Grazing and Border–Collision in Piecewise-smooth Systems: A Unified Analytical Framework

M. di Bernardo*, C.J. Budd†, A.R. Champneys*

*Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, U. K.
E-mail:M.diBernardo@bristol.ac.uk

†Department of Mathematical Sciences
University of Bath, Bath BA2 7AY, U. K.

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A new comprehensive derivation of normal form maps for grazing bifurcations in piecewise-smooth flows is presented. This links grazings with so-called border-collisions in piecewise-smooth maps. Contrary to previous literature, piecewise-linear local maps are derived only at non-smooth discontinuity boundaries. All other maps have either square-root or (3/2)-type singularities.

Many physical systems characterized by the occurrence of non-smooth events, such as discrete state transitions, can exhibit complex dynamics [1,2]. Examples include vibro-impacting machines, switching circuits and physiological systems which are intrinsically non-smooth on macroscopic time-scales [1]. These systems are often piecewise smooth (PWS), and are modelled by sets of ordinary differential equations (ODEs) which are smooth in regions S_i of phase space; smoothness being lost as trajectories cross the boundaries $\Sigma_{i,j}$ between adjacent regions, see Fig. 1. They exhibit a richness of bifurcation phenomena unique to their non-smooth character. These include so-called *grazing* bifurcations occurring when a part of the system trajectory hits tangentially the boundary sets $\Sigma_{i,j}$. Such grazing events frequently lead to a multitude of complex dynamical transitions, such as period-adding cascades and sudden transitions from a periodic orbit to a chaotic attractor [3,2].

To classify the transitions observed in physical systems one must derive appropriate normal form maps describing the system behaviour in a neighborhood of the grazing event. In the literature dealing with bifurcations of non-smooth systems (e.g. [4]), it is often conjectured that such mappings are piecewise linear if the piecewise smooth vector field is continuous across the boundaries. If this conjecture were true then their bifurcations could be studied by using the theory of border collisions of piecewise-linear maps developed in [5] or of C-bifurcations [4]. In contrast, if the system states are discontinuous, such as for a restitution law in impact oscillators, then the maps are known to have a square-root singularity [3,6]. It remains to be proved whether grazing in a PWS system with a continuous vector field leads to a piecewise-linear map in general. Our analysis presented here indicates that this is often not the case (see also [7] for hypotheses that lead to a (3/2)-type map).

In this Letter, we propose a unified analytical framework for studying the local dynamics near grazing of general PWS systems. We establish a clear relationship between the continuity properties of the vector field at the grazing point and the functional form of the local map associated with it. We find that if grazing occurs with a smooth boundary (see Fig. 1(a)) the local map is indeed piecewise smooth but **never** piecewise linear. In contrast, if the boundary is itself non-smooth and grazing takes place at a corner-type singularity where the vector field is discontinuous (see Fig. 1 (b)) then the mapping is piecewise linear. We term this event a *corner-collision* bifurcation and we claim that this implies a border-collision of the corresponding local map.

These findings have immediate theoretical and experimental relevance for understanding phenomena caused by transitions between different smooth systems in macroscopic time-scales. According to the nature of the system under investigation, we show that grazing events yield bifurcation scenarios which can be classified using different local maps. The overall results are summarized in Table 1. Note that while there exist classification strategies for bifurcations in piecewise-linear or square-root maps [5,6], the dynamics of maps with (3/2)-type singularities have not been fully analysed.

The analytical framework we propose uses formal power series expansions and asymptotics which together give a synthetic analytical description of the grazing normal form map for a generic PWS system. We begin by assuming that sufficiently close to the grazing or corner-collision point, the phase space region under consideration is divided into two regions S_1 and S_2 by some boundary, Σ (see Fig. 1). This comprises either a smooth manifold (Fig. 1(a)) or a triangular wedge when projected onto a general plane (Fig. 1(b)). In the former case, the *discontinuity boundary* is described by the zero set of a smooth codimension-one surface $H(x) = 0$. In the latter, the wedge is described instead by two smooth codimension-one surfaces Σ_1 and Σ_2 which are given by the zero sets of differentiable functions $H_1(x)$ and $H_2(x)$. These sets are supposed to intersect along a smooth codimension-two surface C (the corner) at a non-zero angle, i.e. $\nabla H_1 \times \nabla H_2 \neq 0$. In either case, the system near grazing can be described by the PWS ODE:

