Low-Gain Integral Control of Well-Posed Linear Infinite-Dimensional Systems with Input and Output Nonlinearities *

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Abstract: Time-varying low-gain integral control strategies are presented for asymptotic tracking of constant reference signals in the context of exponentially stable, well-posed, linear, infinite-dimensional, single-input, single-output, systems – subject to globally Lipschitz, nondecreasing input and output nonlinearities. It is shown that applying error feedback using an integral controller ensures that the tracking error is small in a certain sense, provided that (a) the steady-state gain of the linear part of the system is positive, (b) the reference value r is feasible in an entirely natural sense, and (c) the positive gain function $t \mapsto k(t)$ is ultimately sufficiently small and not of class L^1. Under a weak restriction on the initial data it is shown that (a), (b) and (c) ensure asymptotic tracking. If, additionally, the impulse response of the linear part of the system is a finite signed Borel measure, the global Lipschitz assumption on the output nonlinearity may be considerably relaxed.

Keywords: Infinite-dimensional well-posed systems; input/output nonlinearities; integral control; robust control; saturation; time-varying control; tracking.

AMS subject classifications: 93B52, 93C10, 93C20, 93C23, 93C25, 93D10, 93D21.

1 Introduction

The present paper extends the line of work on low-gain control of infinite-dimensional systems initiated by the recent contributions [9]–[16]. Consider the feedback configuration shown in Fig. 1, where $\mathbf{\Sigma}$ denotes an exponentially stable, time-invariant, single-input, single-output system with transfer function $\mathbf{G}$, φ and ψ denote static nonlinearities, and r denotes the reference value. In [6] the authors developed a theory of low-gain integral control for the feedback system in Fig. 1 under the assumption that $\mathbf{\Sigma}$ is finite-dimensional. In particular, the following principle was established in [6]; if $\mathbf{G}(0) > 0$ and if the reference value r is feasible in an entirely natural sense, then there exists a number $k^* > 0$ such that, for all nondecreasing, globally Lipschitz input nonlinearities φ and all nondecreasing, locally Lipschitz and affinely sector-bounded output nonlinearities ψ (the sector-bound assumption on ψ can be removed if φ is bounded) the following holds: for all positive, bounded and continuous integrator gains $k(\cdot)$ (thus, in particular, for positive constant gains), the output $y(t)$ of the closed-loop system depicted in Fig. 1 converges to r as $t \to \infty$ provided that $\limsup_{t\to\infty} k(t) < k^*$ and k is not of class L^1.

* This work was supported by the UK Engineering & Physical Sciences Research Council (Grant Ref: GR/L78086).

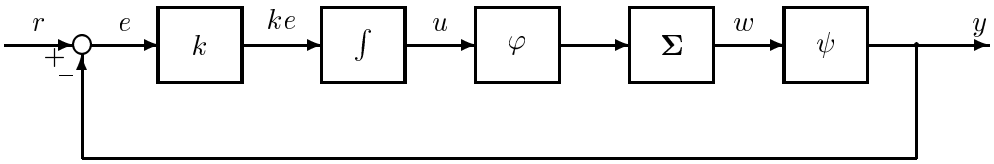

Fig. 1: Low-gain control with input and output nonlinearities

The number k^* is closely related to the supremum κ^* of the set of all numbers $\kappa > 0$ such that the function

$$s \mapsto 1 + \kappa \frac{\mathbf{G}(s)}{s}$$

is positive-real: for example, if both φ and ψ are globally Lipschitz with Lipschitz constants $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then $k^* = \kappa^*/(\lambda_1\lambda_2)$. Formulae for the computation of κ^* have appeared in [13] for various classes of systems.

Under the above assumptions on $\mathbf{\Sigma}$, φ and ψ, the problem of tracking feasible constant reference signals r by low-gain integral control reduces to that of appropriately choosing the gain function k. In a purely linear and finite-dimensional context, such a controller design approach ("tuning regulator theory", see [5], [17], [21]) is well known and has been successfully applied in process control (see, for example, [3] and [18]).

In the present paper we prove that, with suitable modifications, the above principle for finite-dimensional $\mathbf{\Sigma}$ continues to hold when $\mathbf{\Sigma}$ is an exponentially stable, well-posed, infinite-dimensional system. We remark that the class of well-posed, linear, infinite-dimensional systems is the largest class of distributed parameter systems for which satisfactory state-space and frequency-domain theories are currently available (see Curtain and Weiss [4], Salamon [23], [24], Staffans [25] and Weiss [27]–[30]). Well-posed systems allow for considerable unboundedness of the control and observation operators and include most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. The essence of our approach is to invoke a particular coordinate transformation and perform a Lyapunov-type analysis of the transformed systems in which an infinite-dimensional generalization of the Kalman-Yakubovich Lemma plays a central rôle. We emphasize that the finite-dimensional Lyapunov argument developed in [6] does not generalize to the infinite-dimensional case and so, the stability analysis given here is fundamentally different to that in [6].

Finally, we mention that [6] contains applications of the above principle to the problem of tuning the integrator gain adaptively (for other contributions to adaptive tuning regulator theory, see [2], [11], [15], [16], [19] and [20]). Generalizations of the finite-dimensional adaptive results in [6] to the infinite-dimensional case are currently under investigation and will be reported elsewhere.

2 Preliminaries on well-posed linear systems

First, some notation: for a Hilbert space H and $\tau \geq 0$, $\mathbf{R}_\tau$ denotes the operator of right-shift by τ on $L^p_{\mathrm{loc}}(\mathbb{R}_+, H)$, where $\mathbb{R}_+ := [0, \infty)$; the truncation operator $\mathbf{P}_\tau : L^p_{\mathrm{loc}}(\mathbb{R}_+, H) \to L^p(\mathbb{R}_+, H)$ is given by $(\mathbf{P}_\tau u)(t) = u(t)$ if $t \in [0, \tau]$ and $(\mathbf{P}_\tau u)(t) = 0$ otherwise; for $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, H) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}_+, H) |\ f(\cdot)\exp(-\alpha\,\cdot) \in L^p(\mathbb{R}_+, H)\}$; for $t_1 < t_2$, $W^{1,2}([t_1, t_2], H)$ denotes the space of all functions $f : [t_1, t_2] \to H$ for which there

exists $g \in L^2([t_1,t_2],H)$ such that $f(t) = f(t_1) + \int_{t_1}^t g(s)ds$ for all $t \in [t_1,t_2]$; for an unbounded interval $J \subset \mathbb{R}$, $W^{1,2}_{\rm loc}(J,H)$ denotes the space of all functions $f : J \to H$ such that for all $t_1, t_2 \in J$ with $t_1 < t_2$, the restriction of f to $[t_1,t_2]$ belongs to $W^{1,2}([t_1,t_2],H)$; $\mathcal{B}(H_1,H_2)$ denotes the space of bounded linear operators from a Hilbert space H_1 to a Hilbert space H_2; for $\alpha \in \mathbb{R}$, $\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > \alpha\}$; the Laplace transform is denoted by $\mathfrak{L}$; the Fourier transform is denoted by $\mathfrak{F}$; the space of finite signed Borel measures on $\mathbb{R}_+$ is denoted by $\mathfrak{M}$; $H^2(\mathbb{C}_\alpha)$ denotes the Hardy-Lebesgue space of square-integrable holomorphic functions defined on $\mathbb{C}_\alpha$ and $H^\infty(\mathbb{C}_\alpha)$ denotes the space of bounded and holomorphic functions on $\mathbb{C}_\alpha$; finally, 'a.a.' (respectively, 'a.e.') is the abbreviation for 'almost all' (respectively, 'almost everywhere').

Remark 2.1 Each element of $L^1_{\rm loc}(\mathbb{R}_+,H)$ is an equivalence class of locally integrable functions which coincide almost everywhere on $\mathbb{R}_+$. Therefore, if $f \in L^1_{\rm loc}(\mathbb{R}_+,H)$, it is not clear what we mean when we say that $f(t)$ converges to $h \in H$ as $t \to \infty$. In this paper we adopt the following convention: for $f \in L^1_{\rm loc}(\mathbb{R}_+,H)$ and $h \in H$ we say that $\lim_{t\to\infty} f(t) = h$ if there exists a representative $\tilde{f} : \mathbb{R}_+ \to H$ of f such that $\lim_{t\to\infty} \tilde{f}(t) = h$ in the usual sense. This convention guarantees that a given $f \in L^1_{\rm loc}(\mathbb{R}_+,H)$ has at most one limit as $t \to \infty$ and the usual algebra of limits is valid. $\diamond$

The concept of a well-posed, linear, infinite-dimensional system was first defined by Salamon [23], [24]. The following equivalent definition is due to Weiss [30] (see also the forthcoming monograph by Staffans [25]).

Definition 2.2 Let U, X and Y be real Hilbert spaces. A *well-posed* linear system with *state space* X, *input space* U and *output space* Y is a quadruple $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$, where:

$\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X;

$\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+,U)$ to X such that, for all $\tau, t \geq 0$,

$$\mathbf{\Phi}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{T}_t\mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v\,, \quad \forall u, v \in L^2(\mathbb{R}_+,U)\,;$$

$\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+,Y)$ such that $\mathbf{\Psi}_0 = 0$ and, for all $\tau, t \geq 0$,

$$\mathbf{\Psi}_{\tau+t}x^0 = \mathbf{P}_\tau\mathbf{\Psi}_\tau x^0 + \mathbf{R}_\tau\mathbf{\Psi}_t\mathbf{T}_\tau x^0\,, \quad \forall x^0 \in X\,;$$

$\mathbf{F} = (\mathbf{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+,U)$ to $L^2(\mathbb{R}_+,Y)$ such that $\mathbf{F}_0 = 0$ and, for all $\tau, t \geq 0$,

$$\mathbf{F}_{\tau+t}(\mathbf{P}_\tau u + \mathbf{R}_\tau v) = \mathbf{P}_\tau\mathbf{F}_\tau u + \mathbf{R}_\tau(\mathbf{\Psi}_t\mathbf{\Phi}_\tau u + \mathbf{F}_t v)\,, \quad \forall u, v \in L^2(\mathbb{R}_+,U).$$

$\diamond$

For an input $u \in L^2_{\rm loc}(\mathbb{R}_+,U)$ and initial state $x^0 \in X$, the associated state function $x \in C(\mathbb{R}_+,X)$ and output $w \in L^2_{\rm loc}(\mathbb{R}_+,Y)$ of $\mathbf{\Sigma}$ are given by

$$x(t) = \mathbf{T}_t x^0 + \mathbf{\Phi}_t\mathbf{P}_t u\,, \quad \text{for a.a. } t \geq 0\,, \tag{1a}$$

$$\mathbf{P}_t w = \mathbf{\Psi}_t x^0 + \mathbf{F}_t\mathbf{P}_t u\,, \quad \text{for a.a. } t \geq 0\,. \tag{1b}$$

$\mathbf{\Sigma}$ is said to be *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable:

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}_t\| < 0\,.$$

$\mathbf{\Psi}_\infty$ and $\mathbf{F}_\infty$ will denote the unique operators from X to $L^2_{\rm loc}(\mathbb{R}_+, Y)$ and from $L^2_{\rm loc}(\mathbb{R}_+, U)$ to $L^2_{\rm loc}(\mathbb{R}_+, Y)$, respectively, satisfying

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty\,, \quad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty\,, \quad \forall \tau \geq 0\,. \tag{2}$$

If $\mathbf{\Sigma}$ is exponentially stable, then the operators $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are uniformly bounded, $\mathbf{\Psi}_\infty$ is a bounded operator from X to $L^2(\mathbb{R}_+, Y)$ and $\mathbf{F}_\infty$ maps $L^2(\mathbb{R}_+, U)$ boundedly to $L^2(\mathbb{R}_+, Y)$. Since $\mathbf{P}_\tau \mathbf{F}_\infty = \mathbf{P}_\tau \mathbf{F}_\infty \mathbf{P}_\tau$ for all $\tau \geq 0$, $\mathbf{F}_\infty$ is a *causal* operator.

