Extending hysteresis operators to spaces of piecewise continuous functions *

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Abstract: We consider continuous-time hysteresis operators, defined to be causal and rate independent operators mapping input signals $u : \mathbb{R}_+ \to \mathbb{R}$ to output signals $v : \mathbb{R}_+ \to \mathbb{R}$. We show how a hysteresis operator defined on the space of continuous piecewise monotone functions, can be naturally extended to piecewise continuous piecewise monotone functions. We prove that the extension is also a hysteresis operator and that a number of important properties of the original operator are inherited by the extension. Moreover, we define the concept of a discrete-time hysteresis operator and we show that discretizing continuous-time hysteresis operators using standard sampling and hold operations leads to discrete-time hysteresis operators.

Keywords: Causality, hysteresis operators, piecewise continuous functions, piecewise monotone functions, rate independence.

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1 Introduction

Generally speaking, hysteresis is a special type of memory-based relation between a scalar input signal $u(\cdot)$ and a scalar output signal $v(\cdot)$ that cannot be expressed in terms of a single-valued function but takes the form of "hysteresis" loops. The memory effects exhibited by hysteresis phenomena are rate independent in contrast to rate dependent memory which is typically fading and hence scale dependent. The concept of a hysteresis operator has been introduced in Brokate [2], Brokate and Sprekels [3] and Visintin [8], to be an operator which is both causal and rate indepentent. For continuous piecewise monotone input signals u this means that at time $t \in \mathbb{R}_+$, the value $v(t)$ of the output signal v is dependent only on the local extrema of u restricted to the time interval $[0, t]$. Hysteresis operators encompass nonlinearities important in applications such as relay (or passive), backlash (or play) and elastic-plastic hysteresis. This type of behaviour arises in mechanical plays, thermostats, elastoplasticity, ferromagnetism and in smart material structures such as piezoelectric elements and magnetostrictive transducers (see Banks et al. [1] for hysteresis phenomena in smart materials). There exists a substantial literature on mathematical modelling and mathematical theory of hysteresis phenomena, see, for example, Brokate [2], Brokate and Sprekels [3], Krasnosel'skiĭ and Pokrovskiĭ [4], Macki et al. [7] and Visintin [8].

Usually, hysteresis operators are defined on spaces of continuous functions (in particular, continuous piecewise monotone functions). For certain applications such as sampled-data control for systems with hysteresis effects it is desirable to extend hysteresis operators to spaces of piecewise continuous functions. We show in this paper that this can be done in great generality. More precisely, we show that any hysteresis operator defined on the space of continuous piecewise monotone functions can be extended to the space of normalized piecewise continuous piecewise monotone functions and that this extension is a hysteresis operator. Furthermore, we introduce a concept of discrete-time hysteresis and show that discretizing continuous-time hysteresis operators using standard sampling and hold operations yields discrete-time hysteresis operators.

In Section 2 of the paper we first introduce continuous-time hysteresis operators defined on a general domain and then restrict our attention to hysteresis operators on domains of continuous piecewise monotone functions. In contrast to most of the literature (see [3] and [8]), where hysteresis operators act on function spaces with a finite time horizon, we consider hysteresis operators acting on functions defined on the infinite time interval $[0, \infty)$. This is motivated by our interest in asymptotic properties of feedback control systems with hysteresis nonlinearities, see [5], [6]. We show that for every hysteresis operator there exists a representing rate independent functional and derive a number of properties enjoyed by hysteresis operators. In Section 3 we consider hysteresis operators defined on the space of continuous piecewise monotone functions, extend the representing functional of such a hysteresis operator to a domain of piecewise continuous piecewise monotone functions and then use the extended functional to define an extension of the hysteresis operator. We show that the extension itself is a hysteresis operator and that other important properties of the original operator are inherited by the extension. In Section 4 we introduce the concept of a discrete-time hysteresis operator and give a discrete-time analogue of some of the continuous-time hysteresis results of Section 2. We show that the discretization of a continuous-time hysteresis operator obtained by standard sampling and hold operations is a discrete-time hysteresis operator. Some technicalities are relegated to the Appendix.

Notation and terminology: We define

$$\mathbb{Z}_+ := \{x \in \mathbb{Z} \,|\, x \geq 0\}\,, \quad \mathbb{R}_+ := \{x \in \mathbb{R} \,|\, x \geq 0\}\,.$$

For sets M and N we denote the set of all functions $f : M \to N$ by $F(M, N)$. We say that a function $f \in F(\mathbb{R}_+, \mathbb{R})$ is *piecewise monotone* if there exists a sequence $0 = t_0 < t_1 < t_2 < \ldots$ such that $\lim_{i\to\infty} t_i = \infty$ and f is monotone on each of the intervals (t_i, t_{i+1}). We say that a function $f \in F(\mathbb{R}_+, \mathbb{R})$ is *piecewise continuous* if there exists a sequence $0 = t_0 < t_1 < t_2 < \ldots$ such that $\lim_{i\to\infty} t_i = \infty$, f is continuous on each of the intervals (t_i, t_{i+1}) and the right and left limits of f exist and are finite at each t_i. The space of all piecewise continuous functions $f \in F(\mathbb{R}_+, \mathbb{R})$ is denoted by $PC(\mathbb{R}_+, \mathbb{R})$. As usual, for $f \in PC(\mathbb{R}_+, \mathbb{R})$, we define

$$f(t+) = \lim_{\tau \downarrow t} f(\tau) \text{ (for } t \geq 0) \quad \text{and} \quad f(t-) = \lim_{\tau \uparrow t} f(\tau) \text{ (for } t > 0)\,.$$

Let $\mathbb{T} = \mathbb{R}_+, \mathbb{Z}_+$; a function $f \in F(\mathbb{T}, \mathbb{R})$ is called *ultimately constant* if there exists $T \in \mathbb{T}$ such that f is constant on $[T, \infty) \cap \mathbb{T}$.

2 Continuous-time hysteresis operators

In this section we present basic background material on hysteresis operators which is needed for the subsequent developments in Sections 3 and 4. Our treatment of hysteresis operators is strongly influenced by chapter 2 in the book [3] by Brokate and Sprekels. Most of the results in this section can be found in a somewhat different and less general form in chapter 2 of [3].

We call a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ a *time transformation* if f is continuous, non-decreasing and satisfies $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$, in other words $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a time transformation if and only if f is continuous, non-decreasing and surjective. We denote the set of all time transformations $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $\mathcal{T}$. For each $\tau \in \mathbb{R}_+$, we define a projection operator $Q_\tau : F(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$(Q_\tau\, u)(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq \tau\,, \\ u(\tau) & \text{for } t > \tau\,. \end{cases}$$

In the following let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$. We introduce the following two assumptions on $\mathcal{F}$:

(F1) $u \circ f \in \mathcal{F}$ for all $u \in \mathcal{F}$ and all $f \in \mathcal{T}$;

(F2) $Q_t(\mathcal{F}) \subset \mathcal{F}$ for all $t \in \mathbb{R}_+$.

We call an operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ *causal* if for all $u, v \in \mathcal{F}$ and all $\tau \in \mathbb{R}_+$ with $u(t) = v(t)$ for all $t \in [0, \tau]$ it follows that $(\Phi(u))(t) = (\Phi(v))(t)$ for all $t \in [0, \tau]$. An operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is called *rate independent* if $\mathcal{F}$ satisfies (F1) and

$$(\Phi(u \circ f))(t) = (\Phi(u))(f(t)), \quad \forall\, u \in \mathcal{F}, \quad \forall\, f \in \mathcal{T}, \quad \forall\, t \in \mathbb{R}_+ .$$

A functional $\varphi : \mathcal{F} \to \mathbb{R}$ is called *rate independent* if $\mathcal{F}$ satisfies (F1) and

$$\varphi(u \circ f) = \varphi(u), \quad \forall\, u \in \mathcal{F}, \quad \forall\, f \in \mathcal{T}.$$

Definition 2.1 Let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$. An operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is called a *hysteresis operator* if $\mathcal{F}$ satisfies (F1) and Φ is causal and rate independent. $\diamond$

For $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$, let $\mathcal{F}^{\mathrm{uc}}$ denote the set of all ultimately constant $u \in \mathcal{F}$, i.e.

$$\mathcal{F}^{\mathrm{uc}} = \{u \in \mathcal{F} \,|\, u \text{ is ultimately constant}\}.$$

Clearly, if $\mathcal{F}$ satisfies (F2), then $\mathcal{F}^{\mathrm{uc}} \neq \emptyset$. Moreover, if $\mathcal{F}$ satisfies (F1), then so does $\mathcal{F}^{\mathrm{uc}}$.

Theorem 2.2 *Let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$ and assume that (F1) and (F2) hold.*

If $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator, then the following statements hold:

(1) for all $u \in \mathcal{F}$ and all $\tau \in \mathbb{R}_+$

$$(\Phi(Q_\tau u))(t) = (\Phi(u))(\tau), \quad \forall\, t \geq \tau;$$

(2) the functional

$$\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}, \quad u \mapsto \lim_{t \to \infty} (\Phi(u))(t), \tag{2.1}$$

is rate independent and satisfies

$$(\Phi(u))(t) = \varphi(Q_t u), \quad \forall\, u \in \mathcal{F}, \quad \forall\, t \in \mathbb{R}_+ . \tag{2.2}$$

Conversely, if $\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}$ is a rate independent functional, then $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ given by (2.2) is a hysteresis operator and satisfies

$$\lim_{t \to \infty} (\Phi(u))(t) = \varphi(u), \quad \forall\, u \in \mathcal{F}^{\mathrm{uc}}. \tag{2.3}$$

For a hysteresis operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$, we call the rate independent functional $\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}$ defined by (2.1) the *representing functional* of Φ.

