pm\quad (j=1,2). \tag{4.7}$$

Because of relations (4.2), the functions $\varphi_1^+(\alpha),\varphi_1^-(\alpha),\varphi_2^+(\alpha)$ are bounded as $\alpha\to 0$ and $\varphi_2^-(\alpha)$ is unbounded at the point $\alpha=0:\varphi_2^-(\alpha)=O(\alpha^{-1})$. Moreover, due to the representation (4.2), the functions $\varphi_1^-(\alpha),\varphi_2^-(\alpha)$ have to satisfy the additional condition

$$\frac{2i}{\alpha}\varphi_1^-(\alpha)+\varphi_2^-(\alpha)=O(1),\quad \alpha\to 0,\quad \alpha\in\mathbb{C}^-. \tag{4.8}$$

4.2 Study of the characteristic functions of the matrix coefficient

The characteristic functions of the matrix $\mathbf{G}(\alpha)$ are given by

$$\lambda_1(\alpha)=b(\alpha)+c(\alpha)\alpha f^{1/2}(\alpha),\quad \lambda_2(\alpha)=b(\alpha)-c(\alpha)\alpha f^{1/2}(\alpha), \tag{4.9}$$

where $f(\alpha)=-\alpha^2-\varkappa_+$. To fix a branch of the function $f^{1/2}(\alpha)$ we cut the α-plane by a straight line that joins the branch points $i\sqrt{\varkappa_+}$ and $-i\sqrt{\varkappa_+}$ and passes through infinity. In addition, we stipulate that

$$-\frac{\pi}{2}<\theta_1<\frac{3\pi}{2},\quad -\frac{\pi}{2}<\theta_2<\frac{3\pi}{2}, \tag{4.10}$$

where

$$\theta_1=\arg(i\sqrt{\varkappa_+}-\alpha),\quad \theta_2=\arg(i\sqrt{\varkappa_+}+\alpha). \tag{4.11}$$

For the chosen branches, we get

$$\theta_1=\frac{\pi}{2},\quad \theta_2=\frac{\pi}{2}\mp\pi,\quad \alpha=\pm 0+i\Im\alpha,\quad \Im\alpha<-i\sqrt{\varkappa_+},$$

$$\theta_1=\frac{\pi}{2}\pm\pi,\quad \theta_2=\frac{\pi}{2},\quad \alpha=\pm 0+i\Im\alpha,\quad \Im\alpha>i\sqrt{\varkappa_+} \tag{4.12}$$

and

$$f^{1/2}(0)=i\sqrt{\varkappa_+}. \tag{4.13}$$

If $\alpha=t$ and $t\in(-i\sqrt{\varkappa_+},i\sqrt{\varkappa_+})$, then

$$f^{1/2}(t)=i\sqrt{\varkappa_++t^2}. \tag{4.14}$$

At infinity, the behaviour of the branch of the function $f^{1/2}(\alpha)$ is given by

$$f^{1/2}(\alpha)\sim i\alpha\operatorname{sgn}(\Re\alpha),\quad \alpha\to\infty. \tag{4.15}$$

The objective of this section is to define the increments $\Delta_1=[\arg\lambda_1(t)]_{\mathbb{R}_1}$, $\Delta_2=[\arg\lambda_2(t)]_{\mathbb{R}_1}$ of the arguments of the characteristic functions as the point t traverses the real axis $\mathbb{R}_1$ in the positive direction. First, we study the behaviour of the functions $\lambda_1(t),\lambda_2(t)$ as $t\to 0$. Using the series representation for the cylindrical functions $I_0(t),I_1(t),K_0(t),K_1(t)$ we establish that

$$I_0(t)K_0(t)=-\log\frac{t}{2}+\psi(1)+\ldots,$$

$$I_1(t)K_0(t)=-\frac{t}{2}\log\frac{t}{2}+\frac{t}{2}\psi(1)+\ldots,$$

$$I_0(t)K_1(t) = \frac{1}{t} + \frac{t}{2}\log\frac{t}{2} + \frac{t}{4} + \ldots,$$

$$I_1(t)K_1(t) = \frac{1}{2} + \frac{t^2}{4}\log\frac{t}{2} + \ldots, \quad t \to +0, \tag{4.16}$$

where $\psi(t)$ is the psi-function: $\psi(t) = \frac{d}{dt}\log\Gamma(t)$. Therefore, from relations (3.9) we have

$$A(t) = \frac{1}{t} + t\log t + O(t), \quad t \to +0,$$

$$B(t) = -\frac{\varkappa_+}{2t} - \left(1 + \frac{\varkappa_+}{4}\right) t\log t + O(t), \quad t \to +0,$$

$$d(t) = 1 + \frac{t^2}{2}\left[\psi(1) + \frac{1}{2}\right] + \ldots, \quad t \to +0. \tag{4.17}$$

Since

$$f^{1/2}(t) = i\sqrt{\varkappa_+} + O(t^2), \quad t \to +0, \tag{4.18}$$

substituting (4.17) into (4.5) yields

$$b(t) = i\varkappa_+ + i\left(2 + \frac{\varkappa_+}{2}\right) t^2 \log|t| + O(t^2),$$

$$c(t) = \frac{2}{t} + 2t\log|t| + O(t), \quad t \to 0. \tag{4.19}$$

The desirable behaviour of the characteristic functions at the point $t = 0$ can be obtained by substituting formulae (4.18), (4.19) into (4.9)

$$\lambda_1(t) = i(2\sqrt{\varkappa_+} + \varkappa_+) + O(t^2\log|t|), \quad t \to 0,$$

$$\lambda_2(t) = -i(2\sqrt{\varkappa_+} - \varkappa_+) + O(t^2\log|t|), \quad t \to 0. \tag{4.20}$$

Now, let $t \to \infty$. The asymptotic expansion of the cylindrical functions for large $|t|$ (Gradsteyn and Ryzhik, 1980) gives

$$A(t) = \frac{1}{2t^2} + \frac{3}{16t^4} + O\left(\frac{1}{t^6}\right),$$

$$B(t) = \left(\frac{1}{4} - \frac{\varkappa_+}{2}\right)\frac{1}{t^2} + O\left(\frac{1}{t^4}\right), \quad d(t) = 1 + O\left(\frac{1}{t^5}\right), \qquad t \to \infty. \tag{4.21}$$

Therefore the components of the matrix coefficient $\mathbf{G}(t)$, the functions $b(t), c(t)$, vanish at infinity

$$b(t) = \frac{i}{|t|}\left(\varkappa_+ - \frac{1}{2}\right) + O\left(\frac{1}{t^3}\right), \quad c(t) = \frac{\operatorname{sgn} t}{t^2} + O\left(\frac{1}{t^4}\right), \quad t \to \pm\infty. \tag{4.22}$$

Because of the growth at infinity of the function $f^{1/2}(t)$, the characteristic functions are bounded as $t \to \pm\infty$:

$$\lambda_j(t) = i + \frac{i}{|t|}\left(\varkappa_+ - \frac{1}{2}\right) + O\left(\frac{1}{t^2}\right), \quad t \to \pm\infty,$$

$$\lambda_j(t) = -i + \frac{i}{|t|}\left(\varkappa_+ - \frac{1}{2}\right) + O\left(\frac{1}{t^2}\right), \quad t \to \pm\infty. \tag{4.23}$$

Since $\Re f^{1/2}(t) = 0$ as $t \in \mathbb{R}_1$, the function $b(t)$ is imaginary and $c(t)$ is real for $t \in \mathbb{R}_1$, then the both characteristic functions $\lambda_1(t)$ and $\lambda_2(t)$ are imaginary on the real axis. The graphs of the $\Im\lambda_1(t)$ and $\Im\lambda_2(t)$ as the point t traverses the real axis in the positive direction, are represented in Fig.2. The value of Poisson's ratio ν is chosen to be 0.3. Qualitatively the same behaviour of $\lambda_1(t)$ and $\lambda_2(t)$ is observed for other admissible values of the parameter $\nu \in [0, \frac{1}{2}]$: $\Im\lambda_1(t) > 0$, $\Im\lambda_2(t) < 0$ and $\Re\lambda_j(t) = 0$ for all t. Thus,

$$\operatorname{ind}\lambda_j(t) = \frac{1}{2\pi}[\arg\lambda_j(t)]_{\mathbb{R}_1} = 0 \quad (j = 1, 2). \tag{4.24}$$

Moreover, for all real t

$$\arg\lambda_1(t) = \frac{\pi}{2}, \quad \arg\lambda_2(t) = -\frac{\pi}{2},$$

$$\arg[\lambda_1(t)\lambda_2(t)] = 0, \quad \arg\frac{\lambda_1(t)}{\lambda_2(t)} = \pi, \quad \forall t \in \mathbb{R}_1. \tag{4.25}$$

The last relationships will be used for the factorization of the matrix $\mathbf{G}(\alpha)$.

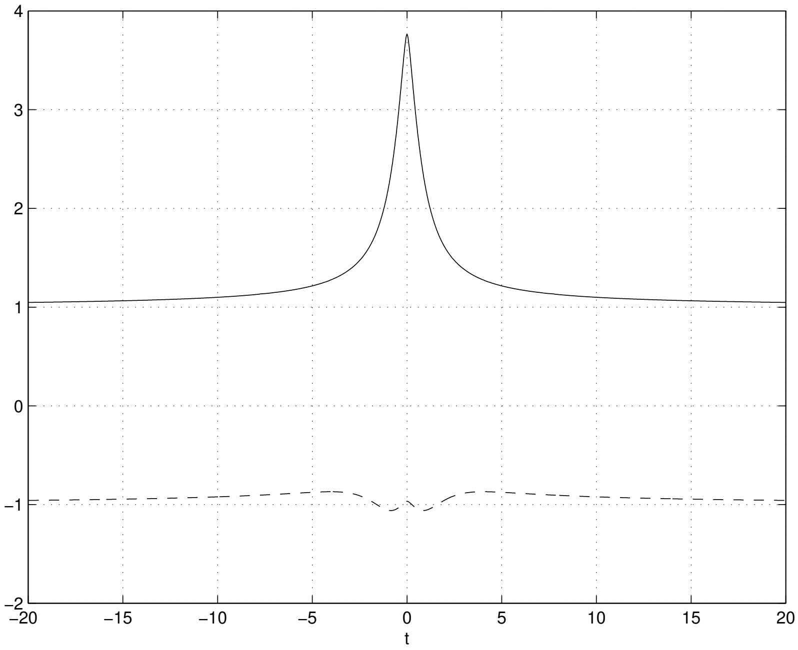

Figure 2: The characteristic functions $-i\lambda_1(t)$ and $-i\lambda_2(t)$ $(-\ -)$.

