EXISTENCE AND BLOW-UP FOR HIGHER-ORDER SEMILINEAR PARABOLIC EQUATIONS: MAJORIZING ORDER-PRESERVING OPERATORS

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Abstract. As a basic example, we establish that in the Cauchy problem for the $2m$-th order semilinear parabolic equation

$$u_t = -(-\Delta)^m u + |u|^p, \quad x \in \mathbf{R}^N, \ t > 0; \quad u(x,0) = u_0(x), \ x \in \mathbf{R}^N,$$

where $m > 1$, $p > 1$, with bounded integrable initial data u_0, the critical Fujita exponent is $p_F = 1 + 2m/N$, so that for $p > p_F$ there exists a class of small global solutions and for $p \in (1, p_F]$ blow-up can occur for arbitrarily small initial data. The analysis of the asymptotics of both classes of global and blow-up solutions is based on comparison with similarity solutions of the majorizing order-preserving equation, which is shown to exist for any $m > 1$. Generalizations of this idea to differential and pseudodifferential evolution equations and relations to positivity sets for higher-order equations are discussed.

1. Introduction: majorizing order-preserving operators and equations

This paper deals with a class of higher-order semilinear parabolic differential and nonlinear integral evolution equations with superlinear lower-order terms. The corresponding evolution processes are not order-preserving and the solutions can blow-up in finite time. We show that the idea of constructing of the corresponding majorizing order-preserving operators and equations plays a key role in establishing of the global existence and the asymptotic behaviour of both global and blow-up solutions.

1.1. Example: critical Fujita exponent, global existence and blow-up.

As a key example, we study global solvability and blow-up of solutions of higher-order semilinear evolution equations of parabolic type. Consider the Cauchy problem for the $2m$-th order equation

$$u_t = \mathbf{A}u + |u|^p \quad \text{in } Q = \mathbf{R}^N \times \mathbf{R}_+, \quad u(x,0) = u_0 \in X = L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N), \tag{1.1}$$

where $p > 1$,

$$\mathbf{A} = -(-\Delta)^m, \quad m > 1 \text{ is integer}, \tag{1.2}$$

and Δ denotes the Laplace operator in $\mathbf{R}^N$. For $m = 1$, this is a classical semilinear reaction-diffusion equation from Combustion Theory. Higher-order generalizations

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of the semilinear heat equations are known in a number of application in thin film theory, flame propagation, phase transition, in bi-stable systems and in processes of higher-order diffusion (e.g., the extended Fisher-Kolmogorov equation or equations of Kuramoto-Sivashinskii type).

The lower-order source-like term $|u|^p$ with $p > 1$ is chosen so that it vanishes sufficiently fast on small solutions $u \to 0$, and conversely it is superlinear on large solutions $u \gg 1$. Therefore, the model equation (1.1), composed from two operators of different signs and monotonicity properties in the right-hand side, can describe both nonlinear phenomena such as existence of global small solutions and finite-time blow up of large ones (the global nonexistence). We will treat both different asymptotics by means of a unified approach. Let us begin with first two results for the Cauchy problem (1.1).

Theorem 1.1. *If*

$$p > p_F = 1 + 2m/N, \tag{1.3}$$

then for any sufficiently small initial data $u_0 \in X$ *the Cauchy problem* (1.1) *admits a global solution satisfying* $u(\cdot, t) \to 0$ *as* $t \to \infty$ *uniformly in* $\mathbf{R}^N$.

Theorem 1.2. *Let* $p \in (1, p_F]$. *If* $u_0 \not\equiv 0$ *and*

$$\int_{\mathbf{R}^N} u_0(x)dx \geq 0, \tag{1.4}$$

then the solution of (1.1) *blows up in finite time* $T > 0$ *in* $L^\infty(\mathbf{R}^N)$ *in the sense that* $u(\cdot, t) \in X$ *for all* $t \in (0, T)$ *and*

$$\|u(\cdot, t)\|_\infty \to \infty, \quad t \to T^-. \tag{1.5}$$

The global nonexistence result for $p \in (1, p_F]$ is already proved in [5] by a test function method and establishes that any weak solution $u \in L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}_+)$ with $u_0 \in L^1_{\rm loc}(\mathbf{R}^N)$ satisfying (1.4) must be trivial: $u \equiv 0$. In Section 4 of the present paper we give a new simple evolution proof of Theorem 1.2 based on the evolution ordinary differential inequality for the first Fourier coefficient, which is an important characteristic of the generic blow-up rate. This eigenfunctions method was introduced by Kaplan [12] for semilinear second-order parabolic equations.

The present result on the global solvability completes the proof that $p = p_F$ is precisely the *critical Fujita exponent* for the higher-order parabolic equation calculated from both sides: (i) blow-up of any small solutions with (1.4) for $1 < p \leq p_F$ and (ii) global existence of small solutions for $p > p_F$. These results are classical [6] for the semilinear second-order parabolic equation with $m = 1$,

$$u_t = \Delta u + u^p, \;\; u \geq 0,$$

and then $p_F = 1 + 2/N$ is the corresponding critical Fujita exponent. Such critical exponents play an important role in different nonlinear evolution equations. See a survey paper [14] and Chapt. 4 in the book [15].

