Fully discrete spectral boundary integral methods on smooth closed surfaces in $\mathbb{R}^3$ *

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Abstract

In this paper we describe and analyse a class of spectral methods, based on spherical polynomial approximation, for second-kind weakly singular boundary integral equations arising from the Helmholtz equation on smooth closed 3D surfaces diffeomorphic to the sphere. Our methods are fully discrete Galerkin methods, based on the application of special quadrature rules for computing the outer and inner integrals arising in the Galerkin matrix entries. For the outer integrals we use, for example, product-Gauss rules. For the inner integrals, a variant of the classical product integration procedure is employed to remove the singularity arising in the kernel. The key to the analysis is a recent result of Sloan and Womersley on the norm of discrete orthogonal projection operators on the sphere. We prove that our methods are stable for continuous data and superalgebraically convergent for smooth data. Our theory includes as a special case a method closely related to one of those proposed by Wienert (1990) for the fast computation of direct and inverse acoustic scattering in 3D.

1 Introduction

The aim of this paper is to give a complete stability and convergence analysis of a spectral method for a class of second-kind boundary integral equations over the boundary ∂D of a smooth bounded domain $D \in \mathbb{R}^3$. Our equations will be of the form

$$v + \mathcal{M}v = f, \tag{1.1}$$

where $\mathcal{M}$ is a linear, weakly singular integral operator on ∂D:

$$\mathcal{M}v(\mathbf{x}) = \int_{\partial D} m(\mathbf{x},\mathbf{y})v(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \partial D. \tag{1.2}$$

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The kernel function $m(\mathbf{x},\mathbf{y})$ is allowed to be any linear combination of kernels arising from the classical boundary integral approach to the Helmholtz equation. More precisely, we allow kernels of the form

$$m(\mathbf{x},\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|} m_1(\mathbf{x},\mathbf{y}) + m_2(\mathbf{x},\mathbf{y}) \tag{1.3}$$

(where $|\mathbf{x}|$ denotes the Euclidean length of $\mathbf{x}$), with m_i, for $i = 1,2$, of the form

$$m_i(\mathbf{x},\mathbf{y}) = m_{i,1}(\mathbf{x},\mathbf{y}) + m_{i,2}(\mathbf{x},\mathbf{y}) \frac{(\mathbf{x}-\mathbf{y})^T \boldsymbol{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} + m_{i,3}(\mathbf{x},\mathbf{y}) \frac{(\mathbf{x}-\mathbf{y})^T \boldsymbol{n}(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}\ , \tag{1.4}$$

with each $m_{i,j}$ infinitely continuously differentiable on $\mathbb{R}^3 \times \mathbb{R}^3$, $i = 1,2$, $j = 1,2,3$, and $\boldsymbol{n}(\mathbf{y})$ denoting the unit outward normal to ∂D at the point $\mathbf{y} \in \partial D$. This includes the single and double layer potentials for the Helmholtz equation on ∂D, together with the adjoint of the double layer and any linear combination of these operators, all of which appear in classical models for direct and inverse acoustic scattering ([5]).

Throughout we will assume that the domain D is connected and has a connected complement $\mathbb{R}^3 \backslash \bar{D}$. We also assume (see (1.6) below) that the surface ∂D is globally parameterised by a smooth (i.e. infinitely continuously differentiable) map $\boldsymbol{q} : \partial B \to \partial D$, where $\partial B = \{\hat{\mathbf{x}} \in \mathbb{R}^3 : |\hat{\mathbf{x}}| = 1\}$ is the unit sphere. Although this assumption is not satisfied by all conceivable boundaries, it is commonly made in scattering problems (see, e.g. [5, §3.6]). The chief advantage of the assumption that ∂D has a global smooth parameterisation is that it allows the application of spectral methods based on spherical polynomial expansions. These have been observed to provide extremely accurate solutions with relatively few degrees of freedom [5, 9, 20], and this is the source of their popularity for inverse problems. The chief disadvantage is that not all boundaries can be handled, and in particular complicated regions with disconnected components or with corners and edges may be excluded. This, however, is considered not to be a major impediment in inverse scattering [5], where general qualitative information about the unknown scatterer is often of greater importance than relatively fine details such as corners and edges. Our analysis here provides for the first time a complete theory of a variant of one of the standard fast numerical methods for 3D scattering (e.g. [5],[20]).

Another area of application for the present results is in the implementation of coupled boundary element / finite element methods, where the coupling is to be imposed on some artificial boundary specified by the user. An example is the case of scattering by an inhomogeneous medium, where an artificial boundary may be used to enclose the inomogeneity, and finite elements are used inside and boundary elements outside the artificial boundary. The artificial boundary may be drawn to satisfy the hypotheses of this paper and the boundary integral equation solved quickly by the spectral method proposed here. In [11] such a method (applied to 2D problems) is analysed in detail.

Throughout this paper, we assume that the integral operator $I + \mathcal{M}$ is injective on the space $C(\partial D)$ of continuous functions on Γ, so that equation (1.1) is well posed on $C(\partial D)$. In applications this is verified by uniqueness results for the underlying PDE.

To introduce our spectral method for equation (1.1), we first use the change of variables $\mathbf{x} = \boldsymbol{q}(\hat{\mathbf{x}})$, to transform to the domain $\hat{\mathbf{x}} \in \partial B$. For any integrable function g

defined on ∂D, we can write

$$\int_{\partial D} g(\mathbf{x}) ds(\mathbf{x}) = \int_{\partial B} g(\boldsymbol{q}(\hat{\mathbf{x}})) J(\hat{\mathbf{x}}) ds(\hat{\mathbf{x}}) \ , \tag{1.5}$$

where J is the Jacobian of $\boldsymbol{q}$. The requirement that $\boldsymbol{q}$ should be a global parametrisation of ∂D can be conveniently expressed as the assumption:

$$J(\hat{\mathbf{x}}) > 0 \ , \tag{1.6}$$

which we shall make throughout the paper.

Now, defining

$$V(\hat{\mathbf{x}}) = J(\hat{\mathbf{x}})^{1/2} v(\boldsymbol{q}(\hat{\mathbf{x}})), \quad F(\hat{\mathbf{x}}) = J(\hat{\mathbf{x}})^{1/2} f(\boldsymbol{q}(\hat{\mathbf{x}})), \quad \hat{\mathbf{x}} \in \partial B,$$

we can rewrite equation (1.1) as

$$V + \mathcal{M}V = F, \tag{1.7}$$

where

$$\mathcal{M}V(\hat{\mathbf{x}}) = \int_{\partial B} M(\hat{\mathbf{x}}, \hat{\mathbf{y}}) V(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}), \quad \hat{\mathbf{x}} \in \partial B, \tag{1.8}$$

and

$$M(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = J(\hat{\mathbf{x}})^{1/2} m(\boldsymbol{q}(\hat{\mathbf{x}}), \boldsymbol{q}(\hat{\mathbf{y}})) J(\hat{\mathbf{y}})^{1/2} \ .$$

This formulation has the advantage that if the original kernel m is symmetric, then so is the transformed kernel M. Moreover the corresponding inner products on ∂B and ∂D are related by:

$$\int_{\partial D} v(\mathbf{x}) \overline{w(\mathbf{x})} ds(\mathbf{x}) = \int_{\partial B} V(\hat{\mathbf{x}}) \overline{W(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}) =: (V, W), \tag{1.9}$$

when W is related to w as V is related to v above .

The equation (1.7) is now ready for the spectral approximation. Let $\mathbb{P}_n$ denote the space of spherical polynomials of degree at most n on ∂B. A convenient basis for $\mathbb{P}_n$ is the set of orthonormal spherical harmonics

$$Y_{l,k}(\hat{\mathbf{x}}) = c_l^k P_l^{|k|}(\cos\theta) \exp ik\phi, \quad 0 \le l \le n, \quad |k| \le l, \tag{1.10}$$

where we have employed spherical polar coordinates:

$$\hat{\mathbf{x}} = \boldsymbol{p}(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}} \ . \tag{1.11}$$

Moreover, $P_l^{|k|}$ are the associated Legendre functions and, with the phase convention of Condon and Shortley [6],

$$c_l^k = \begin{cases} (-1)^k \sqrt{\frac{2l+1}{4\pi} \frac{(l-k)!}{(l+k)!}}, & \text{if } k \ge 0 \ , \\ (-1)^k c_l^{|k|}, & \text{if } k < 0 \ . \end{cases}$$

In particular, if $k = 0$, then $Y_{l,0}(\hat{\mathbf{x}}) = \sqrt{(2l+1)/4\pi}\; P_l(\cos\theta)$, where P_l is the Legendre polynomial of degree l.

The *Galerkin method* seeks an approximation $V_n^{\mathrm{G}} \in \mathbb{P}_n$ to the solution V of (1.7), with the property

$$(V_n^{\mathrm{G}} + \mathcal{M}V_n^{\mathrm{G}}, W_n) = (F, W_n), \quad \text{for all} \;\; W_n \in \mathbb{P}_n \; . \tag{1.12}$$

Its implementation requires the calculation of double integrals:

$$(\mathcal{M}Y_{l,k}, Y_{l',k'}) = \int_{\partial B} \left(\int_{\partial B} M(\hat{\mathbf{x}}, \hat{\mathbf{y}}) Y_{l,k}(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}) \right) \overline{Y_{l',k'}(\hat{\mathbf{x}})} ds(\hat{\mathbf{x}}), \tag{1.13}$$

which usually have to be approximated by suitable quadrature rules on ∂B. In our *semidiscrete Galerkin method* we apply quadrature to the integral arising from the inner product (i.e. to the outer integral in (1.13)). Since we have assumed that ∂D is smooth, this integral has a smooth integrand. Our *fully discrete Galerkin method* is obtained by applying quadrature also in the computation of the weakly singular inner integrals $\mathcal{M}Y_{l,k}$. Here special quadrature rules are needed because of the weak singularity. Note that since the $Y_{l,k}$ have global support on ∂B, the inner integrals in (1.13) are weakly singular for all pairs of indices (l, k) and (l', k').

In this paper, the semidiscrete Galerkin method will be based on an appropriate quadrature rule on ∂B of the form:

$$\int_{\partial B} F(\hat{\mathbf{x}}) ds(\hat{\mathbf{x}}) \approx \sum_{j=1}^{m} w_j F(\hat{\mathbf{x}}_j) =: Q_m F. \tag{1.14}$$

Precise assumptions on (1.14) will be stated in §3. Two key requirements on the quadrature rule (1.14) are that it has positive weights and that it is exact for polynomials of degree $\leq 2n$:

$$w_j > 0 \; , \;\; j = 1, \ldots , m \; , \quad Q_m F = \int_{\partial B} F(\hat{\mathbf{x}}) ds(\hat{\mathbf{x}}) \; , \text{ for all } F \in \mathbb{P}_{2n} \; . \tag{1.15}$$

From (1.14) we build a discrete version of the inner product on ∂B:

$$(F, G)_m := \sum_{j=1}^{m} w_j F(\hat{\mathbf{x}}_j) \overline{G(\hat{\mathbf{x}}_j)} \; , \tag{1.16}$$

and a corresponding discrete orthogonal projection or *hyperinterpolation* operator ([16]):

$$\mathcal{L}_n W \;=\; \sum_{l=0}^{n} \sum_{|k| \leq l} (W, Y_{l,k})_m Y_{l,k} \; . \tag{1.17}$$

It is important for our later argument that (as is easily verified, using assumption (1.15)), $\mathcal{L}_n P = P$ for arbitrary $P \in \mathbb{P}_n$.

Using (1.16), the semidiscrete Galerkin approximation $V_n^{\mathrm{sd}} \in \mathbb{P}_n$ is then defined to be the solution of

$$(V_n^{\mathrm{sd}} + \mathcal{M}V_n^{\mathrm{sd}}, W_n)_m = (F, W_n)_m \; , \quad \text{for all} \quad W_n \in \mathbb{P}_n \; , \tag{1.18}$$

or equivalently

$$V_n^{\rm sd} + \mathcal{L}_n \mathcal{M} V_n^{\rm sd} = \mathcal{L}_n F \ . \tag{1.19}$$

In §4 we show that $V_n^{\rm sd}$ is well-defined and superalgebraically convergent to V (i.e. converges faster then any power of n^{-1}) in the uniform norm, as $n \to \infty$. This new result relies crucially on recent results of Sloan and Womersley [17, 18] which show, under reasonable assumptions, that $\|\mathcal{L}_n\|_\infty$ grows with order $n^{1/2}$ as $n \to \infty$. The norm of the L_2 orthogonal projection onto $\mathbb{P}_n$ has the same rate of growth and so the proof of convergence of $V_n^{\rm sd}$ follows the same lines as the proof of convergence of the true Galerkin method given by Atkinson [1].

Of greater practical significance are the results obtained in §5, which establish the same superalgebraic convergence for a fully discrete analogue of (1.19). To motivate this fully discrete version, observe that the implementation of (1.18) requires the computation of the weakly singular integrals

$$\mathcal{M}V(\hat{\mathbf{x}}) = \int_{\partial B} M(\hat{\mathbf{x}}, \hat{\mathbf{y}}) V(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}), \tag{1.20}$$

with $V = Y_{l,k}$, for $|k| \le l,\ 0 \le l \le n$ and for various $\hat{\mathbf{x}} \in \partial B$.