4.3 Factorization of the matrix coefficient

The function $c(\alpha)$ has a pole at the point $\alpha = 0$, therefore the factors $\mathbf{X}(\alpha)$, $\mathbf{X}^{-1}(\alpha)$ defined by

$$\mathbf{G}(t) = \mathbf{X}^{+}(t)[\mathbf{X}^{-}(t)]^{-1} = [\mathbf{X}^{-}(t)]^{-1}\mathbf{X}^{+}(t), \quad t \in \mathbb{R}_1 \tag{4.26}$$

have a pole as well. The solution of the factorization problem (4.26) is given by (Khrapkov, 1971)

$$\mathbf{X}(\alpha) = \Lambda(\alpha)[\mathbf{I}\cosh\{f^{1/2}(\alpha)\beta(\alpha)\} + \mathbf{B}(\alpha)\sinh\{f^{1/2}(\alpha)\beta(\alpha)\}],$$

$$[\mathbf{X}(\alpha)]^{-1} = \Lambda(\alpha)[\mathbf{I}\cosh\{f^{1/2}(\alpha)\beta(\alpha)\} - \mathbf{B}(\alpha)\sinh\{f^{1/2}(\alpha)\beta(\alpha)\}], \tag{4.27}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B}(\alpha) = \frac{1}{\alpha f^{1/2}(\alpha)} \begin{pmatrix} l(\alpha) & m(\alpha) \\ n(\alpha) & -l(\alpha) \end{pmatrix} \tag{4.28}$$

and $\Lambda(\alpha)$, $\beta(\alpha)$ are required to be the solution of the scalar problems

$$\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)} = D^{1/2}(t), \quad t \in \mathbb{R}_1,$$

$$\beta^{+}(t) - \beta^{-}(t) = \frac{\varepsilon(t)}{f^{1/2}(t)}, \quad t \in \mathbb{R}_1. \tag{4.29}$$

Here we assumed the following notations

$$D(t) = \det \mathbf{G}(t) = \lambda_1(t)\lambda_2(t),$$

$$\varepsilon(t) = \frac{1}{2}\log\frac{\lambda_1(t)}{\lambda_2(t)}. \tag{4.30}$$

Due to (4.25), $\operatorname{ind} D(t) = 0$ and the solution of the first from (4.29) problem is defined by

$$\Lambda(\alpha) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log D^{1/2}(t)}{t-\alpha}dt\right\} = \exp\left\{\frac{\alpha}{\pi i}\int_{0}^{\infty}\frac{\log D^{1/2}(t)}{t^2-\alpha^2}dt\right\}. \tag{4.31}$$

Here we took into account that $D(\alpha) = D(-\alpha)$. The solution of the second problem (4.29) turns out to be

$$\beta(\alpha) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\varepsilon(t)}{f^{1/2}(t)} \frac{dt}{t-\alpha} \tag{4.32}$$

which is bounded as $\alpha \to 0$. As it follows from (4.25) and (4.30)

$$\varepsilon(t) = \frac{i\pi}{2} + \frac{1}{2} \log \left| \frac{\lambda_1(t)}{\lambda_2(t)} \right| \tag{4.33}$$

and therefore integral (4.32) can be rewritten in the form

$$\beta(\alpha) = \mathcal{I}(\alpha) + \beta_0(\alpha), \tag{4.34}$$

where

$$\mathcal{I}(\alpha) = \frac{1}{4} \int\limits_{-\infty}^{\infty} \frac{dt}{f^{1/2}(t)(t-\alpha)},$$

$$\beta_0(\alpha) = -\frac{\alpha}{2\pi} \int\limits_{0}^{\infty} \log \left| \frac{\lambda_1(t)}{\lambda_2(t)} \right| \frac{dt}{\sqrt{t^2 + \varkappa_+}(t^2 - \alpha^2)}. \tag{4.35}$$

The first integral can be calculated in elementary functions if we use formula 2.2.5.23 (Prudnikov et al, 1986). Finally we have

$$\mathcal{I}(\alpha) = \frac{1}{4 f^{1/2}(\alpha)} \log Y(\alpha), \quad Y(\alpha) = \frac{\alpha + i f^{1/2}(\alpha)}{\alpha - i f^{1/2}(\alpha)}, \tag{4.36}$$

where $0 < |\arg \alpha| < \pi$, $|\arg Y(\alpha)| < \pi$.

4.4 Behaviour of the factors at zero and at infinity

Before proceeding further, we should describe the behaviour of the functions $\Lambda(\alpha)$ and $\beta(\alpha)$ as $\alpha \to 0$ and $\alpha \to \infty$. First, we note that $D(\pm 0) = 4(1-\nu^2)$ and $D(t) = 1 + O(t^{-2})$ as $t \to \pm\infty$. Therefore

$$\Lambda(\alpha) = O(1), \quad \alpha \to 0 \quad \text{and} \quad \Lambda(\alpha) = 1 + O\left(\frac{1}{\alpha}\right), \quad \alpha \to \infty. \tag{4.37}$$

Since the limit values $f^{-1/2}(-0)\varepsilon(-0) = f^{-1/2}(+0)\varepsilon(+0)$ and they are bounded, it is clear that the function $\beta(\alpha)$ is bounded at the point $\alpha = 0$. Direct analysis of (4.36) with (4.15) yields the desirable asymptotic expansion for large $|\alpha|$ for the function $\mathcal{I}(\alpha)$

$$\mathcal{I}(\alpha) = \frac{i}{2\alpha} \log \alpha + \frac{1}{4i\alpha} \left(\pm\pi i + \log \frac{\varkappa_+}{4} \right) + O\left(\frac{\log \alpha}{\alpha^3}\right), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm. \tag{4.38}$$

Now it is seen that the behaviour of the function $\beta(\alpha)$ as $\alpha \to \infty$ is described by

$$\beta(\alpha) = \frac{i}{2\alpha} \log \alpha + \frac{A^\pm}{\alpha} + O\left(\frac{\log \alpha}{\alpha^3}\right), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm. \tag{4.39}$$

where

$$A^\pm = \frac{1}{4} \left(\pm\pi - i \log \frac{\varkappa_+}{4} \right) + \frac{1}{2\pi} \int\limits_{0}^{\infty} \log \left| \frac{\lambda_1(t)}{\lambda_2(t)} \right| \frac{dt}{\sqrt{t^2 + \varkappa_+}}. \tag{4.40}$$

Next, we define the behaviour of the factors $\mathbf{X}^\pm(\alpha)$ as $\alpha \to \infty$. From (4.39) and (4.15) we get

$$\cosh\{f^{1/2}(\alpha)\beta(\alpha)\} = \frac{1}{2e_\pm} \alpha^{1/2} + O(\alpha^{-1/2}), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm,$$

$$f^{-1/2}(\alpha) \sinh\{f^{1/2}(\alpha)\beta(\alpha)\} = \frac{i}{2e_\pm} \alpha^{-1/2} + O(\alpha^{-3/2}), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm, \tag{4.41}$$

where $0 < |\arg\alpha| < \pi$ and

$$e_\pm = e^{iA^\pm}. \tag{4.42}$$

By substituting these formulae into (4.27) we obtain

$$\mathbf{X}^\pm(\alpha) = \frac{1}{2e_\pm}\alpha^{1/2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + O(\alpha^{-1/2}\mathbf{C}), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm,$$

$$[\mathbf{X}^\pm(\alpha)]^{-1} = \frac{1}{2e_\pm}\alpha^{1/2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + O(\alpha^{-1/2}\mathbf{C}), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm,$$

$$0 < |\arg\alpha| < \pi, \tag{4.43}$$

where $\mathbf{C}$ is a constant matrix.

Finally, let us write down the representation of the factors in a vicinity of the point $\alpha = 0$:

$$\mathbf{X}^\pm(\alpha) = \Lambda^\pm(0)\begin{pmatrix} c_0^\pm - \varkappa_+^{-1/2}s_0^\pm + O(\alpha) & i\varkappa_+^{-1/2}s_0^\pm\alpha + O(\alpha^2) \\ -i\varkappa_-\varkappa_+^{-1/2}s_0^\pm\alpha^{-1} + O(1) & c_0^\pm + \varkappa_+^{-1/2}s_0^\pm + O(\alpha) \end{pmatrix}, \quad \alpha \to 0, \quad \alpha \in \mathbb{C}^\pm,$$

$$[\mathbf{X}^\pm(\alpha)]^{-1} = \frac{1}{\Lambda^\pm(0)}\begin{pmatrix} c_0^\pm + \varkappa_+^{-1/2}s_0^\pm + O(\alpha) & -i\varkappa_+^{-1/2}s_0^\pm\alpha + O(\alpha^2) \\ i\varkappa_-\varkappa_+^{-1/2}s_0^\pm\alpha^{-1} + O(1) & c_0^\pm - \varkappa_+^{-1/2}s_0^\pm + O(\alpha) \end{pmatrix}, \quad \alpha \to 0, \quad \alpha \in \mathbb{C}^\pm, \tag{4.44}$$

where

$$c_0^\pm = \lim_{\alpha\to 0, \alpha\in\mathbb{C}^\pm} \cosh\{f^{1/2}(\alpha)\beta(\alpha)\},$$

$$s_0^\pm = \lim_{\alpha\to 0, \alpha\in\mathbb{C}^\pm} \sinh\{f^{1/2}(\alpha)\beta(\alpha)\}. \tag{4.45}$$

It is possible to find explicit values of $\Lambda^\pm(0)$, $c_0^\pm$ and $s_0^\pm$. From (4.31) we obtain

$$\Lambda^\pm(0) = D^{\pm 1/4}(0) = [4(1-\nu^2)]^{\pm 1/4}. \tag{4.46}$$

Further, comparing (4.45) and (4.13) we get

$$c_0^\pm(0) = \cos\{\sqrt{\varkappa_+}\beta^\pm(0)\}, \quad s_0^\pm = i\sin\{\sqrt{\varkappa_+}\beta^\pm(0)\}. \tag{4.47}$$