$$\dot{x} := F(x) = \begin{cases} F_1(x), & \text{if } x \in S_1 \\ F_2(x), & \text{if } x \in S_2 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$,$F_1, F_2 : \mathbb{R}^n \mapsto \mathbb{R}^n$ are supposed to be sufficiently smooth and defined over the entire local region under consideration. For the sake of simplicity we further assume that the surfaces defined by the zero-sets of $H(x), H_1(x)$ and $H_2(x)$ are flat up to a sufficiently high order. Note that this may be assumed without loss of generality by making an appropriate sequence of near-identity transformations [8].

We say that a grazing occurs when a trajectory intersects a smooth boundary Σ tangentially. Without loss of generality this can be assumed to occur at the point $x = 0$ at which we further require that (a) $H^0 = 0$; (b) $\nabla H^0 \neq 0$; (c) $\langle \nabla H^0, F_i^0 \rangle = 0$; (d) $\langle \nabla H, F_{ix} F_i^0 \rangle > 0$. Here a superscript 0 represents a quantity evaluated at the grazing point $x = 0$. In contrast, if the discontinuity boundary is non-smooth at $x = 0$, then a *corner-collision* bifurcation is said to occur under similar generic hypotheses, when the trajectory intersects Σ at this point. In both cases, we assume that Σ is never simultaneously attracting from regions S_1 and S_2, so that so-called Filippov solutions (or sliding modes) cannot exist. This final assumption can be similarly expressed by appropriate inequalities which we omit for brevity.

To perform the analysis, we make use of the concept of *discontinuity mapping* (DM), see [7]. This is the local map that describes the correction that must be made to the global Poincaré map from surfaces in S_1 in order to describe trajectories that pass through region S_2 close to $x = 0$. The DM is derived analytically by considering ε-perturbations of a grazing or corner-colliding trajectory in the presence of the discontinuity boundary (see Fig. 2) by considering Taylor expansions of the flows Φ_1 and Φ_2 defined by

$$\frac{\partial \Phi_i}{\partial t} = F_i(\Phi_i(x,t)), \quad \Phi_i(x,0) = x, \quad i = \begin{cases} 1 & \text{if } x \in S_1 \\ 2 & \text{if } x \in S_2 \end{cases}.$$

As the vector fields are smooth, the flows $\Phi_i(x,t)$ can be expanded in Taylor series about the grazing point $(0,0)$:

$$\Phi_i(x,t) = x + F_i^0 t + a_i t^2 + b_i x t + c_i t^3 + d_i x^2 t + e_i x t^2 + f_i t^4 + g_i x^3 t + h_i x^2 t^2 + j_i x t^3 + O(5), \tag{2}$$

where $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, j_i$ are the matrix and tensor coefficients of the expansion and $O(5)$ is a shorthand for terms of order at least 5.

Consider first the case of a grazing bifurcation and let $x_g(t) = \Phi_1(0,t)$ be the trajectory which grazes the boundary at $x = 0$ when $t = 0$. Now, consider perturbations to x_g of size ε such that $x(t) = \Phi_1(\varepsilon x_0, t)$ for some unit vector x_0, which satisfies $\langle \nabla H, x_0 \rangle < 0$. Then, by assumptions (a)-(d), there will exist some $t_1 = -\delta < 0$ at which the perturbed trajectory, $x(t)$, crosses Σ at $x = \bar{x}$ passing from S_1 into S_2. The analysis can be split into three different stages: motion in S_1 before the first crossing of the switching manifold; motion in S_2; and finally motion after the second crossing of Σ from S_2 to S_1.