Weiss [27] has established that, if $\alpha > \omega(\mathbf{T})$ and $u \in L^2_\alpha(\mathbb{R}_+, U)$, then $\mathbf{F}_\infty u \in L^2_\alpha(\mathbb{R}_+, Y)$ and there exists a unique holomorphic $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathcal{B}(U, Y)$ such that

$$\mathbf{G}(s)(\mathfrak{L}u)(s) = [\mathfrak{L}(\mathbf{F}_\infty u)](s)\,, \quad \forall s \in \mathbb{C}_\alpha\,, \tag{3}$$

where $\mathfrak{L}$ denotes Laplace transform. In particular, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ for all $\alpha > \omega(\mathbf{T})$. The function $\mathbf{G}$ is called the *transfer function* of $\mathbf{\Sigma}$.

$\mathbf{\Sigma}$ and its transfer function $\mathbf{G}$ are said to be *regular* if there exists a linear operator D such that

$$\lim_{s \to \infty,\, s \in \mathbb{R}} \mathbf{G}(s)u = Du\,, \quad \forall u \in U\,,$$

in which case, by the principle of uniform boundedness, it follows that $D \in \mathcal{B}(U, Y)$. The operator D is called the *feedthrough operator* of $\mathbf{\Sigma}$.

The generator of $\mathbf{T}$ is denoted by A with domain $\mathrm{dom}(A)$. Let X_1 be the space $\mathrm{dom}(A)$ endowed with the graph norm. The norm on X is denoted by $\|\cdot\|$, whilst $\|\cdot\|_1$ denotes the graph norm. Let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|$, where $\lambda \in \varrho(A)$ is any fixed element of the resolvent set $\varrho(A)$ of A. Then $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0-semigroup on X_1 and extended to a C_0-semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, its generator) and the associated restrictions and extensions. With this convention, we may write $A \in \mathcal{B}(X, X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X.

By a representation theorem due to Salamon [23] (see also Weiss [28, 29]), there exist unique operators $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$ (the *control operator* and the *observation operator* of $\mathbf{\Sigma}$, respectively) such that, for all $t \geq 0$, $u \in L^2_{\rm loc}(\mathbb{R}_+, U)$ and $x^0 \in X_1$,

$$\mathbf{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \quad \text{and} \quad (\mathbf{\Psi}_\infty x^0)(t) = C\mathbf{T}_t x^0\,.$$

B is said to be *bounded* if it is so as a map from the input space U to the state space X, otherwise, B is said to be *unbounded*. C is said to be *bounded* if it can be extended continuously to X, otherwise, C is said to be *unbounded*. The operators A, B, C (and D, in the regular case) are called the *generating operators* of $\mathbf{\Sigma}$.

The *Lebesgue extension* of C was adopted in [29] and is defined by

$$C_L x^0 = \lim_{t \to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\tau x^0\, d\tau\,, \tag{4}$$

where $\mathrm{dom}(C_L)$ is the set of all those $x^0 \in X$ for which the above limit exists. Clearly $X_1 \subset \mathrm{dom}(C_L) \subset X$. Furthermore, for any $x^0 \in X$, we have that $\mathbf{T}_t x^0 \in \mathrm{dom}(C_L)$ for almost all $t \geq 0$ and

$$(\mathbf{\Psi}_\infty x^0)(t) = C_L \mathbf{T}_t x^0 \,, \quad \text{for a.a. } t \geq 0 \,.$$

It is known (see [4] and [23]) that for all $s, s_0 \in \varrho(A)$ with $s \neq s_0$, the transfer function $\mathbf{G}$ and the generating operators are related in the following way:

$$\frac{1}{s - s_0}(\mathbf{G}(s) - \mathbf{G}(s_0)) = -C(sI - A)^{-1}(s_0 I - A)^{-1} B \,. \tag{5}$$

This identity shows that the operators A, B and C determine $\mathbf{G}$ only up to an additive constant. Under the assumption that $\mathbf{\Sigma}$ is regular (with feedthrough operator D), it has been shown in [27] that $(sI - A)^{-1}BU \subset \mathrm{dom}(C_L)$ for all $s \in \varrho(A)$ and the transfer function $\mathbf{G}$ can be expressed as

$$\mathbf{G}(s) = C_L(sI - A)^{-1}B + D \,, \quad \forall s \in \mathbb{C}_{\omega(\mathbf{T})} \,,$$

which is familiar from finite-dimensional systems theory.

If $\mathbf{\Sigma}$ is a well-posed linear system with generating operators (A, B, C), then for any $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the function x given by (1a) is the unique solution of the initial-value problem

$$\dot{x}(t) = Ax(t) + Bu(t) \,, \quad x(0) = x^0 \,, \tag{6}$$

where the derivative on the left-hand side of (6) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (6) in the space X_{-1}, then for any $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, (6) has a unique strong solution (in the sense of Pazy [22], p. 109) given by the variation of parameters formula

$$t \mapsto x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \,. \tag{7}$$

In the case where the input function u has additional smoothness, we have the following result (*cf.* Lemma 2.5 in Salamon [24]).

Lemma 2.3 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be a well-posed system with state space X, input space U and generating operators (A, B, C). If $x^0 \in X$ and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, U)$ are such that $Ax^0 + Bu(0) \in X$, then the unique solution (7) of the initial-value problem (6) is continuously differentiable in X.*

The following lemma is essentially contained in [14]: the proof of statement (a) is implicit in the argument establishing Lemma 2.2 of [14] and statement (b) is part of Lemma 2.2 of [14].

Lemma 2.4 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be an exponentially stable well-posed system with state space X, input space U and generating operators (A, B, C).*

(a) *For all $(x^0, u) \in X \times L^\infty(\mathbb{R}_+, U)$, the function x given by (7) satisfies*

$$x \in L^\infty(\mathbb{R}_+, X) \,.$$

(b) *If $u \in L^\infty(\mathbb{R}_+, U)$ and $\lim_{t\to\infty} u(t) = u_\infty$ exists, then, for all $x^0 \in X$, the function x given by (7) satisfies*

$$\lim_{t\to\infty} \|x(t) + A^{-1}Bu_\infty\| = 0 \,.$$

If $\mathbf{\Sigma}$ is a well-posed system with state space X, input space U, output space Y and generating operators (A, B, C), then for all $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, the corresponding state function x of $\mathbf{\Sigma}$ satisfies (6) in X_{-1} and the corresponding output function $w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ is given by

$$w(t) = C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty u)(t)\,, \quad \text{for a.a. } t \geq 0\,. \tag{8}$$

If $x^0 \in X_1$ and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, U)$ with $u(0) = 0$, it has been shown in [23] that

$$w(t) = C\left(x(t) - (s_0 I - A)^{-1} B u(t)\right) + \mathbf{G}(s_0) u(t)\,, \quad \forall t \geq 0\,, \tag{9}$$

where s_0 is an arbitrary element of $\varrho(A)$ and $\mathbf{G}$ is the transfer function of $\mathbf{\Sigma}$. If $\mathbf{\Sigma}$ is regular (with feedthrough operator D), then the state function x defined by (7) satisfies $x(t) \in \mathrm{dom}(C_L)$ for almost all $t \geq 0$ and the output w given by (8) can be written in the familiar form

$$w(t) = C_L x(t) + D u(t)\,, \quad \text{for a.a. } t \geq 0\,.$$

Lemma 2.5 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be a well-posed system with state space X, input space U, transfer function $\mathbf{G}$ and generating operators (A, B, C) and let $s_0 \in \varrho(A)$. For $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, let x and w denote the state and output functions of $\mathbf{\Sigma}$. Then*

$$C(A - s_0 I)^{-1}\left(x(t) - x(t_0)\right) = \int_{t_0}^{t} \left(s_0 C(A - s_0 I)^{-1} x(\tau) + w(\tau) - \mathbf{G}(s_0) u(\tau)\right) d\tau\,, \quad \forall t \geq t_0 \geq 0\,.$$

In particular, if $0 \in \varrho(A)$, then

$$CA^{-1}\left(x(t) - x(t_0)\right) = \int_{t_0}^{t} \left(w(\tau) - \mathbf{G}(0) u(\tau)\right) d\tau\,, \quad \forall t \geq t_0 \geq 0\,.$$

Proof. Let $t_1 > t_0 \geq 0$ and choose sequences $(x_n^0) \subset X_1$ and $(u_n) \subset W^{1,2}([0, t_1], U)$, with $u_n(0) = 0$ for all n, such that

$$\lim_{n\to\infty} \|x^0 - x_n^0\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u - u_n\|_{L^2(0,t_1;U)} = 0\,. \tag{10}$$

Moreover, define x_n to be the solution of (6) for initial value x_n^0 and input u_n and let w_n be the corresponding output. Since $Ax_n^0 + Bu_n(0) = Ax_n^0 \in X$, x_n is, by Lemma 2.3, continuously differentiable in X and satisfies

$$\dot{x}_n(t) = Ax_n(t) + Bu_n(t) \quad (\text{in } X)\,, \quad \forall t \geq 0\,.$$

It follows that

$$\begin{aligned} C(A - s_0 I)^{-1} \dot{x}_n(t) &= C(A - s_0 I)^{-1}(Ax_n(t) + Bu_n(t)) \\ &= C[x_n(t) + s_0 (A - s_0 I)^{-1} x_n(t) - (s_0 I - A)^{-1} B u_n(t)]\,, \end{aligned}$$

and hence, by (9),

$$C(A - s_0 I)^{-1} \dot{x}_n(t) = s_0 C(A - s_0 I)^{-1} x_n(t) + w_n(t) - \mathbf{G}(s_0) u_n(t)\,.$$

Thus, for all $t \in [t_0, t_1]$,

$$C(A - s_0 I)^{-1}\left(x_n(t) - x_n(t_0)\right) = \int_{t_0}^{t} \left(s_0 C(A - s_0 I)^{-1} x_n(\tau) + w_n(\tau) - \mathbf{G}(s_0) u_n(\tau)\right) d\tau\,.$$

By (10) and the continuity properties of a well-posed system we have that $\lim_{n\to\infty} \|x(t) - x_n(t)\| = 0$ for all $t \in [0, t_1]$ and $\lim_{n\to\infty} \|w - w_n\|_{L^2(0,t_1;Y)} = 0$. Letting $n \to \infty$, we conclude that

$$C(A - s_0 I)^{-1} \left(x(t) - x(t_0)\right) = \int_{t_0}^{t} \left(s_0 C(A - s_0 I)^{-1} x(\tau) + w(\tau) - \mathbf{G}(s_0)u(\tau)\right) d\tau\,, \quad \forall t \in [t_0, t_1]\,.$$

Since t_1 was arbitrary, the claim follows. □

For notational simplicity we assume in the rest of this section that $\mathbf{\Sigma}$ is a single-input, single-output system, *i.e.*, $U = Y = \mathbb{R}$. An inspection of the proofs shows that the results below (Lemma 2.6 and Corollary 2.7) generalize in an obvious manner to well-posed systems with finite-dimensional input and output spaces $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$.