Analysis of (4.34), (4.35) show that

$$\beta^\pm(0) = \pm\frac{1}{4\sqrt{\varkappa_+}}\left(\pi - i\log\frac{2+\sqrt{\varkappa_+}}{2-\sqrt{\varkappa_+}}\right). \tag{4.48}$$

Therefore the quantities $c_0^\pm$, $s_0^\pm$ are expressible in terms of elementary functions:

$$c_0^+ = c_0^- = c_0 = \mu_+ + \mu_-,$$

$$s_0^+ = -s_0^- = s_0 = \mu_+ - \mu_-, \tag{4.49}$$

where

$$\mu_\pm = \frac{1}{2\sqrt{2}}(1\pm i)\left(\frac{2+\sqrt{\varkappa_+}}{2-\sqrt{\varkappa_+}}\right)^{\pm 1/4}. \tag{4.50}$$

4.5 Solution of the vector Riemann-Hilbert problem by quadratures

We have factorized the matrix $\mathbf{G}(\alpha)$. The factors $\mathbf{X}^+(\alpha), \mathbf{X}^-(\alpha)$ are analytic in $\mathbb{C}^+, \mathbb{C}^-$, respectively and at the point $\alpha = 0 \in \mathbb{R}_1$ they have an isolated singularity (a simple pole). If we substitute the representation $\mathbf{G}(t) = [\mathbf{X}^-(t)]^{-1}\mathbf{X}^+(t)$ into the boundary condition (4.3) and take into account that $\mathbf{g}(t) = -\mathbf{M}(t)\mathbf{F}^+(t)$ we obtain

$$\mathbf{X}^-(t)\varphi^-(t) + \mathbf{X}^-(t)M(t)\mathbf{F}^+(t) = \frac{i}{2}\mathbf{X}^+(t)\varphi^+(t), \quad t \in \mathbb{R}_1. \tag{4.51}$$

The vector $\mathbf{X}^-(\alpha)\mathbf{M}(\alpha)\mathbf{F}^+(\alpha)$ has a pole at the point $\alpha = 0$. Moreover, analysis of relations (4.44), (4.1) and (3.13) show

$$\mathbf{X}^-(\alpha)\mathbf{M}(\alpha)\mathbf{F}^+(\alpha) = \begin{pmatrix} O(1) \\ q\alpha^{-1} + O(1) \end{pmatrix}, \quad \alpha \to 0, \tag{4.52}$$

where

$$q = i\nu_0 F_1^+(0)\left(2c_0 - \frac{\varkappa_+ + 1}{\sqrt{\varkappa_+}}s_0\right), \quad \nu_0 = \frac{1}{\sqrt{2}(1-\nu^2)^{1/4}}. \tag{4.53}$$

We introduce the vector $\boldsymbol{\Upsilon}(\alpha)$

$$\boldsymbol{\Upsilon}(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\mathbf{X}^-(t)\mathbf{M}(t)\mathbf{F}^+(t) - \frac{q}{t}\mathbf{I}_0\right]\frac{dt}{t-\alpha}, \tag{4.54}$$

where

$$\mathbf{I}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.55}$$

In terms of the boundary values $\boldsymbol{\Upsilon}^\pm(t)$ of the vector $\boldsymbol{\Upsilon}(\alpha)$, equation (4.51) can be rewritten as follows

$$\mathbf{X}^-(t)\boldsymbol{\varphi}^-(t) - \boldsymbol{\Upsilon}^-(t) = \frac{i}{2}\mathbf{X}^+(t)\boldsymbol{\varphi}^+(t) - \frac{q}{t}\mathbf{I}_0 - \boldsymbol{\Upsilon}^+(t), \quad t \in \mathbb{R}_1. \tag{4.56}$$

In order to apply Liouville's theorem, we need to know the behaviour of the left- and right-hand sides at zero and at infinity. First we analyse the function $\boldsymbol{\Upsilon}(\alpha)$, that admits the representation

$$\boldsymbol{\Upsilon}(\alpha) = \boldsymbol{\Upsilon}^{(0)}(\alpha) + \boldsymbol{\Upsilon}^{(1)}(\alpha), \tag{4.57}$$

where

$$\boldsymbol{\Upsilon}^{(0)}(\alpha) = -\frac{1}{4\pi i e_-}\int_{-\infty}^{\infty}\frac{t^{1/2}}{t-\alpha}dt\int_0^{\infty} e^{itz/a}[p_1(z) + ip_2(z)]dz\mathbf{I}_1,$$

$$\boldsymbol{\Upsilon}^{(1)}(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\boldsymbol{\Omega}(t)}{t-\alpha}dt \tag{4.58}$$

and

$$\mathbf{I}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \boldsymbol{\Omega}(t) = \mathbf{X}^-(t)\mathbf{M}(t)\mathbf{F}^+(t) - \frac{q}{t}\mathbf{I}_0 + \frac{t^{1/2}}{2e_-}[F_1^+(t) + iF_2^+(t)]\mathbf{I}_1. \tag{4.59}$$

Here we used the asymptotic expansion (4.43). The argument of t for $t < 0$ is chosen to be π. It will be shown that the solution is independent of the choice of $\arg t$ (due to inequality (4.43) it can be $-\pi$). Analysis of the function $\boldsymbol{\Upsilon}^{(1)}(\alpha)$ at infinity shows that

$$\boldsymbol{\Upsilon}^{(1)}(\alpha) = \frac{1}{\alpha}\boldsymbol{\Upsilon}^* + o\left(\frac{1}{\alpha}\mathbf{E}\right), \quad \alpha \to \infty, \tag{4.60}$$

where $\mathbf{E}$ is a constant vector and

$$\boldsymbol{\Upsilon}^* = \begin{pmatrix} \Upsilon_1^* \\ \Upsilon_2^* \end{pmatrix} = -\frac{1}{2\pi i}\int_0^{\infty}dz\int_{-\infty}^{\infty}\omega(t,z)e^{itz/a}dt,$$

$$\omega(t,z) = \mathbf{X}^-(t)\mathbf{M}(t)\mathbf{p}(z) + \frac{t^{1/2}}{2e_-}[p_1^+(z) + ip_2^+(z)]\mathbf{I}_1, \quad \mathbf{p}(z) = \begin{pmatrix} p_1(z) \\ p_2(z) \end{pmatrix}. \tag{4.61}$$

The function $\omega(t,z)$ vanishes at infinity: $\omega(t,z) = O(t^{-1/2})$, $|t| \to \infty$. As far as the function $\boldsymbol{\Upsilon}^{(0)}(\alpha)$ is concerned, it behaves at infinity as follows

$$\boldsymbol{\Upsilon}^{(0)}(\alpha) = \frac{\eta_0}{\alpha}\mathbf{I}_1 + o\left(\frac{1}{\alpha}\mathbf{E}\right), \quad \alpha \to \infty, \tag{4.62}$$

where $\eta_0 = const$. We do not specify this constant because the final formulae for the solution do not depend on η_0. Thus, the behaviour at infinity of the function $\boldsymbol{\Upsilon}(\alpha)$ is described by (4.57), (4.60) and (4.62). The asymptotics of the matrices $[\mathbf{X}^\pm(\alpha)]^{-1}$ and the vectors $\boldsymbol{\varphi}^\pm(\alpha)$ at infinity are given by (4.43) and (4.7). Taking into account the behaviour of the matrices $[\mathbf{X}^\pm(\alpha)]^{-1}$ as $\alpha \to 0$ (4.44) and that of the functions $\varphi_j^\pm(\alpha)$

$$\varphi_1^\pm(\alpha) = O(1), \quad \varphi_2^+(\alpha) = O(1), \quad \varphi_2^-(\alpha) = O\left(\frac{1}{\alpha}\right), \quad \alpha \to 0 \tag{4.63}$$

we apply the generalized Liouville theorem (Gakhov, 1966). It yields the formulae for the solution

$$\boldsymbol{\varphi}^-(\alpha) = [\mathbf{X}^-(\alpha)]^{-1}\left[\frac{C_0}{\alpha}\mathbf{I}_0 + C_1\mathbf{I}_1 + \boldsymbol{\Upsilon}^-(\alpha)\right], \quad \alpha \in \mathbb{C}^-,$$

$$\boldsymbol{\varphi}^+(\alpha) = -2i[\mathbf{X}^+(\alpha)]^{-1}\left[\frac{C_0+q}{\alpha}\mathbf{I}_0 + C_1\mathbf{I}_1 + \boldsymbol{\Upsilon}^+(\alpha)\right], \quad \alpha \in \mathbb{C}^+, \tag{4.64}$$

where C_0, C_1 are arbitrary constants. The analysis of formulae (4.64) shows that

$$\boldsymbol{\varphi}^+(\alpha) \sim \frac{2}{i\Lambda^+(0)}\begin{pmatrix} \left(c_0 + \frac{s_0}{\sqrt{\varkappa_+}}\right)[C_1 + \Upsilon_1^+(0)] + \frac{s_0}{i\sqrt{\varkappa_+}}(C_0+q) \\ \left\{-\frac{\varkappa_- s_0}{i\sqrt{\varkappa_+}}[C_1 + \Upsilon_1^+(0)] + \left(c_0 - \frac{s_0}{\sqrt{\varkappa_+}}\right)(C_0+q)\right\}\alpha^{-1} \end{pmatrix}, \quad \alpha \to 0,$$

$$\boldsymbol{\varphi}^-(\alpha) \sim \frac{1}{\Lambda^-(0)}\begin{pmatrix} \left(c_0 - \frac{s_0}{\sqrt{\varkappa_+}}\right)[C_1 + \Upsilon_1^-(0)] - \frac{s_0}{i\sqrt{\varkappa_+}}C_0 \\ \left\{\frac{\varkappa_- s_0}{i\sqrt{\varkappa_+}}[C_1 + \Upsilon_1^-(0)] + \left(c_0 + \frac{s_0}{\sqrt{\varkappa_+}}\right)C_0\right\}\alpha^{-1} \end{pmatrix}, \quad \alpha \to 0. \tag{4.65}$$