Proof of Theorem 2.2: Assume that $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator. To prove statement (1), let $u \in \mathcal{F}$, $\tau \in \mathbb{R}_+$ and $s > \tau$. We define a time transformation $f \in \mathcal{T}$ by

$$f(t) = \begin{cases} t & \text{for } 0 \le t \le \tau\,, \\ \tau & \text{for } \tau < t \le s\,, \\ t + \tau - s & \text{for } t > s\,. \end{cases}$$

Then, using the causality and rate independence of Φ, we have for $t \in [\tau, s]$

$$(\Phi(Q_\tau\, u))(t) = (\Phi(u \circ f))(t) = (\Phi(u))(f(t)) = (\Phi(u))(\tau)\,.$$

Since $s > \tau$ was arbitrary, this yields statement (1). To prove statement (2), we first note that the limit in (2.1) exists since for ultimately constant u, $\Phi(u)$ is ultimately constant by statement (1). Using the rate independence of Φ, we see that for all $u \in \mathcal{F}^{\mathrm{uc}}$ and all $f \in \mathcal{T}$

$$\varphi(u \circ f) = \lim_{t\to\infty} (\Phi(u \circ f))(t) = \lim_{t\to\infty} (\Phi(u))(f(t)) = \lim_{t\to\infty} (\Phi(u))(t) = \varphi(u)\,,$$

showing that φ is rate independent. Using statement (1), we obtain for all $u \in \mathcal{F}$ and all $t \in \mathbb{R}_+$

$$(\Phi(u))(t) = (\Phi(Q_t\, u))(t) = \lim_{s\to\infty} (\Phi(Q_t\, u))(s) = \varphi(Q_t\, u)\,,$$

which is (2.2).

Conversely, assume that $\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}$ is rate independent and define $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ by (2.2), i.e. $(\Phi(u))(t) = \varphi(Q_t\, u)$. Then, trivially, Φ is causal. Moreover, for all $u \in \mathcal{F}$, $f \in \mathcal{T}$ and $t \in \mathbb{R}_+$

$$(\Phi(u \circ f))(t) = \varphi(Q_t\,(u \circ f)) = \varphi((Q_{f(t)}\, u) \circ f) = \varphi(Q_{f(t)}\, u) = (\Phi(u))(f(t))\,,$$

thus Φ is rate independent. Finally, let $u \in \mathcal{F}^{\mathrm{uc}}$, then

$$\lim_{t\to\infty} (\Phi(u))(t) = \lim_{t\to\infty} \varphi(Q_t u) = \varphi(u)\,,$$

which is (2.3). □

Let $\mathcal{S}^r$ denote the set of all right-continuous step functions $u : \mathbb{R}_+ \to \mathbb{R}$, that is there exists a sequence $0 = t_0 < t_1 < t_2 < \ldots$ such that $\lim_{i\to\infty} t_i = \infty$ and u is constant on each of the intervals $[t_i, t_{i+1})$. For $\tau > 0$ define $\mathcal{S}^r_\tau \subset \mathcal{S}^r$ to be the set of all right-continuous step functions $u : \mathbb{R}_+ \to \mathbb{R}$ of step length τ, i.e. u is constant on each interval $[i\tau, (i+1)\tau)$. We note that whilst $\mathcal{S}^r$ satisfies (F1) and (F2), $\mathcal{S}^r_\tau$ satisfies (F2), but not (F1). The following corollary is an immediate consequence of Theorem 2.2, statement (1).

Corollary 2.3 *Let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$ and assume that (F1) and (F2) hold. Let $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Then*

$$\Phi(\mathcal{F}^{\mathrm{uc}}) \subset \mathcal{F}^{\mathrm{uc}}\,, \quad \Phi(\mathcal{F} \cap \mathcal{S}^r) \subset \mathcal{S}^r\,, \quad \Phi(\mathcal{F} \cap \mathcal{S}^r_\tau) \subset \mathcal{S}^r_\tau\,.$$

For any $u \in F(\mathbb{R}_+, \mathbb{R})$ and any $t \in \mathbb{R}_+$, we define

$$M(u,t) := \{\tau \in (t,\infty) \,|\, u \text{ is monotone on } (t,\tau)\}\,.$$

If u is piecewise monotone, then $M(u,t) \neq \emptyset$ for all $t \in \mathbb{R}_+$, and the *standard monotonicity partition* $t_0 < t_1 < t_2 < \ldots$ of u is defined recursively by setting $t_0 = 0$ and $t_{i+1} = \sup M(u,t_i)$ for all $i \in \mathbb{Z}_+$ such that $M(u,t_i)$ is bounded. If u is piecewise monotone and ultimately constant, then the standard monotonicity partition of u is finite.

The space of all piecewise monotone $u \in C(\mathbb{R}_+, \mathbb{R})$ is denoted by $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. We define $C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ to be the space of all ultimately constant $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. We note that $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ both satisfy (F1) and (F2). Let $F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R})$ denote the space of ultimately constant $u : \mathbb{Z}_+ \to \mathbb{R}$. We define the *restriction operator* $R : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R})$ by

$$(R(u))(k) = \begin{cases} u(t_k) & \text{for } k \in [0,m] \cap \mathbb{Z}_+\,, \\ \lim_{t\to\infty} u(t) & \text{for } k \in \mathbb{Z}_+ \setminus [0,m]\,, \end{cases}$$

where $0 = t_0 < t_1 < \ldots < t_m$ is the standard monotonicity partition of u.

The following lemma will be an important tool in the next section.

Lemma 2.4 *Let $u, v \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Then $R(u) = R(v)$ if and only if there exist $f, g \in \mathcal{T}$ such that $u \circ f = v \circ g$.*

The above lemma is due to Brokate and Sprekels [3] (see lemma 2.2.4 in [3]). Since only a sketch of the proof is given in [3], we have included a complete proof in the Appendix.

As an immediate consequence of Lemma 2.4 and Theorem 2.2, statement (2), we obtain the following corollary.

Corollary 2.5 *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator, $u, v \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$. Then*

$$R(Q_t\, u) = R(Q_t\, v) \quad \Longrightarrow \quad (\Phi(u))(t) = (\Phi(v))(t)\,.$$

The above corollary says that the output $(\Phi(u))(t)$ at time $t \in \mathbb{R}_+$ of a hysteresis operator Φ corresponding to a continuous piecewise monotone input u is determined completely by the local extrema of u restricted to the time interval $[0,t]$.

3 Extending hysteresis operators defined on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ to spaces of piecewise continuous functions

Let $NPC(\mathbb{R}_+, \mathbb{R}) \subset PC(\mathbb{R}_+, \mathbb{R})$ denote the space of all *normalised piecewise continuous* functions $u : \mathbb{R}_+ \to \mathbb{R}$, that is u is piecewise continuous and is right-continuous or left-continuous at each point $t \in \mathbb{R}_+$. In particular, if $u \in NPC(\mathbb{R}_+, \mathbb{R})$, then u is right-continuous at $t = 0$. The space of all piecewise monotone functions $u \in NPC(\mathbb{R}_+, \mathbb{R})$

is denoted by $NPC_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, whilst $NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ denotes the space of all ultimately constant $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. We note that $NPC_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and $NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ both satisfy (F1) and (F2). For $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, we define $u(\infty) := \lim_{\tau\to\infty} u(\tau)$.

Lemma 3.1 *Let $u \in NPC(\mathbb{R}_+,\mathbb{R})$, $f \in \mathfrak{T}$ and $t > 0$. Define*

$$\tau = \begin{cases} \max f^{-1}(\{t\}) & \text{if } u(t-) = u(t)\,, \\ \min f^{-1}(\{t\}) & \text{if } u(t-) \neq u(t)\,. \end{cases}$$

Then $(u\circ f)(\tau+) = u(t+)$ and $(u \circ f)(\tau-) = u(t-)$.

Proof: Since f is continuous, non-decreasing and surjective, for all $t \in \mathbb{R}_+$, $f^{-1}(\{t\})$ is a compact interval and therefore τ is well defined. We consider two cases.

Case 1: Suppose that $u(t-) = u(t)$. Then, $f(\tau + h) > t$ for all $h > 0$ and so $(u\circ f)(\tau+) = u(f(\tau)+) = u(t+)$. Moreover, if $f^{-1}(\{t\})$ is a singleton, we have $f(\tau - h) < t$ for all $h \in (0, \tau]$ and so $(u \circ f)(\tau-) = u(f(\tau)-) = u(t-)$. If $f^{-1}(\{t\})$ is not a singleton, we have $f(\tau - h) = t$ for all sufficiently small $h > 0$ and so $(u \circ f)(\tau-) = u(t) = u(t-)$.

Case 2: Suppose that $u(t-) \neq u(t)$. Then, since $u \in NPC(\mathbb{R}_+,\mathbb{R})$, it follows that $u(t+) = u(t)$. Adopting an argument similar to that in Case 1 yields the claim. □

Let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and let $0 = t_0 < t_1 < \ldots < t_m$ be the standard monotonicity partition of u. We define the map $\rho : NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+,\mathbb{R})$ by

$$\rho(u) = (u(t_0), u(t_1-), u(t_1+), u(t_2-), u(t_2+), \ldots, u(t_m-), u(t_m+), u(\infty), u(\infty), \ldots)\,.$$

Let $\tau > 0$. We define the *prolongation operator* $P_\tau : F(\mathbb{Z}_+,\mathbb{R}) \to C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ by letting $P_\tau u$ be the linear interpolate for the values $(P_\tau u)(i\tau) = u(i)$. Moreover, we introduce the operator

$$\tilde{R} : NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+,\mathbb{R})\,, \quad u \mapsto R((P_\tau \circ \rho)(u))\,.$$

Clearly, for any $\tau > 0$

$$R \circ P_\tau \circ R = R\,, \tag{3.1}$$

and using (3.1) it is easy to show that $R \circ P_\tau \circ \tilde{R} = \tilde{R}$.