The next lemma provides a useful formula for the output (1b) if $u \in W^{1,2}_{\rm loc}(\mathbb{R}_+, \mathbb{R})$.

Lemma 2.6 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ be a single-input, single-output, well-posed system with state space X, transfer function $\mathbf{G}$ and generating operators (A, B, C). Let $s_0 \in \varrho(A)$ and set $\mathbf{H}(s) := -C(sI - A)^{-1}(s_0 I - A)^{-1}B$. Let $x^0 \in X$, $u^0 \in \mathbb{R}$ and $u \in W^{1,2}_{\rm loc}(\mathbb{R}_+, \mathbb{R})$ with $u(0) = u^0$ and denote the corresponding output of $\mathbf{\Sigma}$ by w. Then*

$$w(t) = C_L \mathbf{T}_t \left(x^0 - (s_0 I - A)^{-1} B u^0\right) + \left(\mathfrak{L}^{-1}(\mathbf{H}) \star \left(\dot{u} - s_0 u\right)\right)(t) + \mathbf{G}(s_0)u(t)\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,.$$

In particular, if $\mathbf{T}_{t_0}(Ax^0 + Bu^0) \in X$ for some $t_0 \geq 0$, then $w \in W^{1,2}_{\rm loc}([t_0, \infty), \mathbb{R})$.

Proof. Let $x^0 \in X$, $u \in W^{1,2}_{\rm loc}(\mathbb{R}_+, \mathbb{R})$ and $\tau > 0$. Define

$$u_\tau(t) := \begin{cases} u(t)\,, & 0 \leq t \leq \tau \\ u(\tau)\,, & t > \tau\,, \end{cases}$$

and note that $\dot{u}_\tau \in L^2(\mathbb{R}_+, \mathbb{R})$. It is sufficient to show that

$$w(t) = C_L \mathbf{T}_t \left(x^0 - (s_0 I - A)^{-1} B u^0\right) + \left(\mathfrak{L}^{-1}(\mathbf{H}) \star \left(\dot{u}_\tau - s_0 u_\tau\right)\right)(t) + \mathbf{G}(s_0)u_\tau(t)\,, \quad \text{for a.a. } t \in [0, \tau]\,. \tag{11}$$

Let $\alpha > \max(0, \operatorname{Re} s_0, \omega(\mathbf{T}))$. Taking Laplace transform of the right-hand side of (11), we obtain for all $s \in \mathbb{C}_\alpha$, using (5),

$$C(sI - A)^{-1}x^0 + \mathbf{H}(s)u^0 + \mathbf{H}(s)((s - s_0)\hat{u}_\tau(s) - u^0) + \mathbf{G}(s_0)\hat{u}_\tau(s) = C(sI - A)^{-1}x^0 + \mathbf{G}(s)\hat{u}_\tau(s)\,.$$

Inverse Laplace transform, using (3), gives (11). Moreover, assume that $\mathbf{T}_{t_0}(Ax^0 + Bu^0) \in X$ for some $t_0 \geq 0$. In order to prove that $w \in W^{1,2}_{\rm loc}([t_0, \infty), \mathbb{R})$, it is sufficient to show that the function given by the right-hand side of (11) is in $W^{1,2}_{\rm loc}([t_0, \infty), \mathbb{R})$. Since $\mathbf{T}_{t_0}(Ax^0 + Bu^0) \in X$, we have that $\mathbf{T}_{t_0}(x^0 - (s_0 I - A)^{-1}Bu^0) \in X_1$. Consequently, the function

$$t \mapsto C_L \mathbf{T}_t (x^0 - (s_0 I - A)^{-1} B u^0)$$

is continuously differentiable on $[t_0, \infty)$ and hence in $W^{1,2}_{\rm loc}([t_0, \infty), \mathbb{C})$. Therefore it is sufficient to show that the function $\tilde{w} : \mathbb{R} \to \mathbb{C}$ given by

$$\tilde{w}(t) = \begin{cases} e^{-\alpha t}(\mathfrak{L}^{-1}(\mathbf{H}) \star \tilde{u})(t) & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0\,, \end{cases}$$

where $\tilde{u} = \mathbf{P}_\tau(\dot{u}_\tau - s_0 u_\tau)$, is in $W^{1,2}(\mathbb{R},\mathbb{C})$. To this end note that the Fourier transform $\mathfrak{F}(\tilde{w})$ of $\tilde{w}$ is given by

$$(\mathfrak{F}\tilde{w})(\omega) = (\mathfrak{L}\tilde{w})(i\omega) = \frac{\mathbf{G}(\alpha+i\omega)-\mathbf{G}(s_0)}{\alpha+i\omega-s_0}(\mathfrak{L}\tilde{u})(\alpha+i\omega)\,. \tag{12}$$

Since $\tilde{u} \in L^2(\mathbb{R}_+,\mathbb{C})$, it follows that the function $\omega \mapsto (\mathfrak{L}\tilde{u})(\alpha+i\omega)$ is in $L^2(\mathbb{R},\mathbb{C})$. Note also that the function $\omega \mapsto \mathbf{G}(\alpha+i\omega)$ is bounded. Hence, by (12), $\tilde{w} \in L^2(\mathbb{R},\mathbb{C})$, and moreover, the function $\omega \mapsto \omega(\mathfrak{F}\tilde{w})(\omega)$ is in $L^2(\mathbb{R},\mathbb{C})$. It follows from the Fourier characterization of $W^{1,2}(\mathbb{R},\mathbb{C})$ (see, for example, [8], p. 165) that $\tilde{w} \in W^{1,2}(\mathbb{R},\mathbb{C})$. □

The following corollary is an easy consequence of Lemma 2.6.

Corollary 2.7 *Let $\mathbf{\Sigma} = (\mathbf{T},\mathbf{\Phi},\mathbf{\Psi},\mathbf{F})$ be an exponentially stable, single-input, single-output, well-posed system with state space X, transfer function $\mathbf{G}$ and generating operators (A,B,C). Let $x^0 \in X$, $u^0 \in \mathbb{R}$ and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$ with $u(0) = u^0$ and denote the corresponding output of $\mathbf{\Sigma}$ by w. If $\dot{u} \in L^2(\mathbb{R}_+,\mathbb{R})$ and $\lim_{t\to\infty} u(t) =: u_\infty$ exists and is finite, then*

$$\lim_{t\to\infty}[w(t) - C_L\mathbf{T}_t(x^0 + A^{-1}Bu^0)] = \mathbf{G}(0)u_\infty\,.$$

In particular, under the additional assumption that $\mathbf{T}_{t_0}(Ax^0+Bu^0) \in X$ for some $t_0 \geq 0$,

$$\lim_{t\to\infty} w(t) = \mathbf{G}(0)u_\infty\,.$$

Remark 2.8 The above corollary shows that, under certain smoothness assumptions on the input and initial data, the final-value theorem holds for a general exponentially stable, single-input, single-output, well-posed system $\mathbf{\Sigma}$. If $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, the latter smoothness assumptions may be relaxed: if the impulse response of $\mathbf{\Sigma}$ is a finite signed Borel measure on $\mathbb{R}_+$, then for any input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$ with $\lim_{t\to\infty} u(t) = u_\infty$ and any $x^0 \in X$,

$$\lim_{t\to\infty}[w(t) - C_L\mathbf{T}_t x^0] = \mathbf{G}(0)u_\infty\,.$$

Under the additional assumption that $\mathbf{T}_{t_0}x^0 \in X_1$ for some $t_0 \geq 0$, $\lim_{t\to\infty} w(t) = \mathbf{G}(0)u_\infty$. ◇

Proof of Corollary 2.7. By exponential stability, $0 \in \varrho(A)$, and an application of Lemma 2.6 with $s_0 = 0$ leads to

$$w(t) = C_L\mathbf{T}_t\left(x^0 + A^{-1}Bu^0\right) + (\mathfrak{L}^{-1}(\mathbf{H}) \star \dot{u})(t) + \mathbf{G}(0)u(t)\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,, \tag{13}$$

where $\mathbf{H}(s) = (\mathbf{G}(s) - \mathbf{G}(0))/s$. Since $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha)$ for some $\alpha < 0$, it follows that $\mathbf{H} \in H^2(\mathbb{C}_\alpha)$ and hence, by the Paley-Wiener Theorem, $\mathfrak{L}^{-1}(\mathbf{H}) \in L^2_\alpha(\mathbb{R}_+,\mathbb{R}) \subset L^2(\mathbb{R}_+,\mathbb{R})$. Since by assumption $\dot{u} \in L^2(\mathbb{R}_+,\mathbb{R})$, we may infer that

$$\lim_{t\to\infty}(\mathfrak{L}^{-1}(\mathbf{H}) \star \dot{u})(t) = 0\,.$$

Combining this with (13) yields the claim. □

3 Problem formulation

The problem of tracking constant reference signals r will be addressed in a context of uncertain single-input ($u \in L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R})$), single-output ($y \in L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R})$), well-posed linear systems $\mathbf{\Sigma}$ with state space X, transfer function $\mathbf{G}$ and generating operators (A, B, C), having a nonlinearity φ in the input channel and a nonlinearity ψ in the output channel:

$$\dot{x}(t) = Ax(t) + B\varphi(u(t)), \quad \text{for a.a. } t \in \mathbb{R}_+, \quad x(0) = x^0 \in X, \tag{14a}$$

$$w(t) = C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t), \quad \text{for a.a. } t \in \mathbb{R}_+ , \tag{14b}$$

$$y(t) = \psi(w(t)), \quad \text{for a.a. } t \in \mathbb{R}_+ . \tag{14c}$$

3.1 The class $\mathcal{S}$ of linear systems

Let $\mathcal{S}$ denote the class of single-input, single-output well-posed linear systems $\mathbf{\Sigma}$ with state space X and transfer function $\mathbf{G}$ satisfying:

$$\text{(a)} \quad \mathbf{\Sigma} \text{ is exponentially stable} \qquad \text{(b)} \quad \mathbf{G}(0) > 0.$$

If $\mathbf{G}$ is the transfer function of a system $\mathbf{\Sigma} \in \mathcal{S}$, then it is readily shown that the following positive-real condition,

$$1 + \kappa \operatorname{Re} \frac{\mathbf{G}(s)}{s} \geq 0 , \quad \forall s \in \mathbb{C}_0 , \tag{15}$$

holds for all sufficiently small $\kappa > 0$, see [16, Lemma 3.10]. Define

$$\kappa^* := \sup\{\kappa > 0 \,|\, 1 + \kappa \operatorname{Re}(\mathbf{G}(s)/s) \geq 0 , \, \forall s \in \mathbb{C}_0\} . \tag{16}$$

Lower bounds and formulae for κ^* may be found in [13]. The following lemma will be invoked in a later stability analysis.

Lemma 3.1 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ with generating operators (A, B, C) and let $\Delta > 1/\kappa^*$. Then there exist $P \in \mathcal{B}(X)$, $P = P^* \geq 0$, and $M \in \mathcal{B}(X_1, \mathbb{R})$ such that*

$$\begin{aligned} \langle Ax_1, Px_2\rangle + \langle Px_1, Ax_2\rangle &= -\langle Mx_1, Mx_2\rangle , \quad \forall x_1, x_2 \in X_1 , \\ (A^{-1}B)^* Px &= Cx - \sqrt{2\Delta} Mx , \quad \forall x \in X_1 , \end{aligned}$$

where $\langle \cdot , \cdot \rangle$ denotes inner product in X.

