Asymptotics of new blow-up self-similar solutions of the nonlinear Schrödinger equation.

C.J. Budd[1]

Abstract

This paper examines blow-up self-similar solutions of the cubic nonlinear Schrödinger equation close to the critical dimension $d = 2$. It gives a formal asymptotic theory for self-similar solutions with multiple maxima, in which the solution close to each maximum takes the form of a rescaled one-dimensional soliton. As $d \to 2$ the maxima move to infinity and are centred close to the point $-\log(d-2)/\left(2\pi/3 - \sqrt{3/2}\right)$. However, the shape of the maxima is unaltered is unaltered in this limit, leading to an interesting new non-uniform bifurcation. The formulae derived from the asymptotic theory are strongly supported by some numerical calculations.

AMS SUBJECT CLASSIFICATIONS. 35Q55

KEY WORDS: Cubic nonlinear Schrödinger equation, blow-up,self-similar solutions, asymptotics.

October 26, 2000

[1]School of Mathematical Sciences, University of Bath, Claverton Down, Bath,BA2 7AY, UK.

1 Introduction

1.1 Preliminaries

The cubic nonlinear Schrödinger (NLS) equation

$$i\frac{\partial u}{\partial t} + \Delta u + u|u|^2 = 0, \tag{1}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad x \in R^d \tag{2}$$

is a useful model for many physical processes including nonlinear optics, plasma physics and quantum theory, as well as forming the envelope equation for many other important problems such as the transverse modulation of a water wave [Fi96, Ha89, SS99, Za84]. The NLS is an example of a unitary Hamiltonian partial differential equation, and during the evolution of the solution $u(x,t)$ both the mass

$$L = \int_{R^d} |u|^2 \, d\mathbf{x} \tag{3}$$

and the Hamiltonian

$$H = \int_{R^d} \left(|\nabla u|^2 - \frac{1}{2}|u|^4 \right) \, d\mathbf{x} \tag{4}$$

are invariants. Despite the constraints on the solution imposed by these constants of integration, if $d \geqslant 2$ then for suitable initial conditions $u_0(\mathbf{x})$ the solution becomes infinite at a single point in a finite time T, forming, at this point, a growing and increasingly narrow peak. We shall refer to this as a blow-up phenomenon. In plasma physics this singularity is called a collapse and in nonlinear optics it corresponds to a self-focusing of a laser beam. There is significant mathematical interest in the nature of the peak leading to many papers [KL95] and a recent book [SS99]. These papers all look at solutions in which the modulus of $u(\mathbf{x}, t)$ in the neighbourhood of the peak is strictly monotone decreasing. Whilst this appears from numerical calculations to be a stable form of behaviour, in the sense that from a variety of initial data a similar pattern of blow-up is observed, it is by no means unique. The paper [BCR99] described a series of numerical experiments which imply the existence of further non-monotone self-similar blow-up patterns. In a recent paper [PS00] very similar multi-bump solutions Ginsburg-Landau equation have also been computed. Whilst the multi-bump solutions of the NLS appear, from the numerical experiments described in [BCR99], to be unstable, they play a role in describing the transient behaviour toward blow-up of solutions with initially non-monotone profiles $u_0(\mathbf{x})$. In this paper we compliment and extend these numerical calculations by giving a formal asymptotic description of a doubly countable infinite set of non-monotone self-similar blow-up solutions. The predictions of the asymptotic formulae so obtained agree very closely with the numerical calculations.

There is numerical evidence [LPSS88c] that the blow-up of the NLS occurs at a single point $\mathbf{x}^*$ and that close to this point the blow-up solution is radially symmetric and a function of the single variable

$$r = |\mathbf{x} - \mathbf{x}^*|.$$

The function $u(\mathbf{x}, t)$, when considered as a function of r then satisfies the one-dimensional hyperbolic partial differential equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial r^2} + \frac{d-1}{r}\frac{\partial u}{\partial r} + u|u|^2 = 0, \quad u_r(0,t) = 0. \tag{5}$$

In this equation, the dimension d can take non-integer values and following [LPSS88a] we consider it to be a bifurcation parameter, with the solution $u(r,t)$ considered also as a function of d.

It is well known [SS99] that when differentiating between different qualitative forms of behaviour for u the value of $d = 2$ is critical. For $d \geqslant 2$, solutions with a negative Hamiltonian H all blow up in a finite time, whereas this is not the case if $d < 2$. (For example if $d = 1$ then the equation is integrable.) Furthermore, if $2 < d < 4$ then there is strong numerical and asymptotic evidence which implies that the blow-up is self-similar and that this description of the solution breaks down precisely when $d = 2$. Our interest in this paper is to study the non-monotone self-similar solutions in the critical limit of

$$d \to 2^+.$$

To study this case we pose the ansatz, originally due to Zakharov [Za84], that for r close to 0 and t close to T, the equation (5) has a self-similar solution $u(r,t)$ which takes the form

$$u(r,t) = \frac{1}{\sqrt{2a(T-t)}} e^{i\log(T-t)/2a} Q\left(\frac{r}{\sqrt{2a(T-t)}}\right). \tag{6}$$

Here the function $Q(y)$ with $y = r/\sqrt{2a(T-t)}$ satisfies a complex ordinary differential equation which is given in Section 2 (1).

The form of ansatz (6) follows immediately from the observation that the equation (5) is invariant under the action of the two symmetries, the *scaling symmetry*

$$(T-t) \to \lambda(T-t), \quad r \to \lambda^{1/2} r, \quad u \to \lambda^{-1/2} u, \quad \lambda > 0,$$

and the *phase invariance symmetry*

$$u \to e^{i\varphi} u, \quad \varphi \in \Re.$$

The scalar $a \in \Re$ is a nonlinear eigenvalue expressing the coupling between the phase and the amplitude of u. The invariance of the Hamiltonian of $u(r,t)$ places a restriction on the values of both the function $Q(y)$ and the eigenvalue a. As will be shown in Section 2 for $u(r,t)$ to have a constant Hamiltonian we also require that Q satisfies the integral constraint

$$H(Q) \equiv \int_0^\infty \left(|Q_r|^2 - \frac{1}{2}|Q|^4\right) r^{d-1} dr = 0. \tag{7}$$

For any value of d such that $2 < d < 4$ the values of a for which a solution can be constructed satisfying (7) appear from the calculations reported in [BCR99] to lie in a discrete countable set.

In [SS99] it is shown that when $|Q(y)|$ is a monotone decreasing solution, then as $d \to 2$ we have $a \to 0$. Similar behaviour is observed in the non-monotone cases and it is this limit that we will study in this paper. Observe that when $a = 0$ (6) is meaningless and there is strong evidence [FP99] that when $d = 2$, the function $u(r,t)$ has a different form of behaviour such that when t is very close indeed to T

$$u(r,t) \sim \left(\frac{\log\left(\log(1/(T-t))\right)}{(T-t)} \right)^{1/2} Q\left(r \left(\frac{\log\left(\log(1/(T-t))\right)}{(T-t)} \right)^{1/2} \right). \tag{8}$$

1.2 The qualitative form of self-similar blow-up

The form of the function $Q(y)$ as $d \to 2^+$ has been considered by several authors [MPSS86],[LPSS88a],[FP99], who all investigate the case of $|Q(y)|$ monotone decreasing, and numerical computations of the monotone solution have been given in many papers for example [ADKM92], [TS92], [MPSS86], [LPSS88c]. The recent paper [BCR99] gives numerical evidence for the existence of many more, non-monotone multi-bump solutions. To describe the form of these new solutions we first introduce the ordinary differential equation

$$R_{yy} + \frac{1}{y} R_y - R + R^3 = 0, \quad R_y(0) = 0 \quad R \to 0 \quad \text{as} \quad y \to \infty. \tag{9}$$

In Section 2 we show that (9) is the limit of the ordinary differential equation satisfied by $Q(y)$ if $ay \ll 1$ and $d - 2 \ll 1$. The equation (9) has been carefully analysed in [MS81], and it is known that it has a countable set of solutions $R_K(y)$ with $K = 0, 1, 2, ..$ such that

$$R_0(y) \equiv 0, \quad R_1(y) > 0$$

and $R_K(y)$ has $K - 1$ zeros and K turning points, for $y \geqslant 0$. Each such function decays exponentially for large y. The function $R_1(y)$ is unique and is often called the *ground state* solution.

We can now give an asymptotic description of a doubly countable set of multi-bump functions $Q(y) \equiv Q_{K,J}(y)$ all derived from the ansatz (6), which satisfy (7), with $(K,J) \in (0,1,2,3,..) \times (0,1,2,3,..)$ and for which $|Q_{K,J}|$ has $K + J$ local maxima. Corresponding to each such solution is an associated value $a_{K,J}$ of a with $a_{K,J} \to 0$ as $d \to 2$. In a non-uniform sense, each of these multi-bump solutions bifurcates from each of the functions $R_K(y)$ as $d \to 2^+$, and there are continuous branches of solutions parameterised by d which terminate at R_K in this limit. However, this bifurcation is both non-analytic and non-uniform. These multi-bump solutions have the following two asymptotic properties as $d \to 2^+$.

- If y is fixed and $a \to 0, d \to 2$ then $Q_{K,J}(y) \to R_K(y)$, although we note that for very large y, the function $Q(y)$ decays polynomially in y, whereas $R_K(y)$ decays exponentially.
- For small a and for $y = \mathcal{O}(1/a)$,the function $|Q_{K,J}(y)|$ has J bumps with maxima located at the points $y = \kappa_j/a$ (where $j = 1..J$), and to leading order κ_j is $1 + \mathcal{O}(a\log(a))$. At these points

$$|Q(y)| = \sqrt{3/2}\,(1 + \mathcal{O}(a\log(a)))\,.$$

The local form of each such bump in a neighbourhood of the point $y = \kappa_j/a$ is to leading order in a, and up to translation, independent of the value of a and d. As the function $R_K(y)$ is exponentially small for $y = \mathcal{O}(1/a)$ we see immediately that $Q_{K,J}(y)$ converges non uniformly to $R_K(y)$ in this limit.

In this notation, the function $Q_{1,0}(y)$ represents the monotone decreasing in modulus solution which occurs when d is close to 2 and which is described in the earlier literature [SS99]. This function is uniformly convergent to the ground state $R_1(y)$.

We emphasise here that there is no clear evidence that the solutions described above are in any way the only possible self-similar solutions. There may of course be more, taking an entirely different asymptotic form. However the ones we describe asymptotically are the only ones which have so far been observed numerically.

To visualise these self-similar solutions,and in particular to see the multi-bumps, we present numerical computations of the moduli of the functions $Q_{0,1}(y)$, $Q_{0,2}(y)$,$Q_{0,3}(y)$ and $Q_{1,0}(y)$, $Q_{1,1}(y)$ and $Q_{1,2}(y)$ when $d = 2.01$. These solutions are all computed using the numerical methods described in [BCR99].

Whilst the analysis of this paper considers only the case of d close to 2 and the associated values of $a_{K,J}$ close to 0, the values of $a_{K,J}$ all lie on branches parametrised by d. These branches are illustrated by the numerically computed graph given in Figure 1.3. Observe the steep gradient of each branch, which we will show asymptotically is maximally steep as $d \to 2^+$.

The computed curves of solutions all extend to the physically interesting parameter range of $d = 3$. As the maximum principle does not apply directly to complex second order differential equations, there is no guarantee that the solutions for which $|Q|$ is non-monotonic when d is close to 2 will remain non-monotonic along the solution branch. In deed,there is some numerical evidence that indicates that the two solutions $Q_{0,3}$ and $Q_{1,3}$ lose their multi-bump form and are both monotone decreasing in modulus when $d = 3$. The figure below shows the form of some of the solutions when $d = 3$.

