ON A SPECTRUM OF BLOW-UP PATTERNS FOR A HIGHER-ORDER SEMILINEAR PARABOLIC EQUATION

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Abstract. We describe a spectrum of blow-up patterns for the $2m$-th order semilinear parabolic equation

$$u_t = -(-\Delta)^m u + |u|^p, \quad x \in \mathbf{R}^N, \; t > 0; \quad m > 1, \; p > 1.$$

This problem is well understood in the second-order case $m = 1$ (the semilinear heat equation), where for $p \le p_S = (N+2)/(N-2)_+$ the inner space-time structure of blow-up patterns is locally governed by separable Hermite polynomials of arbitrary finite order as eigenfunctions of a linear self-adjoint differential operator.

We consider $m > 1$ and describe a principle idea of constructing of a spectrum of blow-up patterns which in the inner self-similar region are composed of a countable subset of separable Kummer's polynomials which are eigenfunctions of a non self-adjoint linear differential operator. A matching procedure extends such linearized structures to the nonlinear intermediate region where the asymptotic blow-up behaviour is described by a unique self-similar solution of the first-order Hamilton-Jacobi equation.

1. Introduction: blow-up, critical exponents, singularity spectra

We consider the Cauchy problem for the $2m$-th order semilinear parabolic equation

$$u_t = \mathbf{A}_m u + |u|^p, \quad x \in \mathbf{R}^N, \; t > 0; \quad u(x,0) = u_0(x) \quad x \in \mathbf{R}^N, \tag{1.1}$$

where $p > 1$ is a fixed exponent and

$$\mathbf{A}_m = -(-\Delta)^m, \quad m > 1,$$

Δ being the Laplace operator in $\mathbf{R}^N$. We fix initial data $u_0 \in X \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. The lower-order term $f(u) = |u|^p$ is superlinear for $|u| \gg 1$, and we consider solutions $u(x,t)$ which blow-up in finite time: $u(x,t)$ is a bounded classical one in $\mathbf{R}^N \times (0, T']$ for any $T' < T$, and

$$\sup_{x \in \mathbf{R}^N} |u(x,t)| \to \infty, \quad t \to T^-. \tag{1.2}$$

It is known that $p_F = 1 + 2m/N$ is the *critical Fujita exponent* for the Cauchy problem (1.1):

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(i) (see [10] and [16]) If $p \in (1, p_F]$, then blow-up (1.2) occurs for arbitrarily small initial data with nonnegative first Fourier coefficient

$$\int_{\mathbf{R}^N} u_0 \ge 0.$$

(ii) (see [16]) If $p > p_F$, then solutions are global in time for any sufficiently small u_0.

We thus assume that in the supercritical case $p > p_F$, the initial function u_0 is large enough to guarantee blow-up in finite time.

The main goal of the present paper is to describe a discrete spectrum of possible blow-up asymptotic patterns and to study the stable generic one. This problem is well understood in the second-order case $m = 1$ (the semilinear heat equation), where the spectrum of blow-up patterns is locally composed from the separable Hermite polynomials of arbitrary order generated by a linear self-adjoint differential operator. These results are referred in detail in Sections 3 and 4, where we are going to compare them with those for $m > 1$.

We show that for $m > 1$, in the *inner region* governed by self-similar scaling, a spectrum of blow-up patterns is generated by a countable subset of separable Kummer's polynomials, which are eigenfunctions of a linear non self-adjoint differential operator. A matching procedure extends these linearized structures into the *intermediate (nonlinear) region*, where the asymptotic blow-up behaviour is described by a self-similar solution of the first-order Hamilton-Jacobi equation.

We see that the general structure of blow-up patterns for $m > 1$ is similar to that for the second-order heat equation, where the rescaled equation is potential and the linearized operator is self-adjoint. Though for $m > 1$ these potential and self-adjoint properties are not valid, we discover some striking similarities with the case $m = 1$ including the basic mathematical methods (already well established) which are necessary to treat such singularities rigorously. We expect that, in general, parabolic problems with $m \ge 1$ describe some general concepts of the discretization (self-organization) of smooth higher-order evolution flows approaching finite-time blow-up.

Moreover, a similar discrete spectrum of blow-up patterns is observed for the *semilinear wave equation*

$$u_{tt} = \Delta u + |u|^p, \quad x \in \mathbf{R}^N, \ \ t > 0,$$

see [15]. An essential difference between the parabolic case and the hyperbolic one is that for the wave equation the inner Kummer-like polynomial structures are generated by a *quadratic pencil* of non self-adjoint operators.

We thus observe an impressive similarity of a general construction of a discrete spectrum of finite-time singularities for a class of *semilinear* evolution partial differential equations regardless their order and even their type. In all such cases the inner singularity behaviour is governed by a discrete spectrum of eigenfunctions of linear operators or, in general, of pencils of linear operators.

It is important that such *linearized spectra* of singular patterns are available for semilinear equations only. For essentially *quasilinear* evolution equations, the interaction between the diffusivity-like operators and the lower-order term $|u|^p$ can lead to a number of first entirely nonlinear eigenfunctions satisfying quasilinear elliptic problems. Such examples are well-known in the mathematical literature on blow-up. These are (i) the quasilinear heat equation

$$u_t = \Delta u^m + u^p \quad (u \geq 0), \quad \text{with } m > 1,$$

see p. 190 in [34] and references therein, and (ii) the p-Laplacian equation with a source

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^p \quad (u \geq 0), \quad \sigma > 0,$$

see [14]. A natural parameter exchange between such linearized and nonlinear spectra is described in [5] by using the model equation with the exponential nonlinearity (a generalization of Frank-Kamenetskii equation from combustion theory)

$$u_t = (|u_x|^\sigma u_x)_x + e^u, \quad \sigma > 0.$$

It was shown that there exists a countable sequence of critical values $\{\sigma_k\} \to \sigma_\infty \approx 0.60$ such that at each $\sigma = \sigma_k$ exactly one linearized pattern is transformed into the corresponding nonlinear one. For $\sigma \in (0, \sigma_1)$, the first generic blow-up pattern is nonlinear and the rest of other patterns is governed by a discrete spectrum of a linear differential operator obtained via linearization. On the other hand, for $\sigma > \sigma_\infty$ the countable spectrum is entirely nonlinear so that for such an essentially quasilinear equation no linearization techniques apply and no Hermite-Kummer polynomial structures occur. Indeed, such a combination of nonlinear and linearized blow-up singularity spectra can be found for "sufficiently quasilinear" higher-order equations and equations of other types.

Principles of singularity formation in nonlinear media play a fundamental role in Mechanics and Physics. It is well known that such singularity phenomena describe microscale nonlinear effects (often chaotic and turbulent), which eventually lead and explain the laws of the medium self-organization which is responsible for global coherent structures. In the general theory of nonlinear evolution equations, such questions reduce to the study of spectra of different essentially singular (finite-time blow-up) solutions, and this idea was well understood a long time ago, see a survey paper [29] and references to Chap. 4 in [34]. The results of the present paper and those mentioned above, complete a formal description of types of formation of blow-up singularity spectra in nonlinear evolution equations.