For the purpose of this introduction, we here describe our method for computing (1.20) for a special case, in which $m(\mathbf{x}, \mathbf{y})$ is the single-layer kernel for the Laplace operator (i.e. $m_1 \equiv 1$ and $m_2 \equiv 0$ in (1.3)) and where $\hat{\mathbf{x}} = \hat{\mathbf{n}} := (0,0,1)^T$, the north pole on ∂B. (We shall see in §5 that the method sketched here extends also to the general kernels (1.3), (1.4).) In the present special case (1.20) can be written

$$\mathcal{M}V(\hat{\mathbf{n}}) = \int_{\partial B} \frac{1}{|\hat{\mathbf{n}} - \hat{\mathbf{y}}|} R(\hat{\mathbf{n}}, \hat{\mathbf{y}}) M_1(\hat{\mathbf{n}}, \hat{\mathbf{y}}) V(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}) \ , \tag{1.21}$$

where we have expressed the integrand as a product of a simple 'radial' singularity $1/|\hat{\mathbf{n}} - \hat{\mathbf{y}}|$ and a more regular part, containing

$$\begin{aligned} R(\hat{\mathbf{x}}, \hat{\mathbf{y}}) &= \frac{|\hat{\mathbf{x}} - \hat{\mathbf{y}}|}{|\boldsymbol{q}(\hat{\mathbf{x}}) - \boldsymbol{q}(\hat{\mathbf{y}})|} \ , & (1.22)\\ \text{and } M_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) &= J(\hat{\mathbf{x}})^{1/2} m_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) J(\hat{\mathbf{y}})^{1/2} = J(\hat{\mathbf{x}})^{1/2} J(\hat{\mathbf{y}})^{1/2} \ . & (1.23) \end{aligned}$$

To approximate (1.21), recall that the spherical harmonics are eigenfunctions of the single layer potential on the sphere, and specifically that (see [5] or [7])

$$\int_{\partial B} \frac{1}{|\hat{\mathbf{x}} - \hat{\mathbf{y}}|} Y_{l,k}(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}) = \frac{4\pi}{2l+1} Y_{l,k}(\hat{\mathbf{x}}) \ , \quad \hat{\mathbf{x}} \in \partial B. \tag{1.24}$$

This observation prompted Wienert [20] to replace the product $R(\hat{\mathbf{n}}, \hat{\mathbf{y}}) M_1(\hat{\mathbf{n}}, \hat{\mathbf{y}}) V(\hat{\mathbf{y}})$ in (1.21) by its discrete orthogonal projection onto $\mathbb{P}_{n'} = \mathrm{span}\{Y_{l,k} : |k| \le l, l = 0, \ldots, n'\}$ for some n'. This, together with (1.24), yields the fully discrete approximation to the operator $\mathcal{M}$ in (1.21):

$$\begin{aligned} \mathcal{M}_{n'} V(\hat{\mathbf{n}}) &:= \int_{\partial B} \frac{1}{|\hat{\mathbf{n}} - \hat{\mathbf{y}}|} \mathcal{L}_{n'} \{ R(\hat{\mathbf{n}}, \cdot) M_1(\hat{\mathbf{n}}, \cdot) V(\cdot) \}(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}) \\ &= \sum_{l=0}^{n'} \sum_{|k| \le l} \frac{4\pi}{2l+1} (R(\hat{\mathbf{n}}, \cdot) M_1(\hat{\mathbf{n}}, \cdot) V(\cdot), Y_{l,k}(\cdot))_{m'} Y_{l,k}(\hat{\mathbf{n}}). \qquad (1.25) \end{aligned}$$

In the above we have used the following convention:

Notation 1.1 *Throughout the paper, for given positive integers n (respectively n'), our notational convention is that m (respectively m') denotes the number of points in the corresponding quadrature rule (1.14) used in the definition of $\mathcal{L}_n$ (respectively $\mathcal{L}_{n'}$). This rule is assumed to satisfy (1.15) for $F \in \mathbb{P}_{2n}$ (respectively $\mathbb{P}_{2n'}$).*

Later on we shall see that n' in (1.25) has to be taken larger than n in order to ensure stability and convergence as $n \to \infty$. Wienert used $n' = n$, but was not able to prove stability or convergence of the corresponding fully discrete method for (1.7). We shall specify n' more precisely in §5.

The approximation (1.25) is certainly a convenient formula for computation, but it is not obvious that it provides a good approximation to $\mathcal{M}V(\hat{\mathbf{n}})$. This is because $R(\hat{\mathbf{n}}, \hat{\mathbf{y}})$ can have different limits as $\hat{\mathbf{y}} \to \hat{\mathbf{n}}$ along different paths on ∂B, from which it follows that $\mathcal{L}_{n'}\{R(\hat{\mathbf{n}}, \cdot)M_1(\hat{\mathbf{n}}, \cdot)V(\cdot)\}(\hat{\mathbf{y}})$ cannot in general be a good uniform approximation to $R(\hat{\mathbf{n}}, \hat{\mathbf{y}})M_1(\hat{\mathbf{n}}, \hat{\mathbf{y}})V(\hat{\mathbf{y}})$. Nevertheless, Wienert [20] showed (and we shall see more details of this in §5) that $\mathcal{M}_{n'}V(\hat{\mathbf{n}})$ is in fact a surprisingly good approximation to $\mathcal{M}V(\hat{\mathbf{n}})$. However, Wienert's pioneering analysis lacked the ingredient required to prove the convergence of the corresponding fully discrete scheme. Here we are able to obtain such convergence results, under the assumption that $n' = (1+a)n,\ a > 0$. Our argument, which requires a detailed analysis of the approximate operator $\mathcal{M}_{n'}$ (appropriately generalised to general kernels and observation points $\hat{\mathbf{x}} \in \partial B$), depends critically on the rule (1.14) being of tensor product type (as defined in §3). These delicate results for the approximation of $MV(\hat{\mathbf{x}})$ are then used in a straightforward way to prove the stability and convergence of the corresponding fully discrete Galerkin method, using the corresponding properties for the semidiscrete case described above.

Note that the interpretation of (1.25) needs some care if the tensor-product quadrature rule $Q_{m'}$ has a quadrature point at $\hat{\mathbf{n}}$, since $R(\hat{\mathbf{n}}, \hat{\mathbf{y}})$ is discontinuous at $\hat{\mathbf{y}} = \hat{\mathbf{n}}$. There is also a theoretical problem, in that the function $R(\hat{\mathbf{n}}, \cdot)M_1(\hat{\mathbf{n}}, \cdot)V(\cdot)$ is not continuous on ∂B and thus is not strictly in the domain of $\mathcal{L}_{n'}$. We avoid these problems by changing to polar coordinates, in effect enlarging the domain of definition of $\mathcal{L}_{n'}$ without changing its norm. Details of this are given in §3.

The plan of this paper is as follows. In the next section we motivate the choice of kernel functions (1.3), (1.4) and briefly outline the application of the boundary integral equation (1.1) in inverse scattering. In §3 we describe the results of Sloan and Womersley concerning the growth of the uniform norm of $\mathcal{L}_n$ as well as our technique for enlarging the domain of $\mathcal{L}_n$. In §4 we prove the convergence of the semidiscrete Galerkin method (1.18). The fully discrete Galerkin method is formulated and analysed in §5, and then in §6 we apply the analysis to a different but related method due to Wienert [20], which has the character of a Nyström or product-integration method. Finally, in §7 we briefly consider questions relating to the implementation of the methods. An important issue is the complexity of the matrix set-up required for the execution of the methods. Following an idea of Wienert [20] (see also [5]), we show that the cost of setting up the $O(n^2) \times O(n^2)$ dense matrix for the fully discrete Galerkin method can be reduced to $O(n^5)$ operations by careful organisation. (This requires $O(n^8)$ operations *a priori*, since each entry requires a double numerical integration with $O(n^2)$ function evaluations.)

2 Applications

As mentioned in the Introduction, a prime motivation for our numerical method is in the modelling of direct and inverse 3D scattering ([4, 5]). In the classical direct problem, one is interested in computing the scattered wave u^{S} which results when a incident plane wave $u^{\mathrm{I}}(\mathbf{x}) = \exp ik\mathbf{x}\cdot\hat{\mathbf{d}}$, $\mathbf{x}\in\mathbb{R}^3$, with direction $\hat{\mathbf{d}}\in\partial B$ and wavenumber $k>0$, strikes a given body $D\subset\mathbb{R}^3$, which is assumed to satisfy the hypotheses specified in §1. Both the scattered wave u^{S} and the total wave $u = u^{\mathrm{I}} + u^{\mathrm{S}}$ satisfy the exterior Helmholtz equation

$$\triangle u + k^2 u = 0, \qquad \mathbf{x}\in\mathbb{R}^3\setminus\bar{D},$$

and, in addition, u^{S} is a "radiating solution" satisfying $\lim_{r\to\infty} r\left(\partial u^{\mathrm{S}}/\partial r - iku^{\mathrm{S}}\right) = 0$. It is often required to compute the "far field pattern" $u_\infty(\hat{\mathbf{x}},\hat{\mathbf{d}})$ of the scattered wave, which is the coefficient appearing in the asymptotic estimate

$$u^{\mathrm{S}}(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\left\{u_\infty\left(\mathbf{x}/|\mathbf{x}|,\hat{\mathbf{d}}\right) + O\left(1/|\mathbf{x}|\right)\right\}\ , \quad |\mathbf{x}|\to\infty.$$

For illustration, we restrict ourselves here to the "sound-soft" case, in which the total wave u vanishes on ∂D. In this case it can be shown (see [5]) that the far-field pattern satisfies

$$u_\infty(\hat{\mathbf{x}},\hat{\mathbf{d}}) = -\frac{1}{4\pi}\int_{\partial D}\frac{\partial u}{\partial n}(\mathbf{y})e^{-ik\hat{\mathbf{x}}.\mathbf{y}}dy\ , \tag{2.1}$$

where $(\partial u/\partial n)(\mathbf{y})$ denotes the outward normal derivative of u at $\mathbf{y}\in\partial D$. The computation of u_∞ requires the Neumann boundary data $\partial u/\partial n|_{\partial D}$ of the total wave, which may be computed by classical integral equation techniques as follows.

Let $\Phi(\mathbf{x},\mathbf{y}) = (1/4\pi)\exp(ik|\mathbf{x}-\mathbf{y}|)|\mathbf{x}-\mathbf{y}|^{-1}$ denote the fundamental solution of the Helmholtz equation, and define the single- and double-layer potentials by

$$\mathcal{S}\phi(\mathbf{x}) := 2\int_{\partial D}\Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})ds(\mathbf{y}), \quad \mathcal{K}\phi(\mathbf{x}) = 2\int_{\partial D}\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\phi(\mathbf{y})ds(\mathbf{y})\ ,$$

for $\mathbf{x}\in\partial D$. In addition, let $\mathcal{K}'$ denote the adjoint of $\mathcal{K}$. Then the required Neumann data $v = \partial u/\partial n$ can be computed as the (unique) solution of the equation

$$v + \mathcal{M}v = f, \tag{2.2}$$

where $\mathcal{M} = \mathcal{K}' - i\mathcal{S}$, and $f = 2\partial u^{\mathrm{I}}/\partial n - 2iu^{\mathrm{I}}$. The combination of $\mathcal{K}$ and $\mathcal{S}$ is employed in order to eliminate the classical problem of internal resonances ([5, page 49]). It is easily seen that the kernel of the operator $\mathcal{M}$ satisfies the assumptions (1.3),(1.4).

The *direct scattering problem* then starts with given D, k and d and computes the far field pattern u_∞ via the solution of (2.2). In other words, the direct problem involves a single **evaluation** of the far field operator F defined, for fixed k and $\hat{\mathbf{d}}$, by

$$F(\partial D) = u_\infty. \tag{2.3}$$

In the *inverse problem*, on the other hand, we seek the **solution** of (2.3), i.e., for a given far field pattern u_∞, we must determine the scatterer ∂D. This nonlinear ill-posed problem may be approached by a (regularised) Newton approach involving the solution of a sequence of linear problems. To make such an approach tractable, in [5] attention is restricted to the reconstruction of a star-shaped scatterer, i.e. $\partial D = \{r(\hat{\mathbf{x}})\hat{\mathbf{x}} : \hat{\mathbf{x}} \in \partial B\}$, with $r : \partial B \to \mathbb{R}$ a positive-valued radial function to be determined. Then r itself may also be approximated by spherical polynomials. It turns out that formulation of the linear problems for updating r in the Newton scheme requires the solution of a set of direct scattering problems with boundary conditions determined by the basis functions used in the expansion of r. These again may be reduced to the solution of an integral equation similar to (2.2), hence the core numerical task for this reconstruction process is covered by the analysis of the present paper.

The inverse scattering application is described in detail in [5], where the fundamental theory can be found. We have described it briefly here because it has informed the analysis of the present paper in two ways. First, the inverse problem makes clear why it is essential to have a fast algorithm for the forward problem (1.1); and second, it justifies the assumption of smooth surfaces ∂D: for not only is the existing theory of uniqueness and Fréchet differentiability worked out under the assumption that the surfaces have at least C^2 smoothness ([13, 10]), but also a realistic technology for iterative correction of non-smooth $3D$ surfaces does not seem to exist. (Currently, even *two*-dimensional analyses mostly assume smooth scatterers).

3 Hyperinterpolation

3.1 Assumptions and Estimates

As remarked in §1, the analysis of this paper depends crucially on properties of the hyperinterpolation operator defined by (1.17), (1.16). We shall assume throughout that the quadrature rule (1.14) used to define (1.16) satisfies (1.15). (Recall also Notation 1.1.)

We introduce the usual Banach space $C(\partial B)$ of continuous functions on ∂B, equipped with the uniform norm $\|\cdot\|_{\infty,\partial B}$ and we also use the same notation for the corresponding operator norm. The crucial property of (1.17) proved in [17] is that there exist constants c_1, $c_2 > 0$ such that

$$c_1 n^{1/2} \leq \|\mathcal{L}_n\|_{\infty,\partial B} \leq c_2 n^{1/2}. \tag{3.1}$$

In [17] (3.1) is proved for general quadrature rules (1.14) satisfying both the assumption (1.15) and a certain "quadrature regularity" property. It is now known from the subsequent work of Reimer [15] that the quadrature regularity assumption is not needed, thus the estimates (3.1) hold without restriction.

In this paper we shall restrict attention to "tensor-product" rules on ∂B, obtained by introducing polar coordinates, and then setting $z = \cos\theta$, to give

$$\int_{\partial B} F(\hat{\mathbf{x}})\, ds(\hat{\mathbf{x}}) = \int_0^{2\pi}\int_0^{\pi} (F\circ \boldsymbol{p})(\theta,\phi)\sin\theta\, d\theta d\phi = \int_0^{2\pi}\int_{-1}^{1} (F\circ\boldsymbol{p})(\cos^{-1} z,\phi) dz d\phi\ , \tag{3.2}$$

with $\boldsymbol{p}(\theta,\phi)$ defined by (1.11). The right-most integral is then approximated by a tensor product of two 1D rules, of the form

$$\int_0^{2\pi} g(\phi)d\phi \cong \sum_r \mu_r g(\phi_r) \tag{3.3}$$

and

$$\int_{-1}^{1} h(z)dz \cong \sum_s \nu_s h(z_s). \tag{3.4}$$

In this paper, for the integral (3.3) with respect to ϕ we use the $2(n+1)$-point rectangle rule with uniform mesh spacing $\pi/(n+1)$, thus

$$\mu_r = \frac{\pi}{n+1}, \quad \phi_r = \frac{r\pi}{n+1}, \qquad r = 0, \dots, 2n+1\ . \tag{3.5}$$

This ensures that (3.3) is exact for all trigonometric polynomials of degree $\leq 2n+1$. For (3.4), we allow any rule such that

(3.4) has positive weights ν_s and is exact for all polynomials of degree $\leq 2n$. (3.6)

Inserting these into (3.2) induces our rule on ∂B:

$$\int_{\partial B} F(\hat{\mathbf{x}})ds(\hat{\mathbf{x}}) \cong \sum_r \sum_s \mu_r \nu_s F(\boldsymbol{p}(\theta_s, \phi_r)), \tag{3.7}$$

with $\boldsymbol{p}(\theta, \phi)$ defined as in (1.11) and $\theta_s = \cos^{-1} z_s$. It can then be shown that (3.7) satisfies the assumption (1.15) (see e.g. [19] or [17]).

Using (3.7), the corresponding discrete inner product (1.16) is

$$(F, G)_m = \sum_r \sum_s \mu_r \nu_s F(\boldsymbol{p}(\theta_s, \phi_r))\overline{G(\boldsymbol{p}(\theta_s, \phi_r))}, \tag{3.8}$$

which is then inserted into (1.17) to define $\mathcal{L}_n$.