Theorem 1.1 gives a characterization of the critical Fujita exponent for the $2m$-th order semilinear parabolic equation from the *global range*. It states that at $p = p_F$ the *trivial* *stationary solution* $u \equiv 0$ of equation (1.1) *loses its stability*, which takes place in a certain weighted space in the supercritical range $p > p_F$. This provide us with an alternative (more local) definition of the critical exponent p_F which does not include the blow-up property and therefore applies to a wider class of equations with bounded nonlinearities.

1.2. **Majorizing order-preserving operators and equations.** As we have mentioned, the main goal of the paper consists in introducing a general idea of the *majorizing order-preserving equation* for evolution semilinear parabolic PDEs with semigroups which *are not order-preserving*. In particular, the proof of Theorems 1.1 in Section 2 is based on the construction of a global self-similar solution of such a majorizing order-preserving equation. This makes it possible to estimate above the solution $u(x,t)$ for small initial data by using a *standard comparison* as for the quasilinear second-order parabolic equations, see a variety of methods in Chapt. 4 in [15]. The only difference is that the majorizing self-similar solutions satisfy integral equations with positive Hammerstein's operators having nonnegative kernels, which in the particular order-preserving case $m = 1$ are equivalent to the standard well known elliptic equations (which become ODEs in radial setting).

We choose the Cauchy problem (1.1) as a particular but well known example. One can extend the majorizing approach to any differential evolution equations or more general integral equations with the evolution driven by pseudodifferential operators with kernels and symbols changing sign. In order to prove the results, the only thing we need is a basic estimate of the fundamental solution, and then its scaling structure determines the critical Fujita exponent, global existence and blow-up estimates depending on the type of nonlinear perturbations.

We claim that the structural behaviour of self-similar solutions of the majorizing order-preserving equations describes some essential features of global and blow-up solutions of the original PDEs, and this approach makes it possible to *extend some key results from the order-preserving case* $m = 1$, studied in great detail in the mathematical literature, see references in the books [1] and [15], *to the oscillatory non-order-preserving one* $m > 1$. As a second step in this direction, using the majorizing order-preserving equation yields a sharp estimate on the blow-up rate of the L^∞-norm of the solution [2] by means of a standard comparison with a spatially homogeneous solution $u(t)$ governed by the ODE $u' = Du^p$, where $D > 1$ is the so-called order deficiency of the linear operator ($D = 1$ if and only if the linear semigroup is order-preserving). See an application in Section 2.

We discuss some extensions of the results to more general equations. In Section 3 we treat the relation between Theorem 1.1 and positivity sets for higher-order parabolic equations.

2. Majorizing order-preserving operators

2.1. **Nonlinear integral equation.** The operator $\mathbf{A}$ is the infinitesimal generator of a strongly continuous semigroup $e^{\mathbf{A}t}: X \to X$ for $t > 0$. In the corresponding equivalent "evolution" integral equation

$$u(t) = e^{\mathbf{A}t}u_0 + \int_0^t e^{\mathbf{A}(t-s)}|u(s)|^p ds, \tag{2.1}$$

the nonlinear term is a locally Lipschitz map. Let $p(\omega) = -|\omega|^{2m}$ be the characteristic polynomial of $\mathbf{A}$ and

$$e^{\mathbf{A}t}u_0 = b(t) * u_0,$$

where $b(t)$ is the kernel of the operator $e^{\mathbf{A}t}$ (the fundamental solution of the parabolic operator):

$$b(x,t) = \mathcal{F}^{-1}(e^{p(\omega)t}) \equiv (2\pi)^{-N} \int_{\mathbf{R}^N} e^{p(\omega)t - i(\omega.x)} d\omega.$$

The fundamental solution takes the standard self-similar form

$$b(x,t) = t^{-N/2m} f(\eta), \quad \eta = x/t^{1/2m}. \tag{2.2}$$

Substituting $b(x,t)$ into the linear equation

$$u_t = -(-\Delta)^m u, \tag{2.3}$$

we obtain that the radially symmetric profile $f(\eta)$ is a unique solution of a linear ordinary differential equation (ODE), which is the radial restriction of the elliptic equation

$$-(-\Delta_\eta)^m f + \frac{1}{2m}\nabla_\eta f \cdot \eta + \frac{N}{2m} f = 0 \quad \text{in } \mathbf{R}^N, \quad \int_{\mathbf{R}^N} f(\eta) d\eta = 1. \tag{2.4}$$

There holds [4]

$$f(\eta) = (2\pi)^{-N/2} \int_0^\infty e^{-s^{2m}} s^{N/2} J_{(N-2)/2}(s|\eta|) ds, \tag{2.5}$$

where J_ν denotes Bessel's function. The integral equation (2.1) takes the form

$$u(t) = \mathbf{M}(u)(t) \equiv b(t) * u_0 + \int_0^t b(t-s) * |u(s)|^p ds, \tag{2.6}$$

where solution $u(t)$ stays for a fixed point of the operator $\mathbf{M}$ for any $t > 0$. Local solvability and uniqueness for (2.6) are straightforward, see e.g. [16], Chapt. 15. This determines the continuous semigroup $u(t) = S(t)u_0$ defined in X locally.