2. Preliminaries

2.1. **Integral equation and local solvability.** The operator $\mathbf{A}_m$ is known to be the infinitesimal generator of a strongly continuous semigroup $e^{\mathbf{A}_m t}: X \to X$ for $t > 0$. In the corresponding equivalent integral equation

$$u(t) = e^{\mathbf{A}_m t} u_0 + \int_0^t e^{\mathbf{A}_m (t-s)} |u(s)|^p ds \tag{2.1}$$

the nonlinear term is a locally Lipschitz map. Denoting by $p(\omega) = -|\omega|^m$ the characteristic polynomial of $\mathbf{A}_m$, we have

$$e^{\mathbf{A}_m t} u_0 = b(t) * u_0,$$

where $b(t)$ is the kernel of the operator $e^{\mathbf{A}_m t}$ (the fundamental solution of the parabolic operator):

$$b(x,t) = \mathcal{F}^{-1}(e^{p(\omega)t}) \equiv (2\pi)^{-N} \int_{\mathbf{R}^N} e^{-|\omega|^m t - i(\omega . x)} d\omega, \quad b(x,0) = \delta(x).$$

The fundamental solution takes the standard self-similar form

$$b(x,t) = t^{-N/2m} f(\eta), \quad \eta = x/t^{1/2m}. \tag{2.2}$$

Substituting $b(x,t)$ into the linear equation

$$u_t = -(-\Delta)^m u, \tag{2.3}$$

we obtain that the radially symmetric profile $f(\eta)$ is a unique solution of a linear ordinary differential equation (ODE), which is the radial restriction of the elliptic equation:

$$\mathbf{B}^* f \equiv -(-\Delta_\eta)^m f + \frac{1}{2m} \nabla_\eta f \cdot \eta + \frac{N}{2m} f = 0 \;\text{ in } \mathbf{R}^N, \quad \int_{\mathbf{R}^N} f = 1, \tag{2.4}$$

where $\mathbf{B}^*$ has the equivalent divergent representation

$$\mathbf{B}^* f \equiv \mathbf{A}_m f + \frac{1}{2m} \nabla \cdot (\eta f). \tag{2.5}$$

There holds [9]

$$f(\eta) = (2\pi)^{-N/2} \int_0^\infty e^{-s^{2m}} s^{N/2} J_{(N-2)/2}(s|\eta|) ds, \tag{2.6}$$

where J_ν denotes Bessel's function. The integral equation (2.1) takes the form

$$u(t) = b(t) * u_0 + \int_0^t b(t-s) * |u(s)|^p ds. \tag{2.7}$$

Local solvability and uniqueness of the classical solution for (2.7) are straightforward, see e.g. [35], Chapt. 15. This determines the semigroup $u(t) = S(t)u_0$ defined in X locally in time, and by the interior parabolic regularity (or by the equivalent regularity analysis of the integral equation), if a classical solution blows up as $t \to T^-$, then (1.2) holds.

2.2. **On spectral properties of $\mathbf{B}^*$.** For any $m > 1$, operator $\mathbf{B}^*$ is not symmetric and does not admit a self-adjoint extension. The point spectrum $\sigma_p(\mathbf{B}^*)$ was studied in [11]. We first observe that due to (2.5) the eigenvalue problem

$$\mathbf{B}^* \psi^* = \lambda \psi^*, \quad \psi^* \in L^2(\mathbf{R}^N), \tag{2.8}$$

admits the first eigenfunction $\psi_0^* = f$ with $\lambda_0 = 0$.

Proposition 2.1. *Let $m > 1$. (i) The point spectrum of $\mathbf{B}^*$ in $L^2(\mathbf{R}^N)$ is real and discrete:*

$$\sigma_p(\mathbf{B}^*) = \{\lambda_\beta = -|\beta|/2m, \; |\beta| = 0, 1, 2, ...\}. \tag{2.9}$$

Each eigenvalue λ_β has a finite multiplicity and eigenfunctions numerated by a multiindex $\beta = (\beta_1, ..., \beta_N)$, $|\beta| \equiv \beta_1 + ... + \beta_N$, are:

$$\psi_\beta^*(\eta) = D^\beta f(\eta) \equiv \left(\frac{\partial}{\partial \eta_1}\right)^{\beta_1} ... \left(\frac{\partial}{\partial \eta_N}\right)^{\beta_N} f(\eta). \tag{2.10}$$

(ii) The set of eigenfunctions $\{\psi_\beta^*\}$ is complete in $L^2(\mathbf{R}^N)$.

The proof in [11] uses the rescaling of the convolution representation of the solution $u(t) = b(t) * u_0$, which by Taylor expansion of the kernel f gives all possible eigenfunctions. See similar computations in the next section.

In the classical parabolic case $m = 1$, where $f(\eta) = (4\pi)^{-N/2} e^{-|\eta|^2/4}$, we thus obtain the *separable Hermite polynomials*

$$e^{|\eta|^2/4}\, \psi_\beta^*(\eta) = H_\beta(\eta) \equiv H_{\beta_1}(\eta_1)...H_{\beta_N}(\eta_N), \tag{2.11}$$

which form an orthogonal basis in the weighted space $L^2_\rho(\mathbf{R}^N)$ with the associated strictly positive weight function $\rho(\eta) = e^{|\eta|^2/4} = (4\pi)^{-N/2}/f(\eta)$. If $m > 1$, then the operator is not self-adjoint, eigenfunctions are not orthogonal and are not generated by polynomials. Moreover, since f changes sign, no weighted space is available. See some other comments below.

2.3. **Majorizing order-preserving equation.** Using the main idea from [16], given the integral (evolution) equation (2.7), we show how to construct the corresponding *order-preserving majorizing equation.* The following estimate of the rescaled kernel f is well known, see [8].

Proposition 2.2. *Let $m > 1$. There exist constants $D > 1$ and $d > 0$ depending on m and N such that*

$$|f(\eta)| < D\,F(\eta) \equiv D\omega_1 e^{-d|\eta|^\alpha} \ \ \textit{in } \mathbf{R}^N, \tag{2.12}$$

where

$$\alpha = \frac{2m}{2m-1} \in (1,2) \quad \textit{and} \quad \omega_1 = \left(\int_{\mathbf{R}^N} e^{-d|\eta|^\alpha}\, d\eta\right)^{-1}.$$

Due to the definition of the fundamental solution, the constant d in (2.12) characterizes the "minimal" behaviour of solutions of the ODE (2.4) as $\eta \to \infty$. We next introduce the *majorizing kernel*

$$\bar b(x,t) = t^{-N/2m} F(\eta), \quad \eta = x/t^{1/2m}, \tag{2.13}$$

which is *strictly positive* for $t > 0$. Therefore, the corresponding majorizing integral equation with nonlinear Hammerstein's integral operator

$$v(t) = \bar b(t) * v_0 + \int_0^t \bar b(t-s) * D|v(s)|^p ds, \quad t > 0; \quad v_0 \geq 0, \tag{2.14}$$

describes the *order-preserving* evolution of nonnegative orbits with the usual partial order. Since the nonlinear term is a locally Lipschitz map, local existence and uniqueness of the solution $v(t)$ are straightforward, see [35], Chapt. 15. In particular we have the *positivity property*:

$$v_0 \geq 0,\, v_0 \not\equiv 0 \implies v(t) > 0,\, t > 0. \tag{2.15}$$

By comparing (2.7) and (2.14), we deduce the following result on the Maximum Principle (comparison) duality between the semilinear parabolic equation and the integral majorizing one, see [16] and also [6].