An economical example of a rule (3.4) satisfying (3.6) is the $(n+1)$-point Gauss rule, where $z_s,\ s = 1, \dots, n+1$, are the zeros of the Legendre polynomial of degree $n+1$, and $\nu_s,\ s = 1, \dots, n+1$, are the corresponding Gauss-Legendre weights. The hyperinterpolation operator with this rule, combined with (3.3), (3.5), was used by Wienert [20], where the sub-optimal estimate $\|\mathcal{L}_n\|_\infty = O(n^2)$ was proved (see also [5, 8]).

An approximation scheme proposed by Kushpel and Levesley [12] can be interpreted, as shown in [17], as hyperinterpolation defined using a tensor-product rule, with the approximate integral with respect to ϕ again given by (3.5), and the right-hand side of (3.4) this time taken to be:

$$\sum_{j=1}^{2n+1} \left[\frac{2}{n+1} \sin\left(\frac{j\pi}{2n+2}\right) \sum_{l=1}^{n+1} \frac{1}{2l-1} \sin\left(\frac{(2l-1)j\pi}{2n+2}\right) \right] h\left(\cos\frac{j\pi}{2n+2}\right), \tag{3.9}$$

a rule originally proposed by Fejér. For this particular case the optimal order $\|\mathcal{L}_n\|_\infty = O(n^{1/2})$ was already established in [12].

These and other rules satisfying the above assumptions are discussed in [17]. In practical calculations, it makes sense to use the above Gauss-Legendre version of (3.4), since it has a relatively small number of points, namely $m = m(n) = 2(n+1)^2$. The rule based on (3.9) has approximately twice that number of points. Neither of these rules has quadrature points at the north or south pole $\theta = 0$ or π, but there exist rules (for example the product Lobatto and Clenshaw-Curtis rules) which do have quadrature points at the poles.

Before leaving this subsection we remark that the hyperinterpolation approximation $\mathcal{L}_n f$ has also been studied in the L_2 norm. In particular, it was shown in [16] that

$$\sup \{ \|\mathcal{L}_n f\|_{2,\partial B} : f \in C(\partial B),\ \|f\|_{\infty,\partial B} \leq 1 \} = \sqrt{4\pi},$$

where $\|\cdot\|_{2,\partial B}$ denotes the L_2 norm on ∂B. Thus the norm of $\mathcal{L}_n$ as a map from $C(\partial B)$ to $L_2(\partial B)$ shows no growth at all as $n \to \infty$. This might suggest that an analysis of the equations in this paper would be easiest in the L_2 setting, instead of the continuous-function space setting used here, but at present we know of no way of carrying out a complete analysis in the L_2 setting.

As indicated in the Introduction, it may be necessary both theoretically and practically to enlarge the domain of $\mathcal{L}_n$. The next subsection describes the required extension.

3.2 An Extension of the Definition of $\mathcal{L}_n$

We introduce the rectangle

$$A := [0, \pi] \times [0, 2\pi]. \tag{3.10}$$

If $F \in C(\partial B)$, then clearly $F \circ \boldsymbol{p}$ (with $\boldsymbol{p}$ given by (1.11)) is continuous on A, i.e. $F \circ \boldsymbol{p} \in C(A)$. Introduce also a weighted inner product on A,

$$\langle f, g \rangle := \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \overline{g(\theta, \phi)} \sin\theta d\theta d\phi \ , \tag{3.11}$$

which is related to the inner product in (1.9) by

$$(F, G) = \langle F \circ \boldsymbol{p},\ G \circ \boldsymbol{p} \rangle \ .$$

Using the tools which led to the definition of (3.7) (in particular the change of variable $z = \cos\theta$ and the univariate quadrature rules (3.3), (3.4)), we now construct a weighted quadrature rule on A by:

$$\int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \sin\theta \, d\theta d\phi \cong \sum_r \sum_s \mu_r \nu_s f(\theta_s, \phi_r) \ , \tag{3.12}$$

valid for any $f \in C(A)$. This is used to construct a discrete weighted inner product of continuous functions f and g on A:

$$\langle f, g \rangle_m := \sum_r \sum_s \mu_r \nu_s f(\theta_s, \phi_r) \overline{g(\theta_s, \phi_r)}, \tag{3.13}$$

which is connected to the discrete inner product (3.8) on ∂B via

$$(F,G)_m = \langle F \circ \boldsymbol{p}, G \circ \boldsymbol{p} \rangle_m \ , \tag{3.14}$$

which holds for all F and $G \in C(\partial B)$.

The important point for our purposes is that (unlike the left-hand side), the right-hand side of (3.14) is defined even for quadrature rules with points at the north or south pole, and even for functions F and G that are continuous only on the punctured sphere $\partial B \backslash \{\hat{\mathbf{n}}, -\hat{\mathbf{n}}\}$, provided that $F \circ \boldsymbol{p}$ and $G \circ \boldsymbol{p}$ have extensions lying in $C(A)$.

We now define an analogue of $\mathcal{L}_n$ for the rectangle A:

$$\tilde{\mathcal{L}}_n f := \sum_{l=0}^{n} \sum_{|k| \le l} \langle f, Y_{l,k} \circ \boldsymbol{p} \rangle_m Y_{l,k} \circ \boldsymbol{p} \ , \quad f \in C(A) \ . \tag{3.15}$$

The connection between $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ (easily verified using (1.17) (3.14) and (3.15)), is:

$$(\mathcal{L}_n W)(\boldsymbol{p}(\theta,\phi)) = (\tilde{\mathcal{L}}_n (W \circ p))(\theta,\phi), \quad (\theta,\phi) \in A \ , \quad W \in C(\partial B) \ . \tag{3.16}$$

Note that while continuous functions on ∂B are mapped by the polar coordinate transformation to continuous functions on A, the reverse statement is in general not true. In effect (3.15) uses the polar coordinate transformation to enlarge slightly the domain of definition of the hyperinterpolation operator. It is important for our analysis that this extension does not enlarge the norm of $\mathcal{L}_n$:

Lemma 3.1

$$\|\tilde{\mathcal{L}}_n\|_{\infty,A} = \|\mathcal{L}_n\|_{\infty,\partial B} \ .$$

Proof. Using (3.15) and (3.13), we obtain, for $(\theta,\phi) \in A$, and $f \in C(A)$,

$$(\tilde{\mathcal{L}}_n f)(\theta,\phi) = \sum_r \sum_s \mu_r \nu_s f(\theta_s,\phi_r) G_n(\boldsymbol{p}(\theta_s,\phi_r), \boldsymbol{p}(\theta,\phi)),$$

where

$$G_n(\hat{\mathbf{x}}', \hat{\mathbf{x}}) := \sum_{l=0}^{n} \sum_{|k| \le l} \overline{Y_{l,k}(\hat{\mathbf{x}}')} Y_{l,k}(\hat{\mathbf{x}}), \quad \hat{\mathbf{x}}', \hat{\mathbf{x}} \in \partial B \ .$$

From this it follows by an easy argument (using the positivity of μ_r, ν_s) that

$$\begin{aligned} \|\tilde{\mathcal{L}}_n\|_{\infty,A} &= \max_{(\theta,\phi)\in A} \sum_r \sum_s \mu_r \nu_s |G_n(\boldsymbol{p}(\theta_s,\phi_r), \boldsymbol{p}(\theta,\phi))| \\ &= \max_{\hat{\mathbf{x}} \in \partial B} \sum_r \sum_s \mu_r \nu_s |G_n(\boldsymbol{p}(\theta_s,\phi_r), \hat{\mathbf{x}})| = \|\mathcal{L}_n\|_{\infty,\partial B} \ , \end{aligned}$$

in the last step using Lemma 5.1 of [17]. $\square$

Note that this lemma, together with (3.1) implies

$$c_1 n^{1/2} \le \|\tilde{\mathcal{L}}_n\|_{\infty,A} \le c_2 n^{1/2} \ , \tag{3.17}$$

where c_1 and c_2 are as given in (3.1), estimates which are useful later on.

4 The Semidiscrete Galerkin Method

The purpose of this section is to analyse the convergence of the semidiscrete Galerkin method (1.18).

From now on, for $r \geq 0$ and $\alpha \in [0,1]$, $C^{r,\alpha}(\partial B)$ will denote the space of functions whose rth derivatives are Hölder continuous with exponent α, endowed with the usual norm $\|\cdot\|_{r,\alpha,\partial B}$ ([2, p. 498]). When $\alpha = 0$, $C^{r,0}(\partial B)$ denotes the space of functions with r continuous derivatives on ∂B, also denoted $C^r(\partial B)$, and with norm $\|\cdot\|_{r,\partial B}$.

We shall make use of a result of Ragozin [14] (see also [8]), which states that, for each non-negative integer r and $\alpha \in [0,1]$, there exists a constant $C_{r,\alpha} > 0$ such that for $G \in C^{r,\alpha}(\partial B)$, there exists $P \in \mathbb{P}_n$ such that

$$\|G - P\|_{\infty,\partial B} \leq C_{r,\alpha} \frac{1}{n^{r+\alpha}} \|G\|_{r,\alpha,\partial B}. \tag{4.1}$$

Before analysing (1.18), it is useful to note that the true Galerkin approximation, see (1.12), can be defined, in analogy with (1.19), as the function satisfying

$$V_n^{\mathrm{G}} \in \mathbb{P}_n, \quad V_n^{\mathrm{G}} + \mathcal{P}_n \mathcal{M} V_n^{\mathrm{G}} = \mathcal{P}_n F, \tag{4.2}$$

with $\mathcal{P}_n W = \sum_{l=0}^{n} \sum_{|k| \leq l} (W, Y_{l,k}) Y_{l,k}$ denoting the L_2 orthogonal projection of W onto $\mathbb{P}_n$. We shall obtain the uniform convergence of both the true and the semidiscrete Galerkin approximations using the following general result.

Theorem 4.1 *Suppose $\mathcal{R}_n : C(\partial B) \to \mathbb{P}_n$ is a projection operator on $\mathbb{P}_n$, i.e. $\mathcal{R}_n$ is linear and $\mathcal{R}_n P = P$ for all $P \in \mathbb{P}_n$, and suppose $\mathcal{R}_n$ is bounded on $C(\partial B)$ and satisfies the estimate*

$$\|\mathcal{R}_n\|_{\infty,\partial B} \leq C_0 n^{1-\varepsilon} \tag{4.3}$$

for some $C_0 > 0$ independent of n and some positive number ε. Suppose also that an approximation V_n of (1.7) is defined by

$$V_n \in \mathbb{P}_n, \qquad V_n + \mathcal{R}_n \mathcal{M} V_n = \mathcal{R}_n F. \tag{4.4}$$

Then for n sufficiently large (4.4) has a unique solution $V_n \in \mathbb{P}_n$.

If, in addition, $F \in C^{r,\alpha_0}(\partial B)$ with $r \geq 0$ and $\alpha_0 \in [0,1]$, then

$$\|V - V_n\|_{\infty,\partial B} \leq C_{r,\alpha_0} \frac{1}{n^{r+\alpha_0-1+\varepsilon}} \|V\|_{r,\alpha_0,\partial B}. \tag{4.5}$$

with C_{r,α_0} independent of n.

Proof The proof follows the proof of the convergence of the Galerkin method in [1]. From (4.1) and the stated assumption on $\mathcal{R}_n$, for any appropriate $P \in \mathbb{P}_n$ we have, for arbitrary $\alpha \in (0,1]$ and $G \in C^{r,\alpha}(\partial B)$,

$$\begin{aligned} \|G - \mathcal{R}_n G\|_{\infty,\partial B} &= \|(I - \mathcal{R}_n)(G - P)\|_{\infty,\partial B} \leq (1 + \|\mathcal{R}_n\|_{\infty,\partial B}) \|G - P\|_{\infty,\partial B} \\ &\leq (1 + C_0 n^{1-\varepsilon}) \|G - P\|_{\infty,\partial B} \leq C \frac{1}{n^{r+\alpha-1+\varepsilon}} \|G\|_{r,\alpha,\partial B} \,, \end{aligned} \tag{4.6}$$

with C denoting a generic constant independent of n. Now set $r = 0$, restrict α to $\alpha \in (0,1)$, and replace G by $\mathcal{M}G$ in (4.6) with $G \in C(\partial B)$, to give

$$\|(I-\mathcal{R}_n)\mathcal{M}G\|_{\infty,\partial B} \le C\frac{1}{n^{\alpha-1+\varepsilon}}\|\mathcal{M}G\|_{0,\alpha,\partial B} \le C\frac{1}{n^{\alpha-1+\varepsilon}}\|G\|_{\infty,\partial B},$$

where in the last step we used the fact [5, Theorem 3.4] that $\mathcal{M}$ maps $C(\partial B)$ boundedly to $C^{0,\alpha}(\partial B)$.

Now, because $\alpha \in (0,1)$ is arbitrary and $\varepsilon > 0$, we can choose $\alpha > 1-\varepsilon$, thus ensuring that $\|(I-\mathcal{R}_n)\mathcal{M}\|_{\infty,\partial B} \to 0$ as $n \to \infty$. Since $(I+\mathcal{M})^{-1}$ exists on $C(\partial B)$, it follows from the Banach lemma that $(I+\mathcal{R}_n\mathcal{M})^{-1}$ also exists and is uniformly bounded on $C(\partial B)$, for n sufficiently large. Thus (4.4) has a unique solution V_n, and

$$\|V-V_n\|_{\infty,\partial B} = \|(I+\mathcal{R}_n\mathcal{M})^{-1}(V-\mathcal{R}_nV)\|_{\infty,\partial B} \le C\|V-\mathcal{R}_nV\|_{\infty,\partial B}, \tag{4.7}$$

so that (4.5) follows from (4.6) with $\alpha = \alpha_0$. □

Remark 4.2 *For the proof of the preceeding theorem, it is crucial that ϵ in the bound (4.3) be positive.*

Corollary 4.3 *The Galerkin and semidiscrete Galerkin solutions V_n^{G} and V_n^{sd} are uniquely defined for n sufficiently large, and satisfy*

$$\|V-V_n^{\mathrm{G}}\|_{\infty,\partial B} \le C_{r,\alpha_0}\frac{1}{n^{r+\alpha_0-1/2}}\|V\|_{r,\alpha_0,\partial B}, \tag{4.8}$$

$$\|V-V_n^{\mathrm{sd}}\|_{\infty,\partial B} \le C_{r,\alpha_0}\frac{1}{n^{r+\alpha_0-1/2}}\|V\|_{r,\alpha_0,\partial B}. \tag{4.9}$$

Proof The estimate (4.3) holds with $\varepsilon = 1/2$ for both $\mathcal{R}_n = \mathcal{P}_n$ (a result of Grönwall - see [1]) and for $\mathcal{R}_n = \mathcal{L}_n$ (as proved by Sloan and Womersley - see (3.1)) . □

5 The Fully Discrete Galerkin Method

5.1 Definition of the Method

In the semidiscrete approximation V_n^{sd}, defined in (1.18), it is assumed that the inner integrals of the form (1.20) can be evaluated exactly. In practice exact evaluation of such integrals is not generally possible, and a further level of approximation is needed. To compute the weakly singular integrals (1.20), we shall employ a more general version of the strategy outlined in §1. To this end we combine (1.8) and (1.3) to write, for $V \in C(\partial B)$

$$\begin{aligned}\mathcal{M}V(\hat{\mathbf{x}}) &= \int_{\partial B}\left\{\frac{1}{|\boldsymbol{q}(\hat{\mathbf{x}})-\boldsymbol{q}(\hat{\mathbf{y}})|}M_1(\hat{\mathbf{x}},\hat{\mathbf{y}})+M_2(\hat{\mathbf{x}},\hat{\mathbf{y}})\right\}V(\hat{\mathbf{y}})ds(\hat{\mathbf{y}}),\\ &= \int_{\partial B}\left\{\frac{1}{|\hat{\mathbf{x}}-\hat{\mathbf{y}}|}R(\hat{\mathbf{x}},\hat{\mathbf{y}})M_1(\hat{\mathbf{x}},\hat{\mathbf{y}})+M_2(\hat{\mathbf{x}},\hat{\mathbf{y}})\right\}V(\hat{\mathbf{y}})ds(\hat{\mathbf{y}}),\end{aligned} \tag{5.1}$$

with

$$M_i(\hat{\mathbf{x}},\hat{\mathbf{y}}) = J(\hat{\mathbf{x}})^{1/2}m_i(\boldsymbol{q}(\hat{\mathbf{x}}),\boldsymbol{q}(\hat{\mathbf{y}}))J(\hat{\mathbf{y}})^{1/2}\ , \quad i = 1,2\ , \tag{5.2}$$

and R as defined in (1.22).

