Adaptive Low-Gain Integral Control of Linear Systems with Input and Output Nonlinearities *

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Abstract An adaptive low-gain integral control framework is developed for tracking constant reference signals in a context of finite-dimensional, asymptotically stable, single-input, single-output linear systems subject to locally Lipschitz, monotone input and output nonlinearities of a general nature: the input nonlinearity is required to satisfy an asymptotic growth condition (of sufficient generality to accommodate nonlinearities ranging from saturation to exponential growth) and the output nonlinearity is required to satisfy a sector constraint in those cases wherein the input nonlinearity is unbounded.

Keywords Adaptive control, input/output nonlinearities, integral control, servomechanisms, tracking.

1 Introduction

Within the context of the servomechanism problem, integral action is a fundamental technique in the control repertoire. For linear finite-dimensional continuous-time single-input, single-output systems Σ (with transfer function G) and with reference to Figure 1, the following principle is well established (see for example [2], [8] or [9]): if Σ is exponentially stable and $G(0) > 0$, then there exists $k^* > 0$ such that, closing the loop, with constant gain $k > 0$, around Σ compensated by an integrator yields a stable closed-loop system that achieves asymptotic tracking (by the output y) of an arbitrary constant reference signal r provided that $k < k^*$ (an analogous result holds for multivariable systems, under suitable assumptions on the matrix $G(0)$).

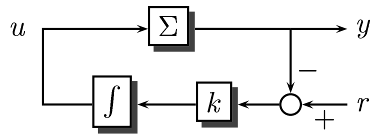

Fig. 1: Constant low-gain integral control

In [3, 4, 5, 6], the above principle is extended to classes of finite-dimensional and infinite-dimensional continuous-time single-input, single-output systems subject to input and/or output nonlinearities. In particular, in [3] it is shown that the principle remains valid if (a) the plant to be controlled is a finite-dimensional exponentially stable linear system Σ, with transfer function G satisfying $G(0) > 0$, subject to a globally Lipschitz non-decreasing input nonlinearity and a locally Lipschitz, non-decreasing sector-bounded output nonlinearity, and (b) the reference value r is feasible in an entirely natural sense; counterparts of these results in an infinite-dimensional setting are contained in [4]. We stress that in these extensions of the above principle, the input nonlinearity is assumed to satisfy a *global* Lipschitz condition: whether or not the principle

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remains valid for input nonlinearities which are merely *locally* Lipschitz is, to the authors' knowledge, an open question to which the present paper provides a partially affirmative answer (in an adaptive context). Specifically, we consider a class of finite-dimensional continuous-time single-input, single-output exponentially stable linear systems Σ having transfer function G with $G(0) > 0$ and subject to non-decreasing locally Lipschitz input nonlinearities φ and non-decreasing locally Lipschitz sector-bounded output nonlinearities ψ (the sector constraint on ψ may be weakened if φ is bounded). Imposing only an asymptotic growth assumption on φ (an assumption sufficiently weak to allow, for example, φ to exhibit exponential growth), we show that, if the reference signal r is feasible (in the natural sense alluded to earlier), then asymptotic tracking (by the output) of r can be achieved by *adaptive* low-gain integral control. In effect, for each feasible r, the adaptive strategy generates an asymptotic gain value sufficiently small to ensure tracking of r: however, we reiterate that the question of existence of a single value $k^* > 0$ (independent of r) such that, for fixed gain $k \in (0, k^*)$, tracking of every feasible r is achieved, remains unanswered. The adaptive approach is constructive: with reference to Figure 2, we provide an explicit class of control strategies of the form (the second equation of which manifests the low-gain structure)

$$\dot{u}(t) = k(t)\sigma(r - y(t)), \quad \dot{k}(t) = -k^2(t)\vartheta(\sigma(|r - y(t)|)), \quad (u(0), k(0)) = (u^0, k^0) \in \mathbb{R} \times (0, \infty)$$

each member of which is characterized by a pair of functions (σ, ϑ) and achieves the tracking objective for all admissible plant triples (Σ, φ, ψ).

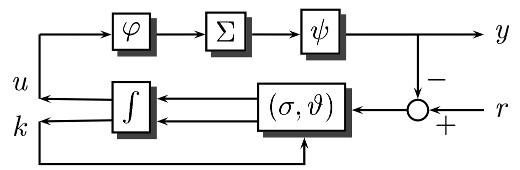

Fig. 2: Adaptive low-gain integral control

The flexibility in controller structure may have practical ramifications *vis à vis* other performance indicators such as, for example, mollifying integrator windup or influencing transient behaviour through appropriate choice of the functions (σ, ϑ). We emphasize the breadth of the class of allowable input nonlinearities which ranges from bounded nonlinearities (with, for example, saturation and deadzone effects) to unbounded nonlinearities with exponential growth. We also stress the rudimentary nature of the adaptive strategy which, at its simplest, takes the form $\dot{u} = ke$, $\dot{k} = -k^2|e|$, where $e(t) = r - y(t)$. Moreover, convergence of the non-increasing gain function k to a *positive* limit is generic behaviour in the sense that convergence to zero cannot occur in all cases other than particular non-generic cases in which the reference value r is, loosely speaking, commensurate with critical values of the plant input nonlinearity (if the latter nonlinearity is strictly increasing, then a positive limit gain is guaranteed in all cases).

2 Problem formulation

The problem of tracking – by adaptive control – constant reference signals r will be addressed in a context of uncertain single-input $u \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, single-output $y \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, finite-dimensional (state space $\mathbb{R}^N$) linear systems $\Sigma = (A, B, C, D)$, having a nonlinearity φ in the input channel and a nonlinearity ψ in the output channel:

$$\dot{x} = Ax + B\varphi(u), \quad x(0) = x^0 \in \mathbb{R}^N, \tag{1a}$$

$$y = \psi(Cx + D\varphi(u)). \tag{1b}$$

2.1 The class $\mathcal{S}$ of linear systems

In (1a), A is assumed to be Hurwitz (that is, every eigenvalue of A is assumed to have negative real part). Furthermore, the transfer function G, given by

$$G(s) = C(sI - A)^{-1}B + D,$$

is assumed to be such that $G(0) > 0$. Thus, the underlying class of finite-dimensional, real, linear systems $\Sigma = (A, B, C, D)$ is

$$\mathcal{S} := \{\Sigma = (A, B, C, D) |\ A \text{ Hurwitz},\ G(0) = D - CA^{-1}B > 0\}.$$

2.1.1 The positive-real condition

The proposition below is implicit in [7, Lemma 3.10].