Lemma 2.1. *The integral equation* (2.14) *is majorizing for the* $2m$*-th order parabolic equation* (1.1) *in the following sense:*

$$D|u_0(x)| \leq v_0(x) \text{ in } \mathbf{R}^N \implies |u(x,t)| \leq v(x,t) \text{ for } x \in \mathbf{R}^N,\ t > 0. \tag{2.16}$$

One can see that, actually, we have the strong comparison property and the inequality in (2.16) is strict: $|u(x,t)| < v(x,t)$ for $t > 0$. It is important that, in general, for $m > 1$, the majorizing integral equation (2.14) is not generated by a semigroup unlike the second-order case $m = 1$, where $b(t) * u_0 = e^{\Delta t} u_0$. The solutions $v(x,t)$ of (2.14) are not time-translational invariant.

3. Self-similar rescaling and linearization

We suppose that $u_0(x)$ is bell-shaped so that the origin $x = 0$ belongs to the corresponding non-empty *blow-up set* of the orbit $\{u(t),\, t \in (0,T)\}$:

$$B(u_0) = \{x \in \mathbf{R}^N :\ \exists\ \{x_n\} \to x,\ \{t_n\} \to T^- \text{ such that } u(x_n,t_n) \to \infty\}. \tag{3.1}$$

By standard regularity results for parabolic equations [8], [13], the set $B(u_0)$ is closed.

We introduce the rescaled variables by setting

$$u(x,t) = (T-t)^{-1/(p-1)}\theta(\eta,\tau), \quad \eta = x(T-t)^{-1/2m}, \quad \tau = -\ln(T-t). \tag{3.2}$$

The rescaled solution θ solves the Cauchy problem

$$\theta_\tau = \mathbf{B}\theta - \frac{\theta}{p-1} + |\theta|^p, \quad \eta \in \mathbf{R}^N,\ \tau > \tau_0 = -\ln T, \tag{3.3}$$

with the initial data $\theta(\eta,\tau_0) = \theta_0(\eta) \equiv T^{1/(p-1)} u_0(T^{1/2m}\eta)$ in $\mathbf{R}^N$, where the linear operator $\mathbf{B}$ is a first-order perturbation of the $2m$-th order elliptic operator:

$$\mathbf{B} = -(-\Delta)^m - \frac{1}{2m}\eta \cdot \nabla\,. \tag{3.4}$$

We will study the asymptotic behaviour of the rescaled orbit $\{\theta(\tau),\, \tau > \tau_0\}$. The corresponding stationary equation

$$\mathbf{B}S - \frac{S}{p-1} + |S|^p = 0 \quad \text{in } \mathbf{R}^N, \tag{3.5}$$

admits two obvious constant solutions: the trivial one $S_0 \equiv 0$ and

$$S_*(\eta) \equiv \theta_* = (p-1)^{-1/(p-1)} > 0.$$

In the present paper we do not consider the problem of the existence of other nontrivial self-similar profiles S satisfying $S(\eta) \to 0$ as $\eta \to \infty$. For (3.5) and for a similar elliptic equation with the monotone nonlinearity $|S|^{p-1}S$, such nontrivial self-similar solutions are expected to play an important role for a general classification of blow-up patterns in the supercritical Sobolev range $p > p_S = (N+2m)/(N-2m)_+$. On the other hand, in the subcritical and critical ranges, Pohozaev's identities for the higher-order elliptic equation (3.5) [33] can be applied in order to derive a nonexistence-like result.

We now consider the stable, centre and unstable manifolds of the constant solution θ_* for the prescribed rescaled smooth parabolic flow. We perform a usual linearization

$$\theta(\tau) = \theta_* + Y(\tau),$$

where Y solves the Cauchy problem

$$Y_\tau = (\mathbf{B} + I)Y + \mathbf{D}(Y), \ \tau > \tau_0; \quad Y(\tau_0) = Y_0 \equiv \theta_0 - \theta_*. \tag{3.6}$$

$\mathbf{D}$ denotes an asymptotically quadratic perturbation of the linear operator $\mathbf{B} + I$:

$$\mathbf{D}(Y) = |\theta_* + Y|^p - \theta_*/(p-1) - pY/(p-1) = d_2 Y^2 + O(Y^3), \ \ Y \to 0, \tag{3.7}$$

where $d_2 = p(p-1)\theta_*^{p-2}/2$.

3.1. **Discrete point spectrum of the non self-adjoint operator B.** We study spectral properties of the linear operator

$$\mathbf{B} = -(-\Delta)^m - \frac{1}{2m}\eta \cdot \nabla. \tag{3.8}$$

If $m = 1$, it admits a natural symmetric representation

$$\mathbf{B} = \frac{1}{\rho} \nabla \cdot (\rho \nabla), \quad \rho(\eta) = e^{-|\eta|^2/4}. \tag{3.9}$$

Therefore, since $-\mathbf{B} \geq 0$ is semibounded from below, there exists a unique Friedrichs extension, which is a self-adjoint operator in the weighted Hilbert space $L^2_\rho(\mathbf{R}^N)$ with the domain $\mathcal{D}(\mathbf{B}) = H^2_\rho(\mathbf{R}^N)$ and a discrete point spectrum. The eigenfunctions form an orthonormal basis in $L^2_\rho(\mathbf{R}^N)$ and the classical Hilbert-Schmidt theory applies. These properties of self-adjoint operator play a fundamental role in the study of blow-up patterns in the case of the second-order semilinear parabolic equation, see e.g. [4], [12], [27], [31], [36], [37] and references in [34], [23].

For any $m > 1$, operator (3.8) is not self-adjoint. In the one-dimensional geometry, $N = 1$, where it becomes the even order ordinary differential operator

$$\mathbf{B}y = (-1)^{m+1} y^{(2m)} - \frac{1}{2m}\eta y', \tag{3.10}$$

it cannot be written down as a self-adjoint differential expression of the form $(p_0 y^{(m)})^{(m)} + \ldots + p_m y$ and is not self-adjoint. Then $\mathbf{B}^*$ is the adjoint differential expression for $\mathbf{B}$ in (3.10). See Chapt. 1 in [32].

In the general case, the following observation establishes the relationship between operators $\mathbf{B}$ and $\mathbf{B}^*$. $\mathcal{L}$ denotes the space of rapidly decreasing functions in $\mathbf{R}^N$.

Proposition 3.1. *Operator* $\mathbf{B}^*$ *in* (2.5) *is adjoint of* $\mathbf{B}$ *in* $\mathcal{L}$.

In order to derive the point spectrum of $\mathbf{B}$, we consider the corresponding parabolic equation

$$w_\tau = \mathbf{B}w, \quad \tau > 0; \quad w(0) = w_0 \in L^1_\rho(\mathbf{R}^N), \tag{3.11}$$

where for simplicity we set $T = 1$. Initial data w_0 are assumed to be bounded in a weighted space with the weight function $\rho(\eta) > 0$ to be naturally associated with the rate of decay of the rescaled kernel $f(\eta)$, see estimate (2.12). Then $w(\eta,\tau) = v(x,t)$ is the rescaled solution of the Cauchy problem for the parabolic equation

$$v_t = \mathbf{A}_m v, \quad t > 0; \quad v(0) = w_0, \tag{3.12}$$

so that

$$v(x,t) = t^{-N/2m} \int_{\mathbf{R}^N} f((x-y)t^{-1/2m}) w_0(y) dy. \tag{3.13}$$

It follows after rescaling that

$$w(\eta,\tau) = \int_{\mathbf{R}^N} f(\zeta - \nu) w_0(\zeta t^{1/2m}) d\zeta, \quad \nu = \eta[(1-t)/t]^{1/2m}. \tag{3.14}$$