To motivate our approximation to (5.1), we note that (1.21) is a special case of (5.1) in which $\hat{\mathbf{x}} = \hat{\mathbf{n}}$, $M_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = J(\hat{\mathbf{x}})^{1/2} J(\hat{\mathbf{y}})^{1/2}$ and $M_2 \equiv 0$. To generalise to (5.1) the procedure outlined in §1, we introduce a change of coordinate system on ∂B, so that $\hat{\mathbf{x}}$ becomes the north pole. To do this we introduce a 3×3 orthogonal matrix $T_{\hat{\mathbf{x}}}$ which carries $\hat{\mathbf{x}}$ to the north pole,

$$T_{\hat{\mathbf{x}}} \hat{\mathbf{x}} = (0, 0, 1)^T = \hat{\mathbf{n}}.$$

If $\hat{\mathbf{x}} = \boldsymbol{p}(\alpha, \beta)$, where $\boldsymbol{p}$ is given by (1.11), then a simple way of achieving this is to choose

$$T_{\hat{\mathbf{x}}} := P(\beta) Q(-\alpha) P(-\beta), \tag{5.3}$$

where $P(\psi)$ and $Q(\psi)$ are 3×3 matrices corresponding to positive rotations by ψ about the z axis and y axis respectively,

$$P(\psi) := \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}. \tag{5.4}$$

The matrix $T_{\hat{\mathbf{x}}}$ is then continuous in $\hat{\mathbf{x}}$.

With the help of $T_{\hat{\mathbf{x}}}$ we can define an induced linear transformation $\mathcal{T}_{\hat{\mathbf{x}}}$ on $C(\partial B)$ by

$$\mathcal{T}_{\hat{\mathbf{x}}} F(\hat{\mathbf{z}}) = F(T_{\hat{\mathbf{x}}}^{-1} \hat{\mathbf{z}}), \quad \hat{\mathbf{z}} \in \partial B, \quad F \in C(\partial B). \tag{5.5}$$

Alternatively, on writing

$$\hat{\mathbf{y}} = T_{\hat{\mathbf{x}}}^{-1} \hat{\mathbf{z}}, \tag{5.6}$$

this definition is equivalent to

$$\mathcal{T}_{\hat{\mathbf{x}}} F(T_{\hat{\mathbf{x}}} \hat{\mathbf{y}}) = F(\hat{\mathbf{y}}), \quad \hat{\mathbf{y}} \in \partial B, \quad F \in C(\partial B).$$

We also need the bivariate analogue of (5.5), also denoted $\mathcal{T}_{\hat{\mathbf{x}}}$:

$$\mathcal{T}_{\hat{\mathbf{x}}} F(\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2) = F(T_{\hat{\mathbf{x}}}^{-1} \hat{\mathbf{z}}_1, T_{\hat{\mathbf{x}}}^{-1} \hat{\mathbf{z}}_2), \quad \hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2 \in \partial B\ , \tag{5.7}$$

for any function $F \in C(\partial B \times \partial B)$. Now observe that (5.6), together with the relation $\hat{\mathbf{x}} = T_{\hat{\mathbf{x}}}^{-1} \hat{\mathbf{n}}$ and the orthogonality of $T_{\hat{\mathbf{x}}}$, imply

$$|\hat{\mathbf{x}} - \hat{\mathbf{y}}| = \left| T_{\hat{\mathbf{x}}}^{-1} (\hat{\mathbf{n}} - \hat{\mathbf{z}}) \right| = |\hat{\mathbf{n}} - \hat{\mathbf{z}}|\,. \tag{5.8}$$

Hence, on recalling that surface measure on ∂B is invariant under orthogonal transformations, we can now write (5.1) as

$$\mathcal{M} V(\hat{\mathbf{x}}) = \int_{\partial B} \left\{ \frac{1}{|\hat{\mathbf{n}} - \hat{\mathbf{z}}|} \mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}}, \hat{\mathbf{z}}) \mathcal{T}_{\hat{\mathbf{x}}} M_1(\hat{\mathbf{n}}, \hat{\mathbf{z}}) + \mathcal{T}_{\hat{\mathbf{x}}} M_2(\hat{\mathbf{n}}, \hat{\mathbf{z}}) \right\} \mathcal{T}_{\hat{\mathbf{x}}} V(\hat{\mathbf{z}}) ds(\hat{\mathbf{z}})\ . \tag{5.9}$$

The weak singularity in the first term of the integrand is now identical to that in (1.21). Proceeding analogously to (1.25), the corresponding discrete operator is taken to be

$$
\begin{aligned}
\mathcal{M}_{n'}V(\hat{\mathbf{x}}) &= \int_{\partial B} \frac{1}{|\hat{\mathbf{n}}-\hat{\mathbf{z}}|}\mathcal{L}_{n'}\left\{\mathcal{T}_{\hat{\mathbf{x}}}R(\hat{\mathbf{n}},\cdot)\mathcal{T}_{\hat{\mathbf{x}}}M_1(\hat{\mathbf{n}},\cdot)\mathcal{T}_{\hat{\mathbf{x}}}V(\cdot)\right\}(\hat{\mathbf{z}})ds(\hat{\mathbf{z}}) \\
&\quad + \int_{\partial B}\mathcal{L}_{n'}\left\{\mathcal{T}_{\hat{\mathbf{x}}}M_2(\hat{\mathbf{n}},\cdot)\mathcal{T}_{\hat{\mathbf{x}}}V(\cdot)\right\}(\hat{\mathbf{z}})ds(\hat{\mathbf{z}}) \ , \qquad (5.10) \\
&= \sum_{l=0}^{n'}\sum_{|k|\le l}\frac{4\pi}{2l+1}\left(\mathcal{T}_{\hat{\mathbf{x}}}R(\hat{\mathbf{n}},\cdot)\mathcal{T}_{\hat{\mathbf{x}}}M_1(\hat{\mathbf{n}},\cdot)\mathcal{T}_{\hat{\mathbf{x}}}V(\cdot),Y_{l,k}(\cdot)\right)_{m'}Y_{l,k}(\hat{\mathbf{n}}) \\
&\quad + (\mathcal{T}_{\hat{\mathbf{x}}}M_2(\hat{\mathbf{n}},\cdot)\mathcal{T}_{\hat{\mathbf{x}}}V(\cdot),1)_{m'} \ , \qquad (5.11)
\end{aligned}
$$

where again we used (1.17) and (1.24). Note that the last term in (5.11) is relatively simple because, after integration over ∂B, only the $\ell=0,\ k=0$ term of (1.17) survives. The first term of (5.11) may be further simplified on noting that

$$
Y_{l,k}(\hat{\mathbf{n}}) = \begin{cases} 0 & \text{if } k\neq 0 \\ \sqrt{\frac{2l+1}{4\pi}} & \text{if } k=0\ . \end{cases}
$$

There are two problems with the approximation (5.10) as it stands. At the algorithmic level, the expression in (5.11) is not well-defined if the m'-point tensor-product quadrature rule uses the north pole $\hat{\mathbf{n}}$ as a quadrature point, since $R(\hat{\mathbf{x}},\hat{\mathbf{y}})$ may be discontinuous at $\hat{\mathbf{x}}=\hat{\mathbf{y}}$ - see (1.22) - and so $\mathcal{T}_{\hat{\mathbf{x}}}R(\hat{\mathbf{n}},\hat{\mathbf{z}})$ may be discontinuous at $\hat{\mathbf{z}}=\hat{\mathbf{n}}$. Similarly, $\mathcal{T}_{\hat{\mathbf{x}}}M_i(\hat{\mathbf{n}},\hat{\mathbf{z}})$ may be discontinuous. In addition there is a theoretical problem in (5.10) in that the domain of $\mathcal{L}_{n'}$ does not include discontinuous functions on ∂B. As indicated in §3, there is a natural way of overcoming both difficulties. By transforming to polar coordinates $\hat{\mathbf{z}}=\boldsymbol{p}(\theta,\phi)$, we can rewrite (5.9) as

$$
\begin{aligned}
\mathcal{M}V(\hat{\mathbf{x}}) &= \int_0^{2\pi}\int_0^{\pi}\mathcal{T}_{\hat{\mathbf{x}}}R(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi))\mathcal{T}_{\hat{\mathbf{x}}}M_1(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi))\mathcal{T}_{\hat{\mathbf{x}}}V(\boldsymbol{p}(\theta,\phi))\cos\theta/2\ d\theta\, d\phi \\
&+ \int_0^{2\pi}\int_0^{\pi}\mathcal{T}_{\hat{\mathbf{x}}}M_2(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi))\mathcal{T}_{\hat{\mathbf{x}}}V(\boldsymbol{p}(\theta,\phi))\sin\theta\, d\theta\, d\phi\ , \qquad (5.12)
\end{aligned}
$$

where in the first term we used $|\hat{\mathbf{n}}-\boldsymbol{p}(\theta,\phi)| = 2\sin\theta/2$ for $\theta\in[0,\pi]$. Note that the adoption of polar coordinates here has eliminated the singularity in (5.9). Similarly, recalling (3.16), we see that, from (5.10), another expression for $\mathcal{M}_{n'}V$ is

$$
\begin{aligned}
&\mathcal{M}_{n'}V(\hat{\mathbf{x}}) = \\
&\int_0^{2\pi}\int_0^{\pi}\tilde{\mathcal{L}}_{n'}\{\mathcal{T}_{\hat{\mathbf{x}}}R(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))\mathcal{T}_{\hat{\mathbf{x}}}M_1(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))\mathcal{T}_{\hat{\mathbf{x}}}V(\boldsymbol{p}(\cdot))\}(\theta,\phi)\cos\theta/2\, d\theta\, d\phi \\
&+\int_0^{2\pi}\int_0^{\pi}\tilde{\mathcal{L}}_{n'}\{\mathcal{T}_{\hat{\mathbf{x}}}M_2(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))\mathcal{T}_{\hat{\mathbf{x}}}V(\boldsymbol{p}(\cdot))\}(\theta,\phi)\sin\theta\, d\theta\, d\phi\ . \qquad (5.13)
\end{aligned}
$$

As we shall see in Lemma 5.6, the right-hand side of (5.13) is well-defined, because $\mathcal{T}_{\hat{\mathbf{x}}}R(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))$ and $\mathcal{T}_{\hat{\mathbf{x}}}M_i(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))$ are continuous (indeed smooth) functions on the set A defined in (3.10). On the other hand, it follows from (3.16) that (5.13) and (5.10) are equivalent when (5.10) is well-defined.

Now that we have settled the definition of $\mathcal{M}_{n'}$, we define our *fully discrete Galerkin approximation* V_n^{fd} by

$$V_n^{\mathrm{fd}} \in \mathbb{P}_n, \qquad V_n^{\mathrm{fd}} + \mathcal{L}_n \mathcal{M}_{n'} V_n^{\mathrm{fd}} = \mathcal{L}_n F. \tag{5.14}$$

Our main result in this paper is the following theorem.

Theorem 5.1 *Let $n' = n'(n)$ satisfy*

$$(1+a)n \le n' \le bn \tag{5.15}$$

for $n \ge n_0$ and some positive constants a, b and n_0, Then for n sufficiently large, (5.14) has a unique solution $V_n^{\mathrm{fd}} \in \mathbb{P}_n$. If in addition $F \in C^{r,\alpha_0}(\partial B)$ with $r \ge 0$ and $\alpha_0 \in (0,1]$, then

$$\|V - V_n^{\mathrm{fd}}\|_{\infty,\partial B} \le C \frac{1}{n^{r+\alpha_0-1/2}} \|V\|_{r,\alpha_0,\partial B}.$$

The proof of this theorem is given in §5.3. Before this we need to give a detailed error analysis of the operator $\mathcal{M}_{n'}$ defined above.

5.2 Error analysis for the inner integral

Theorem 5.2 *Let $\mathcal{M}$ and $\mathcal{M}_{n'}$ be the operators given by (5.12) and (5.13), and assume $n' > n$. Then for $k \in \mathbb{N}$ there exists $C_k > 0$, independent of n and n', such that*

$$\|(\mathcal{M} - \mathcal{M}_{n'})V\|_{\infty,\partial B} \le C_k (n')^{1/2} \frac{\log^2(n'-n)}{(n'-n)^k} \|V\|_{\infty,\partial B} \quad , \tag{5.16}$$

for all $V \in \mathbb{P}_n$.