For illustration, consider the function u shown in Figure 1. Clearly, $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ with standard monotonicity partition $0 = t_0 < t_1 < t_2 < t_3 < t_4$. The sequences $\rho(u)$ and $\tilde{R}(u)$ are given by

$$\rho(u) = (u_0, u_7, u_6, u_4, u_4, u_5, u_3, u_7, u_2, u_1, u_1, u_1, \ldots)\,,$$

and

$$\tilde{R}(u) = (u_0, u_7, u_4, u_5, u_3, u_7, u_1, u_1, u_1, \ldots)\,.$$

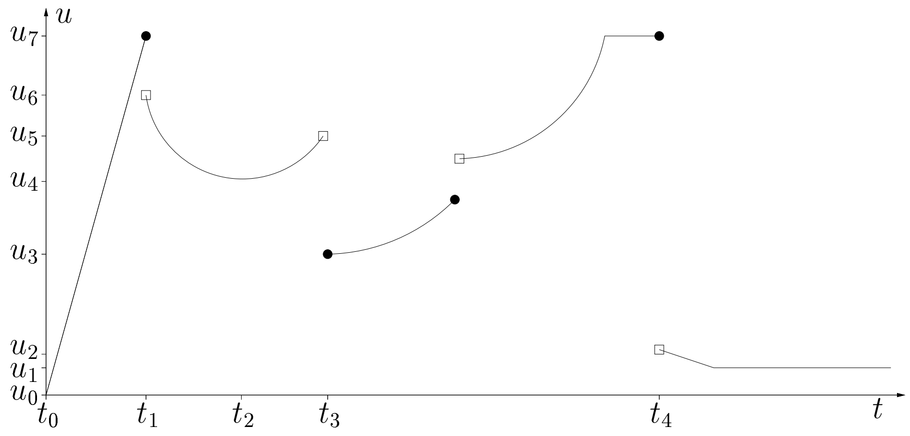

Figure 1: Example of a function in $NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$

Lemma 3.2 *$\tilde{R}$ is an extension of R, the definition of $\tilde{R}$ does not depend upon the choice of $\tau > 0$ and*

$$\tilde{R}(u \circ f) = \tilde{R}(u)\,, \quad \forall\, u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})\,, \quad \forall\, f \in \mathfrak{T}\,. \tag{3.2}$$

Proof: Let $u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, then, since $u(t+) = u(t-)$ for all $t > 0$,

$$\tilde{R}(u) = R((P_\tau \circ \rho)(u)) = R(u)\,,$$

showing that $\tilde{R}$ is an extension of R. Let $\tau_1, \tau_2 > 0$ and $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. Clearly, $R((P_{\tau_1} \circ \rho)(u)) = R((P_{\tau_2} \circ \rho)(u))$, from which it follows that the definition of $\tilde{R}$ does not depend upon the choice of $\tau > 0$.

Finally, let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, $f \in \mathfrak{T}$ and let $0 = t_0 < t_1 < \ldots < t_m$ be the standard monotonicity partition of u. Define $\tau_0 := 0$ and for $i = 1, \ldots, m$

$$\tau_i := \begin{cases} \max f^{-1}(\{t_i\}) & \text{if } u(t_i-) = u(t_i)\,, \\ \min f^{-1}(\{t_i\}) & \text{if } u(t_i-) \neq u(t_i)\,. \end{cases}$$

Then $0 = \tau_0 < \tau_1 < \ldots < \tau_m$ is the standard monotonicity partition of $u \circ f$ and by Lemma 3.1, $(u \circ f)(\tau_i\pm) = u(t_i\pm)$ for $i = 0, 1, \ldots, m$. Hence, $\rho(u) = \rho(u \circ f)$, and therefore, $\tilde{R}(u) = \tilde{R}(u \circ f)$, showing that (3.2) holds. $\square$

For any rate independent $\varphi : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}$ we define

$$\tilde{\varphi} : NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}\,, \quad u \mapsto \varphi((P_\tau \circ \tilde{R})(u))\,, \tag{3.3}$$

where $\tau > 0$. We show that the definition of $\tilde{\varphi}$ does not depend on τ. To this end, let $\tau_1, \tau_2 > 0$ and $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. Then, clearly there exists $f \in \mathfrak{T}$ such that $(P_{\tau_1} \circ \tilde{R})(u) = (P_{\tau_2} \circ \tilde{R})(u) \circ f$ and therefore by the rate independence of φ

$$\varphi((P_{\tau_1} \circ \tilde{R})(u)) = \varphi((P_{\tau_2} \circ \tilde{R})(u))\,.$$

Lemma 3.3 *Let $\varphi : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ be rate independent and define $\tilde{\varphi}$ by (3.3). Then*

(1) $\tilde{\varphi}$ is an extension of φ, i.e.

$$\tilde{\varphi}(u) = \varphi(u)\,, \quad \forall\, u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})\,;$$

(2) for any $\tau > 0$

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \rho)(u))\,, \quad \forall\, u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})\,;$$

(3) for $u, v \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$

$$\tilde{R}(u) = \tilde{R}(v) \quad \Longrightarrow \quad \tilde{\varphi}(u) = \tilde{\varphi}(v)\,;$$

(4) $\tilde{\varphi}$ is rate independent, i.e.

$$\tilde{\varphi}(u \circ f) = \tilde{\varphi}(u)\,, \quad \forall\, u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})\,, \quad \forall\, f \in \mathfrak{T}\,.$$

Proof: Let $\tau > 0$ and $u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Clearly, $R(u) = R((P_\tau \circ R)(u))$, and so using Lemma 2.4, there exist $f, g \in \mathfrak{T}$ such that $u \circ f = (P_\tau \circ R)(u) \circ g$. Thus the rate independence of φ in combination with Lemma 3.2 gives

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \tilde{R})(u)) = \varphi((P_\tau \circ R)(u) \circ g) = \varphi(u \circ f) = \varphi(u)\,,$$

which is statement (1). To prove statement (2), let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. By definition $\tilde{R}(u) = R((P_\tau \circ \rho)(u))$ and therefore,

$$(P_\tau \circ \tilde{R})(u) = P_\tau(R((P_\tau \circ \rho)(u)))\,.$$

Thus, invoking (3.1)

$$R((P_\tau \circ \tilde{R})(u)) = R(P_\tau(R((P_\tau \circ \rho)(u)))) = R((P_\tau \circ \rho)(u))\,,$$

and so using the rate independence of φ and Lemma 2.4

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \tilde{R})(u)) = \varphi((P_\tau \circ \rho)(u))\,.$$

For statement (3), let $u, v \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Suppose that $\tilde{R}(u) = \tilde{R}(v)$; then by the definition of $\tilde{R}$, $R((P_\tau \circ \rho)(u)) = R((P_\tau \circ \rho)(v))$. Since $(P_\tau \circ \rho)(u), (P_\tau \circ \rho)(v) \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ it follows from an application of Lemma 2.4, the rate independence of φ and statement (2) that

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \rho)(u)) = \varphi((P_\tau \circ \rho)(v)) = \tilde{\varphi}(v)\,.$$

Statement (4) follows immediately from (3.2) and statement (3). $\square$

Let $\Phi : C_{\rm pm}(\mathbb{R}_+,\mathbb{R}) \to F(\mathbb{R}_+,\mathbb{R})$ be a hysteresis operator and define

$$\tilde{\Phi} : NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R}) \to F(\mathbb{R}_+,\mathbb{R}),$$

by setting

$$(\tilde{\Phi}(u))(t) = \tilde{\varphi}(Q_t\, u), \quad \forall\, t \in \mathbb{R}_+, \tag{3.4}$$

where φ is the representing functional of Φ and $\tilde{\varphi}$ is the extension of φ to $NPC^{\rm uc}_{\rm pm}(\mathbb{R}_+,\mathbb{R})$ given by (3.3).

Theorem 3.4 *Let $\Phi : C_{\rm pm}(\mathbb{R}_+,\mathbb{R}) \to F(\mathbb{R}_+,\mathbb{R})$ be a hysteresis operator and define $\tilde{\Phi} : NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R}) \to F(\mathbb{R}_+,\mathbb{R})$ by (3.4). Then*

(1) $\tilde{\Phi}$ is an extension of Φ;

(2) $\tilde{\Phi}$ is a hysteresis operator with representing functional $\tilde{\varphi}$;

(3) for $u, v \in NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R})$ and $t \in \mathbb{R}_+$

$$\tilde{R}(Q_t\, u) = \tilde{R}(Q_t\, v) \quad \Longrightarrow \quad (\tilde{\Phi}(u))(t) = (\tilde{\Phi}(v))(t);$$

(4) $\tilde{\Phi}(\mathcal{S}^r) \subset \mathcal{S}^r$ and $\tilde{\Phi}(\mathcal{S}^r_\tau) \subset \mathcal{S}^r_\tau$.