The first step is to use the Taylor expansions (2) in order to derive an asymptotic expression for δ. In so doing, we find a unique positive solution for δ which can be expressed as an asymptotic expansion in $\sqrt{\varepsilon}$ of the form $\delta = \gamma_1 \varepsilon^{\frac{1}{2}} + \gamma_2 \varepsilon + \gamma_3 \varepsilon^{\frac{3}{2}} + O(\varepsilon^2)$, where the coefficients $\gamma_1, \gamma_2, \gamma_3$ are expressed solely in terms of $F_{1,2}$ and their derivatives evaluated at the grazing point. Similarly, knowing δ, one can derive an estimate for $\bar{x}$. That is, $\bar{x} = \chi_1 \varepsilon^{\frac{1}{2}} + \chi_2 \varepsilon + \chi_3 \varepsilon^{\frac{3}{2}} + O(\varepsilon^2)$.

In the second stage, the motion evolves on the other side of the boundary until after some time $t_2 = \Delta > 0$, Σ is crossed again at $x = \hat{x}$. Here, we have $H(\hat{x}) = \Phi_2(\bar{x}, \Delta) = 0$. Using the Taylor series expansion of Φ_2 about the grazing point and the quantities computed in the previous stage, Δ can now be obtained as an asymptotic expansion in ε. Ignoring the trivial solution $\Delta = 0$, we get $\Delta = \nu_1 \varepsilon^{\frac{1}{2}} + \nu_2 \varepsilon + \nu_3 \varepsilon^{\frac{3}{2}} + O(\varepsilon^2)$, where again the coefficients can be expressed in terms of the vector field and its derivatives evaluated at the grazing point.

In order to finally arrive at the DM, we proceed through the third stage as follows. We solve from the point $\hat{x} = \Phi_2(\bar{x}, \Delta)$, backwards in time through a time $-t_2$ using flow Φ_1 until we hit the Poincaré section containing the initial point εx_0. Here, we present the case of relevance to a periodically forced non-autonomous system where the appropriate Poincaré section is defined stroboscopically by $t = 0$. The more general case of autonomous systems can be treated similarly but leads to algebraically more cumbersome expression. The discontinuity mapping is then the map from the initial point εx_0 to the final point $x_f = \Phi_1(\hat{x}, -t_2)$, where for the zero-time Poincaré section $t_2 = \Delta - \delta$. Using the asymptotic expansions for δ, $\bar{x}$, Δ and the expansion for $\hat{x} = \Phi_2(\bar{x}, \Delta)$ we can then systematically express x_f as a Taylor series in $\sqrt{\varepsilon}$.

Considering the case of discontinuity of the vector field at the grazing point $F_1^0 \neq F_2^0$, we find the leading order term in x_f is $O(\varepsilon^{\frac{1}{2}})$. Specifically, we have $x_f = (F_2^0 - F_1^0)\nu_1 \varepsilon^{\frac{1}{2}} + O(\varepsilon)$. (Note that a square-root singularity is also observed in the case where F has a δ-function discontinuity at $x = 0$ [6].)

If, on the contrary, the vector field is continuous at the grazing point but has discontinuous Jacobian, i.e. $F_1^0 = F_2^0$ and $F_{1x}^0 \neq F_{2x}^0$, then the $O(\varepsilon^{\frac{1}{2}})$ contributing to x_f vanishes. In fact, it is possible to show that in this case the discontinuity mapping is the identity up to $O(\varepsilon)$ and the leading-order non-trivial term is at least $O(\varepsilon^{\frac{3}{2}})$. This is also true if the Jacobian is continuous but the Hessian is not, i.e. $F_1^0 = F_2^0$, $F_{1x}^0 = F_{2x}^0$ but $F_{1xx}^0 \neq F_{2xx}^0$.