We now use the analytic expansion of the kernel converging everywhere:

$$f(\zeta - \nu) = \sum_{(\gamma)} \frac{1}{\gamma!} D^\gamma f(\zeta) \nu^\gamma, \quad \nu^\gamma \equiv \nu_1^{\gamma_1} ... \nu_N^{\gamma_N}, \tag{3.15}$$

where $\gamma = (\gamma_1, ..., \gamma_N)$ is the multiindex. Substituting into (3.14), we obtain the following expansion:

$$w(\eta,\tau) = \sum_{(\gamma)} C_\gamma(\tau) \eta^\gamma, \tag{3.16}$$

with the expansion coefficients

$$C_\gamma(\tau) = \frac{1}{\gamma!} e^{-|\gamma|\tau/2m} (1 - e^{-\tau})^{-|\gamma|/2m} \langle \psi^*_\gamma(\zeta), w_0(\zeta(1-e^{-\tau})^{1/2m}) \rangle.$$

Here ψ^*_γ are eigenfunctions (2.10) of the adjoint operator $\mathbf{B}^*$ and $\langle \cdot, \cdot \rangle$ denotes the duality product (with the corresponding adjoint space to be determined). Note that the set of the separable polynomials $\{\eta^\gamma, \, |\gamma| = 0, 1, ...\}$ presented in the expansion (3.16) is complete in, say, $L^2_\rho(\mathbf{R}^N)$ since it contains a complete subset of the separable Hermite polynomials (2.11).

Using a necessary change of variables in the duality product, one obtains that

$$C_\gamma(\tau) = \frac{1}{\gamma!} e^{-|\gamma|\tau/2m} (1 - e^{-\tau})^{-N/2m} \int D^\gamma f(\zeta(1-e^{-\tau})^{-1/2m}) w_0(\zeta) d\zeta. \tag{3.17}$$

In order to obtain the eigenfunctions, we expand all the terms in (3.17):

$$(1 - e^{-\tau})^{-1/2m} = 1 + \sum_{k=1}^\infty b_{k0} e^{-k\tau}, \quad (1 - e^{-\tau})^{-N/2m} = 1 + \sum_{k=1}^\infty c_k e^{-k\tau},$$

and setting $h(\tau) = (1 - e^{-\tau})^{-1/2m} - 1$,

$$D^\gamma f(\zeta(1-e^{-\tau})^{-1/2m}) = D^\gamma f(\zeta + \zeta h(\tau)) = \sum_{(\delta)} \frac{1}{\delta!} D^{\gamma+\delta} f(\zeta) \zeta^\delta h^{|\delta|}(\tau)$$

$$= \sum_{(\delta)} \sum_{k=0}^{\infty} e^{-(|\delta|+k)\tau} b_{k\delta} \frac{1}{\delta!} D^{\gamma+\delta} f(\zeta)\zeta^\delta.$$

We now state the following property of the eigenfunctions $\psi^*_\gamma = D^\gamma f$ of $\mathbf{B}^*$. Applying the differential operator D^γ to equation (2.4), we obtain the following relation between the higher-order eigenfunctions and the lower-order ones:

$$(-1)^{m+1}\psi^*_{\gamma+2mI} + \frac{1}{2m}\nabla\psi^*_\gamma \cdot \eta + \frac{N+|\gamma|}{2m}\psi^*_\gamma = 0 \quad (I = (1,...,1)).$$

This makes it possible to express the first-order operator in (3.15),

$$\nabla\psi^*_\gamma \cdot \eta \equiv \psi^*_{(\gamma_1+1,...,\gamma_N)}\eta_1 + ... + \psi^*_{(\gamma_1,...,\gamma_N+1)}\eta_N,$$

and get rid of some of the odd-order differential expressions. In particular, if $\gamma = (\gamma_1,...,\gamma_N)$ is such that $\gamma_i \geq 2m$ for any $i = 1,2,...,N$, then by setting $\gamma + 2mI = \delta$ we get the recurrent relation

$$(-1)^{m+1}\psi^*_\delta + \frac{1}{2m}\nabla\psi^*_{\delta-2mI}\cdot\eta + \frac{N+|\delta-2m|}{2m}\psi^*_{\delta-2mI} = 0.$$

Integrating in (3.17), substituting in (3.16) and summing up the terms with common exponential multipliers by using the above recurrences, we arrive at the following expression of the solution: for sufficiently smooth initial data $w_0 \in L^1_\rho(\mathbf{R}^N)$ with the weight function

(3.18) $$\rho(\eta) = e^{-d_*|\eta|^\alpha} > 0 \quad (\alpha = 2m/(2m-1)),$$

where $d_* = d_*(m,N) \in (0,d]$ is a small constant, there holds

(3.19) $$w(\eta,\tau) = \sum_{(\beta)} M_\beta(w_0) e^{-|\beta|\tau/2m}\psi_\beta(\eta),$$

where the functions ψ_β are given by special finite linear combinations of the functions from $\{\eta^\gamma\}$ and the multipliers $M_\beta(w_0)$ denote the β-th "momenta" of initial data w_0. Moreover, the general linear separate structure of (3.19) with purely exponential τ-dependent factors implies a complete characterization of the point spectrum of $\mathbf{B}$. Using estimate (2.12) of the rescaled kernel, we fix the space $L^2_\rho(\mathbf{R}^N)$ with the weight function (3.18), where the exponent $d_* > 0$ is always assumed to be sufficiently small, see comments below. We ascribe to the differential operator $\mathbf{B}$ the dense domain $W^{2m,2}_\rho(\mathbf{R}^N)$. As a differential operator with smooth coefficients, $\mathbf{B}$ is closable, and by $\mathbf{B}$ we will also denote its smallest closed extension with the domain $\mathcal{D}(\mathbf{B})$.

Lemma 3.1. *(i) Operator* $\mathbf{B}$ *in* $L^2_\rho(\mathbf{R}^N)$ *has the discrete real point spectrum*

(3.20) $$\sigma_p(\mathbf{B}) = \{\lambda_\beta = -|\beta|/2m, \;\; |\beta| = 0,1,2,...\} \quad (= \sigma_p(\mathbf{B}^*)).$$

Each eigenfunction has the separable decomposition

$$\psi_\beta(\eta) = \psi_{\beta_1}(\eta_1)...\psi_{\beta_N}(\eta_N), \tag{3.21}$$

where $\psi_{\beta_k}(\eta_k)$ *is an eigenfunction of the one-dimensional operator* (3.10) *with* $\eta = \eta_k$.
(ii) The subset $\{\psi_\beta\}$ *is complete in* $L^2_\rho(\mathbf{R}^N)$.

Proof. The result (i) follows from expansion (3.19), which also guarantees that $\sigma_p(\mathbf{B})$ contains no other eigenvalues $\lambda \in \mathbb{C}$. Completeness (ii) follows from (3.21) and the fact that the subset of elementary polynomials $\{\eta^\beta\}$ is complete. In general, regardless the pure polynomial structure of eigenfunctions, completeness properties in weighted spaces can be also obtained from the continuity of the corresponding uniformly parabolic flow (3.12). $\square$

For $m = 1$, both (3.20) and (3.21) are well-known properties of the separable Hermite polynomials, see [3].