Proposition 2.1 *If G is the transfer function of a system $\Sigma = (A, B, C, D) \in \mathcal{S}$, then*

$$1 + \kappa \operatorname{Re} \frac{G(s)}{s} \geq 0 \qquad \forall s \in \mathbb{C} \textit{ with } \operatorname{Re} s > 0 \tag{2}$$

for all $\kappa > 0$ sufficiently small.

We refer to (2) as the positive-real condition. Define

$$\kappa^* := \sup\{\kappa > 0 \,|\, (2) \text{ holds}\} > 0\,. \tag{3}$$

The following is a statement of [3, Lemma 2.1] (a consequence of the positive-real condition in conjunction with a variant [10, Theorem 1] of the Kalman-Yakubovich-Popov lemma).

Lemma 2.2 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$ and $\Delta > 1/\kappa^*$. Then there exists $P \in \mathbb{R}^{N\times N}$ such that $P = P^T \geq 0$ and*

$$\begin{bmatrix} PA + A^TP & PA^{-1}B - C^T \\ (A^{-1}B)^TP - C & -2\Delta \end{bmatrix} < 0\,.$$

2.2 The tracking objective and feasibility

For $\Sigma = (A, B, C, D) \in \mathcal{S}$, nonlinearities $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ (of an admissible class, to be made precise in due course) and reference value $r \in \mathbb{R}$, the tracking objective is to determine an input u such that the output y of (1) has the property $y(t) \to r$ as $t \to \infty$. To insure that this objective is achievable, we will impose a feasibility condition on r, namely,

$$\Psi^r \cap \overline{\Phi} \neq \emptyset\,, \quad \text{where } \Psi^r := \{v \in \mathbb{R} \mid \psi(G(0)v)) = r\}, \quad \Phi := \operatorname{im}\varphi, \quad \overline{\Phi} := \operatorname{clos}(\Phi)$$

and refer to the set

$$\mathcal{R} := \{r \in \mathbb{R} \mid \Psi^r \cap \overline{\Phi} \neq \emptyset\}$$

as the set of *feasible reference values*. The next proposition (a consequence of [4, Proposition 3.4]) asserts that, if ψ is continuous and monotone, then $r \in \mathcal{R}$ is close to being a necessary condition for tracking insofar as, if the tracking objective is achievable whilst maintaining boundedness of $\varphi \circ u$, then $r \in \mathcal{R}$.

Proposition 2.3 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$ and $x^0 \in \mathbb{R}^N$. Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ and let ψ be continuous and monotone. Let $u : \mathbb{R}_+ \to \mathbb{R}$ be such that $\varphi\circ u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and let $x : \mathbb{R}_+ \to \mathbb{R}^N$, $t \mapsto (\exp At)x^0 + \int_0^t (\exp A(t-s))\varphi(u(s))ds$ be the unique solution of the initial-value problem (1a). Then*

$$\lim_{t\to\infty} \psi\left(Cx(t) + D\varphi(u(t))\right) = r \quad \Longrightarrow \quad r \in \mathcal{R}\,.$$

2.3 Admissible input/output nonlinearities

2.3.1 Preliminaries

For notational convenience, the following classes of nonlinearities are introduced:

$$
\begin{aligned}
&\mathcal{L} := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ locally Lipschitz}\}, \quad \mathcal{L}_0 := \{f \in \mathcal{L} \mid f(0) = 0\} \\
&\mathcal{L}(\lambda) := \{f \in \mathcal{L} \mid 0 \le (f(\xi) - f(0))\xi \le \lambda\xi^2 \ \forall \xi \in \mathbb{R}\}, \quad \mathcal{L}_0(\lambda) := \mathcal{L}_0 \cap \mathcal{L}(\lambda) \\
&\mathcal{M} := \{f \in \mathcal{L} \mid f \text{ non-decreasing}\}, \quad \mathcal{M}_0 := \{f \in \mathcal{M} \mid f(0) = 0\} \\
&\mathcal{M}(\lambda) := \mathcal{M} \cap \mathcal{L}(\lambda), \quad \mathcal{M}_0(\lambda) = \mathcal{M}_0 \cap \mathcal{L}(\lambda).
\end{aligned}
$$

Remark 2.4 We apply the terminology "sector bounded" to functions of class $\mathcal{L}(\lambda)$ (and its subclasses $\mathcal{L}_0(\lambda)$, $\mathcal{M}(\lambda)$ and $\mathcal{M}_0(\lambda)$). If $f \in \mathcal{M}(\lambda)$, then, for each $\nu \in \mathbb{R}$, there exists $\tilde{\lambda} \ge 0$ such that the function $\xi \mapsto f(\xi + \nu) - f(\nu)$ is of class $\mathcal{M}_0(\tilde{\lambda})$. $\diamond$

With each function $f \in \mathcal{L}$, we associate functions $f^\dagger : \mathbb{R}_+ \to \mathbb{R}_+$ and $f^-, f^\diamond : \mathbb{R} \to \mathbb{R}$ defined by

$$
\begin{aligned}
f^\dagger(\nu) &:= \inf\{\lambda \in \mathbb{R}_+ \mid |f(u) - f(v)| \le \lambda|u - v| \ \forall u, v \in [-\nu, \nu]\} \\
f^-(\xi) &:= \liminf_{\substack{\alpha \to \xi \\ h \downarrow 0}} \frac{f(\alpha + h) - f(\alpha)}{h} \\
f^\diamond(\xi) &:= \liminf_{n \to \infty} n[f(\xi + 1/n) - f(\xi)]\,.
\end{aligned}
$$

Remark 2.5 It is readily verified that, for each $\nu \in \mathbb{R}_+$, the set $\{\lambda \in \mathbb{R}_+ \mid |f(u) - f(v)| \le \lambda|u - v| \ \forall u, v \in [-\nu, \nu]\}$ is closed (and is evidently bounded from below by 0): therefore, "inf" in the definition of $f^\dagger(\nu)$ may be replaced by "min". Thus, the quantity $f^\dagger(\nu)$ furnishes the smallest Lipschitz constant for the restriction, to the compact set $[-\nu, \nu]$, of a locally Lipschitz function f.
If f is continuously differentiable with derivative f', then $f^- = f^\diamond = f'$. If f is piecewise continuously differentiable (and so its left $f'_-(\xi)$ and right $f'_+(\xi)$ derivatives exist at every point $\xi \in \mathbb{R}$), then $f^-(\xi) \le \min\{f'_-(\xi)\,,\, f'_+(\xi)\}$ and $f^\diamond(\xi) = f'_+(\xi)$. $\diamond$