Proof: Statement (1) is clear since $\tilde{\varphi}$ is an extension of φ and thus $\tilde{\Phi}$ is an extension of Φ. By Lemma 3.3, part (4), $\tilde{\varphi}$ is rate independent. Therefore, by Theorem 2.2, $\tilde{\Phi}$ is a hysteresis operator with representing functional $\tilde{\varphi}$, showing that statement (2) holds. Statement (3) follows from the definition of $\tilde{\Phi}$ and Lemma 3.3, part (3). Since $\mathcal{S}^r_\tau \subset \mathcal{S}^r \subset NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R})$, statement (4) follows from statement (2) combined with Corollary 2.3. $\square$

In the following we define continuous, piecewise monotone "approximations" u_1, u_2, u_3, ... of a given function $u \in NPC^{\rm uc}_{\rm pm}(\mathbb{R}_+,\mathbb{R})$ such that $\Phi(u_k)$ "approximates" $\tilde{\Phi}(u)$ as $k \to \infty$. Let $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of u and set $\tau_0 := 0$. For each $k \in \mathbb{Z}_+$, define

$$\varepsilon_k := \frac{1}{k+2} \min_{1\le i\le n} (\tau_i - \tau_{i-1}). \tag{3.5}$$

For each $k \in \mathbb{Z}_+$, define an operator $C_k : NPC^{\rm uc}_{\rm pm}(\mathbb{R}_+,\mathbb{R}) \to C^{\rm uc}_{\rm pm}(\mathbb{R}_+,\mathbb{R})$ by setting

(1) if $t \in [\tau_j - \varepsilon_k, \tau_j)$ and u is right-continuous at τ_j, then

$$(C_k(u))(t) = \begin{cases} \text{linear interpolant of } u(\tau_j - \varepsilon_k) \text{ and } u(\tau_j-) & \text{for } t \in [\tau_j - \varepsilon_k, \tau_j - \varepsilon_k/2], \\ \text{linear interpolant of } u(\tau_j-) \text{ and } u(\tau_j) & \text{for } t \in [\tau_j - \varepsilon_k/2, \tau_j); \end{cases}$$

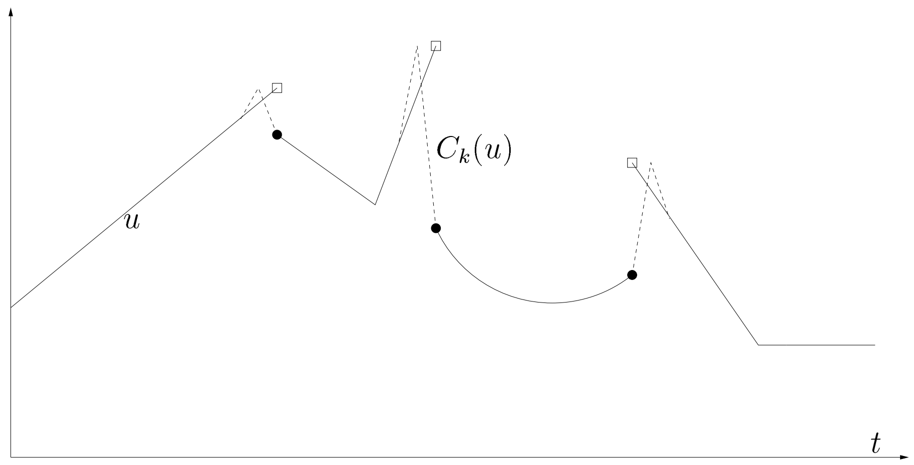

Figure 2: Example of a function $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and its approximation $C_k(u)$

(2) if $t \in (\tau_j, \tau_j + \varepsilon_k]$ and u is left-continuous at τ_j, then

$$(C_k(u))(t) = \begin{cases} \text{linear interpolant of } u(\tau_j) \text{ and } u(\tau_j+) & \text{for } t \in (\tau_j, \tau_j + \varepsilon_k/2]\,, \\ \text{linear interpolant of } u(\tau_j+) \text{ and } u(\tau_j + \varepsilon_k) & \text{for } t \in [\tau_j + \varepsilon_k/2, \tau_j + \varepsilon_k]\,; \end{cases}$$

(3) $(C_k(u))(t) = u(t)$ otherwise

(see Figure 2 for an illustration).

Lemma 3.5 *Let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Then*

(1) for any $t \in \mathbb{R}_+$, there exists $l > 0$ such that

$$\tilde{R}(Q_t\, u) = R(Q_t\, C_k(u))\,, \quad \forall\, k \geq l\,;$$

(2) for any $t_2 > t_1 \geq 0$, if u is continuous on $[t_1, t_2]$, then there exists $l > 0$ such that

$$\tilde{R}(Q_t\, u) = R(Q_t\, C_k(u))\,, \quad \forall\, t \in [t_1, t_2]\,, \quad \forall\, k \geq l\,.$$

Proof: Let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, let $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of u and let $0 = t_0 < t_1 < \ldots < t_m$ be the standard monotonicity partition of u. Define ε_k by (3.5).

To prove statement (1), let $t \in \mathbb{R}_+$ and choose $l > 0$ such that

$$\varepsilon_l < \min\{|t_i - \tau_j| \,|\, 1 \leq i \leq m, 1 \leq j \leq n, t_i \neq \tau_j\}\,,$$

and

$$\varepsilon_l < \min\{|t - \tau_j| \,|\, 1 \leq j \leq n, t \neq \tau_j\}\,.$$

Then $\tilde{R}(Q_t\, u) = R(Q_t\, C_k(u))$ for all $k \geq l$.

To prove statement (2), let $t_2 > t_1 \geq 0$ be such that u is continuous on $[t_1, t_2]$. Hence, there exists $l_1 > 0$ such that

$$C_k(u)|_{[t_1,t_2]} = u|_{[t_1,t_2]}\,, \quad \forall\, k \geq l_1\,. \tag{3.6}$$

Moreover, by statement (1), there exists $l_2 > 0$ such that

$$\tilde{R}(Q_{t_1}\, u) = R(Q_{t_1}\, C_k(u))\,, \quad \forall\, k \geq l_2\,. \tag{3.7}$$

Hence, by (3.6) and (3.7) statement (2) holds for $l := \max(l_1, l_2)$. □

For $u \in NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$, $t > 0$ and $\varepsilon > 0$, we define

$$J_\varepsilon(u,t) := \cup_{i=1}^n (\tau_i - \varepsilon, \tau_i + \varepsilon) \quad \text{and} \quad d(u,t) := \min_{1\leq i\leq n-1} (\tau_{i+1} - \tau_i)/2\,,$$

where $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of $Q_t\, u$.

Proposition 3.6 *Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and $u \in NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. Then*

(1) for all $t \in \mathbb{R}_+$, there exists $l > 0$ such that

$$(\tilde{\Phi}(u))(t) = (\Phi(C_k(Q_t\, u)))(t)\,, \quad \forall\, k \geq l\,;$$

(2) if for $t_3 \geq t_2 > t_1 \geq 0$, u is continuous on $[t_1, t_2]$, then there exists $l > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(Q_{t_3}\, u)))(s)\,, \quad \forall\, s \in [t_1, t_2]\,, \quad \forall\, k \geq l\,;$$

(3) for all $t \in \mathbb{R}_+$ and all $\varepsilon \in (0, d(u,t))$, there exists $l > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(Q_t\, u)))(s)\,, \quad \forall\, s \in [0,t) \setminus J_\varepsilon(u,t)\,, \quad \forall\, k \geq l\,.$$

Proof: Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and $u \in NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. Statement (1) follows from Theorem 3.4 and Lemma 3.5, part (1), and statement (2) follows from Theorem 3.4 and Lemma 3.5, part (2). For statement (3), let $t \in \mathbb{R}_+$, $\varepsilon \in (0, d(u,t))$ and let $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of $Q_t\, u$. Clearly u is continuous on $[\tau_i + \varepsilon, \tau_{i+1} - \varepsilon]$ for $1 \leq i \leq n-1$. Therefore by statement (2) and the causality of Φ, there exists $l_i > 0$ such that for $1 \leq i \leq n-1$

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(Q_t\, u)))(s)\,, \quad \forall\, s \in [\tau_i + \varepsilon, \tau_{i+1} - \varepsilon]\,, \quad \forall\, k \geq l_i\,.$$

To conclude the proof, we distinguish between two cases: $\tau_n + \varepsilon < t$ and $\tau_n + \varepsilon \geq t$.

If $\tau_n + \varepsilon < t$, then u is continuous on $[\tau_n + \varepsilon, t]$ and therefore again by statement (2), there exists $l_n > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(Q_t\, u)))(s)\,, \quad \forall\, s \in [\tau_n + \varepsilon, t]\,, \quad \forall\, k \geq l_n\,.$$

If $\tau_n + \varepsilon \geq t$, then set $l_n := 0$.

In both cases define $l := \max_{1 \leq i \leq n} l_i$ and statement (3) then follows. □

The following example shows that $\tilde{\Phi}$ defined by (3.4) is not the unique hysteresis operator extending a given hysteresis operator $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$.