Let us note that the distribution (3.20) of the eigenvalues as $|\beta| \to \infty$ plays an important role for the completeness of the generalized eigenfunctions. Using Schmidt's Theorem [24], Chapt. 6, one can consider $\mathbf{B} - I$ with the uniformly negative point spectrum $\sigma_p(\mathbf{B} - I) = \{-|\beta|/2m - 1 < 0\}$. Assuming that the resolvent $\mathbf{C} = R_1(\mathbf{B}) = (\mathbf{B} - I)^{-1} : H^{2m}_\rho(\mathbf{R}^N) \to H^{2m}_\rho(\mathbf{R}^N)$ is a compact integral operator (a kernel condition, this can be easily checked for $N = 1$ or in the radial case for the ordinary differential operator, see the results below), we have that $\mathbf{CC}^*$ and $\mathbf{C}^*\mathbf{C}$ are positive compact self-adjoint operators. Consider the ordered s-numbers denoted by μ,

$$\mathbf{C}v = \mu w, \quad \mathbf{C}^* w = \mu v,$$

so that μ^2 are real eigenvalues of $\mathbf{CC}^*$ and $\mathbf{C}^*\mathbf{C}$. The rate of convergence to zero of the characteristic numbers is known to measure the closeness of $\mathbf{C}$ to finite-dimensional compact operators, For instance, if for some $p \geq 1$,

$$|\beta|^{1/p}\mu_\beta \to 0, \quad |\beta| \to \infty,$$

then this rate of convergence corresponds to the completeness of generalized eigenfunctions in $H^{2m}_\rho(\mathbf{R}^N)$ [25]. We will show that in the one-dimensional case or in the radial setting the s-numbers satisfy $|\mu_\beta| \sim 1/|\beta|$ as $|\beta| \to \infty$ and this gives the completeness. On the other hand, in these two cases of ordinary differential operators $\mathbf{B}$ the completeness can be established by the ODEs techniques, see the analysis below.

3.2. **The one-dimensional spectral problem.** It follows from Lemma 3.1 that it is sufficient to study in detail the point spectrum and eigenfunctions of the one-dimensional ordinary differential operator (3.10). We then follow [32]. Consider $\mathbf{B}$ and the adjoint differential expression $\mathbf{B}^*$ in $L^1_\rho(\mathbf{R}_+)$ with domains $W^{2m,1}_\rho(\mathbf{R}_+)$ of functions satisfying the symmetry condition at the origin

$$y'(0) = ... = y^{(2m-1)}(0) = 0. \tag{3.22}$$

In addition, at $\eta = \infty$ the boundary conditions are imposed by the weight function ρ associated with the minimal behaviour of solutions of the ODE. Using a standard asymptotic analysis for the linear ODE

$$\mathbf{B}y = \lambda y, \tag{3.23}$$

we have that for any $\lambda \in \mathbb{C}$, the dominant exponential behaviour

$$y(\eta) \sim e^{a\eta^\alpha}, \quad \eta \to \infty \quad (a \in \mathbb{C}),$$

is governed by the two differential operators in (3.10) and this yields the following algebraic equation on a:

$$(-1)^{m+1}(\alpha a)^{2m-1} = (2m)^{-1}.$$

Here, for odd $m = 1, 3, 5, ...$,

$$a = a_k = \alpha^{-1}(2m)^{-1/(2m-1)}e^{2\pi il/(2m-1)}, \;\; l = 0, 1, ..., 2m-2,$$

and for even $m = 2, 4, ...$,

$$a_k = \alpha^{-1}(2m)^{-1/(2m-1)}e^{i[\pi+2\pi l/(2m-1)]}, \;\; l = 0, 1, ..., 2m-2.$$

It follows that for any $m > 1$, there exist m exponentially growing solutions with $\operatorname{Re} a_k > 0$ and $m-1$ exponentially decaying solutions with $\operatorname{Re} a_k < 0$. The last $2m$-th intermediate type of behaviour corresponds to finite order Kummer's polynomials, see the proof below. In the second-order case $m = 1$, the exponentially decaying solutions are not available. Nevertheless, for $m > 1$, similar to $m = 1$, the non-empty $(m-1)$-dimensional subset of exponentially decaying solutions generates no eigenfunctions.

Proposition 3.2. *For any $m \geq 1$, the operator $\mathbf{B}$ admits no eigenfunctions with exponential decay at infinity.*

Proof. This is again a straightforward consequence of the decomposition (3.19) with finite order separable polynomials ψ_β. $\square$

Using the above analysis, we present the necessary condition at infinity:

$$\rho(\eta)y(\eta) \;\text{ is bounded in }\; \mathbf{R}_+. \tag{3.24}$$

One can see that (3.24) imposes exactly half of boundary conditions at the right-hand end point of the interval $(0, \infty)$, so that both conditions (3.22) and (3.24) correspond to the special case of separable boundary conditions of Sturm type for even-order differential operators [32].

Condition (3.24) naturally corresponds to the weighted space $L^\infty_\rho(\mathbf{R}_+)$ and the domain of $\mathbf{B}$ is, say, $C^{2m}_\rho(\mathbf{R}_+)$, with the norm

$$\|y\|_\rho = \sup_{\eta \in \mathbf{R}_+} \rho(\eta)(|y(\eta)| + ... + |y^{(2m)}(\eta)|).$$

It is dense in $L^1_\rho(\mathbf{R}_+)$ and in $L^2_\rho(\mathbf{R}_+)$. Moreover, the asymptotic behaviour of solutions at $\eta \to \infty$ shows that these weighted spaces are also naturally connected with the condition (3.24) provided that $d_* > 0$ is small enough. Therefore, we can pose the eigenvalue problem for $\mathbf{B}$ in $L^2_\rho(\mathbf{R}^N)$ (or in $L^q_\rho(\mathbf{R}^N)$ with any $q \geq 1$) in the space of symmetric functions satisfying (3.22).

A similar analysis applies to the adjoint operator $\mathbf{B}^*$ and we impose the same condition (3.24). The smallest closed extensions of the operator $\mathbf{B}$ and of the adjoint one $\mathbf{B}^*$ satisfy

$$\langle \mathbf{B}y, z\rangle = \langle y, \mathbf{B}^* z\rangle, \quad y \in \mathcal{D}(B), \;\; z \in \mathcal{D}(B^*). \tag{3.25}$$

Let us state some typical results for ordinary differential operators, see Chapt. 1 in [32].

Proposition 3.3. *The boundary conditions at the origin* (3.22) *and at infinity are self-adjoint.*

Proof. (3.22) are symmetry conditions for the self-adjoint operator $y^{(2m)}$ in $L^2(\mathbf{R}_+)$. Fixing arbitrary solutions y and z of

$$\mathbf{B}y = \lambda y, \quad \mathbf{B}^* z = \mu z,$$

satisfying the boundary conditions, integrating by parts, we obtain

$$\langle \mathbf{B}y, z\rangle = \langle y, \mathbf{B}^* z\rangle + [(y^{(2m-1)} - \frac{1}{2m}\, y)z - y^{(2m-2)}z' + y^{(2m-3)}z'' - ... + yz^{(2m-1)}]|_0^\infty,$$

where all the non-integral terms vanish due to the boundary conditions at $\eta = 0$ and known exponential decay of the admissible solutions $y(\eta)$, $z(\eta)$ and their derivatives as $\eta \to \infty$. This implies (3.25) and completes the proof. $\square$

We now present a general description of the eigenvalue-eigenfunction pairs of operator $\mathbf{B}$ in the weighted space $L^1_\rho(\mathbf{R})$. The symmetry condition at the origin (3.22) restricts to the even-order polynomials only.