Thus, $\varphi_1^\pm(\alpha) = O(1),\ \alpha \to 0$. In order for the function $\varphi_2^+(\alpha)$ to be bounded as $\alpha \to 0$ and the condition (4.8) to be satisfied we have to put

$$C_0 = -\frac{i\varkappa_- s_0 d_2}{2d_1 d_3\sqrt{\varkappa_+}}\left[\Upsilon_1^-(0) - \Upsilon_1^+(0) + \frac{id_3 q\sqrt{\varkappa_+}}{\varkappa_- s_0}\right],$$

$$C_1 = \frac{1}{2d_1}\left[d_2\Upsilon_1^-(0) - \frac{\varkappa_-}{\sqrt{\varkappa_+}}s_0\Upsilon_1^+(0) + id_3 q\right], \tag{4.66}$$

where

$$d_1 = \sqrt{\varkappa_+}s_0 - c_0, \quad d_2 = 2c_0 - \frac{1+\varkappa_+}{\sqrt{\varkappa_+}}s_0, \quad d_3 = c_0 - \frac{s_0}{\sqrt{\varkappa_+}}. \tag{4.67}$$

5 Particular case of the loading

In this section we will simplify the representation of the solution of the Riemann-Hilbert problem for the particular situation when the tangential and normal loads $p_1(z), p_2(z)$ are representable in the form

$$p_k(z) = \sum_{j=1}^{N} A_{kj} e^{-\gamma_j z/a}, \tag{5.1}$$

where N is a positive integer, A_{kj}, γ_j are real and $\gamma_j > 0$. In addition, we shall find the stress-intensity factors

$$K_I = \lim_{z\to -0}\sqrt{-2\pi z}\,\sigma_r(a,z), \quad K_{II} = \lim_{z\to -0}\sqrt{-2\pi z}\,\tau_{rz}(a,z), \tag{5.2}$$

5.1 Solution of the Riemann-Hilbert problem

Formula(5.1) implies that the Fourier transforms (3.10) of the functions $p_k(z)$ are rational functions

$$F_k^+(\alpha) = ai\sum_{j=1}^{N}\frac{A_{kj}}{\alpha + i\gamma_j}. \tag{5.3}$$

Because the positions of the poles of the functions $F_k(\alpha)$ are known, we can rewrite the boundary condition (4.51) as follows

$$\mathbf{X}^-(t)\boldsymbol{\varphi}^-(t) + \mathbf{X}^-(t)M(t)\mathbf{F}^+(t) - \mathbf{H}^+(t) = \frac{i}{2}\mathbf{X}^+(t)\boldsymbol{\varphi}^+(t) - \mathbf{H}^+(t), \quad t \in \mathbb{R}_1. \tag{5.4}$$

where $\mathbf{H}^+(\alpha)$ is the rational vector-function

$$\mathbf{H}^+(\alpha) = \begin{pmatrix} H_1^+(\alpha) \\ H_2^+(\alpha) \end{pmatrix} = ai\sum_{j=1}^{N}\frac{1}{\alpha + i\gamma_j}\begin{pmatrix} h_{1j} \\ h_{2j} \end{pmatrix} \tag{5.5}$$

with the coefficients

$$h_{kj} = -\left(x_{k1}^{(j)} + \frac{2}{\gamma_j}x_{k2}^{(j)}\right)A_{1j} + x_{k2}^{(j)}A_{2j}, \tag{5.6}$$

where $\mathbf{X}^-(-i\gamma_j) = (x_{kn}^{(j)})_{k,n=1,2}$, $j = 1, 2, \ldots, N$. The left-hand side of eqn (5.4) is an analytic vector-function in $\mathbb{R}^-$, the right-hand side is analytic in $\mathbb{C}^+$. The same argument as in Section 4.5 yields the solution of the Riemann-Hilbert problem

$$\varphi^-(\alpha) = -\mathbf{M}^-(\alpha)\mathbf{F}^+(\alpha) + [\mathbf{X}^-(\alpha)]^{-1}\left[\frac{C_0}{\alpha}\mathbf{I}_0 + C_1\mathbf{I}_1 + \mathbf{H}^+(\alpha)\right], \quad \alpha \in \mathbb{C}^-,$$

$$\varphi^+(\alpha) = -2i[\mathbf{X}^+(\alpha)]^{-1}\left[\frac{C_0}{\alpha}\mathbf{I}_0 + C_1\mathbf{I}_1 + \mathbf{H}^+(\alpha)\right], \quad \alpha \in \mathbb{C}^+, \tag{5.7}$$

where the vectors $\mathbf{I}_0$, $\mathbf{I}_1$ are defined in (4.55), (4.59), and the coefficients C_0, C_1 are given by

$$C_0 = 0, \quad C_1 = -H_1^+(0). \tag{5.8}$$

These two conditions are necessary and sufficient in order for the functions $\varphi_1^+(\alpha), \varphi_2^+(\alpha), \varphi_1^-(\alpha)$ to be bounded at $\alpha = 0$ and the functions $\varphi_1^-(\alpha)$ $\varphi_2^-(\alpha)$ to satisfy the additional condition (4.8).

5.2 Stress-intensity factors

Let us now evaluate the stress-intensity factors K_I, K_{II}. From the definition of the factors (5.2) we get

$$\sigma_r(a, z) \sim \frac{K_I}{\sqrt{-2\pi z}}, \quad \tau_{rz}(a, z) \sim \frac{K_{II}}{\sqrt{-2\pi z}}, \quad z \to -0. \tag{5.9}$$

From the integral representation (3.11) of the functions $\Phi_j^-(\alpha a)$ and by Abelian theorem for the Fourier transform we obtain the asymptotics for the functions $\Phi_j^-(\alpha a)$ at infinity

$$\Phi_1^-(\alpha a) \sim \frac{K_{II}}{\sqrt{2}}e^{i\pi/4}(-\alpha)^{-1/2},$$

$$\Phi_2^-(\alpha a) \sim \frac{K_I}{\sqrt{2}}e^{i\pi/4}(-\alpha)^{-1/2}, \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^-. \tag{5.10}$$

If we find the asymptotic representation at infinity of the solution $\Phi_k^-(\alpha a)$ directly from analysis (5.7), we shall obtain the factors K_I, K_{II} by comparing that expression with formula (5.10). Now we need not only the first term in expansions (4.37), (4.41) but the second one as well

$$\cosh\{f^{1/2}(\alpha)\beta(\alpha)\} = \frac{1}{2}\left(\alpha^{1/2}\frac{1}{e_\pm} + \alpha^{-1/2}e_\pm\right) + O\left(\frac{\log\alpha}{\alpha^{3/2}}\right), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm,$$

$$f^{-1/2}(\alpha)\sinh\{f^{1/2}(\alpha)\beta(\alpha)\} = -\frac{1}{2i\alpha}\left(\alpha^{1/2}\frac{1}{e_\pm} - \alpha^{-1/2}e_\pm\right) + O\left(\frac{\log\alpha}{\alpha^{5/2}}\right), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^\pm. \tag{5.11}$$

In addition, we look into the behaviour of the function $\Lambda(\alpha)$ defined by (4.31). It follows

$$\Lambda(\alpha) = 1 + \frac{id_0}{\alpha} + o\left(\frac{1}{\alpha}\right), \quad \alpha \to \infty, \tag{5.12}$$

where

$$d_0 = \frac{1}{2\pi}\int_0^\infty \log D(t)dt. \tag{5.13}$$

Substituting (5.11), (5.12) into (4.27) yields the asymptotic behaviour of the factors $\mathbf{X}^\pm(\alpha)$, $[\mathbf{X}^\pm(\alpha)]^{-1}$ at infinity:

$$\mathbf{X}^\pm(\alpha) = \frac{\alpha^{1/2}}{2e_\pm}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$+\frac{\alpha^{-1/2}}{2}\begin{pmatrix} e_\pm + (id_0+1)e_\pm^{-1} & ie_\pm + d_0e_\pm^{-1} \\ -ie_\pm - d_0e_\pm^{-1} & e_\pm + (id_0-1)e_\pm^{-1} \end{pmatrix} + o(\alpha^{-1/2}),$$

$$[\mathbf{X}^{\pm}(\alpha)]^{-1} = \frac{\alpha^{1/2}}{2e_{\pm}}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$+\frac{\alpha^{-1/2}}{2}\begin{pmatrix} e_{\pm}-(id_0+1)e_{\pm}^{-1} & -ie_{\pm}+d_0e_{\pm}^{-1} \\ ie_{\pm}-d_0e_{\pm}^{-1} & e_{\pm}-(id_0-1)e_{\pm}^{-1} \end{pmatrix} + o(\alpha^{-1/2}), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^{\pm}. \tag{5.14}$$

We proceed finding out the behaviour of the functions $\Phi_1^-(\alpha)$, $\Phi_2^-(\alpha)$:

$$\Phi_1^-(\alpha) = -\varphi_1^-(\alpha), \quad \Phi_2^-(\alpha) = \frac{2i}{\alpha}\varphi_1^-(\alpha) + \varphi_2^-(\alpha) \tag{5.15}$$

at infinity. Comparing formulae (5.15) with (5.7) and (5.5), (5.14) gives the desirable alternative description of the asymptotics of the functions $\Phi_1^-(\alpha)$, $\Phi_2^-(\alpha)$ at infinity

$$\Phi_1^-(\alpha) \sim -\frac{1}{2\alpha^{1/2}}\left[2C_1e_- + \frac{1}{e_-}(aih_1^0 - ah_2^0 - C_1)\right],$$

$$\Phi_2^-(\alpha) \sim \frac{i}{2\alpha^{1/2}}\left[2C_1e_- - \frac{1}{e_-}(aih_1^0 - ah_2^0 - C_1)\right], \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^-, \tag{5.16}$$

where

$$h_k^0 = \sum_{j=1}^{N} h_{kj},$$

$$C_1 = -H_1^+(0) = -ah_*, \quad h_* = \sum_{j=1}^{N}\frac{h_{1j}}{\gamma_j}. \tag{5.17}$$