Lengthy algebraic manipulations [8] allow the analytical derivation of the leading-order part of the discontinuity mapping in the two cases treated above. This in turn allows explicit expressions for the normal forms associated with hyperbolic periodic orbits undergoing grazing in general n-dimensional PWS systems. Specifically, we present here formulae for the case when the grazing trajectory is part of a hyperbolic mT-periodic orbit $p(t)$ of a T-periodically forced system. That is (omitting the superscript 0 on each quantity involving F and H):

I. If the vector field is discontinuous at grazing, we have:

$$x \mapsto \begin{cases} Nx + M\mu, & \text{if } \langle \nabla H, x\rangle > 0 \\ N\mathbf{w}\sqrt{|\langle \nabla H, x\rangle|} + M\mu + \text{h.o.t.} & \text{if } \langle \nabla H, x\rangle < 0, \end{cases}$$

where

$$\mathbf{w} = 2(F_2 - F_1)\frac{\langle \nabla H, F_{2x}F_1\rangle}{\langle \nabla H, F_{2x}F_2\rangle}\left(\frac{2}{\langle \nabla H, F_{1x}F_1\rangle}\right)^{\frac{1}{2}},$$

grazing occurs at $\mu = 0$, and N and M are the linear parts of the Poincaré map calculated using flow Φ_1 alone.

II. If the vector field is continuous at $x = 0$, i.e $F_1 = F_2 := F$, but has discontinuous Jacobian (or Hessian):

$$x \mapsto \begin{cases} Nx + M\mu, & \text{if } \langle \nabla H, x\rangle > 0 \\ N\Big(x + \mathbf{v}_1(|\langle \nabla H, x\rangle|)^{\frac{3}{2}} + \\ V_2 x(|\langle \nabla H, x\rangle|)^{\frac{1}{2}} + \\ \mathbf{v}_3\langle \nabla H, F_{2x}x\rangle(|\langle \nabla H, x\rangle|)^{\frac{1}{2}}\Big) + M\mu & \text{if } \langle \nabla H, x\rangle < 0 \end{cases}$$

where

$$\begin{aligned} \mathbf{v}_1 =& \frac{1}{\langle \nabla H, F_{1x}F_1\rangle^{\frac{3}{2}}}\Big\{\frac{2}{3}\left(F_{2xx} - F_{1xx}\right)F^2 + \\ & 2F_{2x}F_{1x}F - \frac{2}{3}\left[(F_{1x})^2 + 2\left(F_{2x}\right)^2\right]F - \\ & \frac{2}{\langle \nabla H, F_{2x}F_2\rangle}\left(F_{2x} - F_{1x}\right)F \cdot \\ & \Big[\frac{2}{3}\left\langle \nabla H, \left(F_{2xx}F_2^2 + (F_{2x})^2 F_2\right)\right\rangle \\ & + \left\langle \nabla H, \left(F_{2x}F_{1x} - 2\left(F_{2x}\right)^2\right)F\right\rangle + \left\langle \nabla H, F_{2xx}F^2\right\rangle\Big]\Big\} \\ V_2 =& \frac{2}{\sqrt{\langle \nabla H, F_{1x}F_1\rangle}}\left(F_{2x} - F_{1x}\right) \\ \mathbf{v}_3 =& \frac{2}{\langle \nabla H, F_{2x}F_2\rangle\sqrt{\langle \nabla H, F_{1x}F_1\rangle}}\left(F_{2x}F_2 - F_{1x}F_1\right). \end{aligned}$$

So, contrary to what has been assumed in the literature, our results rule out the possibility of piecewise-linear maps associated to grazing events involving a smooth discontinuity boundary. However, experiments on a certain class of electronic circuits, so-called DC/DC power converters, indicate that piecewise-linear maps can indeed be observed at a corner collision point [9]. To prove that this is true, we must adapt the above analytical framework in order to take into account the geometry of the corner. For brevity, we will consider only the so-called external corner-collision depicted in Fig. 1(b); the internal case, where the orbit intersects the corner transversally, can be studied similarly [10].