2.2. **Majorizing kernel and linear majorizing order-preserving operator.** For $m > 1$, the kernel $b(x,t)$ changes sign and is oscillatory as $|x| \to \infty$, and the semigroup $S(t)$ is not order-preserving. On the other hand, for the second-order parabolic equation with $m = 1$, where $b(t) > 0$ for $t > 0$, it is order-preserving and this makes the analysis much easier since the Maximum Principle applies. In particular, due to the order-preserving (comparison) property, Theorems 1.1 and 1.2 are known for the quasilinear second-order equations like $u_t = \Delta u^q + u^p$, $p_F = q + 2/N$ [8] or for other nonlinear second-order reaction-diffusion equations with sufficiently arbitrary nonlinearities, see the results references in [15], Chapt. 4.

For $m > 1$, we begin with constructing of the majorizing kernel. The following estimate of the profile f is well known, see p. 46 in [3]. It also follows from a standard asymptotic analysis of the ODE (2.4).

Proposition 2.1. *Let $m > 1$. There exist constants $D > 1$ and $d > 0$ depending on m and N only such that*

$$|f(\eta)| \le D\, F(\eta), \quad F(\eta) = \omega_1 e^{-d|\eta|^\alpha} \ \textit{in} \ \mathbf{R}^N, \tag{2.7}$$

where

$$\alpha = \frac{2m}{2m-1} \in (1,2) \quad \textit{and} \quad \omega_1 = \left(\int_{\mathbf{R}^N} e^{-d|\eta|^\alpha}\, d\eta \right)^{-1}.$$

Integrating (2.7) over $\mathbf{R}^N$, we obtain that $D \ge \int |f|$, so that $D > 1$ provided that f changes sign. If $f \ge 0$, i.e., the linear semigroup is order-preserving, we have $f = F$ so that $D = 1$. $D > 1$ is said to be the *order deficiency* of the parabolic operator in (2.3) and of the linear convolution operator $\mathbf{B}(t)\phi = b(t) * \phi$ in (2.6). (In fact, the actual order deficiency of the operator is measured by the difference $\delta = D - 1$, but we keep the notation D for convenience.)

A sharp estimate on the asymptotic exponential decay rate of $f(\eta)$ as $|\eta| \to \infty$ can be obtained by a standard asymptotic analysis of the linear ODE. We keep the two main terms, which are responsible for the exponential expansion in the ODE (2.4),

$$(-1)^{m+1} f^{(2m)} + ... + \frac{1}{2m} f' \xi + ... = 0, \quad \xi = |\eta| \gg 1,$$

and looking for a solution in the standard exponential form

$$f(\xi) \sim e^{-a\xi^\alpha}, \quad \xi \gg 1.$$

Then $\alpha = 2m/(2m-1)$ and the algebraic equation for the constant a:

$$(-1)^m (\alpha a)^{2m-1} + \frac{1}{2m} = 0.$$

It has $2m-1$ different roots $\{a_k\}$, and in (2.7) we choose

$$-d > \max\{\mathrm{Re}\, a_k < 0\}.$$

We thus introduce the *positive majorizing kernel*

$$\bar b(x,t) = t^{-N/2m} F(\eta), \quad \eta = x/t^{1/2m}, \tag{2.8}$$

which is majorizing relative to the oscillating kernel $b(t)$ of the parabolic operator:

$$|b(x,t)| \le D\, \bar b(x,t) \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+. \tag{2.9}$$

As a by product, we obtain the *linear majorizing operator* $\bar{\mathbf{A}}(t)$ with the majorizing kernel $\bar b(t)$. The corresponding non-autonomous equation

$$\bar u_t = \bar{\mathbf{A}}(t) \bar u \ \text{ for } t > 0, \quad \bar u(0) = \phi, \tag{2.10}$$

describes the *order-preserving* evolution relative to initial data $\phi \in L^1(\mathbf{R}^N)$. Indeed, consider the convolution operator

$$\bar{u}(x,t) = \bar{\mathbf{B}}(t)\phi \equiv \bar{b}(t) * \phi = t^{-N/2m} \int F((x-y)t^{-1/2m})\phi(y)dy. \tag{2.11}$$

Differentiating (2.11) with respect to t, we get

$$\bar{u}_t = \bar{b}_t(t) * \phi, \tag{2.12}$$

where the right-hand side determines $\bar{\mathbf{A}}(t)u$. The majorizing operator is defined on the domain generated by the convolution operator $\bar{\mathbf{B}}(t)$:

$$\bar{W} = \{U(t) \in L^1,\, t \geq 0 : \; \forall\, t \geq 0 \; \exists \; \phi \in L^1 \text{ such that } U(t) = \bar{b}(t) * \phi\}. \tag{2.13}$$

In general, this means that the orbit $\{U(t),\, t > 0\} \subset \bar{W}$ admits the representation

$$U(t) = \bar{b}(t) * \phi(t); \quad \phi(t) \in L^1, \;\; t \geq 0. \tag{2.14}$$

$\bar{\mathbf{A}}(t)$ is single-valued on $\bar{W}$. Arguing by contradiction, we assume that there exist ϕ_1 and ϕ_2 from L^1 such that for some $t_0 > 0$,

$$U(t_0) = \bar{b}(t_0) * \phi_1 = \bar{b}(t_0) * \phi_2,$$

and the corresponding convolution orbits $u_1(t) = \bar{b}(t) * \phi_1$, $u_2(t) = \bar{b}(t) * \phi_2$ are such that $u_{1t}(t_0) \neq u_{2t}(t_0)$. Then $v = \phi_1 - \phi_2 \in L^1$ satisfies $\bar{b}(t_0) * v = 0$, which on the Fourier Transform implies that $W(\omega, t_0)\mathcal{F}(v)(\omega) = 0$. Since the symbol $W(\omega, t_0) \neq 0$ a.e., we have that $\mathcal{F}(v)(\omega) = 0$ a.e., so that $v \equiv 0$. Since $\bar{b}(t) > 0$ for $t > 0$ the solution $\bar{u}(t)$ depends monotonically on ϕ in $\mathbf{R}^N \times \mathbf{R}_+$ and we have the usual partial ordering $\geq$.