When $d = 3$, the time-dependent solution $u(\mathbf{x}, t)$ corresponding to each of the solutions $Q_{K,J}$, is a multi-bump self-similar solution in which the bumps approach each other (with separation proportional to $\sqrt{T - t}$) as the blow-up time T is approached. This behaviour is to be distinguished from a series of copies of the monotone solution which all blow-up at the same time and which have multiple peaks that do not approach each other.

It is interesting to note that all of the maxima (apart from the single maximum at the origin) of the multi-bump solutions move off to infinity and vanish as $d \to 2$ and the analysis presented in this paper does not help understand the nature (if any) of self-similar multi-bump solutions in two dimensions. We leave this as a subject for future study.

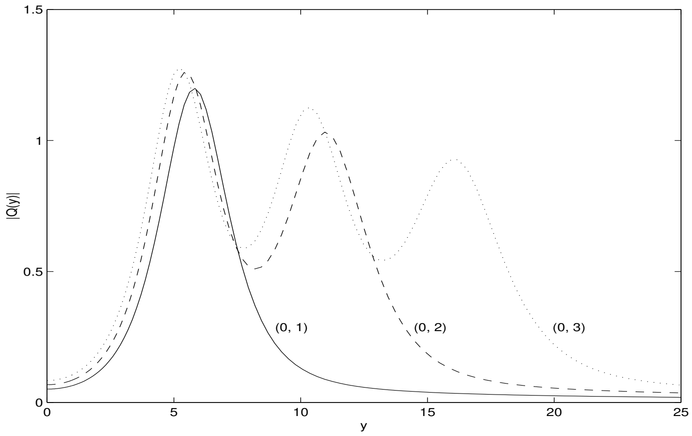

Figure 1: The functions $|Q_{K,J}(y)|$ for $(K,J) = (0,1),(0,2)$ and $(0,3)$ when $d = 2.01$.

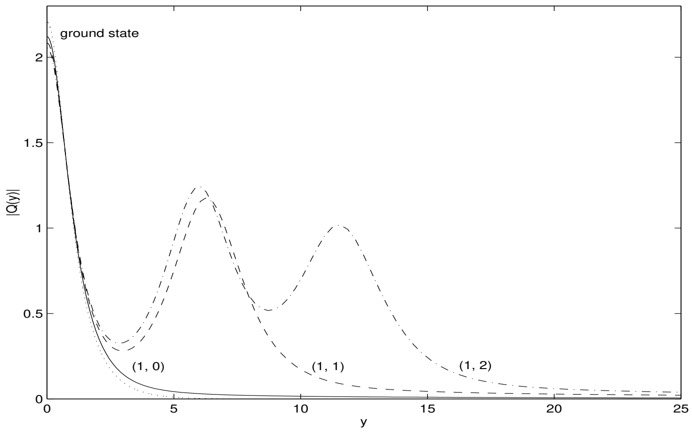

Figure 2: The functions $|Q_{K,J}(y)|$ for $(K,J) = (1,0),(1,1) and (1,2)$ when $d = 2.01$

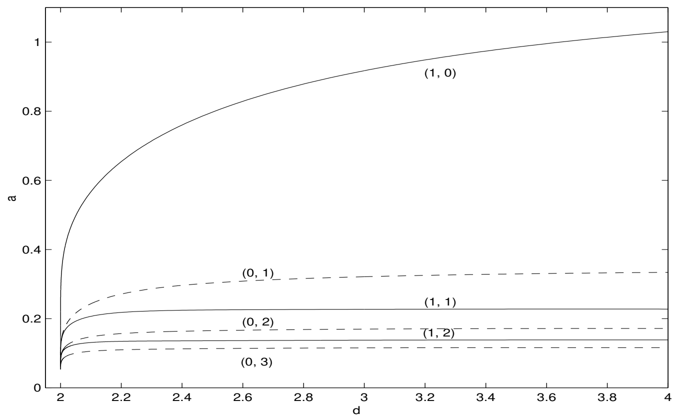

Figure 3: The values of $a_{K,J}$ for $(K,J) = (0,1),(0,2),(0,3),(1,0),(1,1),(1,2)$.

1.3 Statement of results

The main results of this paper are a series of formal asymptotic descriptions of the solutions $Q_{K,J}$ and a calculation of d as a function of $a_{K,J}$ in the limit of $a_{K,J} \to 0$ and $d \to 2$. We

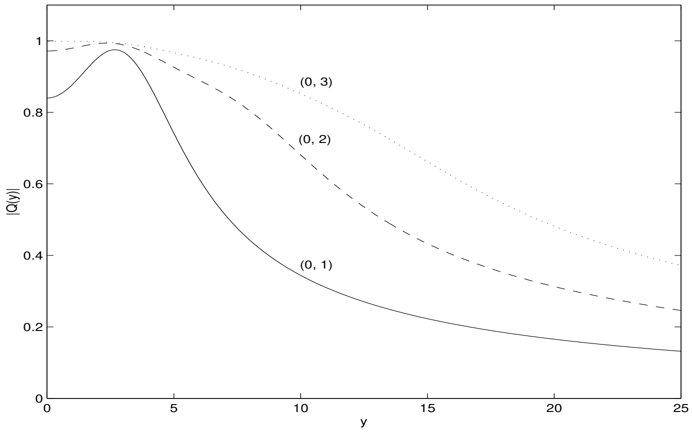

Figure 4: The functions $|Q_{K,J}|$ for $(K,J) = (0,1), (0,2)$ and $(0,3)$ when $d = 3$.

summarise our results as follows.

Formal Proposition 1.1

(i) If $J > 0$ then for each solution pair $\{Q_{K,J}(y), a_{K,J}\}$, we have that in the limit of $d \to 2^+, a_{K,J} \to 0$ there are constants α_J, γ_J and λ such that

$$(d-2) \sim \frac{\alpha_J}{a_{K,J}^{\gamma_J}} e^{-\lambda/a_{K,J}}, \quad \textit{where} \quad \lambda = 2\pi/3 - \sqrt{3}/2. \tag{10}$$

(ii) If $J > 0$ then as $d \to 2^+$ the multi-bump components of $|Q_{K,J}|$ have maxima at the K turning points of R_K and also at the points $y = \kappa_j/a_{K,J}$ where

$$\kappa_j = 1 + \mathcal{O}(a \log(a)) \tag{11}$$

These results are similar to those obtained for the monotone decreasing solution $Q_{1,0}(y)$ reported in [KL95], [SS99] in which we have

$$(d-2) \sim \frac{12.75..}{a} e^{-\pi/a}. \tag{12}$$

In all cases the function $d(a)$ is maximally flat, and the corresponding function $a(d)$ maximally steep, as $a \to 0$ and $d \to 2^+$.

To resolve the individual location of each of the multi-bumps in the requires the use of exponentially small asymptotic estimates, and we do not consider this calculation in this paper. However, in the particular case of $J = 1$ we can make some very precise estimates as to the location of the single bump. We describe these as follows.

Formal Proposition 1.2

(i) If $J = 1$ then

$$\kappa = 1 + \mathcal{O}(a^2) \quad \textit{and} \quad |Q(1/a)| = \sqrt{3/2} + \mathcal{O}(a^2). \tag{13}$$

Furthermore, to leading order $\gamma = 1$ and $\alpha = 3$ so that

$$(d-2) = \frac{3}{a_{0,1}} e^{-(2\pi/3 - \sqrt{3/4})(1+\mathcal{O}(a_{0,1}^2))/a_{0,1}} (1 + \mathcal{O}(a^2)). \tag{14}$$

(ii) In the range $1 \ll y < 2/a_{0,1}$ we have to leading order

$$|Q_{0,1}(y)| \sim \sqrt{\frac{3}{2}}\, sech\left(\sqrt{\frac{3}{4}}(y - 1/a_{0,1})\right). \tag{15}$$

(iii) If $0 \leqslant y \ll 1/a^{2/3}$

$$Q_{0,1}(y) = e^{-iay^2/4} 4(3/4)^{3/4} \sqrt{\frac{\pi}{a_{0,1}}} e^{-\left(\sqrt{3/4}+\pi/3\right)/2a_{0,1}} I_0(y)(1 - a^2 y^3/24..)$$

so that

$$Q_{0,1}(0)^2 = \frac{16\pi(3/4)^{3/2}}{a} e^{-(\sqrt{3/4}+\pi/3)/a_{0,1}} (1 + \mathcal{O}(a^2)). \tag{16}$$

Note that both $d-2$ and $Q(0)$ are exponentially small functions of $a_{0,1}$.

We derive the results in these two propositions by a mixture of scaling arguments, WKB analysis and matched asymptotic expansions. Our numerical numerical computations will give convincing support for the asymptotic results described in Formal Proposition 1.2.

The conclusions of Formal Proposition 1.2 are illustrated by the following computations of the function $Q_{0,1}$. In the first computation we show $|Q_{0,1}(y)|$ for each of the three cases of $d = 2.01, d = 2.0001$ and $d = 2.000001$. Observe that this solution converges to zero for any fixed y. However, we see the additional bump moving to the right as d decreases toward 2. Note that the location of this bump appears, from the figure, to be proportional to $-\log(d-2)$, consistent with the asymptotic formulae. It is also clear from the figure that the form of the solution at the bump, in particular the maximum height of $\sqrt{3/2}$, appears to be almost independent of the value of d.

The layout of the remainder of this paper is as follows. In Section 2 we discuss some of the general known theory about the function $Q(y)$, in particular looking at its behaviour when $ay \ll 1$ and when $ay \gg 1$. This analysis is partly a summary of known results given in [KL95] and [SS99]. We also derive some general results about the scaling of the solutions as $a \to 0$. In Section 3 we look at the solution for the intermediate range $ay = \mathcal{O}(1)$ and determine the structure of the multi-bumps. In Section 4 we use a WKB analysis to match the multi-bump form of the function $Q(y)$ derived in Section 3 when $ay = \mathcal{O}(1)$ to the behaviour for the range $ay \ll 1$ when $Q(y)$ is close to the function $R_K(y)$, and for the

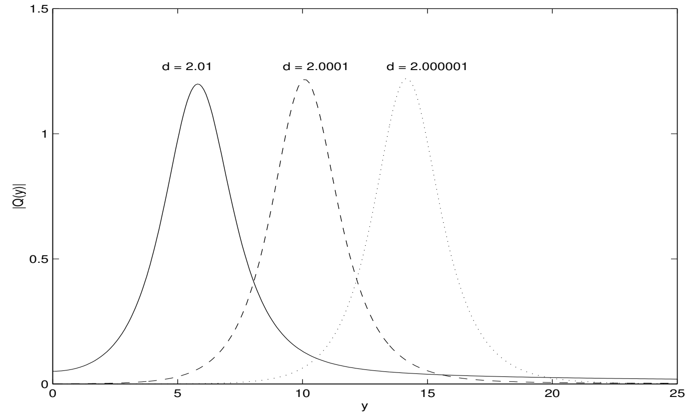

Figure 5: The function $Q_{0,1}$ as $d \to 2$.

range $ay \gg 1$ when $Q(y)$ is polynomially decaying. From this calculation we derive the results in the formal propositions. In Section 5 we compare the asymptotic results with some numerical calculations. Finally in Section 6 we draw some conclusions for this work.

2 Preliminary analysis

In this section we establish some results about the behaviour of the function $Q(y)$ in the two ranges $ay \ll 1$ and $ay \gg 1$. More basic results together with rigorous proofs of certain of them can be found in [SS99] and in [BCR99]. However, certain of the observations in this section, still require rigorous verification.