Lemma 2.6 *Let $f \in \mathcal{L}$. Then*

(i) *$f^\dagger$ is non-decreasing,*

(ii) *f^- is lower semicontinuous,*

(iii) *$f^\diamond$ is Borel measurable,*

(iv) *$f^-(\xi) \le f^\diamond(\xi)$ for all $\xi \in \mathbb{R}$,*

(v) *if $g : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then $f \circ g$ is absolutely continuous with derivative*

$$(f \circ g)'(t) = f^\diamond(g(t))\dot{g}(t) \qquad \text{for a.a. } t \in \mathbb{R}_+\,,$$

(vi) *for every $\epsilon > 0$, $|f^\diamond(\xi)| \le f^\dagger(|\xi| + \epsilon)$ for all $\xi \in \mathbb{R}$.*

Proof:

(i) That $f^\dagger$ is non-decreasing is an immediate consequence of its definition.

(ii) Noting that, for all $\xi \in \mathbb{R}$, $f^-(\xi) = -f^\circ(\xi;-1)$ (the Clarke directional derivative of f at ξ in "direction" -1), the assertion follows by upper semicontinuity of f° (see, [1, Proposition 2.1.1]).

(iii) Being the pointwise lower limit of a sequence of continuous functions, $f^\diamond$ is Borel measurable.

(iv) The claim is an immediate consequence of the definitions of f^- and $f^\diamond$.

(v) Being the composition of a locally Lipschitz function f and an absolutely continuous function g, $f \circ g$ is absolutely continuous. Let $\mathcal{D} \subset \mathbb{R}_+$ denote the set (of full measure) of all points $t > 0$ at which both derivatives $(f \circ g)'(t)$ and $\dot{g}(t)$ exist and define $\mathcal{D}_0 := \{t \in \mathcal{D} |\ \dot{g}(t) \neq 0\}$. By the local Lipschitz property of f, for each $t \in \mathcal{D}$ and $\delta > 0$, there exists $\lambda > 0$ such that

$$|f(g(t+h)) - f(g(t) + h\dot{g}(t))| \leq \lambda |g(t+h) - g(t) - h\dot{g}(t)| \quad \forall h \in (-\delta, \delta)$$

and so $\lim_{h\to 0}[f(g(t+h)) - f(g(t) + h\dot{g}(t))]/h = 0$. Since

$$f(g(t) + h\dot{g}(t)) - f(g(t)) = (f \circ g)(t+h) - (f \circ g)(t) + f(g(t) + h\dot{g}(t)) - f(g(t+h)) \quad \forall t \in \mathcal{D},$$

we may conclude that, for each $t \in \mathcal{D}_0$, f is differentiable at $g(t)$, with derivative $f'(g(t))$ satisfying

$$f'(g(t))\dot{g}(t) = (f \circ g)'(t) \quad \forall t \in \mathcal{D}_0$$

and, for each $t \in \mathcal{D}\backslash\mathcal{D}_0$ (in which case, $\dot{g}(t) = 0$), $(f \circ g)'(t) = 0$. Recalling that $\mathbb{R}_+\backslash\mathcal{D}$ has measure zero and noting that, if f is differentiable at ξ, then $f'(\xi) = f^\diamond(\xi)$, if follows that

$$(f \circ g)'(t) = f^\diamond(g(t))\dot{g}(t) \quad \text{for a.a. } t \in \mathbb{R}_+ .$$

(vi) Let $\epsilon > 0$ and $\xi \in \mathbb{R}$ be arbitrary. Let $N \in \mathbb{N}$ be such that $N > 1/\epsilon$. Then $|f(\xi + (1/n)) - f(\xi)| \leq f^\dagger(|\xi| + \epsilon)$ for all $n \geq N$ and so $|f^\diamond(\xi)| \leq f^\dagger(|\xi| + \epsilon)$. □

2.3.2 Critical values

Let $f \in \mathcal{M}$. A point $\xi \in \mathbb{R}$ is said to be a *critical point* (and $f(\xi)$ is said to be a *critical value*) of f if $f^-(\xi) = 0$. We denote, by $\mathcal{C}(f)$, the set of critical values of $f \in \mathcal{M}$:

$$\mathcal{C}(f) := \{f(\xi) |\ \xi \in \mathbb{R},\ f^-(\xi) = 0\} .$$

For example, if $f \in \mathcal{M}$ is strictly increasing, then $\mathcal{C}(f) = \emptyset$; if $f \in \mathcal{M}$ is piecewise continuously differentiable, with left and right derivatives f'_- and f'_+, then $f(\xi) \in \mathcal{C}(f)$ if $f'_-(\xi)f'_+(\xi) = 0$ (that is, if at least one of the one-sided derivatives is zero at ξ).

2.3.3 The class $\mathcal{N}(\delta)$ of input/output nonlinearities

We are now in a position to define the class $\mathcal{N}(\delta)$ (parameterized by $\delta > 0$ and with the property that $\mathcal{N}(\delta_1) \subset \mathcal{N}(\delta_2)$ if $\delta_1 \leq \delta_2$) of admissible pairs (φ, ψ) of input/output nonlinearities. In essence, the class comprises all pairs (φ, ψ) of locally Lipschitz, non-decreasing functions $\mathbb{R} \to \mathbb{R}$ with the following additional properties: (i) $\varphi^\dagger$ satisfies an exponential growth constraint (quantified by $\delta > 0$); (ii) if φ is unbounded, then ψ is sector bounded. Specifically,

$$\begin{aligned}\mathcal{N}(\delta) := \{(\varphi, \psi) \in \mathcal{M} \times \mathcal{M} |\ &\varphi^\dagger(\nu) = o(\exp(\delta\nu)) \text{ as } \nu \to \infty, \\ &\varphi \text{ unbounded} \ \Rightarrow\ \psi \in \mathcal{M}(\lambda) \text{ for some } \lambda > 0\} . \quad (4)\end{aligned}$$

For example, if $\psi \in \mathcal{M}(\lambda)$ for some $\lambda > 0$ and $\varphi \in \mathcal{M}$ is such that φ is piecewise continuously differentiable with polynomially bounded derivative, viz. for some constants $\alpha > 0$ and $n \in \mathbb{N}$, $\varphi'(\xi) \leq \alpha[1 + |\xi|^n]$ at every point ξ of continuity of φ', then $(\varphi, \psi) \in \mathcal{N}(\delta)$ for all $\delta > 0$ (if φ has exponentially bounded derivative, viz. for some constants $\alpha, \beta > 0$, $\varphi'(\xi) \leq \beta \exp(\alpha|\xi|)$ at every point ξ of continuity of φ', then $(\varphi, \psi) \in \mathcal{N}(\delta)$ for all $\delta > \alpha$).