Actually, equality (2.11) determines an inverse operator for $t > 0$, $\phi = \bar{\mathbf{B}}^{-1}(t)\bar{u}$, acting along the *backward convolution vector field.* Substituting this into (2.12), we arrive at the following formal representation of the evolution majorizing operator:

$$\bar{u}_t = \bar{b}_t(t) * \phi = \bar{b}_t(t) * \bar{\mathbf{B}}^{-1}(t)\bar{u} \equiv \bar{\mathbf{A}}(t)\bar{u}. \tag{2.15}$$

In general, operator $\bar{\mathbf{B}}(t)$ in (2.11) is not invertible on $L^1(\mathbf{R}^N)$ and is not a Fredholm operator. Indeed, its symbol given by the Fourier Transform of the kernel $W(\omega, t) = \mathcal{F}(\bar{b}(t))(\omega) \equiv \int F(\zeta) e^{i t^{1/2m}(\omega.\zeta)} d\zeta$ is not necessarily positive, so that the Convolution Theorem applied to equation (2.11), $\mathcal{F}(\bar{u}(t))(\omega) = W(\omega, t)\,\mathcal{F}(\phi)(\omega)$, does not give a solution $\mathcal{F}(\phi)(\omega) = \mathcal{F}(\bar{u}(t))(\omega)/W(\omega, t)$ if $W(\omega, t)$ vanishes (unless $\mathcal{F}(\bar{u}(t))(\omega)$ vanishes on the zero set of the symbol). Hence, the operator is not invertible [10].

2.3. **Nonlinear majorizing order-preserving operator and equation.** The most of applications are obtained by comparison with positive solutions $v(x,t)$ satisfying the nonlinear majorizing integral equation with the majorizing operator having a positive kernel

$$v(t) = \bar{\mathbf{M}}(v)(t) \equiv \bar{b}(t) * v_0 + \int_0^t \bar{b}(t-s) * (Dv^p(s))ds, \quad t > 0, \tag{2.16}$$

where $v_0 \in X$ and $D = D(m, N) > 1$ for any $m > 1$ is the order deficiency given in (2.7). Local existence and uniqueness are obtained by contractivity properties of such integral operators with locally Lipschitz nonlinearities, see [16]. By comparing (2.6) and (2.16), we deduce the following result.

Theorem 2.1. *The integral equation* (2.16) *is majorizing for the* $2m$*-th order parabolic equation* (1.1) *in the following sense: for any* $u_0 \in X$ *with* $D|u_0| \le v_0$*, there holds*

$$|S(t)u_0| \le v(t), \quad t > 0. \tag{2.17}$$

The proof is straightforward since the difference $w(t) = v(t) - |u(t)|$ satisfies an integral inequality with a positive kernel of the form $(D\bar{b}(t-s) - |b(t-s)|) * (p\xi^{p-1}(s))$, $\xi(s) \in (|u(s)|, v(s))$. In the case of compactly supported data u_0 and v_0 we use approximation. It is important that the order deficiency $D = D(m,N) > 1$, which occurs in both the initial function and in the nonlinear term in (2.16), is necessary to compensate the lack of order-preserving (monotonicity) properties of the higher-order parabolic flow. In the first case, it establishes comparison of initial data: $|u_0(x)| \le D|u_0(x)| \le v_0(x)$ in $\mathbf{R}^N$. In the second operator, it is necessary for comparison in the nonlinear terms.

2.4. **Global self-similarity for the majorizing equation: eigenfunctions of Hammerstein's operator.** As a general property of the majorizing nonlinear integral equation (2.16) with $v_0 = 0$, we now use its scaling invariance. It admits the following global self-similar solution:

$$v_*(x,t) = t^{-1/(p-1)}\theta(\eta), \quad \eta = x/t^{1/2m}, \tag{2.18}$$

with a nontrivial function $\theta \ge 0$, $\theta \not\equiv 0$, to be determined. Let us comment on initial data for v_*. Assuming for this moment that $\theta \in L^1(\mathbf{R}^N)$ and setting $\beta = 1/(p-1) - N/2m > 0$ if $p > p_F$, we have that

$$\|v_*(t)\|_1 = t^{-\beta}\|\theta\|_1 \to 0, \;\; t \to 0,$$

so that $v_*(0) = 0$ in $\mathcal{D}'(\mathbf{R}^N)$. This is a typical phenomenon of "nonuniqueness" in the supercritical range $p > p_F$. Then for any fixed $t > 0$, there holds $\bar{b}(t) * v_*(\tau) \to 0$ as $\tau \to 0$.