Proof. Let $\mathbf{G}$ be the transfer function of $\mathbf{\Sigma}$. By (5),

$$\mathbf{H}(s) := C(sI - A)^{-1}A^{-1}B = \begin{cases} \dfrac{\mathbf{G}(s) - \mathbf{G}(0)}{s} , & s \in \varrho(A), \, s \neq 0 \\ \mathbf{G}'(0) , & s \in \varrho(A), \, s = 0 . \end{cases} \tag{17}$$

By exponential stability of $\mathbf{\Sigma}$, $\mathbf{H}$ is holomorphic in $\mathbb{C}_\alpha$ for some $\alpha < 0$. Since $\Delta > 1/\kappa^*$, the positive-real condition (15) guarantees the existence of a constant $\varepsilon > 0$ such that

$$\Delta + \operatorname{Re} \mathbf{H}(i\omega) \geq \varepsilon , \quad \forall \omega \in \mathbb{R} .$$

By a result in van Keulen [26] (see Theorem 3.10 and Remark 3.14 in [26]), there exists $\tilde{P} \in \mathcal{B}(X)$, $\tilde{P} = \tilde{P}^*$, such that

$$\langle Ax_1, \tilde{P}x_2\rangle + \langle \tilde{P}x_1, Ax_2\rangle = 1/(2\Delta)\langle [(A^{-1}B)^*\tilde{P} + C]x_1, [(A^{-1}B)^*\tilde{P} + C]x_2\rangle\,, \quad \forall x_1, x_2 \in X_1\,. \tag{18}$$

Setting

$$P := -\tilde{P} \in \mathcal{B}(X)\,, \quad M := \sqrt{1/(2\Delta)}[C - (A^{-1}B)^*P] \in \mathcal{B}(X_1, \mathbb{R})\,,$$

we obtain, using (18),

$$\langle Ax_1, Px_2\rangle + \langle Px_1, Ax_2\rangle = -\langle Mx_1, Mx_2\rangle\,, \quad \forall x_1, x_2 \in X_1\,, \tag{19a}$$

$$(A^{-1}B)^*Px = Cx - \sqrt{2\Delta}Mx\,, \quad \forall x \in X_1\,. \tag{19b}$$

By a routine argument it follows from (19a) and exponential stability of $\mathbf{\Sigma}$ that $P \geq 0$. □

3.2 Input/output nonlinearities

In this section we introduce certain sets of nonlinearities which will play a rôle in the sequel. Moreover, some definitions and auxiliary results are collected. The following sets of monotone, nondecreasing nonlinearities are first introduced:

$$\mathcal{M} := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ locally Lipschitz and nondecreasing}\}\,,$$

$$\mathcal{M}(\lambda) := \{f \in \mathcal{M} \mid 0 \leq (f(\xi) - f(0))\xi \leq \lambda\xi^2\,, \quad \forall \xi \in \mathbb{R}\}\,,$$

$$\mathcal{M}_L(\lambda) := \{f \in \mathcal{M} \mid f \text{ is globally Lipschitz with Lipschitz constant } \lambda\}\,.$$

Clearly, $\mathcal{M}_L(\lambda) \subset \mathcal{M}(\lambda) \subset \mathcal{M}$.

Remark 3.2 (i) Let $f \in \mathcal{M}(\lambda)$. Then $f(\xi) = f(0) + \gamma(\xi)\xi$ for all ξ, where

$$\gamma : \xi \mapsto \begin{cases} (f(\xi) - f(0))/\xi\ , & \xi \neq 0 \\ \lambda\ , & \xi = 0 \end{cases}$$

is upper semicontinuous and $\gamma(\xi) \in [0, \lambda]$ for all $\xi \in \mathbb{R}$.

(ii) If $f \in \mathcal{M}(\lambda)$, then, for each $\nu \in \mathbb{R}$, there exists $\tilde{\lambda} \geq 0$ such that the function $\xi \mapsto f(\xi + \nu) - f(\nu)$ is of class $\mathcal{M}(\tilde{\lambda})$.

(iii) If $f \in \mathcal{M}_L(\lambda)$, then, for each $\nu \in \mathbb{R}$, $\xi \mapsto f(\xi + \nu) - f(\nu)$ is also of class $\mathcal{M}_L(\lambda)$. ◇

We say that a pair of nonlinearities (φ, ψ) is in $\mathcal{N}$ if $\varphi \in \mathcal{M}_L(\lambda_1)$ for some $\lambda_1 > 0$, $\psi \in \mathcal{M}$ and at least one of the following holds:

$$\text{(i) } \varphi \text{ is bounded, or (ii) } \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0\,.$$

Equivalently,

$$\mathcal{N} := \{(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M} \mid \lambda_1 > 0,\ \varphi \text{ unbounded } \Rightarrow\ \psi \in \mathcal{M}(\lambda_2) \text{ for some } \lambda_2 > 0\}\,.$$

The proof of the following lemma is given in [14].

Lemma 3.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and (ε_n) be any sequence with $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Define the function $f^\diamond : \mathbb{R} \to \mathbb{R}$ by*

$$f^\diamond(\xi) := \limsup_{n\to\infty} \frac{f(\xi + \varepsilon_n) - f(\xi)}{\varepsilon_n}\,. \tag{20}$$

Then $f^\diamond$ is Borel measurable and $f^\diamond \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$. If $g : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then $f \circ g$ is absolutely continuous and

$$\frac{d}{dt}(f \circ g)(t) = f^\diamond(g(t))\dot{g}(t)\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,.$$

If $f \in \mathcal{M}_L(\lambda)$ for some $\lambda \geq 0$, then

$$0 \leq f^\diamond(\xi) \leq \lambda\,, \quad \forall \xi \in \mathbb{R}. \tag{21}$$

3.3 The tracking objective and feasibility

Given $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ and static nonlinearities $\varphi, \psi : \mathbb{R} \to \mathbb{R}$, the tracking objective is to determine, by feedback, an input u such that, for given $r \in \mathbb{R}$, the output y of (14) has the property $y(t) \to r$ as $t \to \infty$ (*cf.* Remark 2.1). Clearly, if this objective is achievable, then r is necessarily in the closure of $\operatorname{im}\psi$. We will impose a stronger condition, namely,

$$\Psi^r \cap \overline{\Phi} \neq \emptyset\,, \quad \text{where } \Psi^r := \{v \in \mathbb{R} \mid \psi(\mathbf{G}(0)v) = r\}\,, \quad \Phi := \operatorname{im}\varphi\,, \quad \overline{\Phi} := \operatorname{clos}(\Phi)$$

and refer to the set

$$\mathcal{R} := \{r \in \mathbb{R} \mid \Psi^r \cap \overline{\Phi} \neq \emptyset\}$$

as the set of *feasible reference values*. The next proposition shows that, if ψ is continuous and monotone, then $r \in \mathcal{R}$ is close to being a necessary condition for tracking insofar as, if tracking of r is achievable whilst maintaining boundedness of $\varphi \circ u$ together with ultimate continuity and ultimate boundedness of w, then $r \in \mathcal{R}$.

Proposition 3.4 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ and $x^0 \in X$. Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ and let ψ be continuous and monotone. Let $u : \mathbb{R}_+ \to \mathbb{R}$ be such that $\varphi \circ u \in L^\infty(\mathbb{R}_+, \mathbb{R})$. If $w : \mathbb{R}_+ \to \mathbb{R}$ given by* (14b) *is such that, for some $\tau \geq 0$, w restricted to $[\tau, \infty)$ is continuous and bounded with*

$$\lim_{t\to\infty} \psi\left(w(t)\right) = r\,, \tag{22}$$

then $r \in \mathcal{R}$.

Proof. By assumption, $\varphi \circ u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and so, by Lemma 2.4(a), the solution x of the initial-value problem (14a) is bounded. Since by assumption the function w is ultimately continuous and bounded, it has non-empty, compact, connected ω-limit set Ω. By continuity of ψ and (22), it follows that

$$\psi(\omega) = r\,, \quad \forall \omega \in \Omega\,. \tag{23}$$

It suffices to show that $\Omega \cap \mathbf{G}(0)\overline{\Phi} \neq \emptyset$. Indeed, if there exist $\omega \in \Omega$ and $v \in \overline{\Phi}$ such that $\omega = \mathbf{G}(0)v$, then, by (23), $\psi(\mathbf{G}(0)v) = \psi(\omega) = r$, showing that $\Psi^r \cap \overline{\Phi}$ is non-empty.

Seeking a contradiction, suppose that $\Omega \cap \mathbf{G}(0)\overline{\Phi} = \emptyset$. Then Ω and $\mathbf{G}(0)\overline{\Phi}$ are closed disjoint intervals and so there exist $\varepsilon > 0$ and $\beta = \pm 1$ such that

$$\inf\{\beta\omega \mid \omega \in \Omega\} - \sup\{\beta \mathbf{G}(0)v \mid v \in \overline{\Phi}\} = 2\varepsilon\,.$$

Moreover, since $w(t)$ approaches its compact ω-limit set Ω as $t \to \infty$, there exists $T > 0$ such that

$$\inf\Omega - \varepsilon < w(t) < \sup\Omega + \varepsilon\,, \quad \forall t > T\,,$$

and so $\beta w(t) > \inf\{\beta\omega \mid \omega \in \Omega\} - \varepsilon$ for all $t > T$. Using Lemma 2.5 (with $s_0 = 0$), it follows that

$$\begin{aligned}\beta C A^{-1}(x(t) - x(T)) &= \beta \int_T^t (w(\tau) - \mathbf{G}(0)\varphi(u(\tau)))\, d\tau \geq \int_T^t \left(\beta w(\tau) - \sup\{\beta \mathbf{G}(0)v \,|\, v \in \overline{\Phi}\}\right) d\tau \\ &> \int_T^t \left(\inf\{\beta\omega \mid \omega \in \Omega\} - \varepsilon - \sup\{\beta \mathbf{G}(0)v \mid v \in \overline{\Phi}\}\right) d\tau = \varepsilon(t - T)\,, \quad \forall t > T\,,\end{aligned}$$

contradicting the boundedness of x. Therefore, $\Omega \cap \mathbf{G}(0)\overline{\Phi} \neq \emptyset$. □

4 Integral control

Let $\mathbf{\Sigma} \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}$, $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and $r \in \mathbb{R}$. We will investigate integral control action

$$u(t) = u^0 + \int_0^t k(\tau)\,[r - y(\tau)]\; d\tau\,, \tag{24}$$

with prescribed time-varying gain $k(\cdot)$. An application of the control (24) to $\mathbf{\Sigma}$ will lead to the following system of nonlinear non-autonomous differential equations:

$$\dot{x}(t) = Ax(t) + B\varphi(u(t))\,, \quad x(0) = x^0 \in X\,, \tag{25a}$$

$$\dot{u}(t) = k(t)[r - \psi\left(C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t)\right)]\,, \quad u(0) = u^0 \in \mathbb{R}. \tag{25b}$$

For $a \in (0, \infty]$, a continuous function

$$(x, u) : [0, a) \to X \times \mathbb{R}$$

is a *solution* of (25) if (x, u) is absolutely continuous as an $(X_{-1} \times \mathbb{R})$-valued function, $(x(0), u(0)) = (x^0, u^0)$ and (25) is satisfied almost everywhere on $[0, a)$, where the derivative in (25a) should be interpreted in the space X_{-1}. †

4.1 Globally Lipschitz output nonlinearities

If $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}_L(\lambda_2)$, we have the following existence and uniqueness result.

† Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive, and hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration, see [1], Theorem 3.1, p. 10.