Example 3.7 Define $Z_e : NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$(Z_e(u))(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ \sum_{0 < \tau \leq t} (u(\tau) - u(\tau-)) & \text{if } t > 0\,. \end{cases}$$

Clearly, Z_e is a causal extension of the trivial operator

$$Z : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})\,, \quad u \mapsto 0\,.$$

We show that Z_e is rate independent. Let $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $f \in \mathfrak{T}$, $t > 0$ and let $0 < t_1 < \ldots < t_m \leq f(t)$ be the points at which $Q_{f(t)}\, u$ is not left-continuous. Define, for $i = 1, \ldots, m$, $\tau_i := \min f^{-1}(\{t_i\})$. Then $0 < \tau_1 < \ldots < \tau_m \leq t$ are the points at which $Q_t\,(u \circ f)$ is not left-continuous. By Lemma 3.1, $(u \circ f)(\tau_i-) = u(t_i-)$ for $i = 1, \ldots, m$, and thus

$$\begin{aligned} (Z_e(u))(f(t)) &= \sum_{0 < \tau \leq f(t)} (u(\tau) - u(\tau-)) \\ &= \sum_{i=1}^{m} (u(t_i) - u(t_i-)) \\ &= \sum_{i=1}^{m} ((u \circ f)(\tau_i) - (u \circ f)(\tau_i-)) \\ &= \sum_{0 < \tau \leq t} ((u \circ f)(\tau) - (u \circ f)(\tau-)) \\ &= (Z_e(u \circ f))(t)\,, \end{aligned}$$

showing that Z_e is rate independent. Therefore Z_e is a hysteresis operator, but $Z_e \neq \tilde{Z} = 0$. It follows that if $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator, then $\tilde{\Phi} + Z_e$, as well as $\tilde{\Phi}$, are hysteresis operators which extend Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. ◇

The following corollary says that, given a hysteresis operator Φ on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $\tilde{\Phi}$ is the unique operator extending Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and satisfying statement (3) of Theorem 3.4.

Corollary 3.8 *Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Suppose that Φ_e is an extension of Φ to $NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. If for all $u, v \in NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ and all $t \in \mathbb{R}_+$*

$$\tilde{R}(Q_t\, u) = \tilde{R}(Q_t\, v) \quad \Longrightarrow \quad (\Phi_e(u))(t) = (\Phi_e(v))(t)\,, \tag{3.8}$$

then $\Phi_e = \tilde{\Phi}$.

Proof: Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator, Φ_e be an extension of Φ to $NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ satisfying (3.8), let $u \in NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$. By Lemma 3.5, part (1), for all sufficiently large k, we have

$$\tilde{R}(Q_t\, u) = \tilde{R}(Q_t\, C_k(Q_t\, u))\,.$$

Hence, by (3.8), for all sufficiently large k

$$(\Phi_e(u))(t) = (\Phi_e(C_k(Q_t\, u)))(t) = (\Phi(C_k(Q_t\, u)))(t)\,.$$

It follows from Proposition 3.6, part (1), that $(\Phi_e(u))(t) = (\tilde{\Phi}(u))(t)$. □

Corollary 3.9 *Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Assume that $\Phi(C_{\rm pm}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. Then*

$$\tilde{\Phi}(NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})) \subset NPC(\mathbb{R}_+, \mathbb{R})\,,$$

and for $u \in NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$, right-continuity of u at $t \in \mathbb{R}_+$ (respectively, left-continuity at $t > 0$) implies right-continuity of $\tilde{\Phi}(u)$ at $t \in \mathbb{R}_+$ (respectively, left-continuity at $t > 0$).

Proof: Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and assume that $\Phi(C_{\rm pm}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. Let $u \in NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. We proceed in four steps.

Step 1: Let us suppose that u is right-continuous at $t \in \mathbb{R}_+$, then there exists $\tau > t$ such that u is continuous on $[t, \tau]$. So by Proposition 3.6, part (2), there exist $l > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(Q_\tau\, u))))(s)\,, \quad \forall\, s \in [t, \tau]\,, \quad \forall\, k \geq l\,.$$

Since by assumption $\Phi(C_{\rm pm}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$, $\Phi(C_k(Q_\tau\, u))$ is a continuous function and so $\tilde{\Phi}(u)$ is right-continuous at t.

Step 2: Similarly, if u is left-continuous at $t > 0$, then it is easy to show that $\tilde{\Phi}(u)$ is left-continuous at t.

Step 3: Assume that u is left-continuous at $t > 0$. We show that the right limit $\lim_{s\downarrow t}(\tilde{\Phi}(u))(s)$ exists and is finite. To this end, define $w = u$ on $\mathbb{R}_+ \setminus \{t\}$ and $w(t) = \lim_{s\downarrow t} u(s)$. Thus w is right-continuous at t. Now $\tilde{R}(Q_\tau\, u) = \tilde{R}(Q_\tau\, w)$ for all $\tau \in \mathbb{R}_+ \setminus \{t\}$ and therefore by Theorem 3.4, part (3), $\tilde{\Phi}(u) = \tilde{\Phi}(w)$ on $\mathbb{R}_+ \setminus \{t\}$. Thus

$$\lim_{s\downarrow t}(\tilde{\Phi}(u))(s) = \lim_{s\downarrow t}(\tilde{\Phi}(w))(s) = (\tilde{\Phi}(w))(t)\,,$$

since $\tilde{\Phi}(w)$ is right-continuous at t by Step 1.

Step 4: Similarly, if u is right-continuous at $t > 0$, then it is easy to show that the left limit $\lim_{s\uparrow t}(\tilde{\Phi}(u))(s)$ exists and is finite. □

We end this section by considering the extension of the backlash (or play) operator.

Example 3.10 Let $h \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$. Defining the function $b_h : \mathbb{R}^2 \to \mathbb{R}$ by

$$b_h(v, w) = \max\{v - h, \min\{v + h, w\}\}\,, \tag{3.9}$$

we introduce the backlash (or play) operator $\mathcal{B}_{h,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ by setting

$$(\mathcal{B}_{h,\xi}(u))(t) = \begin{cases} b_h(u(0), \xi) & \text{for } t = 0\,, \\ b_h(u(t), (\mathcal{B}_{h,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i \in \mathbb{Z}_+\,, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \ldots$ is such that $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval (t_i, t_{i+1}). We remark that ξ plays the role of an "initial state". It is not difficult to show that the definition is independent of the choice of the partition (t_i), see [5]. It is well-known that $\mathcal{B}_{h,\xi}$ is a hysteresis operator, see, for example, [3]. The backlash operator $\mathcal{B}_{h,\xi}$ is illustrated in Figure 3.

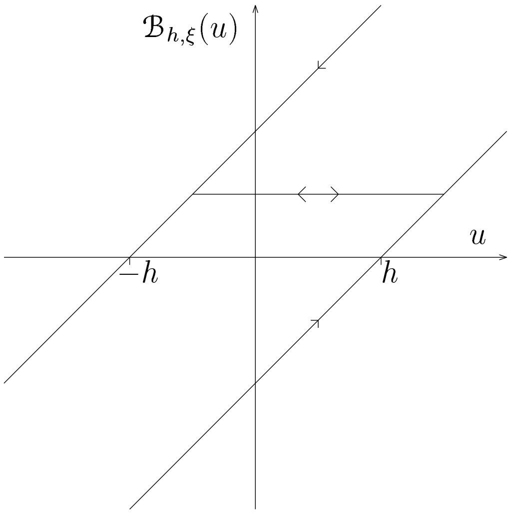

Figure 3: Backlash hysteresis

By Theorem 3.4, the extension $\tilde{\mathcal{B}}_{h,\xi}$ of $\mathcal{B}_{h,\xi}$ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ given by (3.4) is a hysteresis operator. By Corollary 3.9, $\tilde{\mathcal{B}}_{h,\xi}(NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset NPC(\mathbb{R}_+, \mathbb{R})$. It is shown in the Appendix that $\tilde{\mathcal{B}}_{h,\xi}$ can be written recursively as

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = \begin{cases} b_h(u(0), \xi) & \text{for } t = 0\,, \\ b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(0)) & \text{for } 0 < t < t_1\,, \\ b_h(u(t_i), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-)) & \text{for } t = t_i,\ i \in \mathbb{Z}_+ \setminus \{0\}\,, \\ b_h(u(t), b_h(u(t_i+), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-))) & \text{for } t_i < t < t_{i+1},\ i \in \mathbb{Z}_+ \setminus \{0\}\,, \end{cases} \tag{3.10}$$

where $0 = t_0 < t_1 < t_2 < \ldots$ is such that $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval (t_i, t_{i+1}). $\diamond$

4 Discrete-time hysteresis operators

We call a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ a *(discrete-time) time transformation* if f is surjective and non-decreasing. We denote the set of all discrete-time time transformations $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ by $\mathfrak{T}^d$. For each $k \in \mathbb{Z}_+$, we define a (discrete-time) projection operator $Q_k^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$(Q_k^d u)(n) = \begin{cases} u(n) & \text{for } n \in [0, k] \cap \mathbb{Z}_+ \,, \\ u(k) & \text{for } m \in \mathbb{Z}_+ \setminus [0, k] \,. \end{cases}$$

We call an operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ *causal* if for all $u, v \in F(\mathbb{Z}_+, \mathbb{R})$ and all $k \in \mathbb{Z}_+$ with $u(n) = v(n)$ for all $n \in [0, k] \cap \mathbb{Z}_+$ it follows that $(\Phi(u))(n) = (\Phi(v))(n)$ for all $n \in [0, k] \cap \mathbb{Z}_+$. An operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is called *rate independent* if

$$(\Phi(u \circ f))(n) = (\Phi(u))(f(n)) \,, \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R}) \,, \quad \forall\, f \in \mathfrak{T}^d \,, \quad \forall\, n \in \mathbb{Z}_+ \,.$$

Definition 4.1 An operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is called a *(discrete-time) hysteresis operator* if Φ is causal and rate independent. $\diamond$

Recall that $F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R})$ denotes the set of all ultimately constant $u \in F(\mathbb{Z}_+, \mathbb{R})$. A functional $\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ is called *rate independent* if

$$\varphi(u \circ f) = \varphi(u) \,, \quad \forall\, u \in F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \,, \quad \forall\, f \in \mathfrak{T}^d \,.$$

The proof of the following theorem is analogous to the proof of Theorem 2.2 and is therefore omitted.