Now, without any restriction on θ, we put $v_0 = 0$ and substituting the similarity structure (2.18) into (2.16), we get

$$t^{-1/(p-1)}\theta(\eta) = D\int_0^t \int_{\mathbf{R}^N} (t-s)^{-N/2m} F((x-y)/(t-s)^{1/2m}) s^{-p/(p-1)}\theta^p(y/s^{1/2m})dyds.$$

Performing natural changes of variables and setting $s = tz$, we conclude that θ is an eigenfunction with the eigenvalue $\lambda = 1$ of *Hammerstein's operator*:

$$\mathbf{K}(\theta) \equiv \int_{\mathbf{R}^N} K(\eta,\zeta)\theta^p(\zeta)d\zeta = \lambda\theta(\eta) \quad \text{in } \mathbf{R}^N, \tag{2.19}$$

with the *positive kernel*

$$K(\eta,\zeta) = D\int_0^1 z^{N/2m - p/(p-1)}(1-z)^{-N/2m} F[(1-z)^{-1/2m}(\eta - \zeta z^{1/2m})]dz. \tag{2.20}$$

Observe that the convergence of the integral at $z = 0$ imposes the restriction $N/2m - p/(p-1) > -1$, i.e., $p > p_F$. Thus, the majorizing equation inherits the same critical Fujita exponent as its differential parabolic one.

The general theory of eigenfunctions of nonlinear positive operators in Banach spaces with cone [13] applies to (2.19), though we note that the domain of integration is unbounded and the kernel is not uniformly positive in $\mathbf{R}^N$. The continuity and compactness of Hammerstein's operator play a role. Finally, we arrive at the following result.

Proposition 2.2. *For $p > p_F$ the integral equation (2.19) with $\lambda = 1$ has a positive radially symmetric solution $\theta = \theta(\xi)$, $\xi = |\eta|$.*

Proof. We will look for radial θ. Setting $t = \eta/(1+\xi) : \mathbf{R}^N \to B_1 = \{|\eta| < 1\}$, we obtain a standard Hammerstein equation with a compact positive linear integral operator over B_1 with a nonnegative kernel $\tilde{K}(z,t)$ and a monotone nonlinearity θ^p. Then $\tilde{K} > 0$ on $B_1 \times B_1$ and $\tilde{K}$ can vanish on its boundary only. Moreover, the kernel is of positive type, i.e., the iterated kernel in strictly positive, see p. 266 in [13]. This means that for any $\lambda > 0$, (2.19) admits a positive (radial) solution, [13], Chapt. 5, and we set $\lambda = 1$ by scaling. $\square$

In general, equations with Hammerstein's and more general Uryson's integral operators in cones with monotone nonlinearities, admit continuous branches of eigenfunctions of infinite length, see p. 250 in [13]. This is obvious for (2.19) since $c\theta$, $c > 0$, is an eigenfunction with $\lambda = c^{1-p}$. We expect that equation (2.19), $\lambda = 1$, admits a wide family of others non-radial solutions. Even in the radial setting, the global structure of solutions branches of (2.19) remains an open problem. We expect that the corresponding critical Sobolev exponent $p_S = (N+2m)/(N-2m)_+$ can play a role. More is known for the parabolic second-order case, $m = 1$. In the radial setting, such integral equations reduce to second-order ODEs and continuous infinite branches of different solutions were constructed for both the semilinear equation [11] and the quasilinear one, $u_t = \Delta u^q + u^p$, $q > 1$, $p > p_F = q + 2/N$, see [15], Theorem 4 on p. 222. The geometry of such branches changes precisely at the Sobolev critical value $p_S = q(N+2)/(N-2)$.

The global self-similar solution (2.18) satisfies the following L^∞ decay rate:

$$\|v_*(t)\|_\infty = c_* t^{-1/(p-1)} \quad \text{as } t \to \infty,$$

where the exponent $1/(p-1) < N/2m$ for $p > p_F$, so that the decay rate of the fundamental solution $\sim t^{-N/2m}$ is faster than of the similarity solution. Similar to the parabolic equation (1.1) with $p > p_F$ (see below), this decay rate corresponds to the *unstable border behaviour* between classes of global and blow-up solutions of the integral equation (2.16).

Completing the discussion on the majorizing order-preserving equation (2.16), we claim that $p_F = 1 + 2m/N$ is its critical exponent not only from the supercritical range (existence of small global solutions is established in Theorem 1.1), but also from the subcritical blow-up range saying that all nonnegative solution $v \not\equiv 0$ blow up if $1 < p \le p_F$. See a monotone stabilization evolution approach in [7] which can be adapted to the evolution integral equation (2.16) with Hammerstein's operators and positive kernels.

2.5. **Note on global self-similarity for the parabolic equation.** Looking for the solution u_* of the self-similar form (2.18) of the parabolic equation (1.1), we arrive at the elliptic equation for θ:

$$-(-\Delta)^m \theta + \frac{1}{2m}\, \nabla\theta \cdot \eta + \frac{1}{p-1}\, \theta + |\theta|^p = 0 \quad \text{in } \mathbf{R}^N. \tag{2.21}$$

On the other hand, using the integral equation (2.6), one obtains the eigenvalue problem (2.19) for the integral operator $\mathbf{K}$, where θ^p is replaced by $|\theta|^p$ and $F(\eta) > 0$ is replaced by $f(\eta)$. The oscillatory rescaled kernel f changes sign and, in general, so does the kernel K, and the positive operators theory in spaces with cones does not apply. One can expect that such similarity solutions are unstable and play no role for the generic asymptotic behaviour of global solutions.