Lemma 4.1 *Let* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ *with generating operators* (A, B, C). *Let* $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}_L(\lambda_2)$, $r \in \mathbb{R}$ *and* $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$.
For each $(x^0, u^0) \in X \times \mathbb{R}$, *the initial-value problem* (25) *has a unique solution* $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$.

The proof of Lemma 4.1 may be found in the Appendix.

Henceforth, we assume that the gain function $k : \mathbb{R}_+ \to (0, \infty)$ satisfies

$$k \in \mathcal{G} := \{g \mid g : \mathbb{R}_+ \to (0, \infty),\ g \text{ is measurable and bounded}\}.$$

The next theorem forms the core of the paper.

Theorem 4.2 *Let* $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$, $(\varphi, \psi) \in \mathcal{M}_L(\lambda_1) \times \mathcal{M}_L(\lambda_2)$, $r \in \mathcal{R}$ *and* $k \in \mathcal{G}$. *Let* (A, B, C) *be the generating operators of* $\mathbf{\Sigma}$. *For* $(x^0, u^0) \in X \times \mathbb{R}$, *let* $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$ *be the unique solution of the initial-value problem* (25). *Let* $k^* = \kappa^*/(\lambda_1 \lambda_2)$, *where* κ^* *is given by* (16). *Assume that* $\limsup_{t\to\infty} k(t) < k^*$. *Then the following hold.*

A. $\lim_{t\to\infty} \varphi(u(t))$ *exists and is finite.*

B. *If* $K(t) := \int_0^t k \to \infty$ *as* $t \to \infty$, *then*

   (b1) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi^r \cap \overline{\Phi}$,

   (b2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\varphi^r\| = 0$,

   (b3) $|r - y(t)| \le e_0(t) + e_1(t)$, *where* $y(t) = \psi\left(C_L \mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t)\right)$, $e_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}_+)$ *for some* $\alpha < 0$ *and* $e_1 : \mathbb{R}_+ \to \mathbb{R}_+$ *is bounded with* $\lim_{t\to\infty} e_1(t) = 0$,

   (b4) *if* $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ *for some* $t_0 \ge 0$, *then* $\lim_{t\to\infty} y(t) = r$,

   (b5) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, *then* $\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) = 0$,

   (b6) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, *then* $u(\cdot)$ *is bounded*.

Remark 4.3 (i) Statement (b3) says that if $k \in \mathcal{G}$ is *not* of class L^1 and $\limsup_{t\to\infty} k(t) < k^*$, then the tracking error $e(t) = r - y(t)$ resulting from an application of the integral control (24) is small in the sense that $|e| \le e_0 + e_1$, where $e_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}_+)$ for some $\alpha < 0$ and $e_1 \in L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ with $\lim_{t\to\infty} e_1(t) = 0$. This means in particular that for any $\varepsilon > 0$

$$\lim_{\tau\to\infty} \mu_L\left(\{t \ge \tau \mid |e(t)| \ge \varepsilon\}\right) = 0,$$

where μ_L denotes Lebesgue measure on $\mathbb{R}_+$.

(ii) In (b4) an additional assumption on the initial data is identified such that asymptotic tracking is ensured. An inspection of the proof below shows that if C is bounded, then asymptotic tracking occurs for arbitrary initial data.

(iii) An immediate consequence of part B of Theorem 4.2 is the following: if $k \in \mathcal{G}$ is chosen such that $k(t)$ tends to zero sufficiently slowly as $t \to \infty$ (in the sense that $k \notin L^1$), then (b1) – (b6) hold.

(iv) Simple examples show that the assertions (b1)–(b6) of part B of Theorem 4.2 are in general not true if the additional assumption $k \notin L^1$ is violated.

(v) Statements (b5) and (b6) capture aspects of the qualitative behaviour of the function u. For example, if $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, then (b5) implies that $u(t)$ converges as $t \to \infty$, provided that the set $\varphi^{-1}(\Psi^r \cap \Phi)$ is a singleton. ◇

Proof of Theorem 4.2. Let $\mathbf{G}$ denote the transfer function of $\mathbf{\Sigma}$.

A. By assumption, $r \in \mathcal{R}$, and so $\Psi^r \cap \overline{\Phi} \neq \emptyset$. Therefore, there exists $\varphi^* \in \Psi^r \cap \overline{\Phi}$. Since $\psi \in \mathcal{M}_L(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \psi(\xi + \mathbf{G}(0)\varphi^*) - r \tag{26}$$

is in $\mathcal{M}_L(\lambda_2)$ (*cf.* Remark 3.2(iii)) and, hence, in $\mathcal{M}(\lambda_2)$. Therefore, by Remark 3.2(i),

$$\tilde{\psi}(\xi) = \gamma(\xi)\xi\,, \quad \forall \xi \in \mathbb{R}, \tag{27}$$

where the function γ is defined by

$$\gamma(\xi) = \begin{cases} \tilde{\psi}(\xi)/\xi & , \quad \xi \neq 0 \\ \lambda_2 & , \quad \xi = 0\,. \end{cases} \tag{28}$$

and satisfies $0 \leq \gamma(\xi) \leq \lambda_2$ for all $\xi \in \mathbb{R}$. Introduce continuous functions z and v and a function $\tilde{w} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ given by

$$z(t) := x(t) + A^{-1}B\varphi(u(t))\,, \quad v(t) := \varphi(u(t)) - \varphi^*\,; \quad \forall t \geq 0\,, \tag{29}$$

and

$$\tilde{w}(t) := w(t) - \mathbf{G}(0)\varphi^*\,, \quad \text{for a.a } t \geq 0\,, \tag{30}$$

where, as in (14b), $w(t) = C_L\mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi \circ u))(t)$ for almost all $t \geq 0$. By Lemma 3.3, $\dot{v}(t) = \varphi^\diamond(u(t))\dot{u}(t)$ for almost all $t \in [0, \infty)$. Since (z, v) is absolutely continuous as an $(X_{-1} \times \mathbb{R})-$valued function, we obtain by direct calculation, using (26),

$$\dot{z}(t) = Az(t) - k(t)\varphi^\diamond(u(t))A^{-1}B\tilde{\psi}(\tilde{w}(t))\,, \quad \text{for a.a. } t \geq 0\,, \tag{31a}$$

$$\dot{v}(t) = -k(t)\varphi^\diamond(u(t))\,\tilde{\psi}(\tilde{w}(t))\,, \quad \text{for a.a. } t \geq 0\,, \tag{31b}$$

with initial values $z^0 := x^0 + A^{-1}B\varphi(u^0)$ and $v^0 := \varphi(u^0) - \varphi^*$. Invoking (27),

$$\dot{z}(t) = Az(t) + A^{-1}B\zeta(t)\,, \quad \text{for a.a. } t \geq 0\,, \tag{32a}$$

$$\dot{v}(t) = \zeta(t)\,, \quad \text{for a.a. } t \geq 0\,, \tag{32b}$$

where

$$\zeta(t) := \frac{d}{dt}\varphi(u(t)) = \dot{v}(t) = -k(t)\eta(t)\tilde{w}(t)\,, \quad \text{with } \eta(t) := \varphi^\diamond(u(t))\gamma(\tilde{w}(t))\,. \tag{33}$$

Note that

$$0 \leq \eta(t) \leq \lambda_1\lambda_2 =: \lambda\,, \quad \forall t \geq 0\,. \tag{34}$$

By assumption, $\limsup_{t\to\infty} k(t) < k^*$. Hence, there exists $k_* > 0$ such that

$$\limsup_{t\to\infty} k(t) < k_* < k^*\,. \tag{35}$$

Since $1/(k_*\lambda) > 1/\kappa^*$, it follows from Lemma 3.1 that there exist $P \in \mathcal{B}(X)$, $P = P^* \geq 0$ and $M \in \mathcal{B}(X_1, \mathbb{R})$ such that

$$\langle Ax_1, Px_2\rangle + \langle Px_1, Ax_2\rangle = -\langle Mx_1, Mx_2\rangle\,, \quad \forall x_1, x_2 \in X_1\,, \tag{36a}$$

$$(A^{-1}B)^*Px = Cx - \sqrt{2/(k_*\lambda)}Mx\,, \quad \forall x \in X_1\,. \tag{36b}$$

We will investigate asymptotic properties of (z, v) using a Lyapunov approach. In order to make use of equations (36) (which hold in X_1), we will have to use an approximation argument. For this purpose choose $(z_n^0) \subset X_1$ such that

$$\lim_{n\to\infty} \|z_n^0 - z^0\| = 0\,.$$

For all n, consider the initial-value problem

$$\dot{\xi} = A\xi + A^{-1}B\zeta\,, \quad \xi(0) = z_n^0 \in X_1\,,$$

and define $z_n : \mathbb{R}_+ \to X$ by

$$z_n(t) := \mathbf{T}_t z_n^0 + A^{-1}\int_0^t \mathbf{T}_{t-\tau} B\zeta(\tau)d\tau\,.$$

Clearly, $z_n(t) \in X_1$ for all $t \geq 0$ and $Az_n \in L^1_{\rm loc}(\mathbb{R}_+, X)$. Hence, by a standard result on abstract Cauchy problems (see Pazy [22], p. 109), z_n is absolutely continuous as a X-valued function and

$$\dot{z}_n(t) = Az_n(t) + A^{-1}B\zeta(t) \quad \text{in } X\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,. \tag{37}$$

Define the absolutely continuous functions

$$V_n : \mathbb{R}_+ \to \mathbb{R}_+\,, \quad t \mapsto \langle z_n(t), Pz_n(t)\rangle + \mathbf{G}(0)v^2(t)\,.$$

Differentiating V_n and using (32b), (33), (36) and (37), we have for almost all $t \in \mathbb{R}_+$

$$\begin{aligned}
\dot{V}_n(t) &= \langle Az_n(t), Pz_n(t)\rangle + \langle Pz_n(t), Az_n(t)\rangle + 2\langle \zeta(t), (A^{-1}B)^* Pz_n(t)\rangle + 2\mathbf{G}(0)v(t)\zeta(t)\\
&= -\langle Mz_n(t), Mz_n(t)\rangle + 2\langle \zeta(t), Cz_n(t) - \sqrt{2/(k_*\lambda)}Mz_n(t)\rangle + 2\mathbf{G}(0)v(t)\zeta(t)\\
&= -(Mz_n(t))^2 - 2k(t)\eta(t)\left(\langle \tilde{w}(t), Cz_n(t) - \sqrt{2/(k_*\lambda)}Mz_n(t)\rangle + \mathbf{G}(0)v(t)\tilde{w}(t)\right)\,.
\end{aligned} \tag{38}$$

Invoking (34) and (35), we see that there exists $t_0 \geq 0$ such that

$$\sup_{t\geq t_0} k(t)\eta(t) < k_*\lambda\,. \tag{39}$$

Integrating (38) gives

$$\begin{aligned}
V_n(t) - V_n(t_0) = -\int_{t_0}^t (Mz_n(\tau))^2 d\tau - 2\int_{t_0}^t k(\tau)\eta(\tau)[\langle \tilde{w}(\tau), Cz_n(\tau) - \sqrt{2/(k_*\lambda)}Mz_n(\tau)\rangle\\
+ \mathbf{G}(0)v(\tau)\tilde{w}(\tau)]d\tau\,.
\end{aligned} \tag{40}$$

Observe that

$$\lim_{n\to\infty} \|z_n(t) - z(t)\| = 0\,, \quad \lim_{n\to\infty} \|z_n - z\|_{L^2(0,t;X)} = 0\,, \quad \forall t \geq 0\,. \tag{41}$$