3 Adaptive integral control

3.1 The feedback system

In the context of (1), we will investigate adaptive integral control action

$$u(t) = u^0 + \int_0^t k(\tau)\sigma(r - y(\tau))\, d\tau\,, \tag{5}$$

with time-varying gain $k(\cdot)$ generated by the adaptive law

$$k(t) = \frac{1}{l(t)}\,, \quad \dot{l}(t) = \vartheta(|\sigma(r - y(t))|), \quad l(0) = l^0 = \frac{1}{k^0} > 0\,, \tag{6}$$

for appropriate choices of $\sigma, \vartheta \in \mathcal{L}$. Equivalently, we may express the adaptive control in the form of a system of differential equations

$$\dot{u} = k\sigma(r - y), \quad \dot{k} = -k^2\vartheta(|\sigma(r - y)|), \quad (u(0), k(0)) = (u^0, k^0) \in \mathbb{R} \times (0, \infty). \tag{7}$$

An application of the control (5,6) (equivalently, (7)) to (1) leads to the following system of nonlinear autonomous differential equations:

$$\dot{x} = Ax + B\varphi(u)\,, \quad x(0) = x^0 \in \mathbb{R}^N\,, \tag{8a}$$

$$\dot{u} = k\sigma(r - \psi\,(Cx + D\varphi(u)))\,, \quad u(0) = u^0 \in \mathbb{R}, \tag{8b}$$

$$\dot{k} = -k^2\vartheta\,(|\sigma(r - \psi\,(Cx + D\varphi(u)))|)\,, \quad k(0) = k^0 \in (0, \infty)\,. \tag{8c}$$

Lemma 3.1 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, $\varphi, \psi, \sigma, \vartheta \in \mathcal{L}$ and $r \in \mathbb{R}$. Assume further that $\vartheta(\xi) \geq 0$ for all $\xi \in \mathbb{R}_+$.*
For each $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, the initial-value problem (8) has a unique maximal solution $(x, u, k) : [0, T) \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$. Moreover, if $T < \infty$, then $\limsup_{t \to T} |u(t)| = \infty$.

Proof: Noting that the righthand sides of the differential equations in (8) are locally Lipschitz functions, the existence of a unique maximal solution $(x, u, k) : [0, T) \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ of the initial-value problem follows by the classical theory of ordinary differential equations: moreover, if $T < \infty$, then the solution is unbounded. Non-negativity of the values of ϑ on $\mathbb{R}_+$, together with (8c), implies that the solution component k is bounded. By continuity of φ and the Hurwitz property of A, we see that component x is bounded whenever component u is bounded. Therefore, if $T < \infty$, then $\limsup_{t \to T} |u(t)| = \infty$. □

3.2 The controller class

We introduce a set $\mathcal{N}_c(\delta)$ (parameterized by $\delta > 0$) of functions defined by

$$\mathcal{N}_c(\delta) := \{(\sigma, \vartheta) |\ \sigma \in \mathcal{L}_0(\lambda) \text{ for some } \lambda > 0,\ \sigma^{-1}(0) = \{0\},\ \vartheta \in \mathcal{L}_0,\ \vartheta(\nu) \geq \delta\nu\ \forall \nu \in \mathbb{R}_+\}.$$

In words, $\mathcal{N}_c(\delta)$ (with the property that $\mathcal{N}_c(\delta_1) \supset \mathcal{N}_c(\delta_2)$ if $\delta_1 \leq \delta_2$) comprises all pairs (σ, ϑ) of locally Lipschitz functions $\mathbb{R} \to \mathbb{R}$ such that σ is sector bounded, $\sigma^{-1}(0) = \{0\}$, ϑ is bounded from below on $\mathbb{R}_+$ by the linear function $\nu \mapsto \delta\nu$ and $\vartheta(0) = 0$. The controller class (parameterized by $\delta > 0$) consists of all strategies of the form (5,6) (equivalently, (7)) with $(\sigma, \vartheta) \in \mathcal{N}_c(\delta)$.

3.3 The main result

We now arrive at the main result of the paper, the essence of which is an assertion that, for each $\delta > 0$ and all plants $(\Sigma, \varphi, \psi) \in \mathcal{S} \times \mathcal{N}(\delta)$, the objective of tracking any feasible reference signal $r \in \mathcal{R}$ is achieved by control (5,6) (equivalently, (7)) provided that the functions (σ, ϑ) are chosen from $\mathcal{N}_c(\delta)$.

Remark 3.2 If $\sigma : \xi \mapsto \xi$ (the identity map) and $\vartheta : \xi \mapsto \delta\xi$, then it is clear that $(\sigma, \vartheta) \in \mathcal{N}_c(\delta)$ and so the simple control strategy given by $\dot{u} = k(r - y)$ and $\dot{k} = -\delta k^2 |r - y|$ provides $\mathcal{S} \times \mathcal{N}(\delta)$-universal tracking of feasible reference signals $r \in \mathcal{R}$. Nevertheless, the flexibility furnished by $\mathcal{N}_c(\delta)$ permits other choices of (σ, ϑ) which may be preferable with respect to other performance indicators: for example, σ may be chosen to be a bounded function, in which case the function u has at most linear growth – a feature which may help mitigate effects, undesirable from a practical viewpoint, such as "integrator windup".