Theorem 4.2 *If $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is a hysteresis operator, then*

(1) for all $u \in F(\mathbb{Z}_+, \mathbb{R})$ and all $k \in \mathbb{Z}_+$

$$(\Phi(Q_k^d u))(n) = (\Phi(u))(k) \,, \quad \forall\, n \geq k \,;$$

(2) the functional

$$\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R} \,, \quad u \mapsto \lim_{n \to \infty} (\Phi(u))(n) \,, \tag{4.1}$$

is rate independent and satisfies

$$(\Phi(u))(n) = \varphi(Q_n^d u) \,, \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R}) \,, \quad \forall\, n \in \mathbb{Z}_+ \,. \tag{4.2}$$

Conversely, if $\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ is a rate independent functional, then $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ given by (4.2) is a hysteresis operator and satisfies

$$\lim_{n \to \infty} (\Phi(u))(n) = \varphi(u) \,, \quad \forall\, u \in F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \,.$$

For a hysteresis operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$, we call the rate independent functional $\varphi : F^{\rm uc}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ defined by (4.1) the *representing functional* of Φ.

Let $\tau > 0$. The τ*-hold* operator $H_\tau : F(\mathbb{Z}_+, \mathbb{R}) \to \mathcal{S}_\tau^r$ is defined by

$$(H_\tau\, u)(n\tau + t) = u(n)\,, \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,, \quad \forall\, t \in [0, \tau)\,,$$

and the τ*-sampling* operator $S_\tau : F(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$(S_\tau\, u)(n) = u(n\tau)\,, \quad \forall\, u \in F(\mathbb{R}_+, \mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,.$$

The above hold and sampling operations are standard in the context of sampled-data control where continuous-time systems are controlled by discrete-time controllers via hold and sampling mechanisms.

Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a continuous-time hysteresis operator and define $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$\Phi^d := S_\tau\, \tilde{\Phi}\, H_\tau\,, \tag{4.3}$$

where $\tilde{\Phi}$ is the extension of Φ to $NPC_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ defined by (3.4). The definition of Φ^d is independent of the choice of τ due to the rate independence of $\tilde{\Phi}$.

Proposition 4.3 *Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a continuous-time hysteresis operator. Then $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ defined by (4.3) is a discrete-time hysteresis operator.*

Proof: It is clear that Φ^d is causal. It remains to show that Φ^d is rate independent. Let $u \in F(\mathbb{Z}_+, \mathbb{R})$ and $f \in \mathcal{T}^d$, then $f^c := \tau P_\tau(f) \in \mathcal{T}$ and $(H_\tau\, u) \circ f^c = H_\tau(u \circ f)$. Hence, using the rate independence of $\tilde{\Phi}$,

$$\begin{aligned} (\Phi^d(u \circ f))(n) &= (\tilde{\Phi}(H_\tau(u \circ f)))(n\tau) = (\tilde{\Phi}((H_\tau\, u) \circ f^c))(n\tau) \\ &= (\tilde{\Phi}(H_\tau\, u))(f^c(n\tau)) = (\tilde{\Phi}(H_\tau\, u))(f(n)\tau) = (\Phi^d(u))(f(n))\,, \end{aligned}$$

showing that Φ^d is rate independent. $\square$

Let $\mathbb{T} = \mathbb{Z}_+, \mathbb{R}_+$ and $\mathcal{F} \subset F(\mathbb{T}, \mathbb{R})$, $\mathcal{F} \neq \emptyset$; then the *numerical value set* $\mathrm{NVS}\,\Psi$ of an operator $\Psi : \mathcal{F} \to F(\mathbb{T}, \mathbb{R})$ is defined by

$$\mathrm{NVS}\,\Psi := \{(\Psi(u))(t) \mid u \in \mathcal{F},\, t \in \mathbb{T}\}\,.$$

The following proposition shows that for a continuous-time hysteresis operator Φ defined on $C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$, the numerical value sets of Φ and Φ^d coincide. This result is important in the context of sampled-data low-gain control of systems subject to input hysteresis (see [6]), but is also of some interest in its own right.

Proposition 4.4 *Let* $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ *be a continuous-time hysteresis operator and define the operator* $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ *by (4.3). Then*

$$(\Phi^d(u))(n) = (\Phi(P_\tau\, u))(n\tau)\,, \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,, \tag{4.4}$$

and $\mathrm{NVS}\,\Phi^d = \mathrm{NVS}\,\Phi$.

Proof: Let $u \in F(\mathbb{Z}_+, \mathbb{R})$ and $n \in \mathbb{Z}_+$. We note that $\tilde{R}(Q_{n\tau}\, H_\tau\, u) = R(Q_{n\tau}\, P_\tau\, u)$ and so by Theorem 3.4, parts (1) and (3)

$$(\Phi^d(u))(n) = (\tilde{\Phi}(H_\tau\, u))(n\tau) = (\Phi(P_\tau\, u))(n\tau)\,.$$

To prove that $\mathrm{NVS}\,\Phi^d = \mathrm{NVS}\,\Phi$, note first, that by (4.4), $\mathrm{NVS}\,\Phi^d \subset \mathrm{NVS}\,\Phi$. To show the reverse inclusion, let $a \in \mathrm{NVS}\,\Phi$. Then there exist $v \in C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$ such that $a = (\Phi(v))(t)$. Set $w := Q_t\, v \in C^{\rm uc}_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. Clearly

$$Q_{k\tau} w = w\,, \quad \forall\, k \geq t/\tau\,.$$

Moreover, $(P_\tau \circ R)(w) \in C^{\rm uc}_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ and so there exists $k_0 > 0$ such that

$$Q_{k\tau}((P_\tau \circ R)(w)) = (P_\tau \circ R)(w)\,, \quad \forall\, k \geq k_0\,.$$

For $k \geq \max(k_0, t/\tau) =: k_1$ we then have

$$(P_\tau \circ R)(Q_{k\tau}\, w) = (P_\tau \circ R)(w) = Q_{k\tau}((P_\tau \circ R)(w))\,. \tag{4.5}$$

Let φ be the representing functional of Φ, then for $k \geq k_1$

$$a = (\Phi(w))(t) = (\Phi(w))(k\tau) = \varphi(Q_{k\tau}\, w) = \tilde{\varphi}(Q_{k\tau}\, w) = \varphi((P_\tau \circ R)(Q_{k\tau}\, w))\,, \tag{4.6}$$

where we have used Theorem 2.2, parts (1) and (2) and the fact that $\tilde{\varphi}$ is an extension of φ. Combining Theorem 2.2, part (2), (4.4), (4.5) and (4.6), we obtain for any $k \geq k_1$

$$a = \varphi(Q_{k\tau}(P_\tau \circ R)(w)) = (\Phi((P_\tau \circ R)(w)))(k\tau) = (\Phi^d(Rw))(k) \in \mathrm{NVS}\,\Phi^d\,.$$

□

We finally look at the discretization of the backlash operator.

Example 4.5 Let $h \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$. We define the discrete-time backlash operator $\mathcal{B}^d_{h,\xi} : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by setting

$$\mathcal{B}^d_{h,\xi} := S_\tau\, \tilde{\mathcal{B}}_{h,\xi}\, H_\tau\,.$$

Using (4.4), we see that for all $u \in F(\mathbb{Z}_+, \mathbb{R})$, $\mathcal{B}^d_{h,\xi}(u)$ can be expressed recursively as

$$(\mathcal{B}^d_{h,\xi}(u))(n) = \begin{cases} b_h(u(0), \xi) & \text{for } n = 0\,, \\ b_h(u(n), (\mathcal{B}^d_{h,\xi}(u))(n-1)) & \text{for } n \in \mathbb{Z}_+ \setminus \{0\}\,, \end{cases}$$

where $b_h : \mathbb{R}^2 \to \mathbb{R}$ is given by (3.9). ◇

Appendix

Proof of Lemma 2.4

For a set S, let $\#S$ denote the cardinality of S. Fix $t_2 > t_1$ and let $u : [t_1, t_2] \to \mathbb{R}$ be non-decreasing. Define the set of *critical values* of u by

$$\mathcal{C}_u := \{z \in \operatorname{im} u \,|\, \#u^{-1}(\{z\}) > 1\} .$$

If $z \in \mathcal{C}_u$, then $I_z := u^{-1}(\{z\}) \subset [t_1, t_2]$ is an interval with $\operatorname{int}(I_z) \neq \emptyset$. If u is continuous, then I_z is closed. Moreover,

$$I_{z_1} \cap I_{z_2} = \emptyset\,, \quad \forall\, z_1, z_2 \in \mathcal{C}_u \text{ with } z_1 \neq z_2. \tag{A.1}$$

Before proving Lemma 2.4 we prove two auxiliary results.