Moreover, defining M_L by (4) with C and C_L replaced by M and M_L, respectively, it follows from (36b) that

$$M_L = \sqrt{(k_*\lambda)/2}\,\left(C_L - (A^{-1}B)^*P\right)$$

with $\mathrm{dom}(M_L) = \mathrm{dom}(C_L)$. Since for almost all $t \geq 0$

$$Cz_n(t) - C_L z(t) = C\mathbf{T}_t z_n^0 - C_L\mathbf{T}_t z^0\,, \quad Mz_n(t) - M_L z(t) = M\mathbf{T}_t z_n^0 - M_L\mathbf{T}_t z^0\,,$$

we have

$$\lim_{n\to\infty} \|Mz_n - M_L z\|_{L^2(0,t;\mathbb{R})} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Cz_n - C_L z\|_{L^2(0,t;\mathbb{R})} = 0\,, \quad \forall t \geq 0\,. \tag{42}$$

Hence, defining

$$V : \mathbb{R}_+ \to \mathbb{R}_+\,, \quad t \mapsto \langle z(t), Pz(t)\rangle + \mathbf{G}(0)v^2(t)\,,$$

taking limits in (40) as $n \to \infty$ and invoking (41) and (42) yields

$$V(t) - V(t_0) = -\int_{t_0}^{t} (M_L z(\tau))^2 d\tau - 2\int_{t_0}^{t} k(\tau)\eta(\tau)[\langle \tilde{w}(\tau), C_L z(\tau) - \sqrt{2/(k_*\lambda)} M_L z(\tau)\rangle + \mathbf{G}(0)v(\tau)\tilde{w}(\tau)]d\tau\,. \tag{43}$$

Moreover, by (37),

$$Cz_n(t) = C\mathbf{T}_t z_n^0 + \left(\mathfrak{L}^{-1}(\mathbf{H}) \star \zeta\right)(t)\,, \quad \forall t \in \mathbb{R}_+\,, \tag{44}$$

where $\mathbf{H}(s) = C(sI - A)^{-1}A^{-1}B$. Since $\zeta(t) = (d/dt)(\varphi \circ u)(t)$ and $z^0 = x^0 + A^{-1}B\varphi(u^0)$, taking limits in (44) as $n \to \infty$ gives

$$C_L z(t) = C_L \mathbf{T}_t \left(x^0 + A^{-1}B\varphi(u^0)\right) + \left(\mathfrak{L}^{-1}(\mathbf{H}) \star \frac{d}{dt}(\varphi \circ u)\right)(t)\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,.$$

Therefore, by Lemma 2.6,

$$C_L z(t) = w(t) - \mathbf{G}(0)(\varphi \circ u)(t) = \tilde{w}(t) - \mathbf{G}(0)v(t)\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,.$$

Using this in (43) and completing squares, we obtain

$$V(t) = V(t_0) - \int_{t_0}^{t} \left\{ \left(M_L z(\tau) - k(\tau)\eta(\tau)\sqrt{2/(k_*\lambda)}\tilde{w}(\tau)\right)^2 + 2k(\tau)\eta(\tau)\left[1 - \frac{k(\tau)\eta(\tau)}{k_*\lambda}\right]\tilde{w}^2(\tau)\right\} d\tau\,. \tag{45}$$

Note that, by (39), the integrand in (45) is nonnegative implying that V is non-increasing on $[t_0, \infty)$. Since V is bounded below, $\lim_{t\to\infty} V(t)$ exists and is finite. Equation (45) combined with (39) moreover implies that

$$\zeta = -k\eta\tilde{w} \in L^2(\mathbb{R}_+, \mathbb{R})\,. \tag{46}$$

Using this in (32a) and appealing to the fact that the semigroup $\mathbf{T}$ is exponentially stable, we may conclude that $z(t) \to 0$ in X. Combining this with the convergence of V, we may conclude that $\lim_{t\to\infty} \mathbf{G}(0)v^2(t)$ exists and is finite. Hence $\lim_{t\to\infty} \varphi(u(t))$ exists and is finite which is statement A.

B. Define

$$w_0(t) := C_L \mathbf{T}_t(x^0 + A^{-1}B\varphi(u^0))\,, \qquad w_1(t) := \left(\mathfrak{L}^{-1}(\mathbf{H}) \star \frac{d}{dt}(\varphi \circ u)\right)(t) + \mathbf{G}(0)\varphi(u(t))\,,$$

where $\mathbf{H}$ is given by (17). Clearly, $\varphi \circ u \in W^{1,2}_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, and hence an application of Lemma 2.6 shows that

$$y(t) = \psi(w_0(t) + w_1(t))\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,.$$

Since, by (46), $(d/dt)(\varphi \circ u) = \zeta \in L^2(\mathbb{R}_+, \mathbb{R})$, and, by part A, $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r$ exists and is finite, we may infer from Corollary 2.7 that

$$\lim_{t\to\infty} w_1(t) = \mathbf{G}(0)\varphi^r \,. \tag{47}$$

Evidently, $\varphi^r \in \overline{\Phi}$ and so, to establish (b1), it suffices to show that $\varphi^r \in \Psi^r$. Seeking a contradiction, suppose that $\varphi^r \notin \Psi^r$. This implies that

$$\theta := \frac{r - \psi(\mathbf{G}(0)\varphi^r)}{2} \neq 0 \,.$$

Using continuity of ψ and (47), we obtain for sufficiently large $s > 0$

$$|\psi(w_1(t)) - \psi(\mathbf{G}(0)\varphi^r)| \leq |\theta| \,, \quad \forall t \geq s \,,$$

which, together with

$$\dot{u}(t) = k(t)[r - y(t)] = k(t)[2\theta + \psi(\mathbf{G}(0)\varphi^r) - \psi(w_0(t) + w_1(t))] \,,$$

implies that

$$\begin{aligned} -k(t)(|\theta| + |\psi(w_1(t)) - \psi(w_0(t) + w_1(t))|) &\leq \dot{u}(t) - 2\theta k(t) \\ &\leq k(t)(|\theta| + |\psi(w_1(t)) - \psi(w_0(t) + w_1(t))|) \,, \quad \forall t \geq s \,. \end{aligned}$$

Since $\theta \neq 0$, either $\theta > 0$ or $\theta < 0$. Assume $\theta > 0$. Then

$$\dot{u}(t) \geq \theta k(t) - k(t)|\psi(w_1(t)) - \psi(w_0(t) + w_1(t))| \,, \quad \forall t \geq s \,,$$

which, on integration and recalling that $\psi \in \mathcal{M}_L(\lambda_2)$, yields

$$\begin{aligned} u(t) - u(s) &\geq \theta(K(t) - K(s)) - \int_s^t k(\tau)|\psi(w_1(\tau)) - \psi(w_0(\tau) + w_1(\tau))|d\tau \\ &\geq \theta(K(t) - K(s)) - \lambda_2 \int_s^t k(\tau)|w_0(\tau)|d\tau \,, \quad \forall t \geq s \,. \end{aligned}$$

By exponential stability of (32a), $w_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, and thus $w_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, which in turn implies that $kw_0 \in L^1(\mathbb{R}_+, \mathbb{R})$. Since $K(t) \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$, hence

$$\sup \overline{\Phi} = \lim_{t\to\infty} \varphi(u(t)) = \varphi^r \,. \tag{48}$$

Let $\varphi^* \in \Psi^r \cap \overline{\Phi}$. Since $\theta > 0$ and ψ is nondecreasing, it follows $\varphi^* > \varphi^r$ and, consequently, by (48), $\sup \overline{\Phi} < \varphi^*$, a contradiction. A similar argument shows that the assumption $\theta < 0$ also leads to a contradiction. Therefore, we may conclude $\varphi^r \in \Psi^r \cap \overline{\Phi}$ which is statement (b1). Statement (b2) follows from Lemma 2.4, part (b). To prove statement (b3) first note that

$$\begin{aligned} |r - y(t)| &= |r - \psi(w_0(t) + w_1(t))| = |\psi(\mathbf{G}(0)\varphi^r) - \psi(w_0(t) + w_1(t))| \\ &\leq \lambda_2|\mathbf{G}(0)\varphi^r - w_0(t) - w_1(t)| \leq \lambda_2|w_0(t)| + \lambda_2|\mathbf{G}(0)\varphi^r - w_1(t)| \,. \end{aligned} \tag{49}$$

Since, by exponential stability, $w_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$ and $w_1(t) \to \mathbf{G}(0)\varphi^r$ as $t \to \infty$, statement (b3) follows by setting $e_0(t) := \lambda_2|w_0(t)|$ and $e_1(t) := \lambda_2|\mathbf{G}(0)\varphi^r - w_1(t)|$.

If $\mathbf{T}_{t_0}(Ax^0 + B\varphi(u^0)) \in X$ for some $t_0 \geq 0$, then $\mathbf{T}_{t_0}(x^0 + A^{-1}B\varphi(u^0)) \in X_1$. Hence $w_0(t)$ converges (exponentially) to 0 as $t \to \infty$ and (b4) follows from (47) and (49).

Next, we establish statement (b5). Assume $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$ which, together with (b1), implies the existence of $\xi^* \in \mathbb{R}$ such that $\varphi^r = \varphi(\xi^*)$. Seeking a contradiction, suppose that $\operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) \not\to 0$ as $t \to \infty$. Then there exist $\varepsilon > 0$ and a sequence $(t_n) \in \mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$, such that

$$\operatorname{dist}(u(t_n), \varphi^{-1}(\varphi^r)) \geq \varepsilon\,. \tag{50}$$

If the sequence $(u(t_n))$ is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity, $\varphi(u_\infty) = \varphi^r$ and so $u_\infty \in \varphi^{-1}(\varphi^r)$. This contradicts (50). Therefore, we may assume that $(u(t_n))$ is unbounded. Extracting a subsequence if necessary, we may then assume that either $u(t_n) \to \infty$ or $u(t_n) \to -\infty$ as $n \to \infty$: if the former holds, then $u(t_n) > \xi^*$ for all n sufficiently large; if the latter holds, then $u(t_n) < \xi^*$ for all n sufficiently large. In either case, by monotonicity of φ it follows that $\varphi(u(t_n)) = \varphi(\xi^*) = \varphi^r$ for all n sufficiently large. Clearly, this contradicts (50) and so statement (b5) must hold.

To prove (b6), assume that $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$ and, for contradiction, suppose that u is unbounded. Then there exists a sequence $(t_n) \subset (0, \infty)$ with $t_n \to \infty$ and $|u(t_n)| \to \infty$ as $n \to \infty$. By monotonicity of φ and (b1), it then follows that either $\varphi^r = \sup \Phi$ or $\varphi^r = \inf \Phi$, contradicting the fact that $\varphi^r \in \Psi^r \cap \operatorname{int}(\Phi) \subset \operatorname{int}(\Phi)$. Therefore, u is bounded. This completes the proof of part B. □

4.2 Locally Lipschitz output nonlinearities

In this subsection it is shown that the global Lipschitz assumption on ψ may be relaxed to the requirement $(\varphi, \psi) \in \mathcal{N}$, provided that $x^0 \in X_1 = \operatorname{dom}(A)$ and $\mu := \mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, where $\mathcal{N}$ is the set of pairs of nonlinearities introduced in Section 3.2 and $\mathfrak{M}$ denotes the space of all finite signed Borel measures.

The proof of the following lemma can be found in the Appendix.