2.6. **Comparison with majorizing self-similar solution. Proof of Theorem 1.1.** The majorizing equation is not translational invariant in time. In order to use a standard comparison, we need an extra construction. Fix a small constant $\tau > 0$, and consider the solution of the Cauchy problem with initial data at $t = \tau$:

$$u(t) = b(t-\tau) * u_0 + \int_\tau^t b(t-s) * |u(s)|^p ds, \quad t > \tau, \tag{2.22}$$

so that

$$|u(t)| \le |b(t-\tau)| * |u_0| + \int_\tau^t |b(t-s)| * |u(s)|^p ds \equiv J_1 + J_2. \tag{2.23}$$

Consider the similarity solution of the majorizing equation (2.18) satisfying

$$v_*(t) = \int_0^t \bar b(t-s) * D v_*^p(s) ds \equiv \int_0^\tau + \int_\tau^t \equiv \bar J_1 + \bar J_2, \;\; t > \tau. \tag{2.24}$$

We have that

$$\bar J_1 \ge \bar J_{11} = \int_{\tau/2}^\tau \bar b(t-s) * D v_*^p(s) ds,$$

and using necessary uniform bounds of $v_*(t)$ in $[\tau/2, \tau] \times B_R$, $R \in (0, \infty)$, where it is uniformly bounded away from zero, we conclude that there exists a sufficiently small function $\phi(\tau) > 0$ in B_R such that

$$\bar J_{11} \ge \bar b(t-\tau) * D\phi(\tau), \quad t > \tau. \tag{2.25}$$

On the other hand, using (2.9), we have that

$$J_2 \le \bar J_2, \quad t > \tau, \tag{2.26}$$

provided that $|u(s)| \le v(s)$ for all $s > \tau$. It follows from (2.25) and (2.26) that, given any $\tau > 0$, if

$$|u_0| \le D\phi(\tau), \tag{2.27}$$

then, as in comparison Theorem 2.1, the usual comparison holds:

$$|u(t)| \le v_*(t) \quad \text{for any } \; t > \tau. \tag{2.28}$$

Hence, $u(t)$ is a global uniformly bounded solution with the decay rate

$$\|u(t)\|_\infty \le c_* t^{-1/(p-1)}, \quad t \gg 1 \quad (c_* = \|\theta\|_\infty). \tag{2.29}$$

This completes the proof.

2.7. **On the stable manifold of the origin.** Denote by $W_0 \subset X$ a non-empty subset of initial functions u_0 such that $u(x,\tau)$ satisfies (2.27) for some $\tau > 0$. Then the *stability subset* W_0 is contained in the stable manifold of the trivial stationary solution $u \equiv 0$: for any $u_0 \in W_0$ the solution $u(\cdot,t) \to 0$ as $t \to \infty$ uniformly in $\mathbf{R}^N$.

2.8. **A generalization.** The result of Theorem 1.1 is true for more general parabolic equations (1.1) with linear operators

$$\mathbf{A} = \sum_{|\sigma| \le m} D^\sigma (a_\sigma(x,t) D^\sigma)$$

with bounded coefficients a_σ provided that a suitable estimate on the fundamental solution (similar to that in Proposition 2.1) is available. Then in the supercritical case a majorizing linear and nonlinear order-preserving operators exist with a non-trivial global similarity solutions obtained via eigenfunctions of positive Hammerstein's operators.

2.9. **A blow-up rate via the majorizing equation.**

Proposition 2.3. *If T is the finite blow-up time of the solution $u(x,t)$ of the Cauchy problem* (1.1)*, then*

$$\|u(t)\|_\infty > V_T(t) \equiv [D(p-1)(T-t)]^{-1/(p-1)}, \quad t \in [0,T), \tag{2.30}$$

where $D > 1$ is the order deficiency of the parabolic operator.

Proof. The proof [2] is based on the standard comparison of $u(x,t)$ with the flat blowing up solution $V_T(t)$ of the majorizing order-preserving equation (2.16) with constant initial data $v_0 \equiv V_T(0)$ chosen such that $V_T(t)$ has the same blow-up time as u. One can see that, by construction of the majorizing evolution, V_T solves the ODE

$$v' = Dv^p, \quad t \in (0,T); \quad v(T) = \infty.$$

Then, for any $t \in [0,T)$, $u(x,t)$ must intersect $V_T(t)$. Otherwise, if $u(x,t_0) \le V_T(t_0)$ in $\mathbf{R}^N$ for some $t_0 \in (0,T)$, one can check that by the strict order-preserving monotonicity of the majorizing evolution, these solutions cannot have the same blow-up time.

□

Observe, that such a derivation of the blow-up estimate (2.30) just repeats the argument of the proof in the order-preserving case $m = 1$ (i.e., $D = 1$), where the comparison is straightforward and is based on the Maximum Principle. It is worth mentioning that the standard comparison and the intersection comparison of solutions having the same blow-up time play an important role in the blow-up and localization theory for second-order quasilinear reaction-diffusion equations, see Chapt. 4 in [15]. We expect that some of these results, which are well known for $m = 1$, can also be translated to the non-order-preserving case $m > 1$ by means of introducing and studying the corresponding majorizing order-preserving equations.