We shall prove this theorem at the end of this lengthy subsection, after first proving several lemmas. First, we write the exact integral operator $\mathcal{M}V$ (given in (5.12)), as

$$\mathcal{M}V(\hat{\mathbf{x}}) = \mathcal{I}_1(H_{\hat{\mathbf{x}}}) + \mathcal{I}_2(G_{\hat{\mathbf{x}}}),$$

where $\mathcal{I}_1$, $\mathcal{I}_2$ are the linear functionals on $C(A)$ defined by

$$\mathcal{I}_1(f) = \int_0^{2\pi} \int_0^{\pi} f(\theta,\phi) \cos\theta/2 \; d\theta d\phi, \quad \mathcal{I}_2(f) = \int_0^{2\pi} \int_0^{\pi} f(\theta,\phi) \sin\theta \; d\theta d\phi, \tag{5.17}$$

and

$$\left.\begin{aligned} H_{\hat{\mathbf{x}}}(\theta,\phi) &= \mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}}, \boldsymbol{p}(\theta,\phi)) \mathcal{T}_{\hat{\mathbf{x}}} M_1(\hat{\mathbf{n}}, \boldsymbol{p}(\theta,\phi)) \mathcal{T}_{\hat{\mathbf{x}}} V(\boldsymbol{p}(\theta,\phi)) \ , \\ G_{\hat{\mathbf{x}}}(\theta,\phi) &= \mathcal{T}_{\hat{\mathbf{x}}} M_2(\hat{\mathbf{n}}, \boldsymbol{p}(\theta,\phi)) \mathcal{T}_{\hat{\mathbf{x}}} V(\boldsymbol{p}(\theta,\phi)) \ . \end{aligned}\right\} \tag{5.18}$$

Note that both $H_{\hat{\mathbf{x}}}$ and $G_{\hat{\mathbf{x}}}$ depend on V, although we suppress this information in the notation. Note also that, although $H_{\hat{\mathbf{x}}}$ and $G_{\hat{\mathbf{x}}}$ are defined only on the rectangle $A = [0,\pi] \times [0, 2\pi]$, both can also be considered, through the definition (1.11) of $\boldsymbol{p}(\theta,\phi)$, to be 2π-periodic functions of both θ and ϕ.

Let C_p denote the class of continuous functions on $\mathbb{R}^2$ which are 2π-periodic in both variables. Crucial to the subsequent argument will be the following symmetry property.

Definition 5.3 *A function $W \in C_p$ is said to be* appropriate to the sphere *or* spherically appropriate *if it has the symmetry property*

$$W(\theta,\phi) = W(-\theta,\phi+\pi), \qquad \textit{for all} \quad \theta,\phi \in \mathbb{R}\ , \tag{5.19}$$

The geometrical interpretation of (5.19) is, of course, that the coordinates (θ,ϕ) and $(-\theta,\phi+\pi)$ point to the same position on the sphere. This symmetry property was discussed in [20] and studied in detail in [9] (where it was referred to as the *reflectional symmetry* property).

Remark 5.4 *It is easily seen that $\boldsymbol{p}$ defined in (1.11) has the symmetry property (5.19), and hence is spherically appropriate. So too is any function of the form $F \circ \boldsymbol{p}$, with $F \in C(\partial B)$.*

In the proof of Theorem 5.2 we shall use the spherical appropriateness of the functions (5.18), together with other ingredients, including: the fact (proved in Lemma 5.6 below) that $\mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))$ and $\mathcal{T}_{\hat{\mathbf{x}}} M_i(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))$ are *smooth* 2π-periodic function on $\mathbb{R}^2$, and hence can be well approximated by (spherically appropriate) trigonometric polynomials; the elementary fact that when $V \in \mathbb{P}_n$, $\mathcal{T}_{\hat{\mathbf{x}}} V(\boldsymbol{p}(\cdot)) \in \mathbb{P}_n$ for each $\hat{\mathbf{x}} \in \partial B$; and a certain exactness property (proved in Lemma 5.11) for the linear functionals $\mathcal{I}_1$ and $\mathcal{I}_2$.

To discuss the smoothness of $\mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))$ and $\mathcal{T}_{\hat{\mathbf{x}}} M_i(\hat{\mathbf{n}},\boldsymbol{p}(\cdot))$, we first need some notation. Let C_p^∞ denote the space of functions in C_p which are infinitely continuously differentiable and have all derivatives 2π-periodic in each variable.

Definition 5.5 *For any function $v : \partial B \times \mathbb{R}^2 \to \mathbb{R}$, we say that v is C_p^∞* uniformly on ∂B *if for all $\hat{\mathbf{x}} \in \partial B$ the function $(\theta,\phi) \mapsto v(\hat{\mathbf{x}},\theta,\phi)$ is in C_p^∞ and each partial derivative (with respect to (θ,ϕ)) is uniformly bounded with respect to $\hat{\mathbf{x}} \in \partial B$.*

Lemma 5.6 *The functions*

$$\begin{array}{ll} \text{(i)} & (\theta,\phi) \mapsto \mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi)) \\ \text{and} & \\ \text{(ii)} & (\theta,\phi) \mapsto \mathcal{T}_{\hat{\mathbf{x}}} M_i(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi))\ , \quad i = 1,2 \end{array}$$

are C_p^∞ uniformly on ∂B.

Proof The proof follows the same lines as the arguments in [9, §4] (but see also the Remark at the end of this proof). We begin by defining

$$\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi) = \boldsymbol{q}(T_{\hat{\mathbf{x}}}^{-1}\boldsymbol{p}(\theta,\phi))\ . \tag{5.20}$$

Given that $\boldsymbol{q}$ is assumed infinitely continuously differentiable, $\boldsymbol{Q}$ is easily shown to be in C_p^∞ uniformly on ∂B ([9, Lemma 4.1]) .

Consider case (i). From (1.22) and (5.7), we have, for $(\theta,\phi) \in \mathbb{R}^2$,

$$\begin{aligned} \mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi)) &= \frac{|T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{n}} - T_{\hat{\mathbf{x}}}^{-1}\boldsymbol{p}(\theta,\phi)|}{|\boldsymbol{q}(T_{\hat{\mathbf{x}}}^{-1}\hat{\mathbf{n}}) - \boldsymbol{q}(T_{\hat{\mathbf{x}}}^{-1}\boldsymbol{p}(\theta,\phi))|} = \frac{|\hat{\mathbf{n}} - \boldsymbol{p}(\theta,\phi)|}{|\boldsymbol{Q}(\hat{\mathbf{x}},0,0) - \boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)|} \\ &= \frac{2|\sin\theta/2|}{|\boldsymbol{Q}(\hat{\mathbf{x}},0,0) - \boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)|}\ , \end{aligned}$$

which is the positive square root of the function

$$\Psi(\hat{\mathbf{x}},\theta,\phi) := \frac{4\sin^2\theta/2}{|\boldsymbol{Q}(\hat{\mathbf{x}},0,0)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)|^2}\ .$$

Note that Ψ is the quotient of two non-negative functions, both of which are C_p^∞ uniformly on ∂B, and which both vanish if and only if $\theta = 2k\pi\ ,\ k\in\mathbb{Z}$. The result (i) follows provided we can show that for each fixed $\phi\in\mathbb{R}$, $\Psi(\hat{\mathbf{x}},\theta,\phi)$ is smooth as θ passes through 0. To show this, noting that $Q(\hat{\mathbf{x}},0,0) = Q(\hat{\mathbf{x}},0,\phi)$ for $\phi\in\mathbb{R}$, we write

$$\Psi(\hat{\mathbf{x}},\theta,\phi) := \frac{(\sin^2\theta/2)/(\theta/2)^2}{|\boldsymbol{Q}(\hat{\mathbf{x}},0,\phi)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)|^2/\theta^2}\ . \tag{5.21}$$

The numerator in (5.21) is infinitely continuously differentiable in θ and nonzero in a neighbourhood of $\theta = 0$. For the denominator we follow [9, p.794-795] and write

$$\boldsymbol{Q}(\hat{\mathbf{x}},0,\phi)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi) = -\theta\int_0^1 \boldsymbol{Q}_1(\hat{\mathbf{x}},t\theta,\phi)dt\ ,\quad \text{where } \boldsymbol{Q}_1 := \partial\boldsymbol{Q}/\partial\theta\ .$$

This means that the denominator in (5.21) can be written as

$$\left(\int_0^1 \boldsymbol{Q}_1(\hat{\mathbf{x}},t\theta,\phi)dt\right)^T\left(\int_0^1 \boldsymbol{Q}_1(\hat{\mathbf{x}},t\theta,\phi)dt\right)\ ,$$

which is again infinitely continuously differentiable in a neighbourhood of $\theta = 0$.

To show that this quantity does not vanish at $\theta = 0$, it is convenient to now extend the domain of definition of $\boldsymbol{q}$ to a Euclidean neighbourhood of the surface ∂B – say, for definiteness, to the annulus $\{\mathbf{x}\in\mathbb{R}^3 : 1/2 < |\mathbf{x}| < 3/2\}$. This can be done, for example, by defining $\boldsymbol{q}$ to take constant values along radial lines emanating from the origin. This allows us to discuss Euclidean derivatives of $\boldsymbol{q}$. We denote the 3×3 Jacobian matrix $(\partial q_i/\partial x_j)$ of $\boldsymbol{q}$ by $\boldsymbol{q}_{\mathbf{x}}$. Then assumption (1.6) implies

$$|\boldsymbol{q}_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\boldsymbol{v}}_{\hat{\mathbf{x}}}\times\boldsymbol{q}_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\boldsymbol{w}}_{\hat{\mathbf{x}}}| > 0 \tag{5.22}$$

where $\hat{\boldsymbol{v}}_{\hat{\mathbf{x}}}$ and $\hat{\boldsymbol{w}}_{\hat{\mathbf{x}}}$ are any orthonormal basis for the tangent plane to ∂B at $\hat{\mathbf{x}}$. It follows that $\boldsymbol{q}_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\boldsymbol{v}}_{\hat{\mathbf{x}}}\neq\mathbf{0}$ for any arbitrary unit vector $\hat{\boldsymbol{v}}$ in the tangent plane to ∂B at $\hat{\mathbf{x}}$. Now a short calculation shows that

$$\boldsymbol{Q}_1(\hat{\mathbf{x}},0,\phi) = \boldsymbol{q}_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\boldsymbol{v}}\ ,\ \text{where } \hat{\boldsymbol{v}} = T_{\hat{\mathbf{x}}}^{-1}(\cos\phi,\sin\phi,0)^T.$$

Since $\hat{\boldsymbol{v}}$ is a unit vector orthogonal to $\hat{\mathbf{x}} = T_{\hat{\mathbf{x}}}\hat{\mathbf{n}}$, it lies in the tangent plane at $\hat{\mathbf{x}}$ and so $\boldsymbol{Q}_1(\hat{\mathbf{x}},0,\phi)$ cannot vanish. It follows that $\boldsymbol{Q}_1(\hat{\mathbf{x}},\theta,\phi)$ does not vanish in some neighbourhood of $\theta = 0$, completing the proof of (i).

To prove (ii), recall (5.2) and (5.20), from which it follows that

$$\mathcal{T}_{\hat{\mathbf{x}}}M_i(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi)) = J(\hat{\mathbf{x}})^{1/2}m_i(\boldsymbol{Q}(\hat{\mathbf{x}},0,0),\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi))J(T_{\hat{\mathbf{x}}}^{-1}\boldsymbol{p}(\theta,\phi))^{1/2}.$$

Thus the required result will follow provided we show that the functions

$$m_i(\boldsymbol{Q}(\hat{\mathbf{x}},0,0),\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi))\ ,\ i=1,2$$

are C_p^∞ uniformly on ∂B. Recalling (1.4), this is true provided the functions

$$\frac{(\boldsymbol{Q}(\hat{\mathbf{x}},0,0)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi))^T\boldsymbol{n}(\boldsymbol{Q}(\hat{\mathbf{x}},0,0))}{|\boldsymbol{Q}(\hat{\mathbf{x}},0,0)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)|^2} \tag{5.23}$$

and

$$\frac{(\boldsymbol{Q}(\hat{\mathbf{x}},0,0)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi))^T\boldsymbol{n}(\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi))}{|\boldsymbol{Q}(\hat{\mathbf{x}},0,0)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)|^2} \tag{5.24}$$

are C_p^∞ uniformly on ∂B. (Recall that $\boldsymbol{n}(\mathbf{x})$ denotes the unit outward normal to the surface ∂D at $\mathbf{x}\in\partial D$, cf. (1.4)).

For brevity we consider only (5.24), the argument for (5.23) being similar. As before, the numerator and denominator in (5.23) are $\mathcal{C}_p^\infty$ uniformly on ∂B and the denominator vanishes if and only if $\theta = 2k\pi\ ,\ k\in\mathbb{Z}$. To show smoothness as θ passes through 0, we follow the argument in [9, Lemma 4.3]. First observe that

$$\boldsymbol{Q}(\hat{\mathbf{x}},0,\phi)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)=\theta^2\int_0^1\int_t^1 \boldsymbol{Q}_2(\hat{\mathbf{x}},s\theta,\phi)\ dsdt-\theta\boldsymbol{Q}_1(\hat{\mathbf{x}},\theta,\phi)\ .$$

where $\boldsymbol{Q}_2 = \partial^2\boldsymbol{Q}/\partial\theta^2$. Then since, for each $\hat{\mathbf{x}}\in\partial B$, the map $(\theta,\phi)\mapsto\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)$ for $(\theta,\phi)\in[0,\pi]\times[0,2\pi)$ is a parametrisation of ∂D, it follows that $\boldsymbol{Q}_1(\hat{\mathbf{x}},\theta,\phi)$ is orthogonal to $\boldsymbol{n}(\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi))$ for all $(\theta,\phi)\in(0,\pi)\times[0,2\pi)$. Since $\boldsymbol{Q}(\hat{\mathbf{x}},\cdot,\cdot)$ is spherically appropriate, the same property holds when $(\theta,\phi)\in(-\pi,0)\times[0,2\pi)$. Hence the numerator in (5.24) can be written

$$\theta^2\left(\int_0^1\int_t^1 \boldsymbol{Q}_2(\hat{\mathbf{x}},s\theta,\phi)dsdt\right)^T\ \boldsymbol{n}(\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi))\ ,$$

for θ in a neighbourhood of 0. Dividing top and bottom of (5.24) (with $\boldsymbol{Q}(\hat{\mathbf{x}},0,0)$ replaced by $\boldsymbol{Q}(\hat{\mathbf{x}},0,\phi)$) by θ^2 and recalling the argument for (i) yields the required result (ii). □

Remark 5.7 *In [9, Lemma 4.2] an analogous argument to that in Lemma 5.6 is used to prove that the function*

$$\frac{\sin\theta}{|\boldsymbol{Q}(\hat{\mathbf{x}},0,\phi)-\boldsymbol{Q}(\hat{\mathbf{x}},\theta,\phi)|}$$

is C^∞ on the rectangle A, with all derivatives with respect to (θ,ϕ) uniformly bounded with respect to $\hat{\mathbf{x}}\in\partial B$. For clarity, we remark here that this function is not continuous as θ passes through 0 and therefore does not belong to the space C_p^∞ defined above. The statement in [9, Lemma 4.2] that this function is in C_p^∞ is therefore incorrect. However that error has no effect on the results of [9], since only the property that this function is in C^∞ on A is needed, and that is in fact what is proved in [9, Lemma 4.2].

Motivated by the previous lemma, we now consider a space of bivariate trigonometric polynomials which is suitable for approximating the functions $\mathcal{T}_{\hat{\mathbf{x}}}R(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi))$ and $\mathcal{T}_{\hat{\mathbf{x}}}M_i(\hat{\mathbf{n}},\boldsymbol{p}(\theta,\phi))$.