Theorem 3.3 *Let $\Sigma = (A, B, C, D) \in \mathcal{S}$, $\delta > 0$, $(\varphi, \psi) \in \mathcal{N}(\delta)$ and $r \in \mathcal{R}$. If $(\sigma, \vartheta) \in \mathcal{N}_c(\delta)$, then, for each $(x^0, u^0, k^0) \in \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$, the unique maximal solution $(x, u, k) : [0, T) \to \mathbb{R}^N \times \mathbb{R} \times (0, \infty)$ of the initial-value problem* (8) *is such that the following hold.*

(i) $T = \infty$,

(ii) $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r \in \Psi^r \cap \overline{\Phi}$,

(iii) $\lim_{t\to\infty} x(t) = -A^{-1}B\varphi^r$,

(iv) $\lim_{t\to\infty} y(t) = r$, *where* $y(t) = \psi\left(Cx(t) + D\varphi(u(t))\right)$,

(v) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$, *then* $\lim_{t\to\infty} \mathrm{dist}\left(u(t), \varphi^{-1}(\varphi^r)\right) = 0$,

(vi) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \mathrm{int}(\Phi)$, *then u is bounded,*

(vii) *if* $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$ *and* $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$, *then the monotone function k converges to a positive value.*

Proof: Introducing functions $e : \mathbb{R}_+ \to \mathbb{R}$ and $p : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$e(t) := \sigma(r - \psi\left(Cx(t) + D\varphi(u(t))\right)) \quad \text{and} \quad p(t) := \int_0^t k(s)\vartheta(|e(s)|)\, ds\,,$$

we first record that

$$k(t) = k^0 \exp(-p(t)) \quad \forall t \in [0, T) \tag{9}$$

and $(d/dt)|u(t)| \leq k(t)|e(t)| \leq k(t)\vartheta(|e(t)|)/\delta$ for almost all $t \in [0, T)$, whence

$$|u(t)| \leq |u^0| + p(t)/\delta \quad \forall t \in [0, T). \tag{10}$$

Therefore, invoking monotonicity of φ (in which case, $\varphi^\diamond(u(t)) \geq 0$) together with statements (i) and (vi) of Lemma 2.6,

$$0 \leq k(t)\varphi^\diamond(u(t)) \leq k^0 e^{-p(t)}\varphi^\dagger(|u(t)| + 1) \leq k^0 e^{-p(t)}\varphi^\dagger(|u^0| + 1 + p(t)/\delta) \quad \forall t \in [0, T)\,. \tag{11}$$

Step I: We will prove assertion (i) and, in addition, establish that $\lim_{t\to\infty} \varphi(u(t))$ exists and is finite and that $K(t) := \int_0^t k \to \infty$ as $t \to \infty$.

By monotonicity, positivity and boundedness of k, there exists $\hat{k} \geq 0$ such that

$$\lim_{t \uparrow T} k(t) = \hat{k} \geq 0 \,.$$

We will consider separately the two possible cases: $\hat{k} > 0$ or $\hat{k} = 0$.

Case A: Assume $\hat{k} > 0$. By (9), it follows that the monotone function p is bounded and so, by (10), u is bounded. By Lemma 3.1, we may conclude that $T = \infty$. Noting that

$$\frac{1}{k(t)} = \frac{1}{k^0} + \int_0^t \vartheta(|e(\tau)|)d\tau \quad \to \quad \frac{1}{\hat{k}} \quad \text{as } t \to \infty \,,$$

we may conclude that $\vartheta(|e(\cdot)|) \in L^1(\mathbb{R}_+, \mathbb{R})$. By properties of $\mathcal{N}_c(\delta)$, $\delta|e(t)| \leq \vartheta(|e(t)|)$ for all t, whence $e \in L^1(\mathbb{R}_+, \mathbb{R})$. Therefore, by (8b), $u(t)$ converges to a finite limit as $t \to \infty$ and, by continuity of φ, it follows that $\varphi(u(t))$ converges to a finite limit as $t \to \infty$. Since $\hat{k} > 0$, it is clear that $K(t) = \int_0^t k \to \infty$ as $t \to \infty$.

Case B: Assume $\hat{k} = 0$. Then, by (9), $p(t) \to \infty$ as $t \to T$. Setting $\hat{p}(t) := |u^0| + 1 + p(t)/\delta$ and invoking properties of $\mathcal{N}(\delta)$, we have $\exp(-\delta\hat{p}(t))\varphi^\dagger(\hat{p}(t)) \to 0$ as $t \to T$. By (11), it follows that

$$k(t)\varphi^\diamond(u(t)) \to 0 \quad \text{as } t \uparrow T \,. \tag{12}$$

By assumption, $r \in \mathcal{R}$ and so $\Psi^r \cap \overline{\Phi} \neq \emptyset$. Let $\varphi^* \in \Psi^r \cap \overline{\Phi}$. Introduce functions

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \psi(\xi + G(0)\varphi^*) - r \,,$$
$$\gamma : \mathbb{R} \to \mathbb{R}_+ \,, \quad \xi \mapsto \begin{cases} -\sigma(-\tilde{\psi}(\xi))/\xi & , \quad \xi \neq 0 \\ 0 & , \quad \xi = 0 \end{cases}$$

and functions z, v, w, η, ζ defined by

$$z(t) := x(t) + A^{-1}B\varphi(u(t)) \,, \quad v(t) := \varphi(u(t)) - \varphi^* \,, \quad w(t) := Cz(t) + G(0)v(t) \,,$$
$$\eta(t) := \varphi^\diamond(u(t))\gamma(w(t)), \quad \zeta(t) := -k(t)\eta(t)w(t) \qquad \forall t \in [0, T) \,.$$

Note that

$$e(t) = \sigma(r - \psi\,(Cx(t) + D\varphi(u(t)))) = \sigma(-\tilde{\psi}(w(t))) = -\gamma(w(t))w(t) \quad \forall t \in [0, T) \tag{13}$$

and, by statement (v) of Lemma 2.6 and (8b), we have

$$\dot{v}(t) = (d/dt)\varphi(u(t)) = \varphi^\diamond(u(t))\dot{u}(t) = k(t)\varphi^\diamond(u(t))e(t) = -k(t)\eta(t)w(t) = \zeta(t) \quad \text{for a.a. } t \in [0, T).$$

From (8a), it now follows that

$$\dot{z}(t) = Az(t) + A^{-1}B\zeta(t) \qquad \text{for a.a. } t \in [0, T) \,, \tag{14a}$$
$$\dot{v}(t) = \zeta(t) \qquad \text{for a.a. } t \in [0, T) \,, \tag{14b}$$

with initial values $z(0) = z^0 := x^0 + A^{-1}B\varphi(u^0)$ and $v(0) = v^0 := \varphi(u^0) - \varphi^*$.