Thus, from (5.10) we have the formulae for the stress-intensity factors

$$K_I = \frac{i-1}{2}\sqrt{a}\left[2h_*e_- + \frac{1}{e_-}(ih_1^0 - h_2^0 + h_*)\right],$$

$$K_{II} = -\frac{i+1}{2}\sqrt{a}\left[2h_*e_- - \frac{1}{e_-}(ih_1^0 - h_2^0 + h_*)\right]. \tag{5.18}$$

5.3 Numerical example

For the purpose of illustration, consider the particular case of the loading (5.1) when $N = 1$. In this case the tangential and normal loads on the sides of the crack are given by

$$p_1(z) = A_{11}e^{-\gamma_1 z/a}, \quad p_2(z) = A_{21}e^{-\gamma_1 z/a}. \tag{5.19}$$

To compute the factors K_I, K_{II}, one needs to know

$$e_- = \exp(iA^-), \quad h_* = \frac{1}{\gamma_1}h_{11}, \quad h_k^0 = h_{k1} \ (k = 1, 2), \tag{5.20}$$

where

$$A^- = -\frac{1}{4}\left(\pi + i\log\frac{\varkappa_+}{4}\right) + \frac{1}{2\pi}\int_0^{\infty}\log\left|\frac{\lambda_1(t)}{\lambda_2(t)}\right|\frac{dt}{\sqrt{t^2+\varkappa_+}},$$

$$h_{k1} = -\left(x_{k1}^{(1)} + \frac{2}{\gamma_1}x_{k2}^{(1)}\right)A_{11} + x_{k2}^{(1)}A_{21}, \tag{5.21}$$

the coefficients $x_{kn}^{(1)}$ $(k, n = 1, 2)$ are the elements of the matrix $\mathbf{X}^-(-i\gamma_1)$

$$\mathbf{X}^-(-i\gamma_1) = \Lambda(-i\gamma_1)\left\{\mathbf{I}\cosh[f^{1/2}(-i\gamma_1)\beta^-(-i\gamma_1)] + \mathbf{B}(-i\gamma_1)\sinh[f^{1/2}(-i\gamma_1)\beta^-(-i\gamma_1)]\right\}, \tag{5.22}$$

where $\Lambda(-i\gamma_1)$ and $\beta(-i\gamma_1)$ are defined by (4.31) and (4.34)-(4.36). We note that for $y = \Im\alpha > \sqrt{\varkappa_+}$ the function $f^{1/2}(\alpha)$ is discontinuous on the line $x = \Re\alpha = 0$, i.e. $f^{1/2}(+0+iy) = -f^{1/2}(-0+iy)$. However, the elements of the matrix $\mathbf{X}^-(-i\gamma_1)$ are continuous since the functions $\cosh[f^{1/2}(\alpha)\beta(\alpha)]$, $f^{-1/2}(\alpha)\sinh[f^{1/2}(\alpha)\beta(\alpha)]$ and $\mathcal{I}(\alpha)$ are even with respect to $f^{1/2}(\alpha)$. Thus, to implement computations of the stress-intensity factors K_I, K_{II},

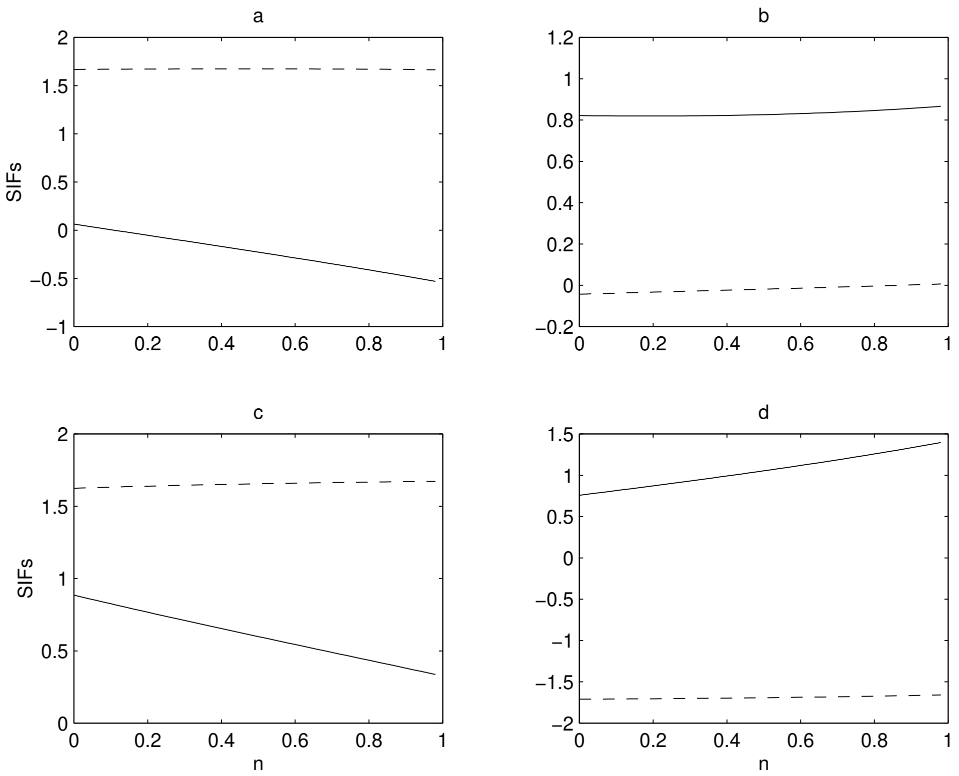

Figure 3: Stress-intensity factors K_I, $K_{II}(-\ -)$ vs $n = 2\nu$ for $a = 1$ and the loads $p_1 = A_{11}e^{-z/a}, p_2 = A_{21}e^{-z/a}$; a: $A_{11} = 1, A_{21} = 0$; b: $A_{11} = 0, A_{21} = 1$; c: $A_{11} = 1, A_{21} = 1$; d: $A_{11} = -1, A_{21} = 1$.

we have to work out the three improper integrals only, namely, the coefficient A^- and the values of the functions (4.31) and (4.35) at the point $\alpha = -i\gamma_1$ $(\gamma_1 > 0)$.

In Fig.3 we present the graphs of the stress-intensity factors K_I, K_{II} versus n, where $n = 2\nu$ and ν is Poisson's ratio for different loads:

$$\begin{aligned}
(a):&\qquad \tau_{rz}(a,z) = e^{-z/a}, \quad \sigma_r(a,z) = 0,\\
(b):&\qquad \tau_{rz}(a,z) = 0, \quad \sigma_r(a,z) = e^{-z/a},\\
(c):&\qquad \tau_{rz}(a,z) = e^{-z/a}, \quad \sigma_r(a,z) = e^{-z/a},\\
(d):&\qquad \tau_{rz}(a,z) = -e^{-z/a}, \quad \sigma_r(a,z) = e^{-z/a}.
\end{aligned}$$

The set of curves shows that the variations of the factor K_{II} are small whereas the factor K_I varies considerably.

6 Weight functions

In the previous section, the stress-intensity factors were found for the special case of loading. The main object of this section is to work out the stress-intensity factors K_I, K_{II} in the general case.

6.1 Stress-intensity factors

To find the factors K_I, K_{II}, we follow the scheme described in Section 5.2. It turns out that the principal term in the asymptotic expansion of the vector $\varphi^-(s)$ is independent of the constants η_0 and d_0 which appeared in expansions (4.62) and (5.12). Indeed, by substituting (5.14), (4.57), (4.60) and (4.62) into (4.64) we obtain

$$\varphi^-(\alpha) = \frac{\alpha^{-1/2}}{2}\left[C_1\begin{pmatrix} 2e_- - e_-^{-1} \\ 2ie_- + ie_-^{-1}\end{pmatrix}\right.$$

$$\left. +e_-^{-1}\begin{pmatrix} C_0 i + \Upsilon_1^* + i\Upsilon_2^* \\ C_0 - i\Upsilon_1^* + \Upsilon_2^*\end{pmatrix}\right] + o(\alpha^{-1/2}), \quad \alpha \to \infty, \quad \alpha \in \mathbb{C}^-, \tag{6.1}$$

and therefore the functions $\Phi_1^-(s), \Phi_2^-(s)$ decay at infinity as follows

$$\Phi_1^-(\alpha) \sim -\frac{\alpha^{-1/2}}{2}\left[(2e_- - e_-^{-1})C_1 + e_-^{-1}(iC_0 + \Upsilon_1^* + i\Upsilon_2^*)\right],$$

$$\Phi_2^-(\alpha) \sim \frac{\alpha^{-1/2}}{2}\left[(2e_- + e_-^{-1})iC_1 + e_-^{-1}(C_0 - i\Upsilon_1^* + \Upsilon_2^*)\right],$$

$$\alpha \to \infty, \quad \alpha \in \mathbb{C}^-. \tag{6.2}$$

On the other hand, for the same functions we have another representation in terms of the factors K_I, K_{II}. By comparing (6.2) with (5.10) we get the exact formulae for the stress-intensity factors

$$K_I = \frac{1-i}{2\sqrt{a}}\left[(2e_- + e_-^{-1})C_1 - e_-^{-1}(iC_0 + \Upsilon_1^* + i\Upsilon_2^*)\right],$$

$$K_{II} = \frac{1+i}{2\sqrt{a}}\left[(2e_- - e_-^{-1})C_1 + e_-^{-1}(iC_0 + \Upsilon_1^* + i\Upsilon_2^*)\right], \tag{6.3}$$

which are valid for the general case of loading.

6.2 Weight functions in terms of the Fourier integrals

To use formulae (6.3) for different loadings $p_1(z)$, $p_2(z)$, we need to re-work out the coefficients $C_0, C_1, \Upsilon_1^*, \Upsilon_2^*$ which depend upon the loading. Let us now find out the stress-intensity factors in the form

$$\frac{1}{\sqrt{a}}K_I = \int_0^\infty \mathfrak{W}_{11}(az)\sigma_r(a, az)dz + \int_0^\infty \mathfrak{W}_{12}(az)\tau_{rz}(a, az)dz,$$

$$\frac{1}{\sqrt{a}}K_{II} = \int_0^\infty \mathfrak{W}_{21}(az)\sigma_r(a, az)dz + \int_0^\infty \mathfrak{W}_{22}(az)\tau_{rz}(a, az)dz, \tag{6.4}$$

where $\mathfrak{W}_{jk}(\zeta)$ are the weight functions for the stress-intensity factors. They are independent of the functions $p_1(az) = \tau_{rz}(a, az)$, $p_2(az) = \sigma_r(a, az)$, $z > 0$. It will be shown that the functions $\mathfrak{W}_{jk}(\zeta)$ depend on Poisson's ratio ν only.