2.1 Preliminaries

Substituting the ansatz (6) into the partial differential equation (1), we find that the function $Q(y)$ satisfies the complex ordinary differential equation

$$Q_{yy} + \frac{(d-1)}{y} Q_y + ia(yQ)_y - Q + Q|Q|^2 = 0, \tag{1}$$

with the boundary condition

$$Q_y(0) = 0. \tag{2}$$

As the phase of Q is arbitrary, we can take

$$Q(0) \in \Re. \tag{3}$$

When considered as an initial value problem, (2) has many solutions. Indeed, it is shown in [BCR99], [W90] that for all $Q(0)$ and $a \in \Re$ there is a global solution $Q(y)$ such that

$$Q(y) \to 0, \quad \text{as} \quad y \to \infty. \tag{4}$$

The solutions of (1) that transform into self-similar solutions of (1) are those which when rescaled have a constant Hamiltonian. Rescaling the Hamiltonian defined in (4) we have

$$H = \int \left(|u_r|^2 - \frac{1}{2}|u|^4 \right) r^{d-1}\, dr = (2a(T-t))^{d-4} \int_0^\infty (|Q_y|^2 - \frac{1}{2}|Q|^4) y^{d-1}\, dy.$$

Thus, if $d < 4$, the Hamiltonian of a self-similar solution $u(r,t)$ can only be constant (indeed bounded), as $t \to T$, if

$$H(Q) \equiv \int_0^\infty \left(|Q_y|^2 - \frac{1}{2}|Q|^4 \right) y^{d-1}\, dy = 0. \tag{5}$$

It is those solutions of (1) which also satisfy (5) that transform to self-similar solutions of the underlying NLS equation. We can then regard the problem of finding self-similar solutions as that of finding (real) values of $Q(0)$ and a such that the solution of (1) then satisfies (5). Numerical evidence implies that for each value of d there is a countable discrete set of values of a and $Q(0)$ such that this is possible. It is shown further in [BCR99] that those solutions of (1) which satisfy the integral condition (5) also satisfy the point condition

$$y(1 + i/a)Q_y + q \to 0 \quad \text{as} \quad y \to \infty. \tag{6}$$

This condition is very useful for numerical computations in which (1) is solved by using a collocation method with (6) imposed as an identity for a large value of y. Details of this computation are given in Section 5.

2.2 The behaviour of $Q(y)$ when $y \gg 1$ and $|Q(y)| \ll 1$.

If the function $|Q(y)|$ is small, the differential equation (1) is closely approximated by the complex ordinary differential equation

$$Q_{yy} + \frac{d-1}{y} Q_y - Q + ia(yQ_y = 0. \tag{7}$$

It was observed in [KL95] that this equation is elliptic in character when $ay > 2$, it is hyperbolic when $ay < 2$ and it has a turning point when $ay = 2$. The equation has exponentially decaying solutions in the hyperbolic range and polynomially decaying solutions in the elliptic range.

To see the form taken by the solutions, we follow [KL95] and set

$$Q(y) = e^{-iay^2/4} y^{(1-d)/2} Z(y). \tag{8}$$

The function $Z(y)$ then satisfies the ordinary differential equation

$$Z_{yy} + Z\left(\frac{a^2y^2}{4} - 1 + \frac{ia}{2}(d-2) - \frac{1}{4y^2}(d-1)(d-3) \right) = 0, \tag{9}$$

which for a small and y large is closely approximated by the parabolic cylinder equation

$$Z_{yy} + Z\left(\frac{a^2y^2}{4} - 1 \right) = 0. \tag{10}$$

Following [KL95], for the elliptic range $ay > 2$, this equation has linearly independent solutions

$$y^{(d-3)/2-i/a} e^{iay^2/4} \; [1 + o(1)] \quad \text{and} \quad y^{(1-d)/2+i/a} e^{-iay^2/4} \; [1 + o(1)] \, . \tag{11}$$

These in turn correspond to solutions $\varphi(y)$ and $\psi(y)$ of (1) which have the form

$$\varphi(y) = y^{-1-i/a} \; (1 + o(1)) \quad \text{and} \quad \psi(y) = y^{(1-d)+i/a} e^{-iay^2/2} \; (1 + o(1)) \, , \tag{12}$$

which we refer to as the *slowly varying* and *rapidly varying* solutions. It is an easy calculation to show that it is only the slowly varying solutions which can satisfy the Hamiltonian condition (5) or the point condition (6). All solutions of the differential equation (1) which correspond to self-similar solutions must be slowly varying for sufficiently large y, placing severe restriction on the admissible values of a. Now, from the result that $|Q(y)| \to 0$ as $y \to \infty$ we deduce that for large y (which must necessarily be in the elliptic range), the function $Q(y)$ must take the asymptotic form

$$Q(y) = \mu\varphi(y) = \mu y^{-1-i/a} \; [1 + o(1)] = \mu y^{-1} e^{-i\log(y)/a} \; [1 + o(1)] \, , \tag{13}$$

for some, as yet, undetermined value of μ. Finding the value of μ plays an important part of our asymptotic calculations, and we obtain an asymptotic estimate in Section 4.3. Observe that in this range the modulus of $Q(y)$ is to leading order given by

$$|Q| = \frac{\mu}{y}$$

which is a monotone function, decreasing at a polynomial rate.

In the hyperbolic range, provided that $|Q(y)|$ is small and that $1 \ll y < 2/a$ the small solutions of (1) take the form

$$\varphi(y) = y^{(1-d)/2} e^{-iay^2/4} e^{-y} \, (1 + o(1)) \quad \text{and} \quad \psi(y) = y^{(1-d)/2} e^{-iay^2/4} e^{y} \, (1 + o(1)) \, . \tag{14}$$

In [KL95] and [SS99] only those exponentially decreasing solutions of the form $\varphi(y)$ were considered, as only these play a role in the discussion of the monotone decreasing solutions of (1). However, both solutions can in principle be admitted as solutions of (1) in the hyperbolic range, provided that $Q(y)$ is small. It is the exponentially growing solutions of the form $\psi(y)$ which, in this range, give rise to the multi-bump components of the function $Q(y)$, for which $|Q(y)| = \mathcal{O}(1)$ when $ay = \mathcal{O}(1)$. In Section 4 we use a WKB analysis to couple the exponentially decaying 'hyperbolic' solutions with the polynomially decaying 'elliptic' solutions.

2.3 The behaviour of the solutions when $ay \ll 1$ as $a \to 0$.

In [SS99] it is proven that the ordinary differential equation (1) can have no non-zero solution satisfying (7) if $a = 0$. Whereas this calculation considered monotone decreasing solutions, the same result applies to non-monotone functions, with an identical proof. A most significant feature of all the solutions $Q(y)$ calculated in this paper is that in the limit of $d \to 2^+$ we have that $a \to 0$. Furthermore, if we consider $d(a)$ to be a function of $a$then this function is maximally flat, with all the derivatives of d with respect to a vanishing when

$a = 0$. This result is proven in [SS99] for the case of a monotone solution profile, and is fully consistent with the asymptotic formula (12). There are severe difficulties in extending this proof to the case of non-monotone solutions. Firstly because the convergence of the non monotone solutions toward the function R_K is non-uniform, and secondly because certain constants assumed non-zero for the proof in [SS99] are zero for the case of non-monotone profiles (in particular those which are perturbations of the zero function R_0). We shall initially make the assumption $a \to 0$ for the case of the non-monotonic profiles and then show that this leads to fully consistent asymptotic formulae.

To consider the behaviour of Q for $ay \ll 1$ we make use of the amplitude and phase decomposition

$$Q(y) = A(y)e^{i\theta(y)}. \tag{15}$$

Then a simple calculation shows that θ satisfies the differential equation

$$\theta_y + \frac{a}{2}y = -\frac{(d-2)}{yA^2}\int_0^y \theta_y A^2 dy'. \tag{16}$$

Noting that $\theta(0) = 0$, if $(d-2)$ is small we then have

$$\theta(y) \approx -\frac{ay^2}{4}. \tag{17}$$

This calculation motivates using the alternative decomposition

$$Q(y) = e^{-iay^2/4}P(y). \tag{18}$$

We then have

$$P_{yy} + \frac{d-1}{y}P_y - P + \frac{a^2y^2}{4}P - ia\left(\frac{d-2}{2}\right) + P|P|^2 = 0, \tag{19}$$

$$P_y(0) = 0, \quad P \to 0 \quad \text{as} \quad y \to \infty.$$

On the assumption that $d-2$ is an exponentially small function of a, then to all algebraic orders in a we have that

$$P_{yy} + \frac{1}{y}P_y - P + \frac{a^2y^2}{4}P + P|P|^2 = 0, \quad P_y(0) = 0, \quad P \to 0 \quad \text{as} \quad y \to \infty. \tag{20}$$

For $a = 0$ and $d = 2$ (20) reduces to

$$P_{yy} + \frac{1}{y}P_y - P + P|P|^2 = 0, \quad P_y(0) = 0 \tag{21}$$

and (20) is a perturbation of (21) if $ay \ll 1$. The behaviour of the solutions of (21) is well known [MS81]. It has a series of decaying solutions $R_K(y)$ that have the form

$$R_K(y) = \frac{\nu_K}{\sqrt{y}}e^{-y} \quad \text{as} \quad y \to \infty. \tag{22}$$

where $R_K(0)$ and ν_K take the following computed values

K	$R_K(0)$	ν_K
0	0	0
1	2.2062	3.45
2	3.3320	82.1
3	4.1501	1910

All other solutions of (21) converge to one of the two fixed points $P = 1, -1$ with

$$P(y) \approx \pm 1 + \alpha J_0(y - y_0) \quad \text{for} \quad y \gg 1,$$

where α and y_0 are appropriate constants. Perturbations from the solutions R_K grow locally as e^y. If these perturbations are not exponentially small in a at the point $y = 0$ (ie. they are not initially of order $\exp(-\mu/a)$ when a is small) then the resulting solution converges toward one of the fixed points for values of y which are large but for which ay is small.

Consider now the equation (20). If ay is small then following [SS99] we may develop a regular perturbed solution to the solutions R_K of (21) which takes the form

$$P(y) = R_K(y) + a^2 P_2(y) + a^4 P_4(y) + ... \tag{23}$$

For large y we may use (22) to estimate the further terms in this expansion, and a simple calculation shows, that provided $y \ll a^{2/3}$, to order a^2 we have

$$P(y) \approx \frac{\nu_K}{\sqrt{y}} e^{-y} \left[1 - \frac{a^2 y^3}{24} \right]. \tag{24}$$

However this expansion for $P(y)$ is, in general, incomplete and must be extended by adding exponentially growing terms. These are excluded in the calculation presented in [SS99] by the assumption of monotonicity, however we cannot exclude them when considering non-monotone solutions. Accordingly we extend the expansion (24) to give, for large y, an expansion of the form

$$P(y) = R_K(y) + a^2 P_2(y) + a^4 P_4(y) + ... + \frac{\alpha}{\sqrt{y}} e^y (1 + a^2 S_2(y) + ..). \tag{25}$$

Here α is an appropriate small constant, and we use the fact that to leading order the function $e^y/\sqrt{y}$ is a solution of (20) for large y. The value of α will be determined by matching in Section 4 and shown to be an exponentially small function of a.