Lemma 3.2. *Let $N = 1$. (i) The ordinary differential operator* (3.10) *in $L^2_\rho(\mathbf{R})$ has the discrete real point spectrum of simple eigenvalues $\sigma_p(\mathbf{B}) = \{\lambda = -\gamma/2m, \;\; \gamma = 0, 1, 2, ...\}$, and each eigenfunction ψ_γ is a γ-th order polynomial.*

(ii) $\sigma_p(\mathbf{B}) = \sigma_p(\mathbf{B}^*)$ and the orthogonality condition holds:

$$\langle \psi_\gamma, \psi^*_\mu\rangle = 0 \quad if \;\; \lambda_\gamma \neq \lambda_\mu. \tag{3.26}$$

For instance, we have:

$$\psi_\gamma(\eta) = c_\gamma \eta^\gamma, \;\; \gamma < 2m; \quad \psi_{2m}(\eta) = c_{2m}\left(1 + (-1)^m \frac{1}{(2m)!}\eta^{2m}\right), \tag{3.27}$$

where c_γ denote normalization constants. Setting $m = 1$ yields $\psi_2 = c_2(1 - \eta^2/2)$, the second Hermite polynomial $H_2(\eta)$.

Proof. (i) Obviously, there exist (degenerate) eigenfunctions (3.27) with any $\gamma < 2m$. We now look for a solution of the ODE (3.23) in the form of Kummer's series

$$Y(\eta) = \sum_{k=0}^{\infty} C_k \eta^k.$$

Substituting into (3.23), one obtains the recurrent formula: for any $k \geq 2m$

$$C_{2m+k} = (-1)^{m+1} \frac{\lambda + \frac{k}{2m}}{(2m+k)(2m+k-1)...(k+1)}\, C_k, \tag{3.28}$$

so that for $k \gg 2m$

$$C_{2m+k} \sim (-1)^{m+1} \frac{1}{k^{2m-1}}\, C_k.$$

Hence, similar to Kummer's polynomials [2], the radius of convergence of the series is $R = \infty$. It follows from the above asymptotic analysis that this gives either exponentially growing solutions (not satisfying (3.24)) or exponentially decaying solutions, which cannot be eigenfunctions by Proposition 3.2, unless $\lambda = -k/2m$ for some $k \geq 2m$. The corresponding eigenfunctions $\psi_\gamma \in L^2_\rho(\mathbf{R})$ are finite polynomials.

(ii) The orthogonality property is standard. Assume $\mathbf{B}\psi_\gamma = \lambda_\gamma \psi_\gamma$ and $\mathbf{B}^* \psi^*_\mu = \lambda_\mu \psi^*_\mu$. Then

$$\langle \mathbf{B}\psi_\gamma, \psi^*_\mu \rangle = \lambda_\gamma \langle \psi_\gamma, \psi^*_\mu \rangle, \quad \langle \psi_\gamma, \mathbf{B}^* \psi^*_\mu \rangle = \lambda_\mu \langle \psi_\gamma, \psi^*_\mu \rangle,$$

so that $(\lambda_\gamma - \lambda_\mu)\langle \psi_\gamma, \psi^*_\mu \rangle = 0$, whence the orthogonality (due to the reality of the spectrum we perform calculations in a space over the field of real numbers). $\square$

3.3. **The radial N-dimensional spectral problem.** The results for $N = 1$ can be translated to the radial N-dimensional ordinary differential operator $\mathbf{B}$ written down in terms of the single variable $\xi = |\eta| > 0$:

$$\mathbf{B}_R = -(-\Delta_\xi)^m - \frac{1}{2m} \xi \frac{d}{d\xi}\, . \tag{3.29}$$

The boundary conditions are

$$y'(0) = ... = y^{(2m-1)}(0) = 0, \tag{3.30}$$

$$\rho(\xi) y(\xi) \quad \text{is bounded in } \mathbf{R}_+ \quad (\rho(\xi) = e^{-d_* \xi^\alpha}). \tag{3.31}$$

Lemma 3.3. *In the radial case, $N \geq 1$, (i) operator $\mathbf{B}_R$ in $L^2_{\bar\rho}(\mathbf{R}_+)$, $\bar\rho(\xi) = \xi^{N-1}\rho(\xi)$, has the real discrete spectrum of simple eigenvalues*

$$\sigma(\mathbf{B}_R) = \{\lambda_{2k} = -k/m, \;\; k = 0, 1, 2, ...\}. \tag{3.32}$$

The eigenfunctions ψ_{2k}, $k \geq m$, are even $2k$-th order polynomials of the form

$$\psi_{2k}(\xi) = c_{2k}(1 + ... + b_{2k}\xi^{2k}) \tag{3.33}$$

(ii) The orthogonality condition (3.26) *holds and $\{c_{2k}\}$ in* (3.33) *are the normalization constants (for convenience, we put $c_{2k} b_{2k} < 0$).*

(iii) The set of eigenfunctions $\{\psi_{2k}\}$ is complete in the weighted space $H^{2m}_{\bar\rho}(\mathbf{R}_+)$ of symmetric functions.

Proof. The statements (i) and (ii) are proved similarly to those in the previous lemma. Such ordinary differential operators with boundary conditions of Sturm type admit point spectrum only [32]. (iii) Let us discuss the completeness using an ODE argument. Consider the operator $\mathbf{B}-I$ with the uniformly negative real point spectrum. Hence, as typical for ordinary differential operators, the resolvent $\mathbf{C} = R_1(\mathbf{B}) = (\mathbf{B} - I)^{-1} : H^{2m}_{\bar\rho}(\mathbf{R}_+) \to H^{2m}_{\bar\rho}(\mathbf{R}_+)$ exists and is bounded and $\sigma(\mathbf{C}) = \{-(k/m+1)^{-1}\}$. We next claim that $\mathbf{C}$ is a compact operator. Since $\sigma(\mathbf{C})$ is composed from simple eigenvalues, the ODEs theory applies. It follows [32] that for any symmetric $g \in H^2_{\bar\rho}(\mathbf{R}_+)$ there exists a uniformly convergent series of eigenfunctions

$$g(\xi) = \sum_{k=0}^{\infty} a_{2k}\psi_{2k}(\xi), \quad \text{with } a_{2k} = \langle g, \psi^*_{2k}\rangle. \tag{3.34}$$

This completes the proof. $\square$

One can calculate the first m (degenerate) eigenfunctions

$$\psi_0 = c_0, \;\; \psi_2 = c_2\xi^2, \;\; ... \;, \psi_{2(m-1)} = c_{2(m-1)}\xi^{2(m-1)}, \tag{3.35}$$

and

$$\psi_{2m}(\xi) = c_{2m}\left(1 + b_{2m}\xi^{2m}\right), \quad b_{2m} = (-1)^m \frac{(N-2)!!}{(2m)!!(2m+N-2)!!}. \tag{3.36}$$

The eigenfunctions are orthonormal:

$$\langle \psi_\lambda, \psi^*_\mu\rangle = \delta_{\lambda\mu}. \tag{3.37}$$

4. **Centre manifold behaviour in the radial setting**

Consider equation (3.6) in the radial case

$$Y_\tau = (\mathbf{B}_R + I)Y + \mathbf{D}(Y), \quad \tau > \tau_0. \tag{4.1}$$

It follows from Lemma 3.3 that $\mathbf{B}_R + I$ has the discrete real spectrum

$$\sigma(\mathbf{B}_R + I) = \{\lambda_{2k} = 1 - k/m, \; k = 0, 1, ...\}. \tag{4.2}$$