Lemma A.1 *Let $t_0, t_1, a, b \in \mathbb{R}$ with $t_1 > t_0$ and $b > a$. If $S \subset [a, b]$ is countable, there exists a surjective continuous non-decreasing function $F : [t_0, t_1] \to [a, b]$, such that $S \subset \mathcal{C}_F$.*

Proof: If S is finite, then the lemma is trivially true. Thus without loss of generality we may assume that S is infinite. Since S is countable we may write $S = \{s_i \,|\, i \in \mathbb{Z}_+\}$ with $s_i \neq s_j$ if $i \neq j$. Without loss of generality we may assume that $a, b \in S$ and that $s_0 = a$ and $s_1 = b$. We recursively define continuous, piecewise linear, non-decreasing, surjective functions $F_j : [t_0, t_1] \to [a, b]$ $(j \in \mathbb{Z}_+ \setminus \{0\})$ such that $\mathcal{C}_{F_j} = \{s_0, \dots, s_j\}$. To this end let $\tau_0, \tau_1 \in (t_0, t_1)$ with $\tau_0 < \tau_1$ and set

$$F_1(t) = \begin{cases} a\,, & t_0 \leq t \leq \tau_0\,, \\ b\,, & \tau_1 \leq t \leq t_1\,, \\ a + \frac{b-a}{\tau_1 - \tau_0}(t - \tau_0)\,, & \tau_0 \leq t \leq \tau_1\,, \end{cases}$$

Suppose that $F_n : [t_0, t_1] \to [a, b]$ is continuous, piecewise linear, non-decreasing and surjective with $\mathcal{C}_{F_n} = \{s_0, \dots, s_n\}$. Define τ_{n+1} by

$$\{\tau_{n+1}\} := F_n^{-1}(\{s_{n+1}\})$$

and set

$$\tau_{n+1}^l := \min\{t \in [t_1, \tau_{n+1}] \,|\, F_n \text{ is strictly increasing on } [t, \tau_{n+1}]\}\,,$$
$$\tau_{n+1}^r := \max\{t \in [\tau_{n+1}, t_2] \,|\, F_n \text{ is strictly increasing on } [\tau_{n+1}, t]\}\,,$$

and

$$l_{n+1} := \frac{1}{2^n} \min\left\{\frac{1}{F_n'(\tau_{n+1})}, \tau_{n+1} - \tau_{n+1}^l, \tau_{n+1}^r - \tau_{n+1}\right\} .$$

We define a function $F_{n+1} : [t_0, t_1] \to [a, b]$ as follows: we set $F_{n+1} = F_n$ on $[t_1, \tau_{n+1}^l] \cup [\tau_{n+1}^r, t_2]$, $F_{n+1} \equiv s_n$ on $[\tau_{n+1} - l_{n+1}, \tau_{n+1} + l_{n+1}]$ and we define F_{n+1} to be affine linear on $[\tau_{n+1}^l, \tau_{n+1} - l_{n+1}]$ and $[\tau_{n+1} + l_{n+1}, \tau_{n+1}^r]$.

The recursively defined functions $F_j : [t_0, t_1] \to [a, b]$ are continuous, piecewise linear, non-decreasing with $F_j(t_0) = a$, $F_j(t_1) = b$ and such that $\mathcal{C}_{F_j} = \{s_0, \dots , s_j\}$. Additionally, by construction we have

$$\sup_{t\in[t_0,t_1]} |F_{j+1}(t) - F_j(t)| \le \frac{1}{2^j} ,$$

which implies that the sequence is Cauchy and therefore convergent. We denote the limit by F. By construction $F : [t_0, t_1] \to [a, b]$ is continuous, non-decreasing and such that $F(t_0) = a$, $F(t_1) = b$ and, moreover, $\#F^{-1}(\{s\}) > 1$ for all $s \in S$. $\square$

Lemma A.2 *Let $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0$.*

(1) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing. Then $\mathcal{C}_u$ is at most countable.

(2) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous. If $I \subset [t_0, t_1]$ is an interval with $\operatorname{int}(I) \ne \emptyset$ and $I \subset u^{-1}(\mathcal{C}_u)$, then u is constant on $\operatorname{clos}(I)$.

(3) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous and set

$$D := [t_0, t_1] \setminus u^{-1}(\mathcal{C}_u) , \quad E := [u(t_0), u(t_1)] \setminus \mathcal{C}_u .$$

Then $\hat{u} := u|_D$ is a bijective map from D to E and $\hat{u}^{-1} : E \to D$ is continuous.

(4) Let $u, v : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous with $u(t_0) = v(t_0)$ and $u(t_1) = v(t_1)$. If $\mathcal{C}_u \subset \mathcal{C}_v$, then there exists a surjective continuous non-decreasing function $f : [t_0, t_1] \to [t_0, t_1]$, such that $u \circ f = v$.

Proof: (1) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing. As above, for $z \in \mathcal{C}_u$, we set $I_z := u^{-1}(\{z\})$. Choose $q_z \in I_z \cap \mathbb{Q}$ for each $z \in \mathcal{C}_u$, which is possible since $\operatorname{int}(I_z) \ne \emptyset$ for $z \in \mathcal{C}_u$. It follows from (A.1) that the map

$$\mathcal{C}_u \to \mathbb{Q} , \qquad z \mapsto q_z$$

is injective, showing that $\mathcal{C}_u$ is at most countable.

(2) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous and $I \subset [t_0, t_1]$ be an interval with $\operatorname{int}(I) \ne \emptyset$ and $I \subset u^{-1}(\mathcal{C}_u)$. Seeking a contradiction, suppose that the claim is not true. Setting $[\alpha, \beta] := \operatorname{clos}(I)$, we have $u(\alpha) < u(\beta)$. Let $z \in (u(\alpha), u(\beta))$. By continuity of u there exists $t \in (\alpha, \beta) \subset I$ such that $u(t) = z$. Hence $z \in u(I)$ and so $z \in \mathcal{C}_u$. Thus

$$(u(\alpha), u(\beta)) \subset \mathcal{C}_u$$

showing that $\mathcal{C}_u$ is uncountable, which is impossible by part (1).

(3) It is clear that $\hat{u} := u|_D : D \to E$ is bijective. It remains to show that $\hat{u}^{-1} : E \to D$ is continuous. Seeking a contradiction, suppose that $\hat{u}^{-1}$ is not continuous. Then there exists $e \in E$, $(e_n) \subset E$ and $\varepsilon > 0$ such that $\lim_{n\to\infty} e_n = e$ and

$$|\hat{u}^{-1}(e_n) - \hat{u}^{-1}(e)| \ge \varepsilon , \quad \forall\, n \in \mathbb{Z}_+ . \tag{A.2}$$

Setting $d_n := \hat{u}^{-1}(e_n)$ and $d := \hat{u}^{-1}(e)$ we have as $n \to \infty$

$$u(d_n) = \hat{u}(d_n) = e_n \to e = \hat{u}(d) = u(d)\,.$$

Moreover, since $\mathrm{clos}\,(D)$ is compact, we may assume without loss of generality that $d_n \to d^* \in \mathrm{clos}\,(D)$ as $n \to \infty$. By (A.2), $|d^* - d| \geq \varepsilon$, and by the continuity of u, $u(d^*) = u(d)$, showing that

$$u(d) \in \mathcal{C}_u\,. \tag{A.3}$$

But by constuction, $d \in D$, implying that

$$u(d) = \hat{u}(d) \in E = [u(t_0), u(t_1)] \setminus \mathcal{C}_u\,,$$

which contradicts (A.3).

(4) Let $u, v : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous with $u(t_0) = v(t_0)$ and $u(t_1) = v(t_1)$ and $\mathcal{C}_u \subset \mathcal{C}_v$. Let $\hat{u}$ be as defined in part (3). Define $\hat{f} : ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \cup \{t_0, t_1\} \to [t_0, t_1]$,

$$\hat{f}(t) := \begin{cases} \hat{u}^{-1}(v(t)) & \text{for } t \in (t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)\,, \\ t_0 & \text{for } t = t_0\,, \\ t_1 & \text{for } t = t_1\,. \end{cases}$$

Obviously, $\hat{f}$ is non-decreasing and, by part (3), $\hat{f}$ is continuous. Now by part (1), there exists $N \subset \mathbb{Z}_+$ such that

$$v^{-1}(\mathcal{C}_u) = \cup_{j \in N} I_j\,,$$

where the $I_j \subset [t_0, t_1]$ are closed intervals with $I_i \cap I_j = \emptyset$ if $i \neq j$. Write $I_j = [a_j, b_j]$. If $a_j \neq t_0$, then for any $\varepsilon > 0$

$$[a_j - \varepsilon, b_j] \cap ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \neq \emptyset\,,$$

since otherwise there would exist $\varepsilon > 0$ such that

$$[a_j - \varepsilon, b_j] \subset v^{-1}(\mathcal{C}_u) \subset v^{-1}(\mathcal{C}_v)\,,$$

and hence, by part (2), $v(t) = v(a_j)$ for all $t \in [a_j - \varepsilon, b_j]$, which is impossible, since $v^{-1}(\{v(a_j)\}) = [a_j, b_j]$.

Similarly, if $b_j \neq t_1$, then for any $\varepsilon > 0$

$$[a_j, b_j + \varepsilon] \cap ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \neq \emptyset\,.$$

Therefore, we may define

$$\alpha_j := \begin{cases} \lim_{t \uparrow a_j} \hat{f}(t) & \text{if } a_j \neq t_0\,, \\ t_0 & \text{if } a_j = t_0\,, \end{cases} \qquad \text{and} \qquad \beta_j := \begin{cases} \lim_{t \downarrow b_j} \hat{f}(t) & \text{if } b_j \neq t_1\,, \\ t_1 & \text{if } b_j = t_1\,. \end{cases}$$

The limits on the right-hand side exist and are finite since $\hat{f}$ is bounded and non-decreasing. Finally, we define $f : [t_0, t_1] \to [t_0, t_1]$ by

$$f(t) = \begin{cases} \hat{f}(t) & \text{for } t \in ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \cup \{t_0, t_1\}\,, \\ \alpha_j & \text{for } t = a_j \neq t_0\,, \\ \beta_j & \text{for } t = b_j \neq t_1\,, \\ \alpha_j + \frac{\beta_j - \alpha_j}{b_j - a_j}(t - a_j) & \text{for } t \in (a_j, b_j)\,. \end{cases}$$

By constuction, f is continuous, non-decreasing with $f(t_0) = t_0$, $f(t_1) = t_1$ and $u \circ f = v$. □

Proof of Lemma 2.4: Note that

$$R(u) = R(u \circ h)\,, \quad \forall\, u \in C^{\text{uc}}_{\text{pm}}(\mathbb{R}_+, \mathbb{R})\,, \quad \forall\, h \in \mathcal{T}\,. \tag{A.4}$$

This follows from the fact that if $0 = t_0 < t_1 < \ldots < t_n$ is the standard monotonicity partition of $u \circ h$, then $0 = h(t_0) < h(t_1) < \ldots < h(t_n)$ is the standard monotonicity partition of u. Let $u, v \in C^{\text{uc}}_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$. We first assume that there exist $f, g \in \mathcal{T}$ such that $u \circ f = v \circ g$. Then, using (A.4), $R(u) = R(u \circ f) = R(v \circ g) = R(v)$, as required.