As previously observed, if α is not exponentially small in a then the solution described by (25) rapidly evolves to one which is a perturbation of that solution of (20) which satisfies $P(0) = 1$. This solution, which we denote as $F(y, a)$, is a development of the fixed point solution of (21) for larger values of y. In particular, if ay is small then to leading order we can ignore the derivative terms and balance the non-derivative terms in (20) to give a leading order expression

$$F(y, a) \approx \sqrt{1 - \frac{a^2 y^2}{4}}.$$

Observe that this function is *not* small in the hyperbolic range $ay < 2$ and becomes singular as $ay \to 2$. Such solutions cannot be matched to the slowly varying solutions in the elliptic range through a WKB analysis given in Section 4. Furthermore numerical calculations all imply that the solutions close to $F(y,a)$ all lead to rapidly varying solutions in the elliptic region. However, a rigorous proof of this result is not yet available.

We infer from this discussion that the only candidates for a self-similar solution of (1) are those solutions which are not close to $F(y,a)$ if $ay \ll 1$. These are precisely those solutions which have an asymptotic expansion of the form (25) for which α is an exponentially function of a. For such solutions the exponentially growing term becomes significant when ay is of order one, as are the terms of the form a^2y^2. It is the interaction between these two which leads to the multi-bump form of the solutions.

2.4 Perturbations from the zero solution

In general, the exponentially small size of α makes the expression(24) hard to work with. However, in the case of a perturbation from $R_0 \equiv 0$, all of the algebraic in a contributions to $P(y)$ vanish and we are left only with the exponentially growing perturbation. The function $P(y)$ satisfies (20) to leading order and is a perturbation of zero so that the term $P|P|^2$ may be neglected. This leads to the equation

$$P_{yy} + \frac{1}{y}P_y - P + \frac{a^2y^2}{4}P = 0,$$

which is the modified Bessel equation. Using the properties of modified Bessel functions it is easy to see that to leading order $P(y)$ has the form

$$P(y) = \alpha\sqrt{2\pi}I_0(y)\left(1 - \frac{a^2y^3}{24} + ..\right) \tag{26}$$

where $I_0(y)$ is the modified Bessel function satisfying

$$I_0(0) = 1 \quad \text{and} \quad \sqrt{2\pi}\, I_0(y) \to \frac{e^y}{\sqrt{y}} \quad \text{as} \quad y \to \infty.$$

In Section 4.4 we develop an asymptotic expansion for α which gives

$$\alpha \sim \exp(-(\sqrt{3/4} + \pi/3)/2a)$$

confirming that the perturbation of $P(y)$ from zero is only significant when $y = \mathcal{O}(1/a)$.

3 The multi-bump form of the solution in the range $y = \mathcal{O}(1/a)$.

In the range $y = \mathcal{O}(1/a)$ the perturbation a^2y^2 of (20) from the equation (21) is of the same order as the other terms in the equation, and we have a large perturbation of the solution from $R_K(y)$ as the terms αe^y become significant. It is in precisely this range that we observe, from the numerical calculations, the existence of the multi-bump solutions with magnitudes of order 1.

3.1 The form of the additional maxima

We analyse the multi-bump solution by considering it to be a rescaled localised solution of Duffings' equation which decays exponentially away from its peaks. Each of these localised solutions exists in the region $y = \mathcal{O}(1/a)$ and strongly resembles a soliton solution of the one dimensional nonlinear Schrödinger equation. From the numerical evidence presented in Section 1, it appears that as $a \to 0$ the peaks in the function $|Q(y)|$ are asymptotically located at points κ/a, but that the solution close to the peak is to leading order independent of a. Now, consider the solution in the neighbourhood of one of these peaks. To do this we rescale the equation (1) by setting

$$y = \kappa/a + s. \tag{1}$$

We presume that $a \ll 1$ and consider an expansion of the solution in terms of a. At present we make no condition on the value of κ, and one purpose of this section is to show that to leading order $\kappa = 1$.

Substituting (1) into (1) gives the system

$$Q_{ss} - Q + i\kappa Q_s + Q|Q|^2 = -ia(sQ)_s - a\frac{(d-1)}{\kappa + as}Q_s \tag{2}$$

Using this expression, we may develop an asymptotic expansion for the rescaled function $Q(s)$ in the neighbourhood of $s = 0$ by setting

$$Q(s) = Q_0(s) + aQ_1(s) + \ldots.$$

The leading order term $Q_0(s)$ then satisfies the reduced equation

$$Q_{0,ss} - Q_0 + i\kappa Q_{0,s} + Q_0|Q_0|^2 = 0. \tag{3}$$

We now rescale the phase of $Q_0(s)$ by setting

$$Q_0(s) = e^{-i\kappa s/2} S_0(s). \tag{4}$$

On substituting into (3) this gives

$$S_{0,ss} - (1 - \kappa^2/4)S_0 + S_0|S_0|^2 = 0. \tag{5}$$

We see that, up to multiplication by an arbitrary constant of modulus one, that the function $S(s)$ is a solution of a scaled form of Duffings' equation. Significantly, this solution is *real*.

We are interested in finding localised solutions which have a maximum at the point $s = 0$ and Duffings' equation has a homoclinic orbit solution with exactly these properties which is precisely the soliton solution of the one-dimensional form of NLS. This solution is unique up to an arbitrary constant of unit modulus, and is given by

$$S_0(s) = \sqrt{2(1-\kappa^2/4)}\ \operatorname{sech}\left(\sqrt{1-\kappa^2/4}\ \ s\right). \tag{6}$$

We note immediately from this result that at $s = 0$ we have to leading order

$$|S_0(0)| = |Q_0(\kappa/a)| = \sqrt{2\,(1-\kappa^2/4)} \tag{7}$$

and that this value is *independent of* a.

For large $|s|$ and small a (such as $|s| = \mathcal{O}(a^{-1/2})$), the solution (6) is exponentially decaying, provided that we are in the hyperbolic region $\kappa < 2$, and we have for large $|s|$

$$|S_0(s)| \sim 2\sqrt{2(1-\kappa^2/4)}e^{-\sqrt{1-\kappa^2/4}\,|s|} \tag{8}$$

3.2 Calculating the location of the maxima

We now show that the location of the maximum follows from a consistency condition at the next order of the asymptotic expansion. To do this we rescale the phase of Q_1 by setting

$$Q_1(s) = e^{-i\kappa s/2} S_1(s)$$

and set $t(s)$ to be the real part of $S_1(s)$.Substituting the expression for S_1 into the rescaled differential equation for Q it follows, after some manipulation, that the function $t(s)$ satisfies the differential equation

$$Lt \equiv t_{ss} - (1-\kappa^2/4)t + 3S_0^3 t = f(s) \equiv \frac{\kappa}{2} sS_0 + \frac{1}{\kappa} S_{0,s}, \quad t(0)=0, \quad |t| \to 0 \quad \text{as} \quad |s| \to \infty. \tag{9}$$

Now, the homogeneous equation

$$Lt = 0$$

has the odd, exponentially decaying, solution $\psi_1(s) = S_{0,s}$ and a further linearly independent, exponentially growing, even valued solution $\psi_2(s)$, with ψ_1 and ψ_2 having constant Wronskian W. It is easy to see that $\psi_2(s)$ has an exponential dependence of the form

$$\psi_2(s) \sim \exp(\sqrt{1-\kappa^2/4}\; s).$$

From the variation of constants formula, applying the condition that $t(0) = 0$, it then follows that we can express the function $t(s)$ in the form

$$t(s) = A\psi_2(s) + \psi_1 \int_0^s \frac{\psi_2 f}{W} ds' - \psi_2 \int_0^s \frac{\psi_1 f}{W} ds'$$

where A is arbitrary at this stage. Using the above expression for I gives

$$t(s) \to (A - I/W)\psi_2(s) \quad \text{as} \quad s \to \infty \quad \text{and} \quad t(s) \to (A + I/W)\psi_2(s) \quad \text{as} \quad s \to -\infty. \tag{10}$$

where I is the integral

$$I \equiv \int_0^\infty f\psi_1 ds = \int_0^\infty \left(\frac{\kappa}{2} sS_0 + \frac{1}{\kappa} S_{0,s}\right) S_{0,s} ds. \tag{11}$$

We now consider the separate cases of solutions with one and several maxima. A localised solution with a *single* maximum ($J = 1$), decays exponentially away from $s = 0$ and for asymptotic consistency we cannot have an exponential growth in the function t for either $s \to \infty$ or $s \to -\infty$. In particular, there can be no contribution due to ψ_2 in these two limits. Hence from (10) it follows that $A = 0$, and $I = 0$. We immediately deduce that $t(s)$ is an odd function of s with $t(0) = 0$ given by

$$t(s) = \psi_1(s) \int_0^s \frac{\psi_2 f}{W} ds' - \psi_2(s) \int_0^s \frac{\psi_1 f}{W} ds'.$$

Using the expression (6) the integral in (11) evaluates to

$$I = \frac{4}{3}\kappa(1-\kappa^2/4)^{1/4}(1-\kappa^2), \tag{12}$$

so that $I = 0$ only if

$$\kappa = 1. \tag{13}$$

This calculation fixes the value of κ to leading order for solutions in which $t \to 0$ as $|s| \to \infty$. We will show presently that in this case $\kappa = 1 + \mathcal{O}(a^2)$.

For a function with *two* maxima we can give a less precise analysis. Suppose that there are two maxima with corresponding values $\kappa_1 < \kappa_2$. We may develop approximations to the solutions as perturbations of the Duffing equation by perturbing around around the peaks at κ_1/a and κ_2/a. Consider the first peak at κ_1/a and set $s = y - \kappa_1/a$. As $s \to -\infty$ we require that $t \to 0$ and hence (with values A and I in expression (10)) we have that $A + I/W = 0$. In contrast as $s \to \infty$ we have an exponentially increasing contribution from $t(s)$ to the local form of $S_0(s)$. We denote $S_0(s)$ in the neighbourhood of κ_1/a by $S_{0,1}(s)$. This contribution then takes the form

$$S_{0,1}(s) + a(A - I/W)\psi_2(s) = S_{0,1}(s) - 2aI\psi_2(s)/W.$$

Now, $\psi_2(s)$ varies as $\exp(\sqrt{1-\kappa_1^2/4}\; s)$ and as s increases toward the second peak we may express this contribution as

$$S_{0,1}(s) - 2aI/W e^{\sqrt{1-\kappa_1^2/4}(\kappa_2-\kappa_1)/a)} e^{\sqrt{1-\kappa_1^2/4}\;(s+\kappa_1/a-\kappa_2/a)}.$$

Conversely, from the previous computation, close to the pointκ_2/a we may express the solution in the form

$$S_{0,2}(s) = 2\sqrt{1-\kappa_2^2/4}\; e^{\sqrt{1-\kappa_2^2/4}s'}$$

where $s' = y - \kappa_2/a = s + \kappa_1/a - \kappa_2/a$. As $S_{0,1}(s)$ is exponentially decreasing, these two expression match to leading order provided that

$$ae^{\sqrt{1-\kappa_1^2/4}\;(\kappa_2-\kappa_1)/a} = \mathcal{O}(1)$$

so that

$$\kappa_2 = \kappa_1 + \mathcal{O}(a\log(a)).$$

A very similar result holds similarly for solutions with more maxima.

By looking at the Hamiltonian, we now refine these two calculations to show that in the case of a solution with a single maximum we have $\kappa = 1 + \mathcal{O}(a^2)$ and for solutions with several maxima that $\kappa = 1 + \mathcal{O}(a\log(a))$.