This gives the following unstable, centre and stable subspaces:

$$E^u = \text{Span}\{\psi_0, \psi_2, ..., \psi_{2(m-1)}\}, \quad E^c = \text{Span}\{\psi_{2m}\}, \quad E^s = \text{Span}\{\psi_{2(m+1)}, ...\}.$$

By completeness of eigenfunctions, there holds

$$H^{2m}_{\bar\rho}(\mathbf{R}_+) \subseteq E^u \oplus E^c \oplus E^s. \tag{4.3}$$

4.1. **On the unstable manifold.** By the spectral properties, the ordinary differential operator $\mathbf{B}_R$ is sectorial, and $\sigma_+(\mathbf{B}_R + I) = \{\lambda \in \sigma(\mathbf{B}_R + I) : \ \mathrm{Re}\,\lambda \ge 0\}$ consists of (finite) $(m+1)$ isolated simple eigenvalues. In this case, a general stable-centre-unstable manifold analysis applies to equation (4.1), see [30].

As usual in blow-up problems, we expect that in the semilinear equation, perturbations on the unstable manifold tangent to the subspace E^u correspond to blow-up evolution with a different blow-up time. For $m = 1$ this is proved by using the standard comparison since the flow is order-preserving. For $m > 1$, where the fundamental solution (and the rescaled kernel f) changes sign and the semigroup is not order-preserving, the proof becomes more involved. Let us note that in the linearized equation

$$Y_\tau = (\mathbf{B}_R + I)Y, \quad \tau > \tau_0, \tag{4.4}$$

which is the rescaled version of the linear parabolic equation for the function $u(x,t) = (T-t)Y(\xi,\tau)$, $\xi = x/(T-t)^{1/2m}$, satisfying

$$u_t = -(-\Delta)^m u, \tag{4.5}$$

the unstable modes occur as a consequence of the obvious invariance of the $2m$-th order parabolic equation under the polynomial translations (radial setting)

$$u \mapsto u+1, \quad u \mapsto u + |x|^2, \ \dots, u \mapsto u + |x|^{2(m-1)}.$$

The unstable manifold of the origin for equation (4.1) in $H^{2m}_{\bar\rho}(\mathbf{R}_+)$ corresponds to the evolution with a different blow-up time and such a behaviour cannot occur if the orbit is rescaled with the correct value of the blow-up time $T = T(u_0)$ (a well established result for the order-preserving semigroup with $m = 1$). Therefore, the stability of the constant stationary solution $\theta \equiv \theta_*$ in the subset of the rescaled orbits $\{\theta(\tau), \ \tau > \tau_0\}$ (the structural stability) depends on the stability properties of the centre manifold behaviour.

4.2. **Stability on the centre manifold.** We will show that for any $m > 1$, the centre manifold behaviour is governed by equations quite similar to those which occur in the second-order self-adjoint (potential) case $m = 1$.

On first results for $m = 1$. The centre manifold analysis of blow-up orbits for the semilinear heat equation

$$u_t = \Delta u + u^p \quad p > 1,$$

has a remarkable history. The exact rate of convergence

$$Y(\xi,\tau) = \tau^{-1}\,\psi_0(\xi) + \dots, \quad \tau \to \infty,$$

corresponding to the first generic blow-up pattern, was first formally derived about 30 years ago in [28] for the exponent $p = 3$ and $N = 1,2$ by using a fundamental idea of the analytic expansion of the rescaled solution as $\tau \to \infty$, which admits a rigorous justification. Since $\tau = -\ln(T-t)$, it then follows that the blow-up behaviour in the intermediate domain is governed by the new rescaled spatial variable

$$\zeta = |x|/\sqrt{(T-t)|\ln(T-t)|}, \quad t \to T^-. \tag{4.6}$$

The paper [28] is one of the first works, where a slow logarithmic factor $\sqrt{|\ln(T-t)|}$, $t \to T^-$, appeared in the mathematical literature on blow-up singularities in nonlinear PDEs (together with the results on self-focusing for the cubic nonlinear Schrödinger equation, see e.g. a survey paper [23]). It turned out that such an analytic approach applies for any exponent $p > 1$ and gives a unique asymptotic profile [17], [18]. Moreover, it was shown that the limit profile $f_*(\zeta)$ rescaled with respect to the variable ζ (it essentially stays the same for any $m > 1$, see the expressions for f_* derived below) occurs in estimates of blow-up solutions for a wide class of quasilinear equations $u_t = \nabla \cdot (k(u)\nabla u) + Q(u)$, see Sect. 4 in [22] and earlier references therein. A detailed centre manifold analysis was performed in [4], [12], [27], [31], [36], [37], see also references in [23]. In the subcritical range $p < p_S = (N+2)/(N-2)_+$ the singularity classification problem is now solved in the maximal generality.

Higher-order equations, $m > 1$**.** In order to study equation (4.1) we fix a complete orthonormal subset of eigenfunctions $\{\psi_{2k}\}$ so that (3.37) holds. Looking for a solution in the form

$$Y(\xi,\tau) = \sum a_{2k}(\tau)\psi_{2k}(\xi),$$

substituting into (4.1) and projecting, we obtain the dynamical system

$$\dot a_{2k} = (1 - k/m)a_{2k} + \langle \mathbf{D}(\textstyle\sum a_{2k}\psi_{2k}), \psi^*_{2k}\rangle, \quad k = 0,1,2,\ldots. \tag{4.7}$$

Assuming that the behaviour on the centre manifold $W^c(0)$ (tangent to E^c) is dominant, setting $k = m$, we have the following asymptotic equation for $\tau \gg 1$:

$$\dot a_{2m} = \langle \mathbf{D}(\textstyle\sum a_{2k}\psi_{2k}), \psi^*_{2m}\rangle = \langle \mathbf{D}(a_{2m}\psi_{2m} + \ldots), \psi^*_{2m}\rangle = a^2_{2m} d_2 \langle \psi^2_{2m}, \psi^*_{2m}\rangle + \ldots,$$

where we have used expansion (3.7) applicable in the weighted space with the exponentially decaying weight function. Denoting $(h_m)^{-1} = -d_2\langle \psi^2_{2m}, \psi^*_{2m}\rangle$ (necessarily, $h_m > 0$), we arrive at the quadratic equation $\dot a_{2m} = -h_m^{-1} a^2_{2m} + \ldots$ for $\tau \to \infty$, and hence

$$a_{2m}(\tau) = h_m \tau^{-1}\,(1 + o(1)) \to 0, \quad \tau \to \infty. \tag{4.8}$$

This describes the centre manifold behaviour in terms of Y,

$$Y(\xi,\tau) = h_m \tau^{-1}\,\psi_{2m}(\xi) + \ldots, \quad \tau \to \infty, \tag{4.9}$$

which is valid on compact subsets in ξ. Since ψ_{2m} is a $2m$-th order polynomial, for large $\xi \gg 1$, we have

$$Y(\xi,\tau) = -\tau^{-1} e_m \xi^{2m} + \ldots\,, \quad e_m = -h_m c_{2m} b_{2m} > 0, \tag{4.10}$$

where c_{2m} is the normalization constant of the eigenfunction ψ_{2m} and b_{2m} is as given in (3.36). Remind that by the construction of eigenfunctions, $c_{2m}b_{2m} < 0$.