Lemma 4.4 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ with transfer function $\mathbf{G}$ and generating operators (A, B, C). Let $r \in \mathbb{R}$ and $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$. Assume that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, $(\varphi, \psi) \in \mathcal{N}$ and $(x^0, u^0) \in X_1 \times \mathbb{R}$. Then the initial-value problem (25) has a unique solution $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$.*

Before presenting the main results of this subsection, we describe a convenient family of projection operators. Specifically, with each $p \in [0, \infty]$, we associate an operator $\Pi_p : \mathcal{M} \to \mathcal{M}$, with the property $\Pi_p \circ \Pi_p = \Pi_p$ (whence the terminology projection operator), defined as follows:

$$\text{if} \quad p < \infty, \ \text{then} \quad \Pi_p f : \xi \mapsto \begin{cases} f(-p) & , \quad \xi < -p \\ f(\xi) & , \quad |\xi| \leq p \\ f(p) & , \quad \xi > p \end{cases} ; \qquad \text{if} \quad p = \infty, \ \text{then} \quad \Pi_p f = \Pi_\infty f := f\,.$$

Theorem 4.5 *Let $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}) \in \mathcal{S}$ with transfer function $\mathbf{G}$ and generating operators (A, B, C). Let $r \in \mathcal{R}$ and $k \in \mathcal{G}$. Assume that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, $(\varphi, \psi) \in \mathcal{N}$ and $(x^0, u^0) \in X_1 \times \mathbb{R}$. Let $(x, u) : \mathbb{R}_+ \to X \times \mathbb{R}$ be the unique solution of the initial-value problem (25). Then there exists $k^* > 0$, independent of (x^0, u^0) and k, such that the following hold.*

A. *If* $\limsup_{t\to\infty} k(t) < k^*$, *then* $\lim_{t\to\infty} \varphi(u(t))$ *exists and is finite.*

B. *If* $\limsup_{t\to\infty} k(t) < k^*$ *and* $K(t) := \int_0^t k \to \infty$ *as* $t \to \infty$, *then*

(b1) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi^r \cap \overline{\Phi}$,

(b2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\varphi^r\| = 0$,

(b3) $\lim_{t\to\infty} y(t) = r$, *where* $y(t) = \psi\left(C_L\mathbf{T}_t x^0 + (\mathbf{F}_\infty(\varphi\circ u))(t)\right)$,

(b4) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, *then* $\lim_{t\to\infty} \operatorname{dist}(u(t), \varphi^{-1}(\varphi^r)) = 0$,

(b5) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \operatorname{int}(\Phi)$, *then* $u(\cdot)$ *is bounded.*

Proof. Set $\mu := \mathfrak{L}^{-1}(\mathbf{G})$ and define

$$p := |\mu|(\mathbb{R}_+) \sup_{\xi\in\Phi} |\xi| \in (0,\infty],$$

where $|\mu|$ denotes the total variation of μ.

A. Let $\varepsilon > 0$. Since $x^0 \in X_1$ it follows form the exponential stability of $\mathbf{T}$ that there exists $t_0 \geq 0$ such that

$$|w(t)| \leq |C_L\mathbf{T}_t x^0| + |(\mu \star (\varphi\circ u))(t)| \leq \varepsilon + |\mu|(\mathbb{R}_+) \sup_{\xi\in\Phi} |\xi|, \quad \forall t \geq t_0.$$

Therefore

$$|w(t)| \leq p + \varepsilon =: q \in (0,\infty], \quad \forall t \geq t_0,$$

and hence

$$\psi(w(t)) = \Pi_q\psi(w(t)), \quad \forall t \geq t_0. \tag{51}$$

Define $\hat{\Psi}^r := \{\xi \in \mathbb{R} \,|\, \Pi_q\psi(\mathbf{G}(0)\xi) = r\}$. We claim that

$$\hat{\Psi}^r \cap \overline{\Phi} = \Psi^r \cap \overline{\Phi}. \tag{52}$$

To see this, note that $\mathbf{G}(0) \leq |\mu|(\mathbb{R}_+)$. Hence, $\mathbf{G}(0)|\xi| \leq p$ for all $\xi \in \overline{\Phi}$, and therefore, $\Pi_q\psi(\mathbf{G}(0)\xi) = \psi(\mathbf{G}(0)\xi)$ for all $\xi \in \overline{\Phi}$, which in turn implies (52).

Since $r \in \mathcal{R}$, we may conclude from (52) that $\hat{\Psi}^r \neq \emptyset$. Let $\varphi^* \in \hat{\Psi}^r$; since $\Pi_q\psi \in \mathcal{M}(\lambda_2)$ for some $\lambda_2 > 0$, the function

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \Pi_q\psi(\xi + \mathbf{G}(0)\varphi^*) - r$$

is in $\mathcal{M}(\tilde{\lambda}_2)$ for some $\tilde{\lambda}_2 > 0$ (*cf.* Remark 3.2(ii)). Therefore, by Remark 3.2(i),

$$\tilde{\psi}(\xi) = \gamma(\xi)\xi, \quad \forall \xi \in \mathbb{R},$$

where the function γ is defined by

$$\gamma(\xi) = \begin{cases} \tilde{\psi}(\xi)/\xi & , \quad \xi \neq 0 \\ \tilde{\lambda}_2 & , \quad \xi = 0 \end{cases}$$

and satisfies $0 \leq \gamma(\xi) \leq \tilde{\lambda}_2$ for all $\xi \in \mathbb{R}$. Let $\lambda_1 > 0$ be a Lipschitz constant for φ and define

$$\lambda := \lambda_1\tilde{\lambda}_2, \qquad k^* := \kappa^*/\lambda,$$

where κ^* is given by (16). Defining z and v by (29) and $\tilde{w}$ by (30), it follows by a direct calculation using (51) that the differential equations (31a) and (31b) hold for almost all $t \geq t_0$. Part A now follows in exactly the same way as the corresponding part in the proof of Theorem 4.2.

B. By part A, $\varphi^r := \lim_{t\to\infty} \varphi(u(t))$ exists and is finite. Moreover, $x^0 \in X_1$, and hence, by the exponential stability of $\mathbf{T}$ and by Theorem 6.1, part (ii) on p. 96 of [7],

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} C\mathbf{T}_t x^0 + \lim_{t\to\infty} (\mu \star (\varphi \circ u))(t) = \mathbf{G}(0)\varphi^r. \tag{53}$$

In particular, w is bounded. Combining this with the local Lipschitz property of ψ and invoking (53), Part B can be derived in a similar way as the corresponding part of Theorem 4.2. □

Remark 4.6 (i) The assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$ implies that $\mathbf{\Sigma}$ is regular and might be difficult to verify for a given well-posed system. However, it seems that it holds for most (if not all) practical examples of exponentially stable well-posed systems. In particular, if B or C is bounded, then $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$ (see Lemma 2.3 in [11]).

(ii) If φ is bounded, then, for any number $q > |\mu|(\mathbb{R}_+)\sup_{\xi\in\Phi}|\xi|$, an inspection of the proof above combined with Remark 3.2(iii) shows that k^* can be chosen as $k^* = \kappa^*/(\lambda_1\lambda_2)$, where κ^* is given by (16), λ_1 is a global Lipschitz constant for φ and λ_2 is a Lipschitz constant for ψ on $[-q, q]$. Under the extra assumption that ψ is continuously differentiable, this statement remains true for $q = |\mu|(\mathbb{R}_+)\sup_{\xi\in\Phi}|\xi|$.

(iii) Parts (iii) to (v) of Remark 4.3 apply to Theorem 4.5. ◇

5 Example: diffusion process with output delay

Consider a diffusion process (with diffusion coefficient $a > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $I = [0, 1]$, with scalar nonlinear pointwise control action (applied at point $x_b \in I$, via a nonlinearity φ with Lipschitz constant $\lambda_1 > 0$) and delayed (delay $h \geq 0$) pointwise scalar observation (output at point $x_c \in I$), with a nonlinearity ψ in the output channel. It is assumed that $x_b < x_c$. We formally write this single-input, single-output system as

$$z_t(t,x) = az_{xx}(t,x) + \delta(x - x_b)\varphi(u(t))\,, \quad w(t) = z(t-h, x_c)\,, \quad y(t) = \psi(w(t))\,,$$

$$z(t,0) = 0 = z(t,1)\,, \quad \forall t > 0\,.$$

For simplicity, we assume zero initial conditions:

$$z(t,x) = 0\,, \quad \forall (t,x) \in [-h, 0] \times [0,1]\,.$$

With input $\varphi(u(\cdot))$ and output $w(\cdot)$, this example qualifies as a well-posed (even regular) linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sh}\sinh(x_b\sqrt{(s/a)}\,)\sinh((1-x_c)\sqrt{(s/a)}\,)}{a\sqrt{(s/a)}\sinh\sqrt{(s/a)}}\,.$$

Since $\mathbf{G}(0) = x_b(1-x_c)/a$, we have, for $r \in \mathbb{R}$,

$$\Psi^r := \{v \in \mathbb{R} \mid (\mathbf{G}(0)v)^3 = r\} = \left\{\frac{ar^{1/3}}{x_b(1-x_c)}\right\}\,.$$

From [13] we know that

$$\kappa^* = \frac{1}{|\mathbf{G}'(0)|} = \frac{6a^2}{x_b(1-x_c)(6ha+1-x_b^2-(1-x_c)^2)}\,.$$

Note the dependence of κ^* on the time-delay h: the larger h, the smaller κ^*.

In the sequel, we choose as output nonlinearity $\psi \in \mathcal{M}$ the cubic $\psi : w \mapsto w^3$, and an input nonlinearity $\varphi \in \mathcal{M}_L(1)$ of saturation type, defined as follows

$$u \mapsto \varphi(u) := \begin{cases} 1, & u \geq 1 \\ u, & u \in (0,1) \\ 0, & u \leq 0 \end{cases}$$

in which case $\overline{\Phi} = [0,1]$. Thus $\Psi^r \cap \overline{\Phi} \neq \emptyset$ if and only if $0 \leq r \leq (x_b(1-x_c)/a)^3$ and so, the set of feasible reference values $\mathcal{R}$ is given by $\mathcal{R} = [0, (x_b(1-x_c)/a)^3]$.

It is not difficult to show that $\mu := \mathfrak{L}^{-1}(\mathbf{G}) \in \mathfrak{M}$, in fact, $\mu(E) = \int_E f(\tau)d\tau$ for some $f \in L^1(\mathbb{R}_+, \mathbb{R})$. Therefore, by Theorem 4.5, the integral control

$$u(t) = \int_0^t k(\tau)[r - y(\tau)]\, d\tau\,,$$

where the gain function k is chosen according to the assumptions of Theorem 4.5, guarantees asymptotic tracking of every constant reference values $r \in \mathcal{R}$.

Further analysis (invoking application of the maximum principle for the heat equation which, for brevity, we omit here) confirms the physical intuition that f is nonnegative valued. This implies $\mathbf{G}(0) = \mu(\mathbb{R}_+) = |\mu|(\mathbb{R}_+)$. Thus, with reference to Remark 4.6(ii), k^* can be chosen as $k^* = \kappa^*/(\lambda_1\lambda_2)$, where $\lambda_1 = 1$ and $\lambda_2 = 3\mathbf{G}(0)^2$.

For purposes of illustration, we adopt the following values

$$a = 0.1\,, \quad x_b = \frac{1}{3}\,, \quad x_c = \frac{2}{3}\,, \quad h = 0.5\,,$$

such that $k^* = 6561/48500 \approx 0.135$. For unit reference signal $r = 1 \in \mathcal{R}$, Fig. 2 depicts the behaviour of the system in each of the following three cases of controller gain functions k,

$$\text{(i)}\ \ k(t) = 1/(t+1)\,, \quad \text{(ii)}\ \ k(t) = \sqrt{1/(t+1)}\,, \quad \text{(iii)}\ \ k(t) \equiv 0.13\,.$$

This figure was generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 20, was adopted to model the diffusion process.