2.10. **Extension of main results to pseudodifferential evolution.** The main idea and the main results on global and blow-up estimates of solutions are true for more general equations (2.6) where the evolution process is governed by an integral operator, say, of the convolution type $\mathbf{B}(t)\phi = b(t) * \phi$, or by a more general pseudodifferential one. Here both the kernel and the symbol $\mathcal{F}(b(t))$ of the operator change sign and $\mathbf{B}(t)$ is not invertible. Assuming the scaling structure (2.2) of the kernel $b(t)$, constructing the corresponding rescaled majorizing kernel, $|f| \le DF$, with the order deficiency $D = D(f) > 1$, and finally introducing the corresponding majorizing equation of the type (2.16), we arrive at similar global existence results and blow-up estimates. The global existence follows by comparison with a self-similar solution $v_*(t)$ obtained from Hammerstein's equation with strictly positive iterated kernel. A sharp lower blow-up estimate is proved by a standard comparison with a spatially homogeneous similarity solution having the same blow-up time.

3. Positivity sets for higher-order parabolic equation

The stability and global existence Theorem 1.1 has a direct relation to the question on *positivity sets* W_+ for higher-order parabolic equations without the Maximum Principle. Consider as a basic example, the equation

$$w_t = -(-\Delta)^m w + |w-1|^p (w-1) \quad \text{in } Q,$$

with initial data $w_0 \in X$. Denote the positivity set

$$W_+ = \{w_0 \in X : \quad w \ge 0 \text{ in } Q\}.$$

Of course, $W_+ \ne \{w \equiv 1\}$ in the case when $w \equiv 1$ is a stable stationary solution, which is the case if $p > p_F$. Setting $w = 1 + u$, we obtain the equation

$$u_t = -(-\Delta)^m u + |u|^p u.$$

Equation (2.16) is the majorizing order-preserving one for the semigroup of the above parabolic flow, so that by Theorem 1.1 we obtain that the positivity set is nontrivial and

$$\{w_0 \in X : \quad w_0 \ge 0, \; w_0 - 1 \in W_0\} \subset W_+,$$

where W_0 is the stable subset of $w \equiv 1$ constructed in the proof of Theorem 1.1. It follows that such a positivity set can be constructed for any semilinear equation of the type

$$w_t = \mathbf{A} w + g(w),$$

with a smooth function $g(w)$ satisfying $g(1) = 0$ provided that $w \equiv 1$ is a stable stationary solution in an appropriate weighted space. For instance, this is always the case if $g'(1) < 0$. Thus, the global existence Theorem 1.1 establishes the structure of the positivity set in the most delicate case $g'(0) = 0$ and $|g(w)| = O(|w-1|^p)$ as $w \to 1$ (the supercritical case $p > p_F$), where the stationary solution $u \equiv 0$ turns out to be asymptotically stable in a weighted space with the weight function somehow related to the majorizing kernel $\bar{b}(x,t)$ for $t \gg 1$. Recall that for any for $p \le p_F$, by Theorem 1.2 this stationary solution is not stable.

4. Blow-up results by ODE analysis

Proof of Theorem 1.2. We will prove that for $p \in (1, p_F]$ there exist arbitrarily small initial data leading to blow-up of sufficiently smooth solutions $u(\cdot,t) \in C^1([0,T] : X)$. Therefore, without loss of generality, we consider initial data $u_0 \in X$ concentrated around the origin, $u_0(x) \to 0$ as $|x| \to \infty$ and $u(x,t)$ and its spatial derivatives vanish as $|x| \to \infty$ for $t > 0$. For instance, this is true for compactly supported data $u_0 \in X$, and the general case is then treated by approximation. We show that the blow-up theorem for higher-order equations can be proved by a special modification of the well-known method of Fourier coefficients (the eigenfunctions method) introduced in [12] for the second-order equations. Moreover, it is sufficient to control the first positive coefficient, and the result is based on blow-up properties of a single ordinary differential inequality (ODI).

We first construct a nonnegative smooth radial function $\psi_1 = \psi_1(|x|) > 0$ satisfying

$$\psi_1(0) = 1, \quad 0 < \psi(r) \le 1 \text{ for } r > 0,$$

$\psi_1(r)$ is decreasing and $\psi_1(r) \to 0$ as $r \to \infty$ sufficiently fast. The analysis is essentially based on a special choice of this test function ψ_1 associated with an "eigenvalue" inequality for the elliptic operator Δ^m. We will use the following estimate on ψ_1: there exists a constant $\lambda_1 > 0$ such that

$$|\Delta^m \psi_1| \le \lambda_1 \psi_1 \quad \text{in } \mathbf{R}^N. \tag{4.1}$$

One can see that in the present geometry such an "eigenfunction" ψ_1 always exists. Actually, we need to check if (4.1) holds for $r \gg 1$ where we put $\psi_1(r) = e^{-r^\beta}$ with an exponent $\beta \in (0,1]$. ψ_1 is extended for all $r \in [0,\infty)$ by a smooth approximation.