Definition 5.8 *We define $\mathbb{T}_n$ to be the space of all complex-valued trigonometric polynomials $t(\theta,\phi)$ of degree $\le n$ with respect to θ and ϕ separately, and which are also spherically appropriate.*

Some elements of $\mathbb{T}_n$, if $n \geq 2$, are

$$\begin{array}{lll} \cos j\theta \cos 2\phi, & \cos j\theta \sin 2\phi, & j = 0, 1, \ldots, n, \\ \sin j\theta \cos \phi, & \sin j\theta \sin \phi, & j = 1, \ldots, n. \end{array}$$

Trigonometric polynomials *not* in $\mathbb{T}_n$ include

$$\begin{array}{lll} \sin j\theta, & & j = 1, \ldots, n, \\ \cos j\theta \cos \phi, & \cos j\theta \sin \phi, & j = 0, 1, \ldots, n, \end{array}$$

since, although they have the correct degree, they do not have the symmetry property (5.19) and therefore are not spherically appropriate.

We observe also the relation between $\mathbb{T}_n$ and the space $\mathbb{P}_n$ of spherical polynomials (introduced in §1) :

$$\{P \circ \boldsymbol{p} \ : \ P \in \mathbb{P}_n\} \subset \mathbb{T}_n. \tag{5.25}$$

The inclusion is strict in (5.25), since (for example) the function $t(\theta, \phi) = \cos\theta \sin 2\phi$ is in $\mathbb{T}_n$ for $n \geq 2$, but does not have a unique value at $\theta = 0$ (corresponding to the north pole), and so cannot be of the form $P \circ \boldsymbol{p}$ where $P \in \mathbb{P}_n$. This relation between the spaces in (5.25) is explored in some detail in [9].

The principal tool for the proof of Theorem 5.2 is Lemma 5.11 below. For this we need the technical results in Lemmas 5.9 and 5.10. To prepare for these, we first introduce the inner product

$$[f, g] = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \overline{g(\theta, \phi)} d\theta d\phi \ , \tag{5.26}$$

defined for all square-integrable functions f, g on A. Then we introduce two subspaces of $\mathbb{T}_n$:

$$\mathbb{T}_n^0 = \left\{ t \in \mathbb{T}_n \ : \ \int_0^{2\pi} t(\theta, \phi) d\phi = 0, \quad \theta \in \mathbb{R} \right\}, \tag{5.27}$$

$$\mathbb{T}_n^1 = \left\{ t \in \mathbb{T}_n \ : \ t(\theta, \phi) = t(\theta, 0), \quad (\theta, \phi) \in \mathbb{R}^2 \right\}, \tag{5.28}$$

For example, $\cos j\theta \, \cos 2\phi \in \mathbb{T}_n^0$, $\cos j\theta \in \mathbb{T}_n^1$.

Our next result establishes that $\mathbb{T}_n^0$ and $\mathbb{T}_n^1$ are orthogonal complements in $\mathbb{T}_n$ with respect to the inner product (5.26).

Lemma 5.9

$$\mathbb{T}_n^1 = \left(\mathbb{T}_n^0\right)^{\perp} \ , \tag{5.29}$$

where $(\mathbb{T}_n^0)^{\perp} = \{t \in \mathbb{T}_n \ : \ [t^0, t] = 0 \text{ for all } t^0 \in \mathbb{T}_n^0\}$. *Moreover,*

$$\mathbb{T}_n = \mathbb{T}_n^0 \oplus \mathbb{T}_n^1. \tag{5.30}$$

Proof First note that if $t^0 \in \mathbb{T}_n^0$ and $t^1 \in \mathbb{T}_n^1$ then

$$\begin{aligned}
[t^0, t^1] &= \int_0^{2\pi} \int_0^{\pi} t^0(\theta,\phi)\overline{t^1(\theta,\phi)}d\theta d\phi \\
&= \int_0^{\pi} \left\{ \int_0^{2\pi} t^0(\theta,\phi)d\phi \right\} \overline{t^1(\theta,0)}d\theta = 0, \qquad (5.31)
\end{aligned}$$

and so $\mathbb{T}_n^1 \subset (\mathbb{T}_n^0)^{\perp}$.

On the other hand, for any $t \in \mathbb{T}_n$ we can write

$$\begin{aligned}
t(\theta,\phi) &= \left\{ t(\theta,\phi) - \frac{1}{2\pi}\int_0^{2\pi} t(\theta,\phi')d\phi' \right\} + \left\{ \frac{1}{2\pi}\int_0^{2\pi} t(\theta,\phi')d\phi' \right\} \\
&=: \tilde{t}^0(\theta,\phi) + \tilde{t}^1(\theta,\phi)\ , \qquad (5.32)
\end{aligned}$$

where we note that $\tilde{t}^i \in \mathbb{T}_n^i$, $i = 0,1$. If in addition $t \in (\mathbb{T}_n^0)^{\perp}$ then

$$0 = [\tilde{t}^0, t] = [\tilde{t}^0, \tilde{t}^0] + [\tilde{t}^0, \tilde{t}^1] = [\tilde{t}^0, \tilde{t}^0],$$

where the last step uses (5.31). This implies that $\tilde{t}^0 = 0$ and hence that $t = \tilde{t}^1 \in \mathbb{T}_n^1$. Hence $(\mathbb{T}_n^0)^{\perp} \subset \mathbb{T}_n^1$, proving (5.29). Because of (5.29) the decomposition of t into a sum of elements of $\mathbb{T}_n^0$ and $\mathbb{T}_n^1$ is unique and so (5.30) follows. □

Next, we establish mapping properties of $\tilde{\mathcal{L}}_n$, defined by (3.15), on these subspaces.

Lemma 5.10 *For $n \geq 0$ we have the mapping properties*

$$\begin{array}{ll}
(i) & \tilde{\mathcal{L}}_n :\ \mathbb{T}_n^0 \to \mathbb{T}_n^0 \\
(ii) & \tilde{\mathcal{L}}_n\ t^1 = t^1, \quad \text{for all } t^1 \in \mathbb{T}_n^1.
\end{array}$$

Proof Consider any $t \in \mathbb{T}_n$. Since t is a trigonometric polynomial of degree $\leq n$ in θ and in ϕ separately, we have, for some coefficients $b_{jk} \in \mathbb{C}$,

$$t(\theta,\phi) = \sum_{|k|\leq n} \sum_{|j|\leq n} b_{jk} \exp(i(j\theta + k\phi)) = \sum_{|k|\leq n} b_k(\theta)\exp(ik\phi), \qquad (5.33)$$

where

$$b_k(\theta) := \sum_{|j|\leq n} b_{jk}\exp(ij\theta),$$

and so $b_k(\theta)$ is a trigonometric polynomial in θ of degree $\leq n$.

Thus, by the definition of $\tilde{\mathcal{L}}_n$ in (3.15), we have

$$\tilde{\mathcal{L}}_n t = \sum_{l=0}^{n} \sum_{|k|\leq l} \sum_{|k'|\leq n} \langle b_{k'}(\theta)\exp(ik'\phi), Y_{l,k}\circ \boldsymbol{p}\rangle_m Y_{l,k}\circ \boldsymbol{p}\ ,$$

and so we see from (5.25) that $\tilde{\mathcal{L}}_n t \in \mathbb{T}_n$. Thus $\tilde{\mathcal{L}}_n : \mathbb{T}_n \to \mathbb{T}_n$.

We can simplify the expression for $\tilde{\mathcal{L}}_n t$ as follows. Using (1.10), (1.11) and (3.13) we see that for all $l = 0, \ldots, n$, $|k| \le l$, and $|k'| \le n$

$$\begin{aligned}
\langle b_{k'}(\theta)\exp(ik'\phi),\, Y_{l,k}\circ\boldsymbol{p}\rangle_m &= c_l^k \sum_r \sum_s \mu_r \nu_s b_{k'}(\theta_s)\exp(ik'\phi_r) P_l^{|k|}(\cos\theta_s)\exp(-ik\phi_r) \\
&= c_l^k \left\{\sum_r \mu_r \exp i(k'-k)\phi_r\right\}\left\{\sum_s \nu_s b_{k'}(\theta_s) P_l^{|k|}(\cos\theta_s)\right\} \\
&= 0 \quad \text{when } k \neq k',
\end{aligned}$$

where the final equality follows because the functions $\{\exp ik\phi \ : \ k = -n, \cdots, n\}$ are an orthogonal set on $[0, 2\pi]$ with respect to the L_2 inner product, and hence also discretely orthogonal with respect to the rectangle rule given in (3.5), given that the rectangle rule is exact for trigonometric polynomials of degree $\le 2n+1$. Hence

$$\tilde{\mathcal{L}}_n t = \sum_{l=0}^{n} \sum_{|k|\le l} \langle b_k(\theta)\exp(ik\phi), Y_{l,k}\circ\boldsymbol{p}\rangle_m Y_{l,k}\circ\boldsymbol{p}. \tag{5.34}$$

To complete the proof we consider separately the cases $t = t^0 \in \mathbb{T}_n^0$ and $t = t^1 \in \mathbb{T}_n^1$. Each of these is assumed to be written in the form (5.33), with coefficients which we denote by $b_k^0(\theta)$ and $b_k^1(\theta)$ respectively. For the case $t = t^0$ we observe that, by the definition (5.27),

$$0 = \int_0^{2\pi} t^0(\theta,\phi)d\phi = \sum_{|k|\le n} b_k^0(\theta)\int_0^{2\pi} \exp(ik\phi)d\phi = 2\pi b_0^0(\theta), \tag{5.35}$$

Moreover, by (1.10) and the orthogonality of the set $\{\exp ik\phi\}$, we find

$$\begin{aligned}
\int_0^{2\pi}\left(\tilde{\mathcal{L}}_n t^0\right)(\theta,\phi)d\phi &= \sum_{l=0}^{n}\sum_{|k|\le l}\langle b_k^0(\theta)\exp(ik\phi), Y_{l,k}\circ\boldsymbol{p}\rangle_m \int_0^{2\pi}\left(Y_{l,k}\circ\boldsymbol{p}\right)(\theta,\phi)d\phi \\
&= 2\pi\sum_{l=0}^{n}\langle b_0^0(\theta), Y_{l,0}\circ\boldsymbol{p}\rangle_m\, Y_{l,0}\circ\boldsymbol{p} \;=\; 0\ ,
\end{aligned}$$

since by (5.35), $b_0^0(\theta) = 0$. This proves the result (i).

Now consider $t = t^1 \in \mathbb{T}_n^1$. Since this function is independent of ϕ, we must have $b_k^1(\theta) = 0$ for $k \neq 0$ in the corresponding expansion (5.33), and so by (5.34),

$$\tilde{\mathcal{L}}_n t^1 = \sum_{l=0}^{n}\langle b_0^1(\theta), Y_{l,0}\circ\boldsymbol{p}\rangle_m Y_{l,0}\circ\boldsymbol{p},$$

which is independent of ϕ, proving that $\tilde{\mathcal{L}}_n t^1 \in \mathbb{T}_n^1$.

Finally note that, since $b_0^1 = t^1 \in \mathbb{T}_n^1$, it follows that $b_0^1(\theta)$ must be spherically appropriate and, since it is independent of ϕ, it must therefore be an even trigonometric polynomial in θ of degree $\le n$ and hence must be expressible as an algebraic polynomial of degree $\le n$ in $\cos\theta$. Recalling (1.10) and the definition of the spherical harmonics, we can deduce that the product $b_0^1(\overline{Y}_{l,0}\circ\boldsymbol{p})$ is an algebraic polynomial of degree $\le 2n$ in $\cos\theta$. By (3.13) and the assumption (3.6), we can in turn conclude that

$$\langle b_0^1(\theta), Y_{l,0}\circ\boldsymbol{p}\rangle_m = \langle b_0^1(\theta), Y_{l,0}\circ\boldsymbol{p}\rangle\ ,$$

where the inner product on the right is defined by (3.11). Hence

$$\tilde{\mathcal{L}}_n t^1 = \sum_{l=0}^{n} \langle b_0^1(\theta), Y_{l,0} \circ \boldsymbol{p} \rangle \, Y_{l,0} \circ \boldsymbol{p} = t^1 \ , \tag{5.36}$$

since the sum is equivalent to the Fourier-Legendre series for the function $(x, y, z)^T \mapsto b_0^1(\cos^{-1} z)$, a polynomial of degree $\le n$. This completes the proof of (ii). □

With the help of the foregoing, we are now able to establish the following key lemma.

Lemma 5.11 *Let h be any continuous function of $\theta \in [0, \pi]$ and define the linear functional on $C(A)$ by*

$$\mathcal{I}_h(f) = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) h(\theta) d\theta d\phi \ .$$

Then, for $n \ge 0$,

$$\mathcal{I}_h\left(\tilde{\mathcal{L}}_n t\right) = \mathcal{I}_h(t), \ \textit{for all } t \in \mathbb{T}_n.$$

Proof Let $t \in \mathbb{T}_n$. By Lemma 5.9 we may write $t = t^0 + t^1$, where $t^i \in \mathbb{T}_n^i$, $i = 0, 1$. Then Lemma 5.10 gives $\tilde{\mathcal{L}}_n t^1 = t^1$ and $\tilde{\mathcal{L}}_n t^0 \in \mathbb{T}_n^0$. From the latter relation we have

$$\mathcal{I}_h\left(\tilde{\mathcal{L}}_n t^0\right) = \int_0^{\pi} \left(\int_0^{2\pi} \left(\tilde{\mathcal{L}}_n t^0\right)(\theta, \phi) d\phi \right) h(\theta) \ d\theta = 0 = \mathcal{I}_h(t^0).$$

Thus, by linearity of $\mathcal{I}_h$ and $\tilde{\mathcal{L}}_n$,

$$\mathcal{I}_h\left(\tilde{\mathcal{L}}_n t\right) = \mathcal{I}_h\left(\tilde{\mathcal{L}}_n t^0\right) + \mathcal{I}_h\left(\tilde{\mathcal{L}}_n t^1\right) = 0 + \mathcal{I}_h(t^1) = \mathcal{I}_h(t),$$

and the lemma is proved. □

Now we can return to the proof of the result stated at the start of this subsection.