First, we transform the expressions (4.66) for the constants C_0, C_1. To do this, we compute $\Upsilon_1^+(0)$ and $\Upsilon_1^-(0)$. By Sokhotski-Plemelj formulae we get

$$\boldsymbol{\Upsilon}^\pm(0) = \pm\frac{1}{2}\mathbf{S}(0) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathbf{S}(t)\frac{dt}{t}, \tag{6.5}$$

where the integral is understood in the sense of the Cauchy principal value and $\mathbf{S}(t)$ is the vector

$$\mathbf{S}(t) = \mathbf{X}^-(t)\mathbf{M}(t)\mathbf{F}^+(t) - \frac{q}{t}\mathbf{I}_0 \tag{6.6}$$

with the components $\S_1(t)$, $\S_2(t)$. By formulae (4.44), (4.46), (4.49), (4.67) and (4.1), (6.5), (6.6) we have

$$\Upsilon_1^\pm(0) = \mp\frac{1}{2}\nu_0 d_3 F_1^+(0) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}[\hat{\chi}_{11}^-(t)F_1^+(t) + \hat{\chi}_{12}^-(t)F_2^+(t)]\frac{dt}{t}, \tag{6.7}$$

where

$$\mathbf{X}^-(t)\mathbf{M}(t) = (\hat{\chi}_{jk}^-(t))_{j,k=1,2}. \tag{6.8}$$

We write down the last relationship in explicit form. Let

$$\chi_{jj}^\pm(t) = \Lambda^\pm(t)\left\{\cosh[f^{1/2}(t)\beta^\pm(t)] + \frac{i(-1)^j}{f^{1/2}(t)}\sinh[f^{1/2}(t)\beta^\pm(t)]\right\}, \quad j = 1, 2,$$

$$\chi_{12}^\pm(t) = -\Lambda^\pm(t)\frac{t}{f^{1/2}(t)}\sinh[f^{1/2}(t)\beta^\pm(t)],$$

$$\chi_{21}^{\pm}(t)=\Lambda^{\pm}(t)\frac{t^2+\varkappa_-}{tf^{1/2}(t)}\sinh[f^{1/2}(t)\beta^{\pm}(t)] \tag{6.9}$$

be the elements of the matrices $\mathbf{X}^{\pm}(t)$. Then

$$\hat{\chi}_{j1}^{-}(t)=-\chi_{j1}^{-}(t)+\frac{2i}{t}\chi_{j2}^{-}(t),\quad \hat{\chi}_{j2}^{-}(t)=\chi_{j2}^{-}(t)\quad (j=1,2). \tag{6.10}$$

Since $\Upsilon_1^-(0)-\Upsilon_1^+(0)=\nu_0 d_3 F_1^+(0)$ then it follows from (4.66) that

$$C_0=-id_2\nu_0F_1^+(0)=-q,$$

$$C_1=\frac{\nu_0}{2}d_3F_1^+(0)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\hat{\chi}_{11}^{-}(t)F_1^+(t)+\hat{\chi}_{12}^{-}(t)F_2^+(t)]\frac{dt}{t}. \tag{6.11}$$

In addition, we need the expression for the quantity $\Upsilon_1^*+i\Upsilon_2^*$

$$\Upsilon_1^*+i\Upsilon_2^*=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{[\hat{\chi}_{11}^{-}(t)+i\hat{\chi}_{21}^{-}(t)]F_1^+(t)+[\hat{\chi}_{12}^{-}(t)+i\hat{\chi}_{22}^{-}(t)]F_2^+(t)\}dt. \tag{6.12}$$

If we take into account the definition (3.10) of the functions $F_1^+(t), F_2^+(t)$ we can rewrite the factors K_I, K_{II} in the form (6.4), with the weight functions defined by

$$\mathfrak{W}_{jk}(\zeta)=-\eta_2^{(j)}\delta_{2k}+\frac{\rho_j}{2\pi i}\int_{-\infty}^{\infty}\chi_{jk*}^{-}(t)e^{it\zeta}dt, \tag{6.13}$$

where δ_{2k} is Kronecker's symbol,

$$\chi_{j1*}^{-}(t)=\frac{1}{t}\eta_1^{(j)}\hat{\chi}_{12}^{-}(t)+\frac{(-1)^j}{e_-}[\hat{\chi}_{12}^{-}(t)+i\hat{\chi}_{22}^{-}(t)],$$

$$\chi_{j2*}^{-}(t)=\frac{1}{t}\eta_1^{(j)}\hat{\chi}_{11}^{-}(t)+\frac{(-1)^j}{e_-}[\hat{\chi}_{11}^{-}(t)+i\hat{\chi}_{21}^{-}(t)],$$

$$\rho_j=-\frac{1+(-1)^j i}{2},\quad \eta_1^{(j)}=2e_--\frac{(-1)^j}{e_-},$$

$$\eta_2^{(j)}=\rho_j\nu_0\left[e_-d_3+\frac{(-1)^j}{e_-}\left(d_2-\frac{1}{2}d_3\right)\right]. \tag{6.14}$$

6.3 Weight functions in terms of exponentially convergent integrals

Although the integral representation (6.13) for the weight functions is convergent, it is not efficient for numerical purposes. Since $\zeta>0$, the function $e^{i\alpha\zeta}$ exponentially decays in the upper α-half plane $\mathbb{C}^+$ as $\Re\alpha\to+\infty$. To reduce integral (6.13) to a rapidly convergent one, we construct the analytical continuation of the functions $\chi_{jk*}^{-}(\alpha)$ into the domain $\mathbb{C}^+$. We start with boundary condition (4.26)

$$\mathbf{X}^-(t)=\mathbf{X}^+(t)[\mathbf{G}(t)]^{-1},\quad t\in\mathbb{R}_1. \tag{6.15}$$

To find the analytical continuation of the functions $\chi_{jk*}^{-}(\alpha)$, we consider the matrix $\mathbf{X}^-(t)\mathbf{M}(t)$

$$\mathbf{X}^-(t)\mathbf{M}(t)=\frac{i\varkappa_+}{2}d(|t|)\mathbf{X}^+(t)\mathbf{G}_0^{-1}(t),\quad t\in\mathbb{R}_1, \tag{6.16}$$

where the inverse matrix $\mathbf{G}_0^{-1}(t)$ is given by

$$\mathbf{G}_0^{-1}(t)=\frac{1}{\mathcal{D}(|t|)}\begin{pmatrix}\Sigma_0(|t|) & -\operatorname{sgn}t\Sigma_2(|t|)\\ -i\operatorname{sgn}t\Sigma_1(|t|) & -\Sigma_0(|t|\end{pmatrix} \tag{6.17}$$

with

$$\mathcal{D}(\alpha)=-(\varkappa_++\alpha^2)^2I_1^2(\alpha)K_1^2(\alpha)-\alpha^4I_0^2(\alpha)K_0^2(\alpha)$$

$$+\alpha^2(\varkappa_+ + \alpha^2)[I_0^2(\alpha)K_1^2(\alpha) + I_1^2(\alpha)K_0^2(\alpha)]. \tag{6.18}$$

Consider the function $(\alpha^2+\delta^2)^{1/2}$ which is analytic in the plane cut along the straight line which joins the branch points $\alpha = i\delta$ and $\alpha = -i\delta$ ($\delta > 0$) and passes through infinity. The branch of the function $(\alpha^2+\delta^2)^{1/2}$ is defined by

$$(\alpha^2+\delta^2)^{1/2} = (\alpha+i\delta)^{1/2}(\alpha-i\delta)^{1/2},$$

$$\theta_+ = \arg(\alpha+i\delta) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \quad \theta_- = \arg(\alpha-i\delta) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right). \tag{6.19}$$

Let $\Gamma^\pm$ be the left and right banks of the cut $\Gamma_\delta = \{\alpha \in \mathbb{C}^+ |\ \Re\alpha = 0, \Im\alpha > \delta\}$, i.e. $\Gamma_\delta^\pm = \{\alpha \in \mathbb{C}^+ |\ \Re\alpha = \pm 0, \Im\alpha > \delta\}$. Then $(\alpha^2+\delta^2)^{1/2} \to \pm iy$, $\delta \to 0, \alpha \in \Gamma_\delta^\pm$, where $\alpha = x+iy$. The analytical continuation into $\mathbb{C}^+$ of the matrix $\mathbf{X}^-(\alpha)\mathbf{M}(\alpha)$ is given by $\Xi(\alpha)$:

$$\Xi(\alpha) = \frac{i\varkappa_+}{2} \lim_{\delta\to+0} \frac{d(w_\delta(\alpha))}{\mathcal{D}(w_\delta(\alpha))} \mathbf{X}^+(\alpha) \begin{pmatrix} \Sigma_0(w_\delta(\alpha)) & -\frac{i\alpha}{w_\delta(\alpha)}\Sigma_2(w_\delta(\alpha)) \\ -\frac{i\alpha}{w_\delta(\alpha)}\Sigma_1(w_\delta(\alpha)) & -\Sigma_0(w_\delta(\alpha)) \end{pmatrix}, \tag{6.20}$$

where $w_\delta(\alpha) = (\alpha^2+\delta^2)^{1/2}$. The matrix $\mathbf{X}^-(\alpha)\mathbf{M}(\alpha)$ is analytic in $\mathbb{C}^+ \setminus \Gamma_\delta$ except for a countable set of isolated singularities, which are the poles of the determinant of the matrix $G_0(\alpha)$, namely, the function $\mathcal{D}(\alpha)$. First, we look into the behaviour of the right-hand side in (6.20) as $\alpha \to 0$. We have

$$d(\pm iy) \sim 1, \quad \mathcal{D}(\pm iy) \sim \varkappa_+ \left(1 - \frac{\varkappa_+}{4}\right), \quad y \to +0,$$

$$\mathbf{X}^+(iy) = \nu_0 \begin{pmatrix} c_0 - \frac{s_0}{\sqrt{\varkappa_+}} + O(y) & -\frac{s_0}{\sqrt{\varkappa_+}}y + O(y^2) \\ -\frac{\varkappa_- s_0}{\sqrt{\varkappa_+}y} + O(1) & c_0 + \frac{s_0}{\sqrt{\varkappa_+}} + O(y) \end{pmatrix}, \quad y \to +0,$$

$$\lim_{\delta\to 0} \begin{pmatrix} \Sigma_0(w_\delta(\alpha)) & -\frac{i\alpha}{w_\delta(\alpha)}\Sigma_2(w_\delta(\alpha)) \\ -\frac{i\alpha}{w_\delta(\alpha)}\Sigma_1(w_\delta(\alpha)) & -\Sigma_0(w_\delta(\alpha)) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 - \frac{\varkappa_+}{2} + O(y^2\log y) & y + O(y^3\log y) \\ -\frac{1}{y} + O(y\log y) & -1 + \frac{\varkappa_+}{2} + O(y^2\log y) \end{pmatrix}, \quad \alpha \in \Gamma_\delta^\pm, \quad y \to +0. \tag{6.21}$$