The condition for the solution of (1) to have zero Hamiltonian is given by

$$H(Q) \equiv \int_0^\infty \left(|Q_y|^2 - \frac{1}{2}|Q|^4\right) y^{d-1}dy = 0. \tag{14}$$

Now, consider the small a limit ($d \to 2$) of this equation. In the neighbourhood of each maxima located at κ_j/a the form of each integrand in the rescaled coordinates becomes

$$\left(|Q_s|^2 - \frac{1}{2}|Q|^4\right)\left(\frac{\kappa_j}{a} + s\right). \tag{15}$$

The dominant term in this expression has magnitude $1/a$, and considering only terms of this order we may simplify (15) to give

$$\left(|Q_{0,s}|^2 - \frac{1}{2}|Q_0|^4\right)\left(\frac{\kappa_j}{a}\right) \tag{16}$$

as the contribution to the integrand close to the peak. We approximate $Q_0(s)$ to leading order in each such integrand by the single function $e^{-i\kappa_j s}S_0(s)$. Differentiating this expression we then have

$$Q_{0,s} = e^{-i\kappa_j s/2}\left(S_{0,s} - \frac{i\kappa_j}{2}S_0\right)$$

so that

$$|Q_{0,s}|^2 = |S_{0,s}|^2 + \frac{\kappa_j^2}{4}|S_0|^2 \quad \text{and} \quad |Q_0|^4 = |S_j|^4.$$

Combining these results, we deduce that the dominant $\mathcal{O}(1/a)$ contribution to the integrand in the Hamiltonian in the neighbourhood of each peak is given by

$$\left(|S_{0,s}|^2 + \frac{\kappa_j^2}{4}|S_0|^2 - \frac{1}{2}|S_0|^4\right)\left(\frac{\kappa_j}{a}\right).$$

When integrating this expression, the limits on the original integral in the neighbourhood of each peak rescale to $-\kappa_j/a$ and ∞. As the solution is exponentially decaying, then to all orders in a we may approximate the integral by taking these limits to be $-\infty$ and ∞ for each individual contribution to H from the behaviour at each peak. Then combining the contribution I_j to H from each of the J peaks with $j = 1...J$ gives the dominant contribution to H as

$$\sum_{j=1}^{J} I_j \equiv \frac{1}{a}\left(\sum_{j=1}^{J}\int_{-\infty}^{\infty}\kappa_j\left(|S_{0,s}|^2 + \frac{\kappa_j^2}{4}|S_0|^2 - \frac{1}{2}|S_0|^4\right)ds.\right)$$

After some manipulation we have that each of the integrals I_j in this expression are given by

$$I_j = \frac{1}{a}\left(2\alpha_j^2\kappa_j\int_{-\infty}^{\infty}\left(\operatorname{sech}^2(\alpha_j s)\left(\alpha_j^2\tanh^2(\alpha_j s) + \frac{\kappa_j^2}{4}\right) - \alpha_j^2\operatorname{sech}^4(\alpha_j s)\right)ds\right),$$

where

$$\alpha_j^2 = 1 - \frac{\kappa_j^2}{4}.$$

Each of these integrals may be evaluated exactly to give

$$I_j = \frac{2}{a}\alpha_j^2\kappa_i\left(\frac{2}{\alpha_j} - \frac{8}{3}\alpha_j\right) = \frac{4}{3a}\alpha_j\kappa_j(\kappa_j^2 - 1).$$

Multiple maxima

Suppose there are $J > 1$ maxima. We have established that the values of κ_j differ by a factor $a\log(a)$. Suppose that each such κ_j and α_j is expressed as

$$\kappa_j = \kappa + \mathcal{O}(a\log(a)) \quad \text{and} \quad \alpha^2 = 1 - \kappa^2/4.$$

Then to leading order

$$H = J\frac{4}{3a}\alpha\kappa(\kappa^2 - 1) + \mathcal{O}(\log(a)).$$

And hence if $H = 0$ we must have

$$\kappa = 1 + \mathcal{O}(a\log(a)) \quad \text{and thus} \quad \kappa_j = 1 + \mathcal{O}(a\log(a)).$$

Single maximum

In the case of a single maximum we have shown that $\kappa = 1$, and hence $H = 0$ to leading order. We can now make a much more precise estimate as to the location of the maximum.To do this we must now determine the correction to the estimate of H from that given above. This correction comes from three main additional terms: the calculation of H for y small, for y large and the additional contribution from the further terms in the asymptotic expansion calculated close to the maximum.

Consider first the contribution to H for smaller values of y so that $ay \ll 1$. This is the range of y values considered in Section 2. In this range $Q(y)$ is approximated exponentially closely by a function of the form $e^{-iay^2/4}R_K(y)(1+\mathcal{O}(a^2))$. Thus

$$|Q_y|^2 \approx R_{K,y}^2 + a^2y^2R_K^2/4 + \mathcal{O}(a^2).$$

Hence the contribution to H from this range of y values is simply

$$I_0 \equiv \int_0^\infty (R_{K,y}^2 - \frac{1}{2}R_K^4)y\ dy + a^2\int_0^\infty R_K^2y^3\ dy + \mathcal{O}(a^2).$$

It is shown in [SS99] that

$$\int_0^\infty (R_{K,y}^2 - \frac{1}{2}R_K^4)\ y\ dy = 0.$$

Hence

$$I_0 = \mathcal{O}\left(a^2\right)$$

and the contribution to H from the small y behaviour of $Q(y)$ is at most $\mathcal{O}(a^2)$.

Now consider the large values of y, in particular for $ay > 2$. In this range, $Q(y)$ is slowly varying and is approximated by

$$Q(y) = \mu y^{-1-i/a}.$$

Thus

$$\int_{2/a}^\infty \left(|Q_y|^2 - \frac{1}{2}|Q|^4\right) y dy \sim \frac{\mu^2}{8}.$$

However, we will show in the matching arguments in Section 4 that μ is an exponentially small function of a and hence this contribution to H is negligible to all orders.

The next correction to the value of H comes from the terms neglected when (15) is approximated by (16). To estimate this correction we must find the contribution the integral for H from the term $Q_1(s)$. To do this we make use of the fact that $S(s)$ is a real function of s. It then follows that in the neighbourhood of the maximum, the integrand for H has the form

$$\left(|Q_s|^2 - \frac{1}{2}|Q|^4\right)\left(\frac{\kappa}{a}+s\right) =$$

$$\left(S_{0,s}^2 + \frac{\kappa^2}{4}S_0^2 - \frac{1}{2}S_0^4 + a\left(2S_{0,s}t_s + \frac{\kappa^2}{4}S_0t + 4S_0^3t\right) + \mathcal{O}(a^2)\right)\left(\frac{\kappa}{a}+s\right). \qquad (17)$$

Integrating this expression gives

$$H = \frac{1}{a}\left(\frac{4}{3}\alpha\kappa(\kappa^2-1) + a\int_{-\infty}^\infty \left(S_{0,s}^2 + \frac{\kappa^2}{4}S_0^2 - \frac{1}{2}S_0^4\right)s + \kappa\left(2S_{0,s}t_s + \frac{\kappa^2}{4}S_0t + 4S_0^3t\right)ds + \mathcal{O}(a^2)\right).$$

Now, the function S_0 is *even* and (from the previous calculation) the function $t(s)$ is *odd* in this case, hence the above integral reduces to

$$H = \frac{1}{a}\left(\frac{4}{3}\alpha\kappa(\kappa^2-1) + \mathcal{O}(a^2)\right).$$

We deduce from the condition $H = 0$ that in this special case

$$\kappa = 1 + \mathcal{O}(a^2).$$

These analytical results are completely in agreement with the numerical calculations.

4 Matching through a WKB analysis

The analysis of the previous section has looked at the behaviour close to the maxima in $|Q|$ located at the points $y = \kappa/a$ (where $\kappa = 1 + \mathcal{O}(a\log(a)))$, at which the solution is of $\mathcal{O}(1)$ in the limit of small a. As the value of $|y - \kappa/a|$ becomes large,for example if $|y - \kappa/a| = \mathcal{O}(a^{-1/2})$ in the limit of small a, then the solution computed in the previous section decays exponentially. When this happens we have simultaneously that

$$y \gg 1 \quad \text{and} \quad |Q| \ll 1 \quad a \ll 1.$$

The whole system then becomes amenable to the WKB type of analysis described in [SS99] and in [KL95] when considering perturbations of the ground state.

In this analysis we determine the form of $Q(y)$ both in the hyperbolic region $y < 2/a$ and also in the elliptic region $y > 2/a$, matching the two solutions so calculated. This allows us to match the following solutions, (i) the exponentially decaying solution for $y > \kappa/a$ with the polynomially decaying solution in the elliptic region. (ii) The exponentially decaying solution for $y < \kappa/a$ with the solution determined in Section 2 for smaller values of y.

Our analysis proceeds as follows. Firstly we construct the WKB solution for $y \gg 1$ on the assumption that $|Q|$ is small. For each such interval, this solution has two free parameters. These are determined by matching with the solution of the rescaled Duffing equation calculated in Section 3 in the region

$$y = \kappa/a + \mathcal{O}(a^{-1/2}).$$

4.1 The WKB Solution

Let us assume that $Q(y) \ll 1$. Following the analysis in Section 2.2 we rescale $Q(y)$ by setting

$$Q(y) = e^{-iay^2/4}y^{(1-d)/2}Z(y). \tag{18}$$

so that, as before, the rescaled function $Z(y)$ satisfies the complex ordinary differential equation

$$-Z'' + \left(-\frac{a^2y^2}{4} + 1 - ia\left(\frac{d-2}{2}\right) - y^{1-d}|Z|^2 + \frac{(d-1)(d-3)}{4y^2}\right)Z = 0. \tag{19}$$

Now, make the assumption, consistent with the behaviour described in the previous section, that for a range of y satisfying $|y - \kappa/a| \gg 1$ we have simultaneously $|Q|$ small and y large. On this assumption, equation (19) simplifies at leading order to

$$Z'' + \left(\frac{a^2y^2}{4} - 1\right) Z = 0. \tag{20}$$

On setting

$$x = \frac{ay}{2}$$

we have

$$Z_{xx} + \frac{4}{a^2}(x^2 - 1)Z = 0. \tag{21}$$

Under this scaling, the maxima in $|Q(y)|$ occur at the points $x_j = \kappa_j/2$. For $x < 1$ and not close to 1, the WKB approximation of the function $Z(x)$ is

$$Z(x) = \frac{C_+}{(1-x^2)^{1/4}} e^{\frac{2}{a}f(x)} + \frac{C_-}{(1-x^2)^{1/4}} e^{-\frac{2}{a}f(x)} \tag{22}$$

where

$$f(x) = \int_0^x \sqrt{1-s^2}ds = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x). \tag{23}$$

The values of the constants C_+ and C_- in this expression depend upon which of the two regions $\kappa/2 < x$ or $x < \kappa/2$ the WKB expression is calculated in.