Proof of Theorem 5.2

On recalling (5.12), (5.13) and (5.17) we write, for $V \in \mathbb{P}_n$,

$$(\mathcal{M} - \mathcal{M}_{n'})V(\hat{\mathbf{x}}) = \mathcal{I}_1\left((I - \tilde{\mathcal{L}}_{n'})\{r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} v_{\hat{\mathbf{x}}}\}\right) + \mathcal{I}_2\left((I - \tilde{\mathcal{L}}_{n'})\{m_{2,\hat{\mathbf{x}}} v_{\hat{\mathbf{x}}}\}\right), \tag{5.37}$$

where we have introduced the notation

$$r_{\hat{\mathbf{x}}}(\theta, \phi) := \mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}}, \boldsymbol{p}(\theta, \phi)) \ , \quad m_{i,\hat{\mathbf{x}}}(\theta, \phi) := \mathcal{T}_{\hat{\mathbf{x}}} M_i(\hat{\mathbf{n}}, \boldsymbol{p}(\theta, \phi)) \ , \quad i = 1, 2$$

and

$$v_{\hat{\mathbf{x}}}(\theta, \phi) := \mathcal{T}_{\hat{\mathbf{x}}} V(\boldsymbol{p}(\theta, \phi)).$$

We come now to the heart of the analysis. Observing that $\mathcal{T}_{\hat{\mathbf{x}}} V \in \mathbb{P}_n$ (because the space $\mathbb{P}_n$ is invariant under rotation), it follows from (5.25) that $v_{\hat{\mathbf{x}}} \in \mathbb{T}_n$. Thus using Lemma 5.11 we can write

$$\mathcal{I}_1\left((I - \tilde{\mathcal{L}}_{n'})\{r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} v_{\hat{\mathbf{x}}}\}\right) = \mathcal{I}_1\left((I - \tilde{\mathcal{L}}_{n'})\{(r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} - t_{\hat{\mathbf{x}},n'-n}) v_{\hat{\mathbf{x}}}\}\right),$$

where for each $\hat{\mathbf{x}} \in \partial B$, $t_{\hat{\mathbf{x}},n'-n}$ represents an arbitrary element of $\mathbb{T}_{n'-n}$. This follows because $t_{\hat{\mathbf{x}},n'-n} v_{\hat{\mathbf{x}}} \in \mathbb{T}_{n'}$, and because Lemma 5.11 gives $\mathcal{I}_1((I - \tilde{\mathcal{L}}_{n'})t') = 0$ for $t' \in \mathbb{T}_{n'}$.

Clearly there exists an entirely analogous expression for the second term on the right-hand side of (5.37). From these expressions combined with (3.17) we have, for each $\hat{\mathbf{x}} \in \partial B$, and arbitrary $t_{\hat{\mathbf{x}},n'-n}$ and $\tau_{\hat{\mathbf{x}},n'-n} \in \mathbb{T}_{n'-n}$,

$$\begin{aligned} & |(\mathcal{M} - \mathcal{M}_{n'})V(\hat{\mathbf{x}})| \\ \leq\ & C(n')^{1/2}\{\|r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} - t_{\hat{\mathbf{x}},n'-n}\|_{\infty,A} + \|m_{2,\hat{\mathbf{x}}} - \tau_{\hat{\mathbf{x}},n'-n}\|_{\infty,A}\}\|v_{\hat{\mathbf{x}}}\|_{\infty,A} \\ =\ & C(n')^{1/2}\{\|r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} - t_{\hat{\mathbf{x}},n'-n}\|_{\infty,A} + \|m_{2,\hat{\mathbf{x}}} - \tau_{\hat{\mathbf{x}},n'-n}\|_{\infty,A}\}\|V\|_{\infty,\partial B}. \end{aligned} \tag{5.38}$$

To complete the proof, note that (by Lemma 5.6) the functions $r_{\hat{\mathbf{x}}}$ and $m_{i,\hat{\mathbf{x}}}$ are in C_p^∞, all their derivatives are uniformly bounded with respect to $\hat{\mathbf{x}}$, and moreover they are also spherically appropriate. To approximate $r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}}$ and $m_{2,\hat{\mathbf{x}}}$ we can employ the projection operator $\mathcal{E}_n$ discussed in [9, §2]. This operator, which is nothing other than a discrete Fourier projection onto tensor-product trigonometric polynomials, has the property that its uniform norm has a bound which grows with $O(\log^2 n)$, see ([9, Lemma 2.1]). Most importantly for our purposes, $\mathcal{E}_n$ maps any spherically appropriate function into an element of $\mathbb{T}_n$ ([9, Lemma 2.2]). So if we set $t_{\hat{\mathbf{x}},n'-n} = \mathcal{E}_{n'-n}\{r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}}\}$, then we obtain

$$\begin{aligned} \|r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} - t_{\hat{\mathbf{x}},n'-n}\|_{\infty,A} &= \|r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} - \mathcal{E}_{n'-n}\{r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}}\}\|_{\infty,A} \\ &\leq C\log^2(n'-n)\|r_{\hat{\mathbf{x}}} m_{1,\hat{\mathbf{x}}} - \xi_{\hat{\mathbf{x}},n'-n}\|_{\infty,A} \end{aligned} \tag{5.39}$$

for all trigonometric polynomials $\xi_{\hat{\mathbf{x}},n'-n}$ of degree $\leq n' - n$. An entirely analogous estimate holds for $m_{2,\hat{\mathbf{x}}} - \tau_{\hat{\mathbf{x}},n'-n}$, with $\tau_{\hat{\mathbf{x}},n'-n} = \mathcal{E}_{n'-n}\{m_{2,\hat{\mathbf{x}}}\}$. In view of the regularity of $r_{\hat{\mathbf{x}}}$ and $m_{i,\hat{\mathbf{x}}}$ and the usual approximation properties of trigonometric polynomials, we can combine (5.39) with (5.38) to obtain, for arbitrary $k \geq 0$,

$$|(\mathcal{M} - \mathcal{M}_{n'})V(\hat{\mathbf{x}})| \leq C_k (n')^{1/2} \frac{\log^2(n'-n)}{(n'-n)^k}\|V\|_{\infty,\partial B}.$$

This estimate is uniform in $\hat{\mathbf{x}} \in \partial B$, yielding the theorem. □

The following corollary follows easily, with the aid of (3.1) .

Corollary 5.12 *Let $\mathcal{M}$ and $\mathcal{M}_{n'}$ be the integral operators given by (5.12) and (5.13), and assume $n' > n$. Then for $k \in \mathbb{N}$ there exists $C_k > 0$, independent of n and n', such that*

$$\|(\mathcal{M} - \mathcal{M}_{n'})\mathcal{L}_n\|_{\infty,\partial B} \leq C_k (nn')^{1/2} \frac{\log^2(n'-n)}{(n'-n)^k}. \tag{5.40}$$

Proof From Theorem 5.2 and (3.1) we have

$$\|(\mathcal{M} - \mathcal{M}_{n'})\mathcal{L}_n V\|_{\infty,\partial B} \leq C_k (n')^{1/2} (\log^2(n'-n))(n'-n)^{-k} n^{1/2}\|V\|_{\infty,\partial B}, \quad V \in C(\partial B),$$

which is equivalent to the stated result. □

Finally for this section, we need the following lemma, which provides a bound on the growth of $\mathcal{M}_{n'}$.

Lemma 5.13

$$\|\mathcal{M}_{n'}\|_{\infty,\partial B} \le C(n')^{1/2}, \quad n' \in \mathbb{N}. \tag{5.41}$$

Proof For simplicity we give the proof under the assumptions that $M_1 \equiv 1$ and $M_2 \equiv 0$ in (5.12). The general case is no more difficult. From (5.13) and (3.17) we then have, for $\hat{\mathbf{x}} \in \partial B$ and C denoting a generic constant,

$$\begin{aligned}
|\mathcal{M}_{n'} V(\hat{\mathbf{x}})| &\le C\|\tilde{\mathcal{L}}_{n'}\{\mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}}, \boldsymbol{p}(\cdot))\mathcal{T}_{\hat{\mathbf{x}}} V(\boldsymbol{p}(\cdot))\}\|_{\infty,A} \\
&\le C(n')^{1/2}\|\mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}}, \boldsymbol{p}(\cdot))\mathcal{T}_{\hat{\mathbf{x}}} V(\boldsymbol{p}(\cdot))\|_{\infty,A} \\
&\le C(n')^{1/2}\|\mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}}, \boldsymbol{p}(\cdot))\|_{\infty,A} \max_{\hat{\mathbf{z}} \in \partial B} |\mathcal{T}_{\hat{\mathbf{x}}} V(\hat{\mathbf{z}})| \\
&\le C(n')^{1/2}\|V\|_{\infty,\partial B},
\end{aligned}$$

where the last inequality follows from Lemma 5.6, and the result follows. □

5.3 Proof of Theorem 5.1

To prove Theorem 5.1 , first we prove that there exists a unique solution to (5.14). To show this, note first that the proof of Theorem 4.1 (with $\mathcal{R}_n = \mathcal{L}_n$) shows that $(I + \mathcal{L}_n\mathcal{M})^{-1}$ exists and satisfies $\|(I + \mathcal{L}_n\mathcal{M})^{-1}\|_{\infty,\partial B} \le C$, for n sufficiently large. The readily verified identity

$$(I + \mathcal{M}\mathcal{L}_n)^{-1} = I - \mathcal{M}(I + \mathcal{L}_n\mathcal{M})^{-1}\mathcal{L}_n \tag{5.42}$$

then shows that $I + \mathcal{M}\mathcal{L}_n$ is invertible for n sufficiently large, and moreover (using the bound for $\|\mathcal{L}_n\|_{\infty,\partial B}$ in (3.1)) that

$$\|(I + \mathcal{M}\mathcal{L}_n)^{-1}\|_{\infty,\partial B} \le Cn^{1/2}. \tag{5.43}$$

This together with (5.40) (choosing $k \ge 2$ and with n' related to n by (5.15)), gives

$$\|(\mathcal{M} - \mathcal{M}_{n'})\mathcal{L}_n\|_{\infty,\partial B}\|(I + \mathcal{M}\mathcal{L}_n)^{-1}\|_{\infty,\partial B} \to 0 \quad \text{as } n \to \infty\ ,$$

and so, by the Banach lemma, $(I+\mathcal{M}_{n'}\mathcal{L}_n)^{-1}$ exists for n sufficiently large, and satisfies

$$\begin{aligned}
\|(I + \mathcal{M}_{n'}\mathcal{L}_n)^{-1}\|_{\infty,\partial B} &= \|(I + \mathcal{M}\mathcal{L}_n)^{-1}[I + (\mathcal{M}_{n'} - \mathcal{M})\mathcal{L}_n(I + \mathcal{M}\mathcal{L}_n)^{-1}]^{-1}\|_{\infty,B} \\
&\le C\|(I + \mathcal{M}\mathcal{L}_n)^{-1}\|_{\infty,\partial B} \le Cn^{1/2}.
\end{aligned} \tag{5.44}$$

The identity

$$(I + \mathcal{L}_n\mathcal{M}_{n'})^{-1} = I - \mathcal{L}_n(I + \mathcal{M}_{n'}\mathcal{L}_n)^{-1}\mathcal{M}_{n'} \tag{5.45}$$

now shows that $(I + \mathcal{L}_n\mathcal{M}_{n'})$ is invertible, and hence that (5.14) has a unique solution $V_n^{\rm fd}$ for n sufficiently large.

To estimate the error in $V_n^{\rm fd}$, observe that

$$\begin{aligned}
V_n^{\rm fd} - V_n^{\rm sd} &= (I + \mathcal{L}_n\mathcal{M}_{n'})^{-1}\{\mathcal{L}_n F - V_n^{\rm sd} - \mathcal{L}_n\mathcal{M}_{n'}V_n^{\rm sd}\} \\
&= (I + \mathcal{L}_n\mathcal{M}_{n'})^{-1}\mathcal{L}_n(\mathcal{M} - \mathcal{M}_{n'})V_n^{\rm sd} \\
&= \mathcal{L}_n(I + \mathcal{M}_{n'}\mathcal{L}_n)^{-1}(\mathcal{M} - \mathcal{M}_{n'})V_n^{\rm sd},
\end{aligned}$$

where the last step follows from the identity $(I + AB)^{-1}A = A(I + BA)^{-1}$. Hence, using (3.1) and (5.44), and the fact that $V_n^{\mathrm{sd}} = \mathcal{L}_n V_n^{\mathrm{sd}}$, we have

$$\begin{aligned}
\|V_n^{\mathrm{fd}} - V_n^{\mathrm{sd}}\|_{\infty,\partial B} &\leq Cn\|(\mathcal{M} - \mathcal{M}_{n'})\mathcal{L}_n\|_{\infty,\partial B}\|V_n^{\mathrm{sd}}\|_{\infty,\partial B} \\
&\leq Cn\|(\mathcal{M} - \mathcal{M}_{n'})\mathcal{L}_n\|_{\infty,\partial B}\|V\|_{\infty,\partial B},
\end{aligned}$$

where in the last step we used the stability of the semidiscrete Galerkin method following from Corollary 4.3. Hence using Corollary 5.12 with (5.15) we have

$$\|V_n^{\mathrm{fd}} - V_n^{\mathrm{sd}}\|_{\infty,\partial B} \leq C_k \frac{\log^2 n}{n^{k-2}} \|V\|_{\infty,\partial B} \tag{5.46}$$

for all $k \geq 2$.

Since k in (5.46) is arbitrary, we may choose $k > r + \alpha_0 + 3/2$. Then the bound (5.46) combined with Corollary 4.3 yields

$$\begin{aligned}
\|V - V_n^{\mathrm{fd}}\|_{\infty,\partial B} &\leq \|V - V_n^{\mathrm{sd}}\|_{\infty,\partial B} + \|V_n^{\mathrm{fd}} - V_n^{\mathrm{sd}}\|_{\infty,\partial B} \\
&\leq C\frac{1}{n^{r+\alpha_0-1/2}}\|V\|_{r,\alpha_0,\partial B}. \qquad (5.47)
\end{aligned}$$

This completes the proof of Theorem 5.1. □

6 Wienert's variant

Instead of the fully discrete Galerkin approximation V_n^{fd} satisfying (5.14), Wienert [20] (see also [5]) considered the solution $V_n^{\mathrm{w}} \in C(\partial B)$ of the related equation

$$V_n^{\mathrm{w}} + \mathcal{M}_{n'}\mathcal{L}_n V_n^{\mathrm{w}} = F. \tag{6.1}$$

On comparing to equation (5.14) for V_n^{fd}, namely

$$V_n^{\mathrm{fd}} + \mathcal{L}_n\mathcal{M}_{n'}V_n^{\mathrm{fd}} = \mathcal{L}_n F,$$

we observe immediately a one-to-one correspondence between the solutions to these two equations, namely

$$\begin{aligned}
V_n^{\mathrm{fd}} &= \mathcal{L}_n V_n^{\mathrm{w}}, \\
V_n^{\mathrm{w}} &= F - \mathcal{M}_{n'}V_n^{\mathrm{fd}},
\end{aligned}$$

so that each of these equations has a (unique) solution if and only if the other does also.