5. Stable manifold patterns, completeness of the spectrum

The patterns on the stable manifold $W^s(0)$ tangent to E^s are generated by the corresponding eigenfunctions with the asymptotic behaviour for $\tau \gg 1$ on compact subsets

$$Y(\xi,\tau) = Ce^{(1-k/m)\tau}\psi_{2k}(\xi) + \ldots, \quad k = m+1, m+2, \ldots\,, \tag{5.1}$$

where $C > 0$ depends on the initial data. Similar to the case $m = 1$, all such patterns are unstable.

The completeness of the spectrum of such blow-up patterns assumes that solutions to equation (4.1) in $H^{2m}_{\bar\rho}(\mathbf{R}_+)$ cannot decay to zero faster than exponentially. For $m = 1$ this problem has been solved by a number of different approaches in the papers mentioned above. A similar problem was analyzed in detailed earlier in [26] in connection with Morse-Smale systems generated by parabolic equations. It is important that the general approach in [26] is stated in a maximal generality for rather arbitrary non self-adjoint operators and is based on general Agmon's estimates. See also [1]. We expect that such an approach can be applied to the present equation which is a quadratic perturbation of a smooth parabolic flow associated with an analytic semigroup generated by a non self-adjoint operator.

6. **Extension to the intermediate Hamilton-Jacobi region**

6.1. **Extension of the centre manifold pattern.** Since $\psi_{2m}(\xi) = c_{2m} b_{2m} \xi^{2m} + ...$ as $\xi \to \infty$, using (4.9), we have that for large $\xi \gg 1$, (4.10) holds. This suggests to introduce the new rescaled variable

$$\zeta = \xi \tau^{-1/2m} \equiv |x|[(T-t)|\ln(T-t)|]^{-1/2m}, \quad t \to T^-, \tag{6.1}$$

so that (4.10) implies that on the lateral boundary of the domain of the centre manifold expansion (4.9)

$$Y = -e_m \zeta^{2m} + ..., \quad \tau \to \infty \quad (\xi \to \infty). \tag{6.2}$$

It follows from (3.3) that the rescaled function $\theta = \theta(\zeta, \tau)$ satisfies

$$\theta_\tau = \mathbf{H}_m(\theta) + \frac{1}{\tau} \mathbf{F}\theta, \quad \tau > \tau_0, \tag{6.3}$$

$$\mathbf{H}_m(\theta) = -\frac{1}{2m} \zeta \theta_\zeta - \frac{\theta}{p-1} + |\theta|^p, \quad \mathbf{F} = -(-\Delta)^m - \frac{1}{2m} \zeta \frac{d}{d\zeta}.$$

Hence, as $\tau \to \infty$, (6.3) is an asymptotically small perturbation of the order $O(1/\tau)$ ($\notin L^1((1,\infty))$, i.e., a non-integrable one) of the first-order Hamilton-Jacobi equation

$$h_\tau = \mathbf{H}_m(h), \quad \tau > \tau_0. \tag{6.4}$$

Such singularly perturbed dynamical systems occur in several reaction-diffusion equations, see [19] and [21], [22], where a general stability theorem covering this special case of nonintegrable asymptotically small singular perturbations of dynamical systems was proved. The main hypothesis of this stability theory is the uniform Lyapunov stability of the ω-limit set of the unperturbed equation (6.4) established in [21], Sect. 3, for general equations like (6.4) in a Banach space $C_\rho(\mathbf{R}_+)$ with a singular weight $\rho(\xi) = |\xi|^{-2}$.

Finally, we conclude that the orbit approaches the stationary profiles satisfying

$$\mathbf{H}_m(f) = 0, \ \ \zeta > 0; \quad f(0) = \theta_*,$$

and this gives a one-parameter family of the limit profiles:

$$f(\zeta) = [(p-1)(1+A\zeta^{2m})]^{-1/(p-1)}, \quad A \ge 0. \tag{6.5}$$

Since $f(\zeta) = \theta_* - A(p-1)^{-p/(p-1)}\zeta^{2m} + ...$ as $\zeta \to 0$, comparing with (6.2) for the intermediate values of ζ yields the unique stable profile (6.5), $f_*(\zeta)$, with the constant

$$A = A_* \equiv e_m(p-1)^{p/(p-1)} > 0. \tag{6.6}$$

6.2. **Extension of the stable manifold patterns.** This analysis is quite similar. It follows from (5.1) that for $\xi \gg 1$

$$Y(\xi,\tau) = Cc_{2k}b_{2k}e^{(1-k/m)\tau}\xi^{2k} + ... \equiv Cc_{2k}b_{2k}\zeta^{2k} + ... \quad (C>0), \tag{6.7}$$

where ζ is the new spatial variable

$$\zeta = \xi e^{-\tau(k-m)/2km} \equiv |x|(T-t)^{-1/2k}, \quad k = m+1, m+2, ... \tag{6.8}$$

Then $\theta(\zeta,\tau)$ solves the exponentially perturbed equation

$$\theta_\tau = \mathbf{H}_k(\theta) - e^{-(1-m/k)\tau}(-\Delta)^m\theta, \tag{6.9}$$

with the Hamilton-Jacobi operator

$$\mathbf{H}_k(\theta) = -\frac{1}{2k}\,\zeta\theta_\zeta - \frac{\theta}{p-1} + |\theta|^p. \tag{6.10}$$

Using the same stability theory [19], [21], similar to the above case $k = m$, we conclude that, under necessary compactness hypotheses, the orbit $\{\theta(\tau)\}$ uniformly approaches the stationary subset of equation (6.9), $\tau = \infty$, consisting in the smooth profiles satisfying $\mathbf{H}_k(f) = 0$, i.e., (cf. (6.5))

$$f(\zeta) = [(p-1)(1+A\zeta^{2k})]^{-1/(p-1)}, \quad A \ge 0, \tag{6.11}$$

where the constant A is uniquely determined from the asymptotics (6.7):

$$A = A_C \equiv -Cc_{2k}b_{2k}(p-1)^{p/(p-1)} > 0. \tag{6.12}$$

This completes the classification of the blow-up patterns in the intermediate Hamilton-Jacobi region.

6.3. **On the extension to the outer region.** The extension of blow-up patterns into the outer region $\{0 < |x| \ll 1\}$, which determines the asymptotic final-time profile $u(x,T^-)$ of the blow-up solutions of the semilinear parabolic (1.1), is also well-understood and is similar to that for $m = 1$. See [22], [27] and references in [34]. Such an extension based on a compactness argument applied to a rescaled parabolic flow shows that the behaviour in the intermediate domain, given by (6.5) or (6.11) for $\zeta \gg 1$, can be extended for $|x| \ll 1$. We then obtain the following asymptotics of the final-time profiles as $x \to 0$:

(i) the generic blow-up pattern formed on the centre manifold with the intermediate scaling (6.1) exhibits the behaviour

$$u(x,T^-) = C_m|x|^{-2m/(p-1)}|\ln|x||^{1/(p-1)}(1+o(1)),$$

where the constant $C_m = [(p-1)^{(2p-1)/(p-1)} e_m/2m]^{1/(p-1)}$ depends on m and the dimension N only;

(ii) the unstable patterns on the stable manifold with the variable (6.8) give for $k = m+1, m+2, ...$

$$u(x, T^-) = \tilde{C}|x|^{-2k/(p-1)}(1 + o(1)),$$

where $\tilde{C}$ depends also on initial data u_0.

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