4.2 Matching the WKB solution with the multi-bump solutions

We determine the values of the constants in (23) by matching with the multi-bump solutions when we simultaneously satisfy the two conditions

$$|x - \kappa/2| \ll 1 \quad \text{and} \quad |y - \kappa/a| \gg 1.$$

Accordingly we set

$$t = x - \kappa/2$$

and assume that in the limit of $a \to 0$ that both t and $Z(x)$ are small. For example we take

$$t = \mathcal{O}(a^{1/2}) \quad \text{so that} \quad |y - \kappa/a| = \mathcal{O}(a^{-1/2}).$$

Expanding the expression (23) as a Taylor series in t we have

$$f(t) = \frac{1}{4}\kappa\sqrt{1-\kappa^2/4} + \frac{1}{2}\sin^{-1}\left(\frac{\kappa}{2}\right) + \sqrt{1-\kappa^2/4}\ \ t + \mathcal{O}(t^2). \tag{24}$$

Now, combining the expressions (18),(22) and(24) the WKB expression for $Q(y)$ becomes

$$Q(t) = \frac{e^{-i\kappa t/a}}{(1-\kappa^2/4)^{1/4}} \left(\frac{a}{\kappa}\right)^{1/2} [C_- e^{-\frac{2}{a}\left(\frac{\kappa}{4}\sqrt{1-\kappa^2/4} + \frac{1}{2}\sin^{-1}(\kappa/2) + \sqrt{1-\kappa^2/4}t\right)}$$

$$+C_+ e^{\frac{2}{a}\left(\frac{\kappa}{4}\sqrt{1-\kappa^2/4} + \frac{1}{2}\sin^{-1}(\kappa/2) + \sqrt{1-\kappa^2/4}\, t\right)}]. \tag{25}$$

Now consider the soliton like solution described in the previous section for the same range of the independent variable t. We have shown that, up to a constant K of unit modulus, the function $Q(y)$ is to leading order

$$Q(y) = Ke^{-i\kappa(y-\kappa/a)/2}\sqrt{2(1-\kappa^2/4)}\ \mathrm{sech}\left(\sqrt{1-\kappa^2/4}(y-\kappa/a)\right).$$

Now, if $x = \kappa/2 + t$ we have immediately

$$y = \frac{\kappa}{a} + \frac{2t}{a}.$$

Thus for this same range

$$Q(t) = Ke^{-i\kappa t/a}\sqrt{2(1-\kappa^2/4)}\ \mathrm{sech}\left(\frac{2}{a}\sqrt{1-\kappa^2/4}\ t\right).$$

Taking $0 < t = \mathcal{O}(a^{1/2})$ this expression simplifies to

$$Q(t) = Ke^{-i\kappa t/a}2\sqrt{2(1-\kappa^2/4)}\ e^{-\frac{2}{a}\sqrt{1-\kappa^2/4}\ t}. \tag{26}$$

The contribution to (25) due to the coefficient C_+ is exponentially increasing and only matches with the expression (26) if we set $C_+ = 0$ to this level of asymptotic approximation. The two expressions (26) and (25) then match perfectly provided that

$$C_-(1-\kappa^2/4)^{-1/4}\left(\frac{a}{\kappa}\right)^{1/2} e^{-\frac{2}{a}\left(\frac{\kappa}{4}\sqrt{1-\kappa^2/4}+\frac{1}{2}\sin^{-1}(\kappa/2)\right)} = 2\sqrt{2(1-\kappa^2/4)},$$

so that

$$C_- = 2\sqrt{2}(1-\kappa^2/4)^{3/4}\left(\frac{\kappa}{a}\right)^{1/2} e^{\frac{2}{a}\left(\frac{\kappa}{4}\sqrt{1-\kappa^2/4}+\frac{1}{2}\sin^{-1}(\kappa_j/2)\right)}. \tag{27}$$

If we perform a very similar calculation, setting

$$y = \frac{\kappa}{a} - \frac{2t}{a}, \quad t > 0 \quad t \ll 1$$

then similarly $C_- = 0$ and

$$C_+ = 2\sqrt{2}(1-\kappa^2/4)^{3/4}\left(\frac{\kappa}{a}\right)^{1/2} e^{-\frac{2}{a}\left(\frac{\kappa}{4}\sqrt{1-\kappa^2/4}+\frac{1}{2}\sin^{-1}(\kappa/2)\right)}. \tag{28}$$

4.3 Matching the WKB solution with the solution at infinity

The previous calculation identifies the coefficients of the WKB expansion in the elliptic range $x < 1$. Now consider the behaviour close to the turning point at $x = 1$. Our discussion of this closely follows that in [SS99]. From (22) we have

$$f(x) = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x).$$

In the limit of $x \to 1$ this takes the form

$$f(x) = \frac{\pi}{4} - \frac{2\sqrt{2}}{3}(1-x)^{3/2},$$

so that as $x \to 1^-$

$$Z(x) \sim \frac{C_-}{(1-x^2)^{1/4}} e^{-\pi/2a} e^{-4\sqrt{2}(1-x)^{3/2}/3a}. \tag{29}$$

For $x \approx 1$, the function $Z(x)$ may be expressed in terms of Airy functions so that there are constants α_1 and α_2 with

$$Z(x) = \alpha_1 \mathrm{Ai}(\zeta) + \alpha_2 \mathrm{Bi}(\zeta), \qquad \zeta = 2(1-x)a^{-2/3}. \tag{30}$$

This solution is exponential in form for $\zeta > 0$ and is oscillatory for $\zeta < 0$. In the limit of large ζ so that $x = 1^-$ it follows from the asymptotic form of $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ that

$$Z(x) = \frac{1}{\sqrt{\pi}} \frac{a^{1/6}}{2^{1/4}} \frac{1}{(1-x)^{1/4}} \left(\frac{1}{2} \alpha_1 e^{-4\sqrt{2}(1-x)^{3/2}/3a} + \alpha_2 e^{-4\sqrt{2}(1-x)^{3/2}/3a} \right). \tag{31}$$

On matching (29) and (31) in the range $x = 1^-$, the contribution due to α_1 is negligible and we have

$$\frac{C_-}{2^{1/4}} e^{-\pi/2a} = \frac{1}{\sqrt{\pi}} \frac{a^{1/6}}{2^{1/4}} \alpha_2. \tag{32}$$

Now, if $x > 1$, the Airy function solution for large $-\zeta$ is asymptotic to

$$Z(x) \sim \frac{1}{\sqrt{\pi}} \frac{1}{(-\zeta)^{1/4}} \left(\alpha_1 \sin(\zeta + \pi/4) + \alpha_2 \cos(\zeta + \pi/4) \right). \tag{33}$$

However, we know from earlier analysis, that for large$x$ we have $|Z|$ monotone decreasing. It is noted in [SS99] that this is consistent with (33) only when $\alpha_1 = i\alpha_2$ so that

$$Z(x) \sim \frac{\alpha_2}{\sqrt{\pi}} \frac{a^{1/6}}{(2(x-1))^{1/4}} e^{i\left(\frac{\pi}{4} + \frac{4\sqrt{2}}{3a}(x-1)^{3/2}\right)}. \tag{34}$$

Similarly in this range, the WKB approximation for $Z(x)$ is given by

$$Z(x) \approx \frac{D}{(1-x^2)^{1/4}} \exp\left(\frac{2i}{a} \int_1^x \sqrt{s^2-1} ds \right). \tag{35}$$

where D is an appropriate (complex) constant. The two expressions (35) and (34) match in amplitude and phase provided that

$$\frac{\alpha_2}{\sqrt{\pi}} \frac{a^{1/6}}{2^{1/4}} = \frac{D}{2^{1/4}}. \tag{36}$$

Combining the expressions (32),(36) then gives

$$D = C_- e^{-\pi/2a}. \tag{37}$$

Now, if x is large, the expression (35) becomes

$$Z(x) \approx D e^{-i\pi/4} x^{-1/2} \exp\left(\frac{2i}{a} \left(\frac{x^2}{2} - \frac{1}{2} \log(2x) \right) \right). \tag{38}$$

However, we have already determined the large y behaviour of $Q(y)$ and this is given in expression (13) so that

$$Q(y) \approx \mu e^{-i \log(y)/a} y^{-1}.$$

Rescaling this expression to give $Z(x)$ as a function of x we have

$$Z(x) \approx \mu e^{i(x^2/a - \log(2x/a)/a)} \left(\frac{2x}{a}\right)^{-1/2}. \tag{39}$$

The two expressions (38),(39) match perfectly if

$$D = \left(\frac{2}{a}\right)^{-1/2} \mu \quad \text{or} \quad \mu = \left(\frac{2}{a}\right)^{1/2} D.$$

Combining this with (37) gives

$$\mu = \left(\frac{2}{a}\right)^{1/2} e^{-\pi/2a} C_-. \tag{40}$$

Observe that μ is exponentially small in a. Thus the amplitude of $|Q|$ for large y is also exponentially small and hence the contribution to the Hamiltonian H is as well. This completes the estimate of H made in Section 3.

To complete this calculation and relate a to d we estimate the value of μ from the large y description of the function $Q(y)$. To do this we modify the argument in [SS99], although it turns out that the calculation is rather simpler for the multi-bump solutions. We make use of the amplitude and phase decomposition of $Q(y)$

$$Q(y) = A(y) e^{i\theta(y)}$$

given in Section 2.3 from which it follows that A and θ satisfy the relation

$$\theta_y + \frac{a}{2} y = -\frac{(d-2)}{yA^2} \int_0^y \theta_y A^2 dy'. \tag{41}$$

If $ay \gg 1$ we have that

$$A \approx \mu y^{-1} \quad \text{and} \quad \theta_y \approx -1/ay.$$

Hence, to leading order

$$\frac{a}{2} y = -\frac{(d-2)y}{\mu^2} \int_0^y \theta_y A^2 \, dy',$$

where we observe that the integral converges as $y \to \infty$. Thus asymptotically we have

$$\frac{a}{2} = -\frac{(d-2)}{\mu^2} \int_0^\infty \theta_y A^2 \, dy',$$

so that

$$\mu^2 = -\frac{2(d-2)}{a} \int_0^\infty \theta_y A^2 \, dy'. \tag{42}$$

We now estimate the integral in the above expression.

We start by looking at a solution with a single peak at $1/a$. For such a solution in the neighbourhood of $1/a$

$$A^2(y) = 2(1 - \kappa^2/4) \operatorname{sech}^2 \left(\sqrt{1 - \kappa^2/4}\, (y - \kappa/4)\right), \quad \text{and} \quad \theta_y = -\kappa/2.$$

Consequently the contribution to the total integral from this peak is given by

$$-\int_{-\infty}^{\infty} \kappa(1-\kappa^2/4)\,\mathrm{sech}^2(\sqrt{1-\kappa^2/4}\;s)ds = -2\kappa\sqrt{1-\kappa^2/4}.$$

The other contributions to the integral come from arise for values of y before the first maximum and for the larger values of y after the maximum. In the latter case $A^2\theta_y \approx -\mu^2/(ay^3)$ so that the total contribution over the elliptic range of $y > 2/a$ is of order a. Similarly, for the smaller values of y we either have $A(y)$ exponentially small or $A(y) \approx R_K$ and $\theta_y \approx -ay/2$. In the latter case the contribution to the integral is given approximately by

$$-a\int_0^{\infty} R_K^2\; y\; dy = \mathcal{O}(a).$$

We conclude, that to leading order we have

$$\int_0^{\infty} \theta_y A^2 dy' = -2\kappa\sqrt{1-\kappa^2/4} + \mathcal{O}(a^2).$$

Substituting this result into the expression (42) gives

$$\mu^2 = \frac{4(d-2)}{a}\kappa\sqrt{1-\kappa^2/4} + \mathcal{O}(1). \tag{43}$$

We now conclude our arguments by estimating d in terms of a. From (43) and the established identity

$$\mu = \left(\frac{2}{a}\right)^{1/2} e^{-\pi/2a} C_-$$

we have

$$\frac{2}{a} e^{-\pi/a} C_-^2 = \frac{4(d-2)}{a}\kappa\sqrt{1-\kappa^2/4}$$

so that

$$e^{-\pi/a} C_-^2 = 2(d-2)\kappa\sqrt{1-\kappa^2/4}.$$

Using the expression (27) for C_- gives

$$8(1-\kappa^2/4)^{3/2}\left(\frac{\kappa}{a}\right) e^{(\kappa\sqrt{1-\kappa^2/4}+2\sin^{-1}(\kappa/2)-\pi)/a}$$

$$= 2(d-2)\kappa\sqrt{1-\kappa^2/4}.$$

Thus

$$(d-2) = \frac{\alpha}{a} e^{-\lambda/a} \tag{44}$$

where

$$\lambda = \pi - 2\sin^{-1}(\kappa/2) - \kappa\sqrt{1-\kappa^2/4} \tag{45}$$

and

$$\alpha = \frac{4(1-\kappa^2/4)^{3/2}\kappa}{\kappa\sqrt{1-\kappa^2/4}} \tag{46}$$