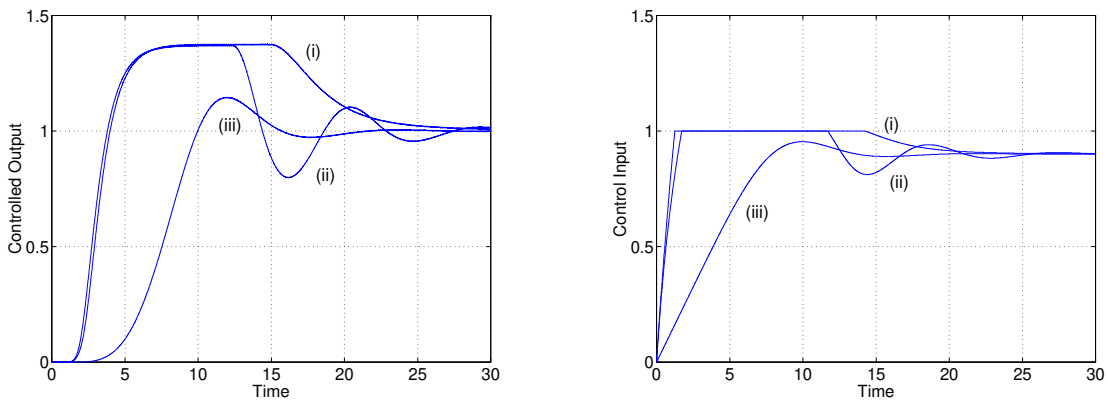

Fig. 2: Performance under low-gain control

Note that, while the constant gain case (iii) results in the best transient response, it requires structural information on the system in order to compute k^*, which may not be available in general. For gain functions $k \in \mathcal{G}$ with $k(t) \to 0$ as $t \to \infty$ and $k \notin L^1$ (as in cases (i) and (ii)), knowledge of k^* is not required.

Appendix: proof of Lemmas 4.1 and 4.4

In proving Lemmas 4.1 and 4.4, we shall apply a result from [11] on the existence of a unique maximal solution u of the following abstract Volterra integro-differential equation:

$$\dot{u}(t) = (Fu)(t)\,, \quad t \geq t_0\,, \tag{54a}$$

$$u(t) = u_{t_0}(t)\,, \quad t \in [0, t_0]\,, \tag{54b}$$

where $t_0 \geq 0$, $u_{t_0} \in C([0, t_0], \mathbb{R})$ and the operator $F : C(\mathbb{R}_+, \mathbb{R}) \to L^1_{\rm loc}(\mathbb{R}_+, \mathbb{R})$ is causal and *weakly Lipschitz* in the following sense:

For all $\alpha \geq 0$, $\delta > 0$, $\rho > 0$ and $\theta \in C([0, \alpha], \mathbb{R})$, there exists a continuous function $f : [0, \delta] \to \mathbb{R}_+$, with $f(0) = 0$, such that

$$\int_\alpha^{\alpha+\varepsilon} \|(Fv)(t) - (Fw)(t)\|\, dt \leq f(\varepsilon) \sup_{\alpha \leq t \leq \alpha+\varepsilon} \|v(t) - w(t)\|$$

for all $\varepsilon \in [0, \delta]$ and for all $v, w \in \mathcal{C}(\alpha, \delta, \rho, \theta)$, where

$$\mathcal{C}(\alpha, \delta, \rho, \theta) := \{w \in C([0, \alpha + \delta], \mathbb{R}) \mid w(t) = \theta(t)\ \forall t \in [0, \alpha], \\ \|w(t) - \theta(\alpha)\| \leq \rho\ \forall t \in [\alpha, \alpha + \delta]\}\,.$$

A *solution* of the initial-value problem (54) on an interval $[0, t_1)$, where $t_1 > t_0$, is a function $u \in C([0, t_1), \mathbb{R})$, with $u(t) = u_{t_0}(t)$ for all $t \in [0, t_0]$, such that u is absolutely continuous on $[t_0, t_1)$ and (54a) is satisfied for almost all $t \in [t_0, t_1)$.

Strictly speaking, to make sense of (54), we have to give a meaning to $(Fu)(t)$, $t \in [0, t_1)$, when u is a continuous function defined on a *finite* interval $[0, t_1)$ (recall that F operates on the space of continuous functions defined on the *infinite* interval $\mathbb{R}_+$). This can be easily done using causality of F: for all $t \in [0, t_1)$, $(Fu)(t) := (Fu^*)(t)$, where $u^* : \mathbb{R}_+ \to \mathbb{R}$ is any continuous function with $u^*(s) = u(s)$ for all $s \in [0, t]$.

The proof of the following result can be found in [11].

Proposition 5.1 *For every $t_0 \geq 0$ and every $u_{t_0} \in C([0, t_0], \mathbb{R})$, there exists a unique solution u of (54) defined on a maximal interval $[0, t_{\max})$, with $t_{\max} > t_0$. Moreover, if $t_{\max} < \infty$, then*

$$\limsup_{t \to t_{\max}} |u(t)| = \infty\,. \tag{55}$$

In the sequel we shall invoke Proposition 5.1 only in the special case $t_0 = 0$.

For $(x^0, u^0) \in X \times \mathbb{R}$, define the causal map $F : C(\mathbb{R}_+, \mathbb{R}) \to L^1_{\rm loc}(\mathbb{R}_+, \mathbb{R})$ by

$$(Fu)(t) := k(t)[r - \psi\left((\mathbf{\Psi}_\infty x^0)(t) + (\mathbf{F}_\infty(\varphi \circ u))(t)\right)]\,. \tag{56}$$

In order to proceed we need the following lemma (see [11]).

Lemma 5.2 *For all $\alpha \geq 0$, $v \in C([0,\alpha],\mathbb{R})$, $\delta > 0$ and $\rho > 0$, there exists $\gamma > 0$ such that for all $\varepsilon \in [0,\delta]$ and $u, w \in \mathcal{C}(\alpha,\delta,\rho,v)$*

$$\int_{\alpha}^{\alpha+\varepsilon} |(\mathbf{F}_\infty(\varphi\circ u))(\tau) - (\mathbf{F}_\infty(\varphi\circ w))(\tau)|\, d\tau \leq \varepsilon\gamma \sup_{\alpha\leq\tau\leq\alpha+\varepsilon} |u(\tau)-w(\tau)|\,. \tag{57}$$

It follows via a routine argument involving Lemma 5.2 that, under the assumptions of Lemma 4.1 and Lemma 4.4, respectively, the operator F as defined in (56) is also weakly Lipschitz. Hence it follows from Proposition 5.1 that the initial-value problems considered in Lemmas 4.1 and 4.4 have a unique solution u on a maximal interval of existence $[0, t_{\max})$.

To prove that $t_{\max} = \infty$, let $T \in [0, t_{\max})$ be arbitrary and let κ be such that $0 \leq |k(t)| \leq \kappa$ for all $t \geq 0$. Multiplying (25b) by u, we obtain

$$\begin{aligned} u(\tau)\dot{u}(\tau) &= u(\tau)k(\tau)\left(r - \psi[(\mathbf{\Psi}_\infty x^0)(\tau) + (\mathbf{F}_\infty(\varphi\circ u))(\tau)]\right) \\ &\leq \kappa|u(\tau)|\left(|r-\psi(0)| + |\psi[(\mathbf{\Psi}_\infty x^0)(\tau) + (\mathbf{F}_\infty(\varphi\circ u))(\tau)] - \psi(0)|\right). \end{aligned} \tag{58}$$

If the hypotheses of Lemma 4.1 hold, then $\psi \in \mathcal{M}_L(\lambda_2)$ for some $\lambda_2 \geq 0$ and

$$|\psi[(\mathbf{\Psi}_\infty x^0)(\tau) + (\mathbf{F}_\infty(\varphi\circ u))(\tau)] - \psi(0)| \leq \lambda_2|(\mathbf{\Psi}_\infty x^0)(\tau) + (\mathbf{F}_\infty(\varphi\circ u))(\tau)| \;\forall \tau \in [0,T). \tag{59}$$

Next we show that, for some $\lambda_2 \geq 0$, (59) remains valid if the hypotheses of Lemma 4.1 are replaced by the hypotheses of Lemma 4.4. Let the latter hypotheses hold and note that $(\mathbf{\Psi}_\infty x^0)(\tau)$ is bounded on $\mathbb{R}_+$. If φ is bounded, then choose $\lambda_2 \geq 0$ to be a Lipschitz constant for ψ on the interval $[-a,a]$, where

$$a := \sup_{t\in\mathbb{R}_+} |(\mathbf{\Psi}_\infty x^0)(t)| + |\mu|(\mathbb{R}_+) \sup_{\xi\in\Phi} |\xi|\,,$$

and (59) holds. If φ is unbounded, then $\psi \in \mathcal{M}(\lambda_2)$ for some $\lambda_2 \geq 0$, in which case

$$0 \leq (\psi(\xi) - \psi(0))\xi \leq \lambda_2\xi^2\,, \quad \forall \xi \in \mathbb{R},$$

and, again, (59) holds.

Combining (58), (59) and estimating we obtain that,

$$u(\tau)\dot{u}(\tau) \leq \kappa[(r-\psi(0))^2 + \lambda_2^2(\mathbf{\Psi}_\infty x^0)^2(\tau) + u^2(\tau) + \lambda_2|(\mathbf{F}_\infty\varphi(u))(\tau)u(\tau)|] \;\forall \tau \in [0,T) \tag{60}$$

for some $\lambda_2 \geq 0$. Integrating (60) and invoking the estimate

$$\int_0^t |(\mathbf{F}_\infty\varphi(u))u| \;\leq \int_0^t |\mathbf{F}_\infty(\varphi(u)-\varphi(0))|\,|u| \;+ \frac{1}{2}\left(\int_0^t (\mathbf{F}_\infty\varphi(0))^2 \;+ \int_0^t u^2\right),$$

together with the Cauchy-Schwarz inequality and the global Lipschitz property of φ, it can be readily shown that there exist positive constants β_1 and β_2 such that

$$u^2(t) \leq \beta_1 + \beta_2 \int_0^t u^2(\tau)\, d\tau\,, \quad \forall t \in [0,T)\,.$$

An application of Gronwall's lemma then shows that $u^2(t) \leq \beta_1 e^{\beta_2 t}$ for all $t \in [0,T)$. Hence u is bounded on $[0,T)$. Since this holds for all $T \in [0, t_{\max})$, it follows by Proposition 5.1 that $t_{\max} = \infty$.

Finally, to obtain a solution of (25), define

$$x(t) := \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B\varphi(u(\tau))\, d\tau\,, \quad \forall t \geq 0\,. \tag{61}$$

By well-posedness, x is a continuous X-valued function and, moreover, since A, considered as a generator on X_{-1} is in $\mathcal{B}(X, X_{-1})$, the function $t \mapsto Ax(t)$ is a continuous X_{-1}-valued function. Consequently, by Pazy [22] (Theorem 2.4, p. 107), in X_{-1} we have

$$\dot{x}(t) = Ax(t) + B\varphi(u(t))\,, \quad \forall t \in \mathbb{R}_+\,.$$

It follows that (x, u) is the unique solution of (25) defined on $\mathbb{R}_+$. □

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