Next, we claim that, for some $\hat{\lambda} > 0$,

$$0 \leq \eta(t) \leq \varphi^\diamond(u(t))\hat{\lambda} \qquad \forall t \in [0, T) \,. \tag{15}$$

To establish the claim we consider separately the two cases: φ bounded and φ unbounded. If the former is the case, then, by the Hurwitz property of A, w is bounded: therefore, since $\tilde{\psi} \in \mathcal{M}_0$, there exists $\tilde{\lambda}$ such that $|\tilde{\psi}(w(t))| \leq \tilde{\lambda}|w(t)|$ for all $t \in [0, T)$. If the latter is the case, then, by properties of $\mathcal{N}(\delta)$, $\psi \in \mathcal{M}(\lambda)$ for some $\lambda > 0$: hence, $\tilde{\psi} \in \mathcal{M}_0(\tilde{\lambda})$ for some $\tilde{\lambda} > 0$ (recall Remark 2.4) and so $|\tilde{\psi}(w(t))| \leq \tilde{\lambda}|w(t)|$ for all $t \in [0, T)$. Recalling that, by properties of $\mathcal{N}_c(\delta)$, $\sigma \in \mathcal{L}_0(\lambda)$, we may conclude that, in each case,

$$0 \leq \gamma(w(t))|w(t)| = |\sigma(-\tilde{\psi}(w(t)))| \leq \lambda|\tilde{\psi}(w(t))| \leq \lambda\tilde{\lambda}|w(t)| \quad \forall t \in [0, T).$$

Therefore, $0 \leq \gamma(w(t)) \leq \lambda\tilde{\lambda}$ for all $t \in [0, T)$, which, on setting $\hat{\lambda} = \lambda\tilde{\lambda}$, yields (15).

Fix $\Delta > 1/\kappa^*$, where κ^* is given by (3). By Lemma 2.2 there exists $P \in \mathbb{R}^{N \times N}$, $P = P^T \geq 0$ such that

$$\Lambda := \begin{bmatrix} PA + A^T P & PA^{-1}B - C^T \\ (A^{-1}B)^T P - C & -2\Delta \end{bmatrix} < 0\,.$$

Define the absolutely continuous function

$$V : \mathbb{R}_+ \to \mathbb{R}_+\,, \quad t \mapsto \langle z(t), Pz(t)\rangle + G(0)v^2(t)\,.$$

Then,

$$\begin{aligned} \dot{V}(t) &= \langle z(t), (PA + A^T P)z(t)\rangle + 2(A^{-1}B)^T Pz(t)\zeta(t) + 2G(0)v(t)\zeta(t) \\ &= \langle z(t), (PA + A^T P)z(t)\rangle + 2[(A^{-1}B)^T P - C]z(t)\zeta(t) + 2w(t)\zeta(t) \\ &= \langle [z^T(t), \zeta(t)]^T, \Lambda[z^T(t), \zeta(t)]^T\rangle + 2\Delta\zeta^2(t) - 2k(t)\eta(t)w^2(t) \\ &\leq -\alpha[\|z(t)\|^2 + \zeta^2(t)] - 2k(t)\eta(t)\,[1 - k(t)\eta(t)\Delta]\,w^2(t) \quad \text{for a.a. } t \in [0, T)\,, \end{aligned}$$

where $\alpha := 1/\|\Lambda^{-1}\|$. Invoking (12) and (15), we see that there exists $t_0 \in [0, T)$ such that

$$k(t)\eta(t)\Delta < \frac{1}{2} \quad \forall t \in [t_0, T)\,. \tag{16}$$

Therefore,

$$\dot{V}(t) \leq -\alpha[\|z(t)\|^2 + \zeta^2(t)] - k(t)\eta(t)w^2(t) \quad \text{for a.a. } t \in [t_0, T)\,. \tag{17}$$

In particular, it follows that V is bounded which, in turn, implies boundedness of $\varphi \circ u$, z and e. By (5), we see that u is bounded on bounded intervals and so, by Lemma 3.1, we may conclude that $T = \infty$.

Introduce functions $q : \mathbb{R}_+ \to \mathbb{R}_+$ and $W_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ (parameterized by $\rho \geq 0$) defined by

$$q(t) := \int_0^t k\eta \qquad \text{and} \qquad W_\rho(t) := \exp(2\rho q(t))V(t).$$

We remark that the function γ is lower semicontinuous and, by statement (iii) of Lemma 2.6, the function $\varphi^\diamond$ is Borel measurable, whence (Lebesgue) measurability of the composition $\varphi^\diamond \circ u$: therefore, $k\eta = k(\varphi^\diamond \circ u)(\gamma \circ w)$, being the product of (Lebesgue) measurable functions, is (Lebesgue) measurable and so q is well defined. Noting that, for some constant $\beta > 0$,

$$V(t) = \langle z(t), Pz(t)\rangle + [w(t) - Cz(t)]^2/G(0) \leq \beta[\|z(t)\|^2 + w^2(t)] \quad \forall t \in \mathbb{R}_+$$

and invoking (16) and (17), we have

$$\begin{aligned}\dot W_\rho(t) &= \exp(2\rho q(t))[\dot V(t) + 2\rho k(t)\eta(t)V(t)]\\ &\le \exp(2\rho q(t))[-(\alpha-(\beta\rho)/\Delta)\|z(t)\|^2 - k(t)\eta(t)(1-2\beta\rho)w^2(t)] \quad \text{for a.a. } t \ge t_0\,.\end{aligned}$$

Choose $\rho > 0$ sufficiently small so that $0 < \beta\rho \le \min\{\alpha\Delta\,,\,1/4\}$, in which case

$$\dot W_\rho(t) \le -\frac{1}{2}\exp(2\rho q(t))k(t)\eta(t)w^2(t) \quad \text{for a.a. } t \ge t_0\,,$$

whence

$$\frac{1}{2}\int_{t_0}^\infty \exp(2\rho q(t))k(t)\eta(t)w^2(t)\,dt \le W_\rho(t_0) < \infty\,.$$

Therefore,

$$\begin{aligned}\int_{t_0}^\infty k(t)\eta(t)|w(t)|dt &= \int_{t_0}^\infty \exp(-\rho q(t))\sqrt{k(t)\eta(t)}\sqrt{k(t)\eta(t)}\exp(\rho q(t))|w(t)|dt\\ &\le \frac{1}{2}\int_{t_0}^\infty \exp(-2\rho q(t))\dot q(t)dt + \frac{1}{2}\int_{t_0}^\infty \exp(2\rho q(t))k(t)\eta(t)w^2(t)dt\\ &\le \frac{\exp(-2\rho q(t_0))}{4\rho} + W_\rho(t_0) < \infty\end{aligned}$$

and so $\zeta = -k\eta w \in L^1(\mathbb{R}_+,\mathbb{R})$. By (14b), we may now infer that $v(t)$, and hence $\varphi(u(t))$, converges to a finite limit as $t \to \infty$.