With careful consideration of higher-order terms, the discontinuity mapping can again be constructed by expanding the system flows about the corner-collision point. Now, though, the linear terms in the expansions can be shown to be sufficient to completely describe the local dynamics near the corner. This can be explained heuristically that in the grazing case a locally parabolic tangency occurs while at a corner collision the time spent "inside" the corner varies linearly with ε. Specifically, it is possible to show that, taking into account trajectories that do not cross the wedge, the entire DM can be simplified to read

$$x \mapsto \begin{cases} x, & \text{if non-crossing,} \\ x + (F_1^0 - F_2^0)\langle \alpha, x\rangle + o(|x|) & \text{if crossing} \end{cases}$$

where $\alpha = J_2 - \langle J_2, F_1^0\rangle J_1$ with $J_i = \nabla H_i^0/\langle \nabla H_i^0, F_i^0\rangle$.

Example 1. As a simple representative example we consider the case of one-degree-of-freedom forced damped harmonic motion in a medium whose characteristics change at $x = 0$:

$$\ddot{x} + \zeta_i\dot{x} + k_i^2 x = \beta_i\cos(t), \quad i = \begin{cases} 1 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases} \qquad (3)$$

In this case, the boundary between the two regions of smooth dynamics S_1 and S_2 is the line $\Sigma := \{x = 0\}$. The change in the medium is modelled by a variation of the linear stiffness ($k_1 \neq k_2$), damping coefficient ($\zeta_1 \neq \zeta_2$) or amplitude of the forcing term ($\beta_1 \neq \beta_2$). Recasting, (3) as a set of first-order ODEs, it is possible to see that the vector field is continuous but has discontinuous Jacobian if $\beta_1 = \beta_2$ and $k_1 \neq k_2$ or $\zeta_1 \neq \zeta_2$ while it is discontinuous if $\beta_1 \neq \beta_2$. Therefore, the above analysis predicts that the local behaviour near grazing is described by a map with a square-root singularity in the latter case or a (3/2)-type singularity in the former. This agrees perfectly with the numerics depicted in Figs. 3-4.

Example 2. To illustrate the corner-collision case, we now take system (3) but suppose that the discontinuity boundary is now a non-smooth wedge defined by the zero-sets of $H_{1,2}(x) = x \mp \dot{x}$ (as for instance in the so-called tri-linear oscillator [1]). In this case, the analysis gives local dynamics described by a piecewise-linear map, which is indeed confirmed by numerical results (Fig. 4(b)).

In conclusion, we have presented a rigorous analytical framework for deriving normal form maps of grazing bifurcations in any PWS dynamical system. In so doing, we have for the first time provided a consistent link between the concepts of grazing bifurcations in PWS flows and border-collisions in PWS maps. We have additionally defined corner-collision as the special case involving grazing with a boundary that itself has a corner-type singularity. Our principal result is that piecewise-linear normal forms can be observed only in this latter case. In all other circumstances, the behaviour near grazing is described by mappings which reduce the dynamics to its essentials: namely the occurrence of a square-root or a (3/2)-type singularity as given in Table 1. The functional forms of these mappings are essential in order to understand and classify the bifurcation scenarios following grazings. A complete classification of the dynamics associated with these maps remains an open question.

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[8] The detailed proofs of the results announced here, together with further numerical examples illustrating the theory will be published in a future paper, which will also contain details of all the algebraic manipulations.

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[10] detailed proofs may be found in [M. di Bernardo, C.J. Budd and A.R. Champneys, *Corner-collision implies border–collision bifurcation*, preprint (2000)].

System at grazing pt.		**Map Singularity**
non smooth boundary		piecewise-linear
smooth boundary:		
F	**discontinuity in**	
	x	square-root [6]
C^0	F	square-root
C^1	F_x	(3/2)-type
C^2	F_{xx}	(3/2)-type

TABLE I. Relationship between the properties of the system at the grazing point and the type of singularity in the corresponding local map.

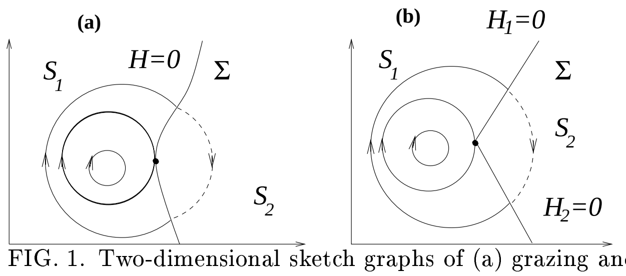

FIG. 1. Two-dimensional sketch graphs of (a) grazing and (b) corner-collision bifurcations.