To prove the converse, assume that $R(u) = R(v)$. By (A.4), without loss of generality we may assume that u and v have the same standard monotonicity partition $0 = t_0 < t_1 < \ldots < t_n$. For $i \in \{0, 1, \ldots, n-1\}$ we define $I_i := [t_i, t_{i+1}]$. Without loss of generality we may assume that both u and v are non-decreasing on I_i. Define $u_i := u|_{I_i}$, $v_i := v|_{I_i}$ and $J_i := [u(t_i), u(t_{i+1})] = [v(t_i), v(t_{i+1})]$.

By Lemma A.2, part (1), $\mathcal{C}_{u_i} \cup \mathcal{C}_{v_i}$ is at most countable and so by Lemma A.1 there exists a surjective continuous non-decreasing function $F_i : I_i \to J_i$, with $\mathcal{C}_{u_i} \cup \mathcal{C}_{v_i} \subset \mathcal{C}_{F_i}$. Therefore by Lemma A.2, part (4), there exist non-decreasing continuous functions $f_i, g_i : I_i \to I_i$ with $f_i(t_i) = t_i$, $f_i(t_{i+1}) = t_{i+1}$, $g_i(t_i) = t_i$ and $g_i(t_{i+1}) = t_{i+1}$ and such that $u_i \circ f_i = F_i = v_i \circ g_i$.

Define functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(t) = \begin{cases} f_i(t) & \text{if } t \in I_i\,, \\ t & \text{if } t \geq t_n\,, \end{cases}$$

and

$$g(t) = \begin{cases} g_i(t) & \text{if } t \in I_i\,, \\ t & \text{if } t \geq t_n\,. \end{cases}$$

By construction $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, non-decreasing and such that $f(0) = g(0) = 0$, $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty$ and $u \circ f = v \circ g$. □

Derivation of (3.10)

First we prove a simple lemma.

Lemma A.3 *Let $\Phi : C_{\rm pm}(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ be a hysteresis operator and let $u \in NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R})$. Assume that $\tau > 0$ is such that u is not left-continuous at τ. Let $0 < \tau_1 < \tau_2 < \ldots < \tau_n = \tau$ denote the points of of discontinuity of $Q_\tau u$, set $\tau_0 = 0$ and define ε_k by (3.5). Then, for all sufficiently large k*

$$(\Phi(C_k(Q_\tau u)))(\tau - \varepsilon_k/2) = (\tilde{\Phi}(u))(\tau-)\,.$$

Proof: Define $v \in NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R})$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \in [0,\tau)\,, \\ u(\tau-) & \text{if } t \geq \tau\,. \end{cases}$$

Set $s_k := \tau - \varepsilon_k/2$ and $w_k := Q_{s_k} C_k(Q_\tau u)$. There exists $l_1 > 0$ such that

$$R(Q_\tau C_k(v)) = R(C_k(v)) = R(w_k) = R(Q_\tau w_k)\,, \quad \forall\, k \geq l_1\,. \tag{A.5}$$

Moreover, by Lemma 3.5, statement (1), there exists $l_2 > 0$ such that

$$\tilde{R}(Q_\tau v) = R(Q_\tau C_k(v))\,, \quad \forall\, k \geq l_2\,. \tag{A.6}$$

Setting $l := \max(l_1, l_2)$, it follows from (A.5) and (A.6) that for all $k \geq l$, $\tilde{R}(Q_\tau v) = R(Q_\tau w_k)$. Hence, since v is left-continuous at τ we may conclude using Theorem 2.2, Theorem 3.4 and Corollary 3.9 that for all $k \geq l$

$$(\tilde{\Phi}(u))(\tau-) = (\tilde{\Phi}(v))(\tau) = (\Phi(w_k))(\tau) = (\Phi(Q_{s_k} C_k(Q_\tau u)))(\tau) = (\Phi(C_k(Q_\tau u)))(s_k)\,.$$

$\square$

Let $u \in NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R})$ and let $0 = t_0 < t_1 < t_2 < \ldots$ be such that $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each of the intervals (t_i, t_{i+1}). It is clear that $(\tilde{\mathcal{B}}_{h,\xi}(u))(0) = b_h(u(0),\xi)$. Let $t > t_0 = 0$; then there exists $i \in \mathbb{Z}_+ \setminus \{0\}$ such that $t \in (0,t_1) \cup [t_i, t_{i+1})$. There exists $l > 0$ such that

$$(C_k(Q_t u))(t) = u(t)\,, \quad \forall\, k \geq l\,, \tag{A.7}$$

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(\tau) = (\mathcal{B}_{h,\xi}(C_k(Q_\tau u)))(\tau)\,, \quad \forall\, k \geq l\,, \quad \forall\, \tau \in \{t_i, t\}\,, \tag{A.8}$$

where (A.8) follows from Proposition 3.6, part (1). We consider three cases.

Case 1: Suppose that $t \in (0, t_1)$.
Clearly, u is monotone on $[0,t]$ and so is $C_k(Q_t u)$. Hence, by (A.7) and (A.8)

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(Q_t u)))(0)) = b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(0))\,.$$

Case 2: Suppose that $t = t_i$.
There are two subcases which need to be distinguished.

Subcase a: u is left-continuous at $t = t_i$.
There exists $\varepsilon > 0$ such that u is monotone on $[t-\varepsilon, t]$. It follows that there exists $l_1 \geq l$ such that $C_k(Q_t\, u)$ is monotone on $[t-\varepsilon, t]$ for all $k \geq l_1$. Choose $(s_n) \subset [t-\varepsilon, t)$ with $\lim_{n\to\infty} s_n = t$. Using (A.8) and Corollary 3.9, we have

$$(\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(t) = (\tilde{\mathcal{B}}_{h,\xi}(u))(t) = (\tilde{\mathcal{B}}_{h,\xi}(u))(t-)\,, \quad \forall\, k \geq l_1\,. \tag{A.9}$$

Using (A.7) and (A.8) it follows for $k \geq l_1$

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(s_n))\,,$$

and hence, since $\lim_{n\to\infty}(\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(s_n) = (\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(t)$

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(t))\,. \tag{A.10}$$

Therefore, combining (A.9) and (A.10) gives

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(t-))\,.$$

Subcase b: u is not left-continuous at $t = t_i$.
Define for each $k \in \mathbb{Z}_+$, $s_k := t - \varepsilon_k/2$, where ε_k is defined by (3.5). $C_k(Q_t\, u)$ is monotone on $[s_k, t]$ and therefore using (A.7), (A.8) and Lemma A.3, for all sufficiently large k

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(s_k)) = b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(t-))\,.$$

Case 3: $t \in (t_i, t_{i+1})$
We consider the same two subcases as in Case 2.
Subcase a: u is left-continuous at t_i.
Define for each $k \in \mathbb{Z}_+$, $s_k := t_i + \varepsilon_k/2$, where ε_k is defined by (3.5). There exists $l_1 \geq l$ such that for $k \geq l_1$, $C_k(Q_t\, u)$ is monotone on $[s_k, t]$. Therefore, using (A.7) and (A.8)

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(s_k))\,, \quad \forall\, k \geq l_1\,. \tag{A.11}$$

Moreover, for all sufficiently large k, $C_k(Q_t\, u)$ is monotone on $[t_i, s_k]$. Hence, using (A.8) and Corollary 3.9, we may conclude that for all sufficiently large k

$$\begin{aligned}(\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(s_k) &= b_h((C_k(Q_t\, u))(s_k), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i)) \\ &= b_h((C_k(Q_t\, u))(s_k), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-))\,.\end{aligned}$$

Combining this with (A.11) gives, for all sufficiently large k

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), b_h((C_k(Q_t\, u))(s_k), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-)))\,.$$

Since $\lim_{k\to\infty}(C_k(Q_t\, u))(s_k) = u(t_i+)$, we have

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), b_h(u(t_i+), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-)))\,.$$

Subcase b: u is not left-continuous at t_i.
Therefore, u is right-continuous at t_i and hence u is monotone on $[t_i, t]$ and so is $C_k(Q_t\, u)$ for all sufficiently large k. By (A.7) and (A.8), for all sufficiently large k

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(Q_t\, u)))(t_i)) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(Q_{t_i}\, u)))(t_i)) \,.$$

Applying (A.8) again, we obtain

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i)) \,,$$

and so, by Case 2,

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), b_h(u(t_i), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-))) = b_h(u(t), b_h(u(t_i+), (\tilde{\mathcal{B}}_{h,\xi}(u))(t_i-))) \,.$$

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