By the Hurwitz property of A, it follows that x (and hence $\vartheta(|e(\cdot)|)$) is bounded and so, for some constant $c > 0$,

$$\frac{1}{k(t)} = \frac{1}{k^0} + \int_0^t \vartheta(|e(\tau)|)d\tau \le c(1+t) \quad \forall t \in \mathbb{R}_+\,.$$

Therefore, $K(t) = \int_0^t k \to \infty$ as $t \to \infty$.

We have now established assertion (a1) and, moreover, that $\lim_{t\to\infty}\varphi(u(t)) =: \varphi^r$ exists and is finite and that the monotone function K is unbounded. Furthermore, we have established the following fact which we record for later reference

$$k(t) \downarrow 0 \text{ as } t \to \infty \quad \Longrightarrow \quad \zeta = -k\eta w \in L^1(\mathbb{R}_+,\mathbb{R}). \tag{18}$$

This completes Step I.

Step II. We will prove the truth of statements (ii)-(vi). By Step I, there exists $\varphi^r \in \mathbb{R}$ such that $\lim_{t\to\infty}\varphi(u(t)) = \varphi^r$ which, together with the Hurwitz property of A, implies that

$$\lim_{t\to\infty}[Cx(t) + D\varphi(u(t))] = G(0)\varphi^r\,. \tag{19}$$

Evidently, $\varphi^r \in \overline{\Phi}$ and so, to establish (a2), it suffices to show that $\varphi^r \in \Psi^r$. Seeking a contradiction, suppose that $\varphi^r \notin \Psi^r$. This supposition implies that

$$c := r - \psi(G(0)\varphi^r) \ne 0\,.$$

Recalling that, by properties of $\mathcal{N}_c(\delta)$, $\sigma \in \mathcal{L}_0(\lambda)$ and $\sigma^{-1}(0) = \{0\}$, we have $c\sigma(c) > 0$ and so, by continuity and (19), there exists $s > 0$ such that

$$c\sigma(r - y(t)) = c\sigma(r - \psi(Cx(t) + D\varphi(u(t)))) \ge c\sigma(c)/2 =: c^* > 0 \qquad \forall t \ge s\,.$$

Noting that $\dot{u}(t) = k(t)\sigma(r - y(t))$, we have $c\dot{u}(t) \geq c^* k(t)$ for all $t \geq s$, which, on integration, yields

$$c[u(t) - u(s)] \geq c^*(K(t) - K(s)) \quad \forall t \geq s\,.$$

Since $K(t) \to \infty$ as $t \to \infty$, we conclude that $cu(t) \to \infty$ as $t \to \infty$. Therefore,

$$\lim_{t\to\infty} \varphi(u(t)) = \varphi^r = \begin{cases} \sup \overline{\Phi} & \text{if } c > 0 \\ \inf \overline{\Phi} & \text{if } c < 0\,. \end{cases} \tag{20}$$

Let $\varphi^* \in \Psi^r \cap \overline{\Phi}$. Then, $c\varphi^* \leq c\varphi^r$ and $\psi(G(0)\varphi^*) = r$ which, together with the non-decreasing property of ψ, yields the contradiction

$$0 = c[r - \psi(G(0)\varphi^*)] \geq c[r - \psi(G(0)\varphi^r)] = c^2 > 0\,.$$

Therefore, we may conclude $\varphi^r \in \Psi^r \cap \overline{\Phi}$ which is statement (ii). Statement (iii) follows from (ii) and the Hurwitz property of A. Statement (iv) is a consequence of (i), (ii) and continuity of ψ.

Next, we establish statement (v). Assume $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$ which, together with (ii), implies the existence of $\xi^* \in \mathbb{R}$ such that $\varphi^r = \varphi(\xi^*)$. Seeking a contradiction, suppose that $\mathrm{dist}(u(t), \varphi^{-1}(\varphi^r)) \not\to 0$ as $t \to \infty$. Then there exist $\varepsilon > 0$ and a sequence $(t_n) \in \mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$, such that

$$\mathrm{dist}\,(u(t_n), \varphi^{-1}(\varphi^r)) \geq \varepsilon\,. \tag{21}$$

If the sequence $(u(t_n))$ is bounded, we may assume without loss of generality that it converges to a finite limit u_∞. By continuity, $\varphi(u_\infty) = \varphi^r$ and so $u_\infty \in \varphi^{-1}(\varphi^r)$. This contradicts (21). Therefore, we may assume that $(u(t_n))$ is unbounded. Extracting a subsequence if necessary, we may then assume that either $u(t_n) \to \infty$ or $u(t_n) \to -\infty$ as $n \to \infty$: if the former holds, then $u(t_n) > \xi^*$ for all n sufficiently large; if the latter holds, then $u(t_n) < \xi^*$ for all n sufficiently large. In either case, by monotonicity of φ it follows that $\varphi(u(t_n)) = \varphi(\xi^*) = \varphi^r$ for all n sufficiently large. Clearly, this contradicts (21) and so statement (v) must hold.

To establish (vi), assume that $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \mathrm{int}(\Phi)$ and, for contradiction, suppose that u is unbounded. Then there exists a sequence $(t_n) \subset (0, \infty)$ with $t_n \to \infty$ and $|u(t_n)| \to \infty$ as $n \to \infty$. By monotonicity of φ and (ii), it then follows that either $\varphi^r = \sup \Phi$ or $\varphi^r = \inf \Phi$, contradicting the fact that $\varphi^r \in \Psi^r \cap \mathrm{int}(\Phi) \subset \mathrm{int}(\Phi)$. Therefore, u is bounded.

This completes Step II.

STEP III. It remains only to establish statement (vii).