Theorem 6.1 *Let n' and n_0 be as in Theorem 5.1. Then for n sufficiently large, (6.1) has a unique solution $V_n^{\mathrm{w}} \in C(\partial B)$. If in addition $F \in C^{r,\alpha_0}(\partial B)$ with $r \geq 0$ and $\alpha_0 \in (0,1]$, then*

$$\|V - V_n^{\mathrm{w}}\|_{\infty,\partial B} \leq C\frac{1}{n^{r+\alpha_0-1}}\|V\|_{r,\alpha_0,\partial B}.$$

Proof The one-to-one correspondence noted above ensures existence and uniqueness under exactly the same conditions as for V_n^{fd}, see Theorem 5.1. From (6.1) and the corresponding exact equation we have

$$\begin{aligned} V - V_n^{\mathrm{w}} &= (I+\mathcal{M})^{-1}F - (I+\mathcal{M}_{n'}\mathcal{L}_n)^{-1}F \\ &= (I+\mathcal{M}_{n'}\mathcal{L}_n)^{-1}(\mathcal{M}_{n'}\mathcal{L}_n - \mathcal{M})V, \end{aligned}$$

so that

$$\begin{aligned} \|V - V_n^{\mathrm{w}}\|_{\infty,\partial B} &\leq \|(I+\mathcal{M}_{n'}\mathcal{L}_n)^{-1}\|_{\infty,\partial B}\Big(\|(\mathcal{M}_{n'}-\mathcal{M})\mathcal{L}_n\|_{\infty,\partial B}\|V\|_{\infty,\partial B} \\ &\qquad + \|\mathcal{M}\|_{\infty,\partial B}\|\mathcal{L}_n V - V\|_{\infty,\partial B}\Big), \end{aligned}$$

and the result follows from (5.44) combined with (3.1), (4.1) and Corollary 5.12 (with $k > r + \alpha_0 + 1/2$). □

Remark 6.2 *To be more precise, Wienert's method is (6.1) in the particular case $n' = n$. The methods developed here prove convergence only when n' is chosen as in Theorem 5.1; the convergence for the case $n' = n$ remains an open question. This question is of limited practical importance, however, since the method which is proved here to converge is of the same complexity as Wienert's method, as we shall now show.*

7 Implementation

In this section we consider the implementation of both the fully discrete method (5.14) and the variant (6.1). The treatment is related to that of [20] and [5]. To keep the formulae as simple as possible, we shall here restrict to the case when $M_1 \equiv 1$ and $M_2 \equiv 0$ in (5.12). The general case is entirely analogous.

First we need to write the n'-point quadrature rule in the discrete inner product $(\cdot,\cdot)_{m'}$ more explicitly. By analogy with (3.8), we write

$$(F,G)_{m'} = \sum_r \sum_s \xi_r \eta_s F(\boldsymbol{p}(\Theta_s,\Phi_r))\overline{G(\boldsymbol{p}(\Theta_s,\Phi_r))}, \tag{7.1}$$

where

$$\xi_r = \frac{\pi}{n'+1},\ \Phi_r = \frac{r\pi}{n'+1}, \quad r = 0,\dots,2n'+1,$$

and where (cf. (3.4)) $\cos\Theta_s$ and η_s are the points and weights of a positive-weight quadrature rule on $[-1,1]$ that is exact for all polynomials of degree $\leq 2n'$; for example, the $(n'+1)$-point Gauss-Legendre rule.

We see that $\mathcal{M}_{n'}V(\hat{\mathbf{x}})$ is, in effect, obtained by applying a special quadrature rule (depending on $\hat{\mathbf{x}}$) to the integral $\mathcal{M}V(\hat{\mathbf{x}})$: for with the aid of (7.1) and the addition theorem for spherical harmonics, namely

$$\sum_{|k|\leq l} \overline{Y_{l,k}(\hat{\mathbf{z}})}Y_{l,k}(\hat{\mathbf{y}}) = \frac{2l+1}{4\pi}P_l(\hat{\mathbf{z}}\cdot\hat{\mathbf{y}}), \qquad \hat{\mathbf{z}},\hat{\mathbf{y}} \in \partial B,\ l \geq 0,$$

equation (5.11) for $\mathcal{M}_{n'}V(\hat{\mathbf{x}})$ can be written as

$$\mathcal{M}_{n'}V(\hat{\mathbf{x}}) \;\; = \sum_r \sum_s \xi_r \tilde{\eta}_s \mathcal{T}_{\hat{\mathbf{x}}} R(\hat{\mathbf{n}}, \boldsymbol{p}(\Theta_s, \Phi_r)) \mathcal{T}_{\hat{\mathbf{x}}} V(\boldsymbol{p}(\Theta_s, \Phi_r)) \;, \tag{7.2}$$

where

$$\tilde{\eta}_s := \eta_s \sum_{l=0}^{n'} P_l(\cos\Theta_s).$$

The fully discrete approximation (5.14) is equivalent to (cf. (1.18),(1.19))

$$V_n^{\mathrm{fd}} \in \mathbb{P}_n, \quad (V_n^{\mathrm{fd}}, W_n)_m + (\mathcal{M}_{n'}V_n^{\mathrm{fd}}, W_n)_m = (F, W_n)_m \quad \text{for all } W_n \in \mathbb{P}_n.$$

Thus on setting

$$V_n^{\mathrm{fd}} = \sum_{l=0}^{n} \sum_{|k|\le l} v_{lk} Y_{l,k},$$

and choosing $W_n = Y_{l',k'}$ for $l' = 0, \dots, n,\ |k'| \le l'$, we obtain, with the aid of (3.8), the explicit linear system

$$\sum_{l=0}^{n} \sum_{|k|\le l} [\delta_{l'l}\delta_{k'k} + \mathbf{M}_{l'k',lk}] v_{lk} = (F, Y_{l',k'})_m \quad \text{for } l' = 0, \dots, n,\ |k'| \le l',$$

where

$$\begin{aligned} \mathbf{M}_{l'k',lk} &= (\mathcal{M}_{n'}Y_{l,k}, Y_{l',k'})_m \\ &= \sum_{r'} \sum_{s'} \mu_{r'}\nu_{s'} \mathcal{M}_{n'} Y_{l,k}(\boldsymbol{p}(\theta_{s'}, \phi_{r'})) \overline{Y_{l',k'}(\boldsymbol{p}(\theta_{s'}, \phi_{r'}))} \\ &= \sum_{r'} \sum_{s'} \mu_{r'}\nu_{s'} \sum_r \sum_s \xi_r \tilde{\eta}_s \mathcal{T}_{\boldsymbol{p}(\theta_{s'},\phi_{r'})} R(\hat{\mathbf{n}}, \boldsymbol{p}(\Theta_s, \Phi_r)) \\ &\quad \times \mathcal{T}_{\boldsymbol{p}(\theta_{s'},\phi_{r'})} Y_{l,k}(\boldsymbol{p}(\Theta_s, \Phi_r)) \overline{Y_{l',k'}(\boldsymbol{p}(\theta_{s'}, \phi_{r'}))}, \end{aligned} \tag{7.3}$$

and where we used, (3.8) and (7.2).

The efficient computation of the matrix $\mathbf{M}$ as given by (7.3) is complicated by the appearance of $\mathcal{T}_{\boldsymbol{p}(\theta_{s'},\phi_{r'})} Y_{l,k}(\boldsymbol{p}(\Theta_s, \Phi_r))$, depending on the six labels s', r', l, k, s and r. Wienert [20] observed that the computation of $\mathbf{M}$ can be simplified by recognising that (because the space $\mathbb{P}_n$ is invariant under arbitrary rotations) the rotated spherical harmonic $\mathcal{T}_{\boldsymbol{p}(\theta_{s'},\phi_{r'})} Y_{l,k}$ is itself a spherical harmonic of degree l, and hence is expressible as a linear combination of $Y_{l,\tilde{k}}$ with $|\tilde{k}| \le l$. To allow us to state this precisely, we introduce an alternative notation $\mathcal{T}(\Omega)$ for the linear transformation induced by the real orthogonal matrix Ω, so that (cf. (5.5))

$$\mathcal{T}(\Omega)F(\hat{\mathbf{z}}) = F(\Omega^{-1}\hat{\mathbf{z}}), \quad \hat{\mathbf{z}} \in \partial B, \quad F \in C(\partial B). \tag{7.4}$$

From this definition it follows easily that the linear transformations $\mathcal{T}(\Omega)$ form a *representation* of the rotations Ω, in the sense that $\mathcal{T}(\Omega_1\Omega_2) = \mathcal{T}(\Omega_1)\mathcal{T}(\Omega_2)$. Thus if we now consider $T_{\hat{\mathbf{x}}}$ defined by (5.3), we can write, for $\hat{\mathbf{x}} = \boldsymbol{p}(\alpha, \beta)$,

$$\mathcal{T}_{\hat{\mathbf{x}}} = \mathcal{T}(T_{\hat{\mathbf{x}}}) = \mathcal{T}(P(\beta)Q(-\alpha)P(-\beta)) = \mathcal{T}(P(\beta))\mathcal{T}(Q(-\alpha))\mathcal{T}(P(-\beta)). \tag{7.5}$$

It is easily verified from (7.4) and the definition of $P(\beta)$ given in (5.4), that

$$\mathcal{T}(P(\beta))Y_{l,k} = e^{-ik\beta} Y_{l,k}. \tag{7.6}$$

For the rotations $\mathcal{T}(Q(\alpha))$ applied to $Y_{l,k}$, we exploit the fact that $\mathcal{T}(Q(\alpha))Y_{l,k}$ is a spherical polynomial of degree l, allowing us to write

$$\mathcal{T}(Q(\alpha))Y_{l,k} = \sum_{|\tilde{k}|\le l} d^{(l)}(\alpha)_{\tilde{k}k} Y_{l,\tilde{k}} \ . \tag{7.7}$$

It may be easily verified that the ordering of the subscripts on the coefficients on the right-hand side ensures that the $(2n+1)\times(2n+1)$ matrices $d^{(l)}(\alpha)$ form a matrix representation of the rotations (i.e. the matrix for the product of two rotations is the product of the corresponding matrices). Using this fact and [3, p.22] we have, explicitly,

$$d^{(l)}(\alpha)_{\tilde{k}k} = \sum_t (-1)^t \frac{[(l+\tilde{k})!(l-\tilde{k})!(l+k)!(l-k)!]^{1/2}}{(l+\tilde{k}-t)!(l-k-t)!t!(t+k-\tilde{k})!} \left(\cos\frac{\alpha}{2}\right)^{2l+\tilde{k}-k-2t} \left(\sin\frac{\alpha}{2}\right)^{2t+k-\tilde{k}} ,$$

where the sum is over all $t \in \mathbb{Z}$, with the convention that the sum is taken over all values of t which lead to non-negative factorials.

Using (7.5), (7.6), (7.7), we obtain immediately

$$\mathcal{T}_{\boldsymbol{p}(\theta_s,\phi_r)} Y_{l,k} = \sum_{|\tilde{k}|\le l} e^{i(k-\tilde{k})\phi_r} d^{(l)}(-\theta_s)_{\tilde{k}k} Y_{l,\tilde{k}}. \tag{7.8}$$

Combining this with (1.10) allows us to write (7.3) in the form

$$\begin{aligned}\mathbf{M}_{\ell' k',lk} = &\sum_{r'}\sum_{s'} \mu_{r'}\nu_{s'} \sum_r\sum_s \xi_r\tilde{\eta}_s R(\boldsymbol{p}(\theta_{s'},\phi_{r'}), T^{-1}_{\boldsymbol{p}(\theta_{s'},\phi_{r'})}\boldsymbol{p}(\Theta_s,\Phi_r)) \\ &\times \sum_{|\tilde{k}|\le l} e^{i(k-\tilde{k})\phi_{r'}} d^{(l)}(-\theta_{s'})_{\tilde{k}k} c_l^{\tilde{k}} P_l^{|\tilde{k}|}(\cos\Theta_s) e^{i\tilde{k}\Phi_r} c_{l'}^{k'} P_{l'}^{|k'|}(\cos\theta_{s'}) e^{-ik'\phi_{r'}}.\end{aligned}$$

The matrix $\mathbf{M}$, with $O(n^4)$ elements, can be computed in a time of order $O(n^5)$ by computing successively the following arrays:

$$\begin{aligned}
E_{s'r's\tilde{k}} &= \sum_r \xi_r R(\boldsymbol{p}(\theta_{s'},\phi_{r'}), T^{-1}_{\boldsymbol{p}(\theta_{s'},\phi_{r'})}\boldsymbol{p}(\Theta_s,\Phi_r)) e^{i\tilde{k}\Phi_r}, \\
D_{s'r'l\tilde{k}} &= \sum_s \tilde{\eta}_s E_{s'r's\tilde{k}} c_l^{\tilde{k}} P_l^{|\tilde{k}|}(\cos\Theta_s), \\
C_{s'r'lk} &= \sum_{|\tilde{k}|\le l} D_{s'r'l\tilde{k}} e^{i(k-\tilde{k})\phi_{r'}} d^{(l)}(-\theta_{s'})_{\tilde{k}k}, \\
B_{s'k'lk} &= \sum_{r'} C_{s'r'lk}\mu_{r'} e^{-ik'\phi_{r'}} \ .
\end{aligned}$$

With careful organisation, each array can be computed in a time $O(n^5)$, since each array has $O(n^4)$ elements, each of which is a sum with $O(n)$ elements. It is desirable to precompute the arrays

$$R(\boldsymbol{p}(\theta_{s'},\phi_{r'}), T^{-1}_{\boldsymbol{p}(\theta_{s'},\phi_{r'})}\boldsymbol{p}(\Theta_s,\Phi_r)),\ c_l^{\tilde{k}} P_l^{|\tilde{k}|}(\cos\Theta_s),\ e^{i(k-\tilde{k})\phi_{r'}} \text{ and } d^{(l)}(-\theta_{s'})_{\tilde{k}k}$$

are precomputed. Finally, we compute the array $\mathbf{M}$ in $O(n^5)$ operations by

$$\mathbf{M}_{l'k',lk} = \sum_{s'} B_{s'k'lk} \nu_{s'} c_{l'}^{k'} P_{l'}^{|k'|}(\cos\theta_{s'}) \ ,$$

with the array $c_{l'}^{k'} P_{l'}^{|k'|}(\cos\theta_{s'})$ precomputed.

Whereas the method above is a discrete form of the Galerkin method, the method of Wienert [20] defined in (6.1) takes the form of a Nyström or product-integration method, as can be easily seen from examining (7.2) and recalling the definition (1.17) of $\mathcal{L}_n$. In this case the linear system (obtained by collocating equation (6.1) at the points $\boldsymbol{p}(\theta_{s'}, \phi_{r'})$) takes the form

$$\sum_r \sum_s [\delta_{s'r'}\delta_{sr} + \tilde{\mathbf{M}}_{s'r',sr}] v_{sr} = F(\boldsymbol{p}(\theta_{s'}, \phi_{r'})) \ ,$$

where $v_{s,r} = V_n^{\mathrm{W}}(\boldsymbol{p}(\theta_s, \phi_r))$. The matrix $\tilde{\mathbf{M}}$, again with $O(n^4)$ elements, like the matrix $\mathbf{M}$ can (as pointed out by Wienert [20]) be evaluated in $O(n^5)$ operations with the aid of (7.8). We refer to [20] for the details.

We conclude with a brief comparison between our fully discrete Galerkin method (5.14) and Wienert's method (6.1). The formulas in the two cases are broadly similar, but it is worth noting one significant difference: *The size of the linear system in the fully discrete Galerkin method is only half of that for Wienert's method.* (We assume here that the Gauss-Legendre rule is used in (3.4); the ratio is even smaller if a different rule is used.) Fundamentally, the reason is that (for $n \geq 3$) the number of points in the hyperinterpolation quadrature rule Q_m (see (1.14)) is necessarily greater than the dimension of the space $\mathbb{P}_n$; and for the case of tensor-product rules satisfying the assumptions in §3.1, the factor is at least 2.

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