Therefore the jumps of the limit values of the elements of the matrix $\Xi(\alpha)$ tend to zero as $\alpha \to 0$:

$$\Xi(\alpha)|_{\alpha\in\Gamma_\delta^+} - \Xi(\alpha)|_{\alpha\in\Gamma_\delta^-} = \begin{pmatrix} O(y^2\log y) & O(y^3\log y) \\ O(y\log y) & O(y^2\log y) \end{pmatrix}, \qquad y \to +0. \tag{6.22}$$

Let α_m be the poles of the function $\mathcal{D}(\alpha)$ in the quarter-plane $\mathbb{C}_+^+ = \{\alpha \in \mathbb{C} |\ \Im\alpha > 0, \Re\alpha > 0\}$. Then the points $\alpha = -\bar\alpha_m$ ($m = 1, 2, \ldots$) are the poles of the function $\mathcal{D}(\alpha)$ in $\mathbb{C}_-^+ = \{\alpha \in \mathbb{C} |\ \Im\alpha > 0, \Re\alpha < 0\}$. We note that the function $\mathcal{D}(\alpha)$ do not have poles in the imaginary and real axes. By the Cauchy theorem we express the weight functions through a sum of residues and an integral which converges exponentially. Finally, we get

$$\mathfrak{W}_{jk}(\zeta) = -\eta_2^{(j)}\delta_{2k} + \rho_j \left\{ \frac{1}{2\pi} \int_0^\infty \left[\frac{\Omega_{jk}^*(iy)}{\mathcal{D}(iy)} - \frac{\Omega_{jk}^*(-iy)}{\mathcal{D}(-iy)} \right] e^{-y\zeta} dy \right.$$

$$\left. + \sum_{m=1}^\infty \frac{\Omega_{jk}^*(\alpha_m)}{\mathcal{D}'(\alpha_m)} e^{i\alpha_m\zeta} + \frac{\Omega_{jk}^*(-\alpha_m)}{\mathcal{D}'(-\alpha_m)} e^{-i\bar\alpha_m\zeta} \right\}, \tag{6.23}$$

where

$$\Omega_{jk}^*(\alpha) = \pm\frac{\eta_1^{(j)}}{\alpha}\Omega_{1k}(\alpha) + (-1)^j \frac{1}{e_-}[\Omega_{1k}(\alpha) + i\Omega_{2k}(\alpha)], \quad \alpha \in \mathbb{C}_\pm^+,$$

$$\Omega_{j1}(\alpha) = \frac{i\varkappa_+}{2} d(\alpha)[\chi_{j1}^+(\pm\alpha)\Sigma_0(\alpha) \mp i\chi_{j2}^+(\pm\alpha)\Sigma_1(\alpha)], \quad \alpha \in \mathbb{C}_\pm^+,$$

$$\Omega_{j2}(\alpha) = \frac{i\varkappa_+}{2} d(\alpha)[\mp i\chi_{j1}^+(\pm\alpha)\Sigma_2(\alpha) - \chi_{j2}^+(\pm\alpha)\Sigma_2(\alpha)], \quad \alpha \in \mathbb{C}_\pm^+. \tag{6.24}$$

6.4 The roots of the function $\mathcal{D}(\alpha)$

To derive an asymptotic formula for the roots of the function $\mathcal{D}(\alpha)$, we use the asymptotic expansions for the cylindrical functions for large $|\alpha|$

$$
\begin{aligned}
I_0(\alpha) &= \frac{e^{\alpha}}{\sqrt{2\pi\alpha}}Q_0(\alpha) + \frac{ie^{-\alpha}}{\sqrt{2\pi\alpha}}Q_1(\alpha),\\
I_1(\alpha) &= \frac{e^{\alpha}}{\sqrt{2\pi\alpha}}R_0(\alpha) - \frac{ie^{-\alpha}}{\sqrt{2\pi\alpha}}R_1(\alpha),\\
K_0(\alpha) &= \sqrt{\frac{\pi}{2\alpha}}e^{-\alpha}Q_1(\alpha),\\
K_1(\alpha) &= \sqrt{\frac{\pi}{2\alpha}}e^{-\alpha}R_1(\alpha),
\end{aligned}
\tag{6.25}
$$

where

$$
\begin{aligned}
Q_j(\alpha) &= 1 + \frac{(-1)^j}{8\alpha} + \frac{9}{128\alpha^2} + \frac{75(-1)^j}{1024\alpha^3} + \frac{3675}{32768\alpha^4} + \dots,\\
R_j(\alpha) &= 1 - \frac{3(-1)^j}{8\alpha} - \frac{15}{128\alpha^2} - \frac{105(-1)^j}{1024\alpha^3} - \frac{4725}{32768\alpha^4} - \dots.
\end{aligned}
\tag{6.26}
$$

The equation $\mathcal{D}(\alpha) = 0$ can be rewritten in the form $e^{2\alpha} = E(\alpha)$ for large $|\alpha|$, where

$$
\begin{aligned}
E(\alpha) &= [-(\varkappa_+ + \alpha^2)^2 R_0^2 R_1^2 - \alpha^4 Q_0^2 Q_1^2 + \alpha^2(\varkappa_+ + \alpha^2)(Q_0^2 R_1^2 + R_0^2 Q_1^2)]^{-1}\\
&\times[-(\varkappa_+ + \alpha^2)^2 R_* R_1^2 + \alpha^4 Q_* Q_1^2 - \alpha^2(\varkappa_+ + \alpha^2)(Q_* R_1^2 - R_* Q_1^2)],\\
R_* &= (2iR_0 + e^{-2\alpha}R_1)R_1, \quad Q_* = (2iQ_0 - e^{-2\alpha}Q_1)Q_1,
\end{aligned}
\tag{6.27}
$$

and therefore $2\alpha_k = 2\pi k i + \log E(\alpha_k)$. Let $\alpha_k^{(m-1)}$ be the $(m-1)$-th approximation of the root α_k. Then starting with

$$
\alpha_k^{(0)} = \pi k i, \quad k = 1, 2, ... \tag{6.28}
$$

we find the m-th approximation of α_k by the following iterative procedure

$$
\alpha_k^{(m)} = \pi k i + \frac{1}{2}\log E(\alpha_k^{(m-1)}), \quad m = 1, 2, \dots; \quad k = 1, 2, \dots. \tag{6.29}
$$

In the table below, we write down the values of the first 15 roots which were computed by formulae (6.28), (6.29) (the second and third columns of the table).

	α_k (asymptotic procedure)			α_k (precise values)	
k	$\Re\alpha_k$	$\Im\alpha_k$	m	$\Re\alpha_k$	$Im\alpha_k$
1	1.36827	2.72708	9	1.36220	2.72218
2	1.63794	6.06023	7	1.63762	6.06008
3	1.82832	9.26686	6	1.82830	9.26684
4	1.96726	12.44254	6		
5	2.07629	15.60544	5		
6	2.16594	18.76174	5		
7	2.24203	21.91414	5		
8	2.30811	25.06403	5		
9	2.36650	28.21222	5		
10	2.41881	31.35919	5		
11	2.46618	34.50525	5		
12	2.50946	37.65062	4		
13	2.54930	40.79545	4		
14	2.58620	43.93985	4		
15	2.62057	47.08391	4		

In the fifth and sixth columns, we present the precise values of the real and imaginary parts of the first, second and third roots of the function $\mathcal{D}(\alpha)$. To work out these roots, we took the exact series representation of the cylindrical functions. It is seen that even for the small numbers k of the roots α_k the asymptotic formulae (6.27) - (6.29) provide good accuracy.

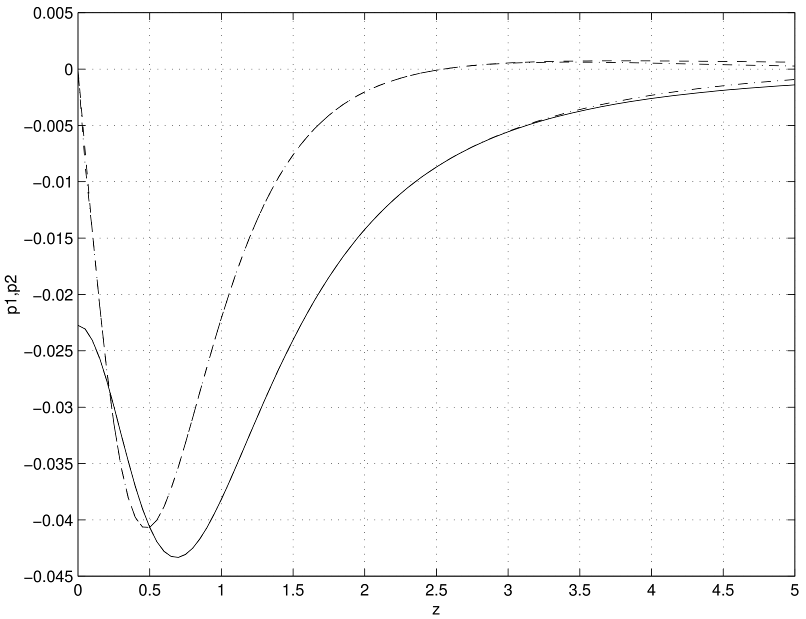

Figure 4: Kelvin's problem. The functions $p_1(z), p_2(z)$: the exact representation (–, – –) and approximation by the exponential functions ($\cdot - \cdot - \cdot$) for $Q = -1, a = 1, c = 0, \nu = 0.3$.