By (ii),

$$\varphi(u(t)) \to \varphi^r \in \Psi^r \cap \overline{\Phi} \quad \text{as } t \to \infty\,.$$

By hypothesis, $\Psi^r \cap \overline{\Phi} = \Psi^r \cap \Phi$ and $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$. Therefore, $\varphi^r \in \Phi$ and $\varphi^r \notin \mathcal{C}(\varphi)$. Thus, the preimage $\varphi^{-1}(\varphi^r)$ is a singleton $\{u^r\}$ and $\varphi^-(u^r) > 0$. By statement (v), $u(t) \to u^r$ as $t \to \infty$. Invoking statements (ii) and (iv) of Lemma 2.6, there exists $s > 0$ such that

$$\varphi^\diamond(u(t)) \geq \varphi^-(u(t)) \geq \varphi^-(u^r)/2 = c > 0 \quad \forall t \geq s\,. \tag{22}$$

Seeking a contradiction, suppose $k(t) \downarrow 0$ as $t \to \infty$. Then, by (9), $p(t) \to \infty$ as $t \to \infty$ and so $k(\cdot)\vartheta(|e(\cdot)|) \notin L^1(\mathbb{R}_+, \mathbb{R})$. By (iv), e is bounded and so, since $\vartheta \in \mathcal{L}_0$ (and hence is locally Lipschitz with $\vartheta(0) = 0$), there exists a constant $\hat{\lambda} > 0$ such that $\vartheta(|e(t)|) = |\vartheta(|e(t)|) - \vartheta(0)| \leq \hat{\lambda}|e(t)|$ for all $t \in \mathbb{R}_+$. Therefore, $ke \notin L^1(\mathbb{R}_+, \mathbb{R})$. Since $\varphi^\diamond(u(t)) \geq c > 0$ for all $t \geq s$ and invoking (13), we may conclude that $-k\varphi^\diamond(u)e = k\eta w \notin L^1(\mathbb{R}_+, \mathbb{R})$, which contradicts (18). Therefore, k converges to a positive limit. This completes the proof. □

Remark 3.4 If $\operatorname{im}\varphi$ is compact, then the assertions of the above theorem remain true when the hypothesis on the growth of $\vartheta \in \mathcal{L}_0$ is replaced by the weaker hypothesis: $\liminf_{\nu\downarrow 0} \vartheta(\nu)/\nu \geq \delta > 0$). For brevity, we omit details here. $\diamond$

4 Example

Consider the second-order system

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = -ax_2 - bx_1 + \varphi(u)\,, \quad y = \psi(x_1)\,,$$

with $a, b > 0$ and $(\varphi, \psi) \in \mathcal{N}(1)$. Choosing $\vartheta : \xi \mapsto \xi$ and $\sigma \in \mathcal{M}_0(1)$ of saturation type, defined as follows

$$\sigma : \ \xi \mapsto \operatorname{sat}(\xi) := \begin{cases} \operatorname{sgn}(\xi), & |\xi| > 1 \\ \xi, & |\xi| \leq 1 \end{cases}$$

then $(\sigma, \vartheta) \in \mathcal{N}_c(1)$.

For purposes of illustration, assume that the input and output nonlinearities take the forms

$$\varphi : \xi \mapsto \xi^3, \quad \psi : \xi \mapsto \arctan(\xi).$$

The transfer function G of the associated linear system Σ is given by

$$G(s) = \frac{1}{s^2 + as + b}, \quad \text{with } G(0) = 1/b > 0\,.$$

Since $\Psi^r = \{v \in \mathbb{R} \mid \psi(G(0)v) = r\} \neq \emptyset$ if and only if $r \in (-\pi/2, \pi/2)$, we have $\Psi^r \cap \overline{\Phi} \neq \emptyset$ if and only if $r \in (-\pi/2, \pi/2)$. Thus, the set $\mathcal{R}$ of feasible reference values is given by $\mathcal{R} = (-\pi/2, \pi/2)$.

By Theorem 3.3, it follows that the adaptive controller

$$\dot{u} = k \operatorname{sat}(r - y), \quad \dot{k} = -k^2|r - y|, \quad (u(0), k(0)) = (u^0, k^0) \in \mathbb{R} \times (0, \infty)$$

achieves the tracking objective for each feasible reference value $r \in \mathcal{R}$. Moreover, if $r \neq 0$ (so that $\Psi^r \cap \mathcal{C}(\varphi) = \emptyset$), then the adapting gain k converges to a positive value. For plant parameter values $a = 2$, $b = 1$, initial data $(x_1(0), x_2(0), u_0, l_0) = (0, 0, 0, 1)$ and the feasible reference value $r = 1$, Fig. 3 (generated using SIMULINK Simulation Software under MATLAB) depicts the system performance under adaptive control. In this case, $\Psi^r \cap \mathcal{C}(\varphi) = \psi^{-1}(1) \cap \{0\} = \emptyset$ and so the gain converges to a positive limit, as is evident in Fig. 3.

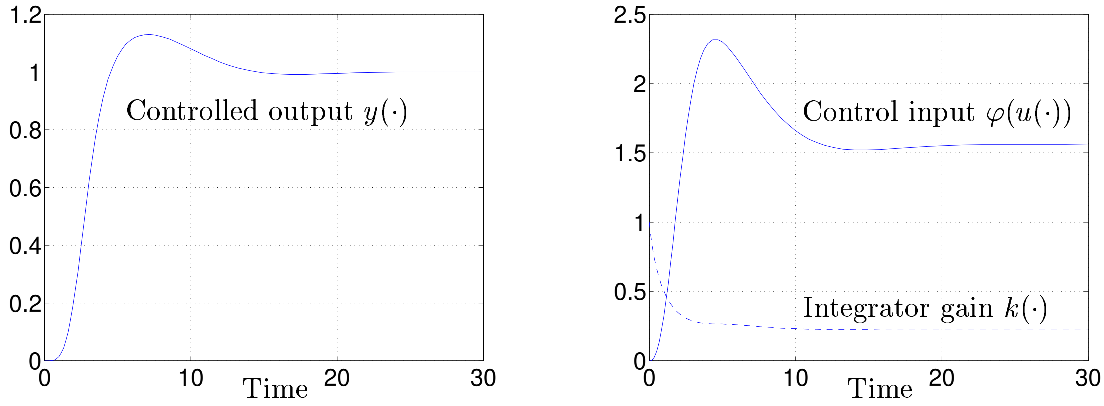

Fig. 3: Performance under adaptive control

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