Now using the precise estimate $\kappa = 1 + \mathcal{O}(a^2)$ we get

$$\lambda = \pi - 2\sin^{-1}(1/2) - \sqrt{3/4} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} + \mathcal{O}(a^2) = 1.22836969.. + \mathcal{O}(a^2) \tag{47}$$

and

$$\alpha = \frac{4(1-1/4)^{3/2}}{\sqrt{3/4}} + \mathcal{O}(a^2) = 3 + \mathcal{O}(a^2). \tag{48}$$

Thus from (44) we have

$$(d-2) = \frac{3}{a} e^{-(\pi - 2\sin^{-1}(1/2) - \sqrt{3/4})/a + \mathcal{O}(a)} \left(1 + \mathcal{O}(a^2)\right). \tag{49}$$

If the solution has multiple maxima, we may perform similar calculations to the above but with less precision. In particular it is much harder to estimate μ for such solutions as the integrands will tend to interfere with each other. In this case we can only say that

$$\mu^2 = \frac{M}{a}$$

for an appropriate constant M. The equations (44), (45) then hold with $\kappa = 1 + \mathcal{O}(a\log(a))$ but we can not give an explicit expression for α. Hence

$$\lambda = \pi - 2\sin^{-1}(1/2) - \sqrt{3/4} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} + \mathcal{O}(a\log(a)) \tag{50}$$

Thus from (44) we have

$$(d-2) = \frac{\alpha_J}{a} e^{-(\pi - 2\sin^{-1}(1/2) - \sqrt{3/4})/a + \mathcal{O}(\log(a))} \left(1 + \mathcal{O}(a^2)\right) = \frac{\alpha_J}{a^{\gamma_J}} e^{-(2\pi/3 - \sqrt{3}/2)/a}, \tag{51}$$

where γ_J is to be determined. We conjecture from the numerical computations that γ_J is always an integer.

4.4 Matching the multi-bump solutions with the solutions for smaller values of y

For smaller values of y, ie. for $ay \ll 1$, it follows from Section 2 that the solutions that we are investigating approximate $e^{-iay^2/4} R_K(y)$ together with a perturbation of the form (25) comprising both a regular expansion in powers of a^2 of functions which decay exponentially in y and an exponentially small, but growing term. Following (25) we have the following expression for large y

$$Q(y) = e^{-iay^2/4} \left[\frac{\nu_K e^{-y}}{\sqrt{y}} \left(1 - \frac{a^2 y^3}{24} + ..\right) + \frac{\alpha e^y}{\sqrt{y}} (1 + ...)) \right]. \tag{52}$$

We now determine the value of α by matching the expression of the solution given by (52) with the WKB solution given earlier. To do this we consider the range $1 \ll y \ll a^{-1}$ in which both the expression (52) and the expression (25) are valid. In the limit of ay small the WKB expression when rescaled becomes

$$Q(y) = e^{-iay^2/4} y^{(1-d)/2} \left(C_- e^{-y} + C_+ e^{y} \right).$$

This matches the solution described by (52) perfectly provided that

$$\nu_K = C_- \quad \text{and} \quad \alpha = C_+.$$

Now, consider solutions which have at least one maximum located at $y = \kappa/a$. Exactly as before, we can extend the WKB analysis to a neighbourhood of this maximum and conclude that

$$\alpha = C_+ = 2\sqrt{2}(1-\kappa^2/4)^{3/4} \left(\frac{\kappa}{a}\right)^{1/2} e^{-\frac{2}{a}\left(\frac{\kappa}{4}\sqrt{1-\kappa^2/4}+\frac{1}{2}\sin^{-1}(\kappa/2)\right)}.$$

Observe that, as required in Section 2.3, α is an exponentially small function of a.

In the case of $Q_{0,1}(y)$ which is a perturbation of $R_0(y) \equiv 0$ with one maximum, we can be very much more precise. Using the result that $\kappa = 1 + \mathcal{O}(a^2)$ we then have

$$\alpha = C_+ = 2\sqrt{2}(3/4)^{3/4} \frac{1}{a^{1/2}} e^{-\frac{2}{a}\left(\frac{1}{4}\sqrt{3/4}+\frac{1}{2}\sin^{-1}(1/2)\right)} \left(1 + \mathcal{O}(a^2)\right).$$

Furthermore, it follows from (26) that for the whole range $0 \leqslant y \ll 1/a^{2/3}$ that

$$Q_{0,1}(y) = e^{-iay^2/4} \alpha \sqrt{2\pi} \mathrm{I}_0(y)(1 - a^2 y^3/24..)$$

so that in this range

$$Q_{0,1}(y) = 4 e^{-iay^2/4} (3/4)^{3/4} \frac{\sqrt{\pi}}{a^{1/2}} e^{-\frac{1}{a}\left(\frac{1}{2}\sqrt{3/4}+\frac{\pi}{6}\sin^{-1}(1/2)\right)} \mathrm{I}_0(y)(1 - a^2 y^3/24..)$$

Combining results then gives

$$Q_{0,1}(0)^2 = \frac{16\pi(3/4)^{3/2}}{a} e^{-(\sqrt{3/4}+\pi/3)/a}(1 + \mathcal{O}(a^2)). \tag{53}$$

We compare this with the earlier estimate of

$$(d-2) = \frac{3}{a} e^{-(2\pi/3-\sqrt{3/4})/a}(1 + \mathcal{O}(a)).$$

Hence, if we make an approximation to this formula by ignoring logarithmic terms, we have that approximately

$$Q_{0,1}(0)^2 \sim (d-2)^{(\pi/3+\sqrt{3/4})/(2\pi/3-\sqrt{3/4})},$$

which is the approximate polynomial law originally identified in [BCR99].

5 Numerical results

In this section we compare the results of some numerical computations with the earlier asymptotic estimates. These results show very close agreement, which is remarkable given the nature of the approximations made. The numerical results are obtained using the methods described in [BCR99] in which the function $Q(y)$ is calculated by solving the differential equation (1) with the constraint (5) by using a high order collocation method as described in [ACR81]. Full details of the numerical procedures are omitted here but may be found in [BCR99].

Starting from an initial value of $d = 2.01$ the value of d was reduced slowly toward 2 and the corresponding values of a and $Q(0)$ determined. In [BCR99] the function $Q_{K,J}$ was calculated for the cases $K = 0, 1$ and $J = 0, 1, 2, 3$. So far no direct numerical evidence has been obtained for solutions with $K > 1$, although it is likely that such solutions will be very hard to compute.

Dependence of a and $Q(0)$ on d

We consider the two cases of $(K, J) = (0, 1)$ and $(1, 2)$. In the case $(K, J) = (0, 1)$ we can use expression (49) to estimate d and $Q(0)$ asymptotically by the quantities

$$d_* = 2 + \frac{3}{a} e^{-(2\pi/3 - \sqrt{3/4})/a} \quad \text{and} \quad Q_* = \frac{4\sqrt{\pi}(3/4)^{3/4}}{\sqrt{a}} e^{-(\sqrt{3/4}+\pi/3)/2a}.$$

The values of a and $Q(0)$ determined in the numerical computations are given as follows

d	a	$Q(0)$
2.01	0.1661249	0.0509578
2.001	0.1230330	0.0074828
2.0001	0.0979989	0.0011216
2.00001	0.0817000	0.0001728
2.000001	0.0701886	0.0000271

In the following two figures we compare the values of $\log(d)$ and $\log(Q(0))$ plotted as functions of the calculated value of $1/a$ compared with the asymptotic values of $\log(d_*)$ and $\log(Q_*)$.

The agreement between the numerical and the asymptotic results is very good in both cases.

When $(K, J) = (1, 2)$ we have the less precise estimate (51) for $(d - 2)$ for which we must estimate the values of α_2 and γ_2. The computed values of d, a and $Q(0)$ are as follows

d	a	$Q(0)$
2.01	0.1089721	2.0546804
2.001	0.0880842	2.1786380
2.0001	0.0732661	2.2001517
2.00001	0.0628771	2.2036942
2.000001	0.0552327	2.2045519

Manipulating this data, an excellent fit is obtained by the expression

$$d_{**} = 2 + \frac{0.78}{a^3} e^{-(2\pi/3 - \sqrt{3/4})/a}$$

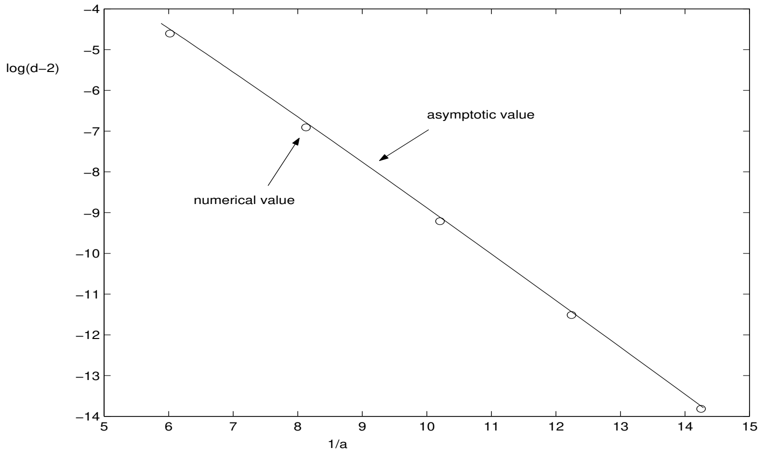

Figure 1: A plot oflog$(d-2)$ as a function of $1/a$ comparing numerical and asymptotic values

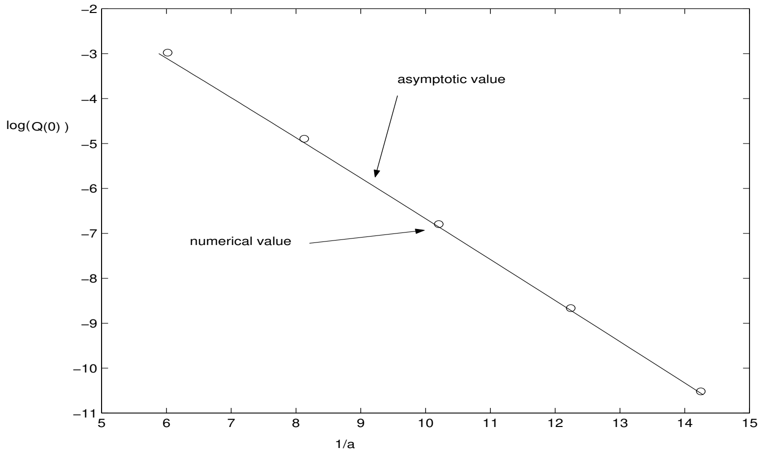

Figure 2: A plot oflog$(Q(0))$ as a function of $1/a$ comparing numerical and asymptotic values

and we present a figure showing $\log(d-2)$ and $\log(d_{**}-2$ plotted as functions of $1/a$ in this case.