Multiplying equation (1.1) by $\psi_R(x) = \psi_1(x/R) > 0$ in $L^2(\mathbf{R}^N)$ with a parameter $R > 0$ and integrating by parts, we obtain that the first Fourier coefficient $E(t) = (u(t), \psi_R)$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbf{R}^N)$, satisfies

$$E'(t) = (u(t), (-1)^{m+1}\Delta^m \psi_R) + (|u|^p(t), \psi_R), \quad t > 0. \tag{4.2}$$

We have $\|\psi_R\|_1 = a_1 R^N$, where $a_1 = \|\psi_1\|_1$, so that by the Hölder inequality

$$(|u|^p(t), \psi_R) \ge \|\psi_R\|_1^{1-p} (|u|, \psi_R)^p = a_2 R^{(1-p)N} \bar{E}^p, \;\; a_2 = a_1^{1-p},$$

where we set $\bar{E}(t) = (|u(t)|, \psi_R) \ge |E(t)|$.

Subcritical range $p < p_F$. Since $\Delta^m \psi_R(x) = R^{-2m} \Delta^m \psi_1(\eta), \; \eta = x/R$, using (4.1), we deduce that

$$(u(t), (-1)^{m+1}\Delta^m \psi_R) \ge -(|u(t)|, |\Delta^m \psi_R|) \ge -\lambda_1 R^{-2m} (|u(t)|, \psi_R) = -\lambda_1 R^{-2m} \bar{E}(t).$$

Substituting these two estimates into (4.2), we arrive at the ODI

$$E' \ge a_2 R^{(1-p)N} \bar{E}(\bar{E}^{p-1} - a_3 R^\mu) \equiv H(\bar{E}), \quad a_3 = \lambda_1/a_2, \tag{4.3}$$

where $\mu = (p-1)N - 2m = N(p-p_F) < 0$. Using the equality

$$(d/dt) \int u(t) = \int |u(t)|^p > 0, \;\; t > 0 \quad (u(t) \not\equiv 0)$$

and (1.4), by slight shifting the origin in time, we may assume that (1.4) holds with the strict positivity. Then one can fix an $R \gg 1$ such that $\bar{E}^{p-1}(0) \geq E^{p-1}(0) > a_3 R^\mu$, and then (4.3) implies that $E'(0) > 0$, so that by continuity $E(t) > E(0) > 0$ for small $t > 0$. Let us show that

$$E(t) \geq E(0) > (a_3 R^\mu)^{1/(p-1)} \quad \text{for all } t > 0. \tag{4.4}$$

Indeed, if $E(t_1) < E(0)$ for some $t_1 > 0$ then there exists $t_2 \in (0, t_1)$ such that $E'(t_2) = 0$ and $E(t) \geq E(0)$ on $(0, t_2]$. Then (4.3) implies that $\bar{E}^{p-1}(t_2) \leq a_3 R^\mu$, contradicting the inequality $\bar{E}^{p-1}(t_2) \geq E^{p-1}(t_2) > a_3 R^\mu$. Thus, (4.4) holds for all $t > 0$ so that both values $\bar{E}(t)$ and $E(t)$ always belong to the monotone increasing positive branch of the function H. We then obtain the ODI

$$E'(t) \geq H(\bar{E}(t)) \geq H(E(t)) > 0, \quad t > 0. \tag{4.5}$$

Integrating this ODI, it follows that $u(x,t)$ cannot be global, otherwise it is not bounded in $L^1(\mathbf{R}^N)$. Next, by standard local properties of smooth parabolic flows governed by the integral equation (2.6), we have that $\|u(t)\|_\infty$ blows up first in the sense of the usual limit as in (1.5) for some finite T. Indeed, if it is $\limsup$ and there exists a sequence $\{t_k\} \to T^-$ such that $\|u(t_k)\|_\infty \leq c$, then by the uniqueness for (2.6) a bounded solution can be extended beyond $t = T$.

The critical case $p = p_F$. In this special case, where $\mu = 0$ in (4.3), we need a slight modification of the "eigenfunction": we set $\psi_1(x) \equiv 1$ in $B_1 = \{|x| < 1\}$. Then (4.3) with $\mu = 0$ implies that $E(t)$ does not blow-up if

$$|E(t)| \leq \bar{E}(t) = \int_{\mathbf{R}^N} |u(t)| \psi_R \leq a_4 = a_3^{1/(p-1)} \quad \text{for all } t \geq 0. \tag{4.6}$$

On the other hand, since $\Delta^m \psi_R \equiv 0$ in B_R, we conclude that as $R \to \infty$

$$(u(t), (-1)^{m+1} \Delta^m \psi_R) \geq -\lambda_1 R^{-2m} \int_{|x|>R} |u(t)| \psi_R = -\lambda_1 R^{-2m} o(1),$$

since by by (4.6)

$$\int_{|x|>R} |u(t)| \psi_R = o(1) \to 0, \quad R \to \infty.$$

Therefore for $R \gg 1$, we arrive at the ODI

$$E' \geq a_2 R^{(1-p)N} \bar{E}(\bar{E}^{p-1} - o(1)) \equiv H(\bar{E}),$$

and the same analysis as for (4.5) applies proving blow-up (1.5). This completes the proof of Theorem 1.2.

Note that the resulting ODIs establish blow-up of solutions with large initial data for any $p > 1$. Moreover, it is important that, besides blow-up itself in a suitable metric, such evolution ODIs can provide us with delicate estimates on the asymptotic blow-up behaviour of general solutions as $t \to T^-$, see e.g. an application in [9], Sect. 3.

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