7 Kelvin's problem for a cylindrical crack

Assume that a point force $Q\mathbf{k}$ acts at the point $(0,0,c)$ along the z-axis, and the surfaces of the crack $\{r = a \pm 0, 0 \leq \varphi \leq 2\pi, 0 < z < \infty\}$ are free of tractions. The problem is axi-symmetric. The resulting stress tensor components $\sigma_r^{(\Sigma)}$, $\tau_{rz}^{(\Sigma)}$ are represented as a sum of two model fields $\sigma_r^{(\Sigma)} = \sigma_r^{(K)} + \sigma_r$, $\tau_{rz}^{(\Sigma)} = \tau_{rz}^{(K)} + \tau_{rz}$, where $\sigma_r^{(K)}, \tau_{rz}^{(K)}$ are the solution of Kelvin's problem (Westergaard, 1952) on the point force $Q\mathbf{k}$ in an unbounded elastic space

$$\sigma_r^{(K)} = Q_* \left(\frac{\varkappa_- z_c}{R^3} - \frac{3r^2 z_c}{R^5} \right), \quad \tau_{rz}^{(K)} = -Q_* \left(\frac{\varkappa_- r}{R^3} + \frac{3rz_c^2}{R^5} \right), \tag{7.1}$$

with $z_c = z - c$, $R^2 = r^2 + z_c^2$, $Q_* = (4\varkappa_+ \pi)^{-1} Q$. The functions σ_r, τ_{rz} are the components of the stress vector, the solution of problem (2.1)-(2.4), (2.6) with the functions $p_1(z), p_2(z)$ defined by

$$p_1(z) = \frac{Q_* a}{R_a^3} \left(\varkappa_- + \frac{3z_c^2}{R_a^2} \right),$$

$$p_2(z) = \frac{Q_* z_c}{R_a^3} \left(-\varkappa_- + \frac{3a^2}{R_a^2} \right), \tag{7.2}$$

where $R_a = \sqrt{a^2 + z_c^2}$.

There are two ways to work out the numerical values of the stress-intensity factors for this problem. The first one is to apply the weight functions constructed in Section 6. The second one is based on an approximation of the functions $p_1(z), p_2(z)$ by the exponentially decaying functions. From numerical point of view the second way is simpler. We choose the basis

$$\{e^{-z/a}, e^{-2z/a}, \ldots, e^{-Nz/a}\} \tag{7.3}$$

and approximate the functions $p_1(z), p_2(z)$ on the finite segment $[0, h]$, where given $\varepsilon > 0$ the parameter h is defined as the smallest number among those which satisfy the inequality

$$\max\{|p_1(h)|, |p_2(h)|\} < \varepsilon. \tag{7.4}$$

We put

$$p_k(z_m) = \sum_{j=1}^{N} A_{kj} e^{-jz_m/a}, \quad m = 1, 2, \ldots N, \tag{7.5}$$

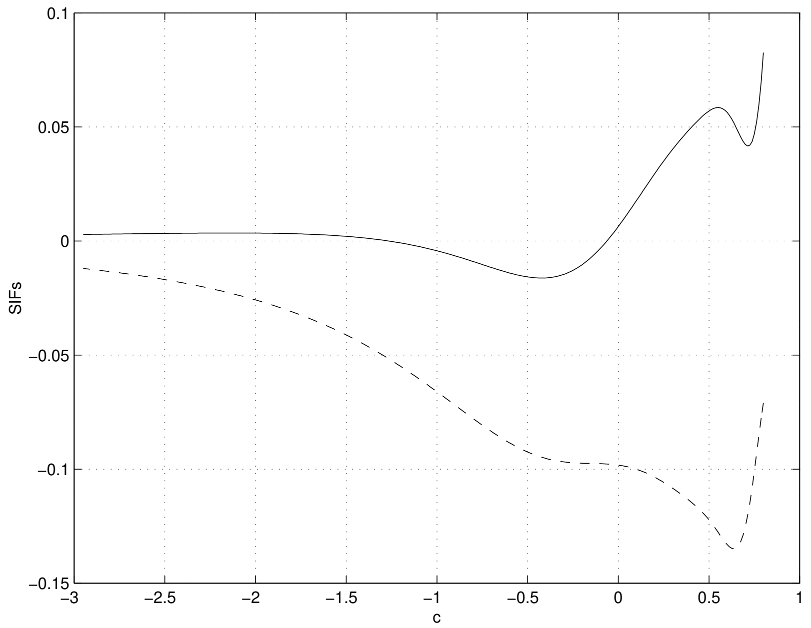

Figure 5: Stress-intensity factors $K_I,\ K_{II}$ $(-\ -)$ vs c for $a = 1$ and a point force $\mathbf{Q} = -\mathbf{k}$.

where $0 = z_1 < z_2 < \ldots < z_N = h$. The coefficients A_{kj} are computed by solving the linear algebraic system (7.5). Then we change the loads $p_1(z), p_2(z)$ by the approximate representations

$$p_k(z) \simeq \sum_{j=1}^{N} A_{kj} e^{-jz/a}, \quad 0 \le z < \infty. \tag{7.6}$$

Obviously, the total error of the approximation depends upon the discrepancy of the exponential approximation on the segment $[0, h]$ and the discrepancy at infinity. The graphs on Fig.4 correspond to the exact representation (7.2) of the loads $p_1(z)$ (the continuous curve) and $p_2(z)$ (the broken line). The dot lines show the exponential approximation (7.6). To construct these graphs, we took $h = 2, N = 20$ and $a = 1, Q = -1, \nu = 0.3$. The discrepancy is visual for $z > 3.5$ and very small. To compute the factors K_I, K_{II} we use the technique of Section 5. The dependence of the factors K_I, K_{II} on the position of the point c, the point of application of the force $Q\mathbf{k}$, is illustrated by the graphs in Fig.5 and Fig.6, where we picked up the following values of the parameters: $\nu = 0.3, Q = -1, a = 1$ (Fig.5) and $a = 5$ (Fig.6).

It is seen that for $a = 1$ $K_I < 0$ if $c \in (-1.26, -0.075)$ and for $a = 5$ $K_I < 0$ if $c \in (-6.45, -0.38)$. It indicates that for these positions of the point c in a vicinity of the point $r = a,\ z = 0$ the crack is closed.

Conclusion

The axi-symmetric problem on a semi-infinite cylindrical crack was reduced to a matrix Riemann-Hilbert problem at the contour $\mathbb{R}_1$ with a 2×2 matrix coefficient $\mathbf{G}_0(\alpha)$ of the structure which was different from the well-known Chebotarev-Khrapkov type. However, it was shown that there existed such a rational matrix $\mathbf{M}(\alpha)$ with a pole on the real axis that $\mathbf{G}_0(\alpha) = \mathbf{M}(\alpha)\mathbf{G}(\alpha)$, where $\mathbf{G}(\alpha)$ admitted a factorization by the Khrapkov method. The solution of the factorization problem has been constructed by quadratures. It is therefore revealed that the method of matrix factorization does not restrict itself to applying to the infinite or semi-infinite bodies described by Cartesian coordinates but is potentially applicable to some situations when defects are given in curvilinear coordinates.

In the particular case of the loading when the traction components on the surface of the crack are linear combinations of the exponentially decaying functions, the solution of the problem was simplified to the form efficient for numerical calculations.

The stress-intensity factors K_I and K_{II} have been found for both general and particular cases of the loading. In addition, the weight functions for the stress-intensity factors which allow to write down the factors in closed

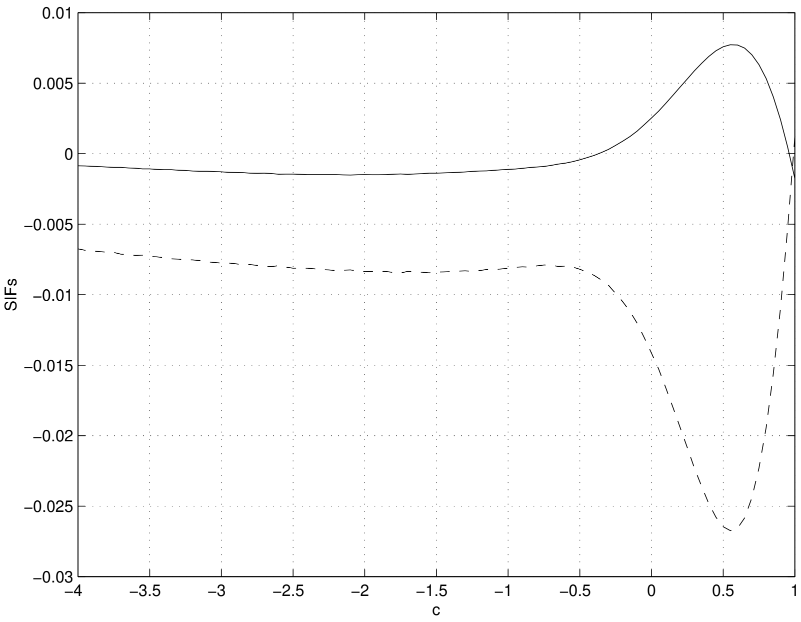

Figure 6: Stress-intensity factors K_I, K_{II} $(-\ -)$ vs c for $a = 5$ and a point force $\mathbf{Q} = -\mathbf{k}$.

form for arbitrary sensible loads, were constructed. The expressions for the weight functions were transformed into an exponentially convergent integral and a sum of residues which decay at infinity exponentially as well.

The problem on a point force (Kelvin's problem) was analysed numerically. To implement calculations, the loads were approximated by a linear combination of the exponential functions which allowed to work out the stress-intensity factors in a simple way. The dependence of the factors on the position of the point force has been studied. It has been found out that there are zones where the crack is closed and when it is open.

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