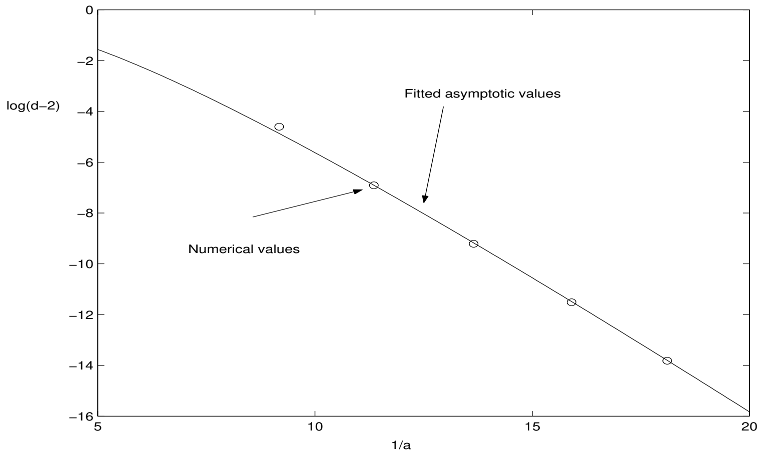

Figure 3: A plot oflog$(d-2)$ as a function of $1/a$ comparing numerical and asymptotic values

Dependence of the location of the maximum on a

For our second computation we look at the location of the maximum point as a function of a. To do this we again look at the two cases of $(K, J) = (0, 1)$ and $(1, 2)$. Firstly consider the case of $(K, J) = (0, 1)$ where we have a single maximum asymptotically located at the point $1/a + \mathcal{O}(a)$. Accordingly, we calculate the location x of the maximum and determine $\kappa = ax = 1 + \mathcal{O}(a^2)$ and $|Q(\kappa/a)|$ as $d \to 2$.

d	a	κ	$\lvert Q(\kappa/a)\rvert$
2.01	0.166125	0.962927	1.99067
2.0001	0.097999	0.984576	1.21642
2.000001	0.070188	0.991541	1.2201

Plotting κ and $|Q(\kappa/a)$ as functions of a^2 we see that as predicted these points lie asymptotically on straight lines through the point $(a, \kappa) = (0, 1)$ and $(a, |Q(\kappa/a)|) = (0, \sqrt{3/2})$.

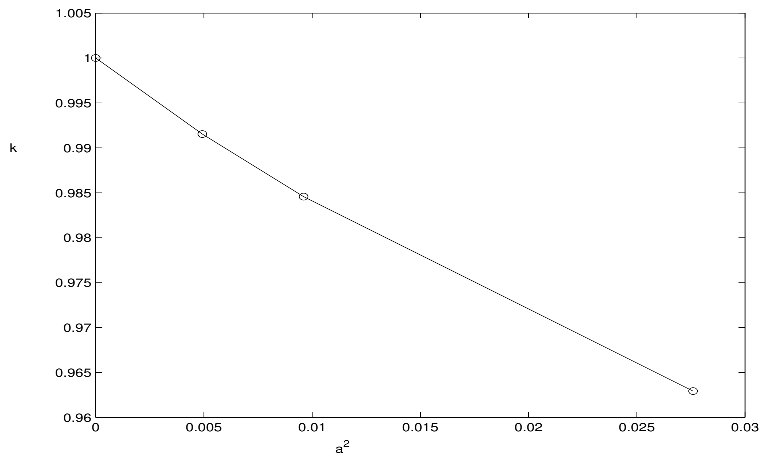

Figure 4: A plot of κ as a function of a^2.

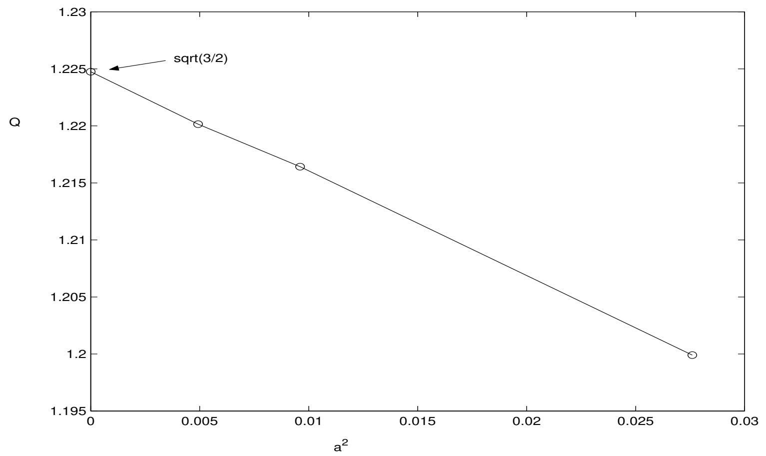

Figure 5: A plot of $|Q(\kappa/a)|$ as a function of a^2.

Now consider the case of $(K, J) = (1, 2)$ for which the solution has two maxima asymptotically located at the two points $\kappa_{1,2}/a = 1/a + \mathcal{O}(\log(a))$. In the following figure we show the resulting form of the function $|Q_{1,2}(y)|$ for different values of d.

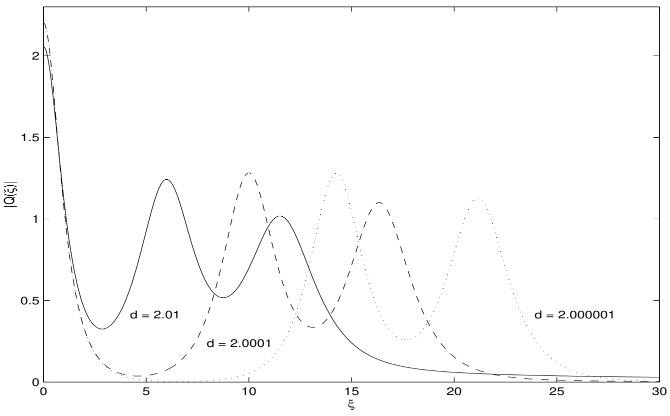

Figure 6: A plot of $|Q_{1,2}(y)|$ as a function of y for $d = 2.01, 2.0001, 2.000001$.

The computed values of $\kappa_{1,2}$ at the maximum points are given as follows

d	a	κ_1	κ_2
2.01	0.10897	0.6499	1.25590
2.0001	0.07326	0.7361	1.19327
2.000001	0.05523	0.7875	1.16362

A plot of $\kappa_{1,2}$ as functions of $a\log(a)$ is given below, showing a good fit to two straight lines passing through $(0, 1)$.

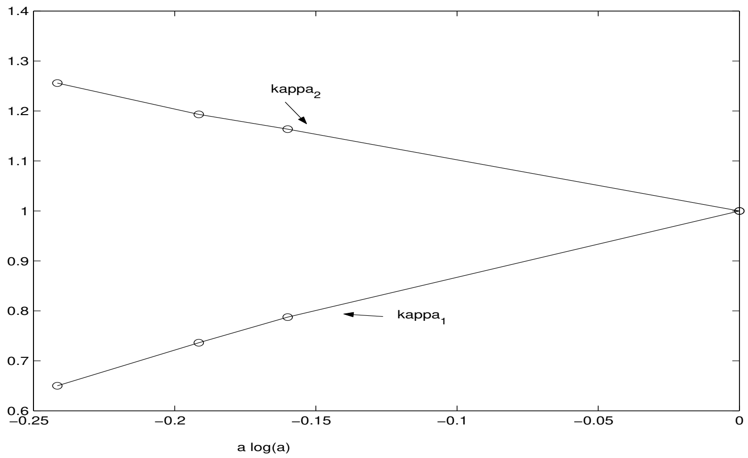

Figure 7: A plot of$\kappa_{1,2}$ as functions of $a\log(a)$.

6 Conclusions

The asymptotic and numerical investigations lead to a consistent picture for the emergence of the multi-bump solutions at the critical dimension $d = 2$. Whilst the asymptotic analysis is formal and relies on the WKB approximation of the solution, the agreement between the numerical and asymptotic calculations is very good. Much further work needs to be done, especially in determining the stability of these solutions (we conjecture that all apart from the monotone solution are unstable) and investigating the way in which they determine the overall dynamics of the solutions of the NLS from arbitrary initial data. For example, they may still be appropriate to act as intermediate blow-up solutions of the NLS which are valid up to the point where the NLS ceases to be a valid physical model. We leave all of these as questions for further research.

References

[ACR81] U. Ascher, J. Christiansen and R.D. Russell, *Collocation Software for Boundary-Value ODEs*, ACM Trans. Math. Software, **2**, (1981), pp. 209-222.

[ADKM92] G.D. Akrivis, V.A. Dougalis, O.A. Karakashian and W.R. McKinney, *Galerkin-finite element methods for the nonlinear Schrödinger equation*, Hellenic Research in Mathematics and Informatics '92 (Athens, 1992), 421–442, Hellenic Math. Soc., Athens, (1992).

[BCR99] C.J. Budd, S. Chen and R.D. Russell, *New self-similar solutions of the nonlinear Schrödinger equation with moving mesh computations*,J. Comp. Phys., **152**, (1999), 756–789.

[Bu83] T.A. Burton, *Volterra Integral and Differential Equations*, Mathematics in Science and Engineering, Vol. 167, Academic Press, INC., (1983).

[DD87] E.A. Dorfi and L.O. Drury, *Simple adaptive grids for 1-D initial value problems*, J. Comp. Phys., **69**, (1987), 175–195.

[Fi96] G. Fibich, *Adiabatic law for self-focusing of optical beams*, Optics Letters, **21**, (1996), 1735–1737.

[FP99] G. Fibich and G. C. Papanicolaou, *Self-focusing in the perturbed and unperturbed nonlinear Schrödinger equation in critical dimension*, SIAM J. Appl. Math. **60**, (2000), 183–240.

[Ha89] A. Hasegawa, *Optical Solitons in Fibers*, Springer-Verlag, Berlin, (1989).

[KL95] N. Kopell and M. Landman, *Spatial structure of the focusing singularity of the nonlinear Schrödinger equation: a geometrical analysis*, SIAM J. Appl. Math., **55**, (1995), 1297–1323.

[LPSS88a] B.J. LeMesurier, G.C. Papanicolaou, C. Sulem and P.L. Sulem, *Focusing and multi-focusing solutions of the nonlinear Schrödinger equation*, Physica D, **31**, (1988), 78–102.

[LPSS88b] B. J. LeMesurier, G. C. Papanicolaou, C. Sulem and P. L. Sulem, *Local structure of the self-focusing singularity of the nonlinear Schrödinger equation*, Physica D, **32**, (1988), 210–226.

[LPSS88c] M. J. Landman, G. C. Papanicolaou, C. Sulem and P. L. Sulem, *Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension*, Phys. Review A, **38**, (1988), 3837–3843.

[MPSS86] D. W. McLaughlin, G. C. Papanicolaou, C. Sulem and P. L. Sulem, *Focusing singularity of the cubic Schrödinger equation*, Phys. Rev. A, **34**, (1986), 1200–1210.

[MS81] K. McLeod and J. Serrin, *Uniqueness of solutions of semilinear Poisson equations*, Proc. Natl. Acad. Sci. USA, **78**, (1981), 6592–6595.

[PS00] P. Plechac and V. Sverac, *On self-similar solutions of the complex Ginzburg-Landau equation*, (2000), submitted.

[TS92] Y. Tourigny and J. M. Sanz-Serna, *The numerical study of blowup with application to a nonlinear Schrödinger equation*, J. Comput. Phys., **102**, (1992), 407–416.

[SS99] C. Sulem and P-L Sulem, *The nonlinear Schrödinger equation, self-focusing and wave collapse*, Applied Math. Sciences **139**, (1999), Springer.

[W90] X. P. Wang, *On singular solutions of the nonlinear Schrödinger and Zakharov equations*, Ph.D. thesis, New York University, 1990.

[Za84] V. E. Zakharov, *Handbook of Plasma Physics*, edited by M. N. Rosenbluth and R. Z. Sagdeev, Elsevier, New York, 1984, Vol. 2.