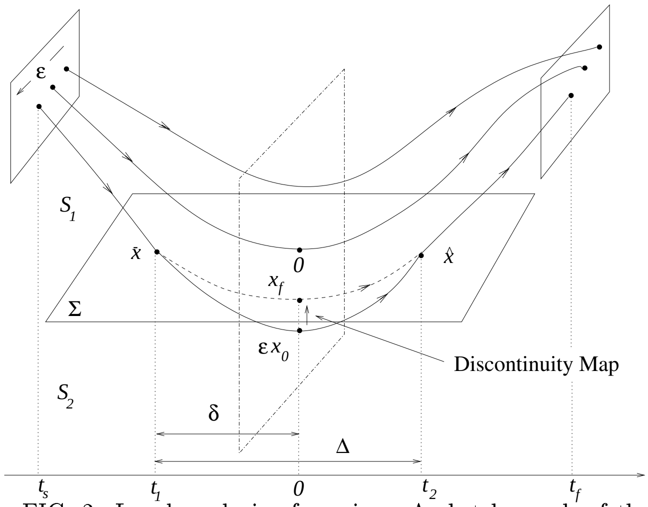

FIG. 2. Local analysis of grazing. A sketch graph of the three-dimensional case

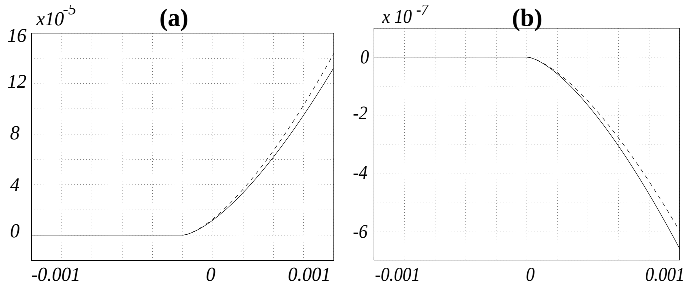

FIG. 3. Theoretical prediction (dashed line) and numerical simulation (solid line) of the change to the local behaviour of Eq. (3) near grazing, when (a) the stiffness or (b) the damping is varied across Σ. $x_f - \varepsilon x_0$ is plotted against ε. The parameters are set to be: (a) $k_1 = 1$, $k_2 = 2$, $\zeta_1 = \zeta_2 = 0.1$; (b) $k_1 = k_2 = 2$, $\zeta_1 = 1$, $\zeta_2 = 0.1$; while $\beta_1 = \beta_2 = 1$.

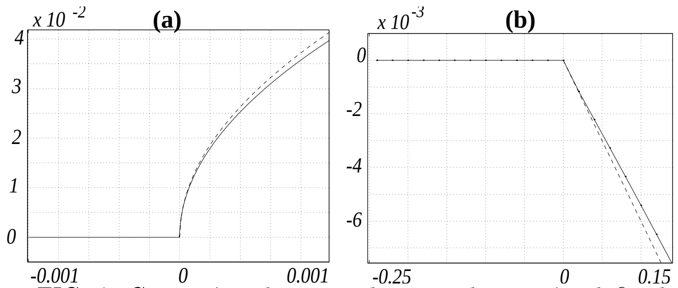

FIG. 4. Comparison between theory and numerics defined similarly to Fig. 3 for: (a) Eq. (3) near grazing, when the amplitude of the forcing term is varied across Σ ($k_1 = k_2 = 2$, $\zeta_1 = \zeta_2 = 0.1$, $\beta_1 = 1, \beta_2 = 2$); (b) for a corner-collision in the modified Eq. (3) (tri-linear oscillator) with $k_1 = k_2 = \sqrt{5}$, $\zeta_1 = \zeta_2 = 0.55$, $\beta_1 = 4.04$, $\beta_2 = 6.04$.