FAST DIFFUSION EQUATION WITH CRITICAL SOBOLEV EXPONENT IN A BALL

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ABSTRACT. We study the asymptotic extinction behaviour for the fast diffusion equation in the unit ball $\Omega = \{|x| < 1\} \subset \mathbf{R}^N$, $N \ge 3$, with the Dirichlet boundary conditions

$$u_t = \Delta u^m \quad \text{in } \Omega \times (0,T), \quad u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T),$$

and positive symmetric initial data $u_0(r)$, $r = |x|$. Let $T \in (0,\infty)$ be the extinction time: $u > 0$ in $\Omega \times (0,T)$ and $u(\cdot,T) = 0$. We show that in the critical Sobolev case

$$m = m_s \equiv (N-2)/(N+2)$$

the asymptotic behaviour as $t \to T^-$ near the origin $x = 0$ is essentially nonself-similar (unlike the cases $m \in (m_s, 1)$ and $m \in (0, m_s)$) and is constructed by matching the expansions in the inner and boundary (outer) domains. This gives the extinction rate as $t \to T^-$:

$$\|u(\cdot,t)\|_\infty = \gamma_0 (T-t)^{(N+2)/4} |\ln(T-t)|^{(N+2)/2(N-2)}(1+o(1)),$$

where $\gamma_0 = \gamma_0(N) > 0$ is a constant. In addition, the extinction behaviour in the supercritical case $m \in (0, m_s)$, which has also not previously addressed, is noted. We discuss other nonlinear reaction-diffusion (blow-up) problems admitting such a critical asymptotic behaviour.

1. Three asymptotic ranges

Let $\Omega = \{|x| < 1\}$ be the unit ball in $\mathbf{R}^N$, $N \ge 3$, with the boundary $\partial\Omega = \{|x| = 1\}$, and $m \in (0,1)$ be a fixed constant. Consider the Dirichlet problem for the fast diffusion equation

$$u_t = \Delta u^m \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

with the boundary and initial conditions

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u(x,0) = u_0(x) > 0 \quad \text{in } \Omega, \tag{1.2}$$

where u_0 is a bounded continuous function. Equation (1.1) is a singular non-degenerate parabolic one, and the Dirichlet problem admits a unique local in time positive classical solution; see references in [9]. The solution is known to vanish in finite time as $t \to T^- < \infty$, so that $u > 0$ in $\Omega \times (0,T)$ and $u(x,T) \equiv 0$. This singular nonlinear effect of finite time

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extinction was first proved by Sabinina [13], [14] for a class of one-dimensional parabolic equations of fast diffusion type in bounded intervals.

We study the asymptotic behaviour of such singular solutions.

1.1. **Subcritical range.** As far as the asymptotic behaviour of the solutions as $t \to T^-$ is concerned, the first important result was established by Berryman and Holland [1] in the *subcritical* parameter range

$$0 < m < 1, \;\; N = 1, 2 \quad \text{and} \quad m_s \equiv (N-2)/(N+2) < m < 1, \;\; N \geq 3.$$

The extinction behaviour was proved to be described by a separable self-similar solution of the form

$$u_*(x,t) = (T-t)^{1/(1-m)} S(x). \tag{1.3}$$

Substituting this solution into (1.1) and taking into account the Dirichlet boundary condition, we arrive at a nonlinear elliptic problem for the function $S \geq 0$:

$$\Delta S^m + (1-m)^{-1} S = 0 \;\text{ in }\; \Omega, \quad S = 0 \;\text{ on }\; \partial\Omega. \tag{1.4}$$

Setting $S = v^{1/m}$ gives the classical semilinear elliptic problem

$$\Delta v + (1-m)^{-1} v^p = 0 \;\text{ in }\; \Omega, \quad v = 0 \;\text{ on }\; \partial\Omega, \tag{1.5}$$

where

$$p = 1/m > 1. \tag{1.6}$$

Therefore, the separable asymptotic behaviour depends essentially on the solvability of the elliptic problem (1.5). In the unit ball Ω such results are well known: in the subcritical Sobolev range $1 < p < p_s$,

$$p_s = (N+2)/(N-2)_+, \tag{1.7}$$

there exists a unique classical bounded positive solution, while in the critical, $p = p_s$, and the supercritical Sobolev range, $p > p_s$, no classical bounded positive solutions exist. On the other hand, for $p \geq p_s$ equation (1.5) in $\mathbf{R}^N$ admits a one-parameter family of strictly positive radial solutions, which do not satisfy the boundary condition on $\partial\Omega$. Nevertheless, these solutions play an important role in our analysis of the critical Sobolev case.

Thus, bearing in mind the above uniqueness result for $S(x)$ and natural regularity results for non-degenerate parabolic equations, in the subcritical range $m_s < m < 1$, the solution, rescaled in accordance with the self-similar structure (1.3), satisfies as $t \to T^-$ (cf. [1])

$$\theta(x,t) \equiv (T-t)^{-1/(1-m)} u(x,t) \to S(x) \quad \text{uniformly in } \; \Omega, \tag{1.8}$$

where $S(x) > 0$ in Ω is the unique profile satisfying (1.4), which is a symmetric ($S = S(|x|)$) monotone decreasing function. This means that the self-similar solution (1.3) describes not only the rate of extinction as $t \to T^-$,

$$\|u(\cdot,t)\|_\infty = S(0)(T-t)^{1/(1-m)}(1+o(1)), \tag{1.9}$$

but also the spatial shape of extinction uniformly in $x \in \Omega$.

1.2. **Supercritical range.** We note that in the (non-existence) range $0 < m \le m_s$, the profile $S(x)$ in (1.8) is a non-negative weak solution and the convergence takes place in the Sobolev space $H_0^1(\Omega)$ [1]. If we treat the convergence in (1.8) in the pointwise sense, the limit function $S(x)$ is then the singular one at the origin: $S(0) = \infty$ and $S \equiv 0$ in $\Omega \setminus \{0\}$; see Section 2. This uniform stabilization result does not specify either the rate of extinction or the spatial structure of the singularity as $t \to T^-$, and a more delicate analysis is necessary.

On the other hand, we claim that in the *supercritical* Sobolev range $p > p_s$ (i.e., $m \in (0, m_s)$) the asymptotic extinction behaviour is the same as for the Cauchy problem in $\mathbf{R}^N$. The fast diffusion equation in $\mathbf{R}^N$ is known to admit a unique "minimal" self-similar solution of the second kind, $u_*(\cdot, t) \in L^1(\mathbf{R}^N)$ for any $t \in (0, T)$; see Section 4 in [8]. This self-similar solution is proved to be asymptotically stable as $t \to T^-$ in the symmetric geometry [5].

The symmetric self-similar behaviour as $t \to T^-$,

$$u(x,t) \sim (T-t)^\alpha F(|x|/(T-t)^\beta), \quad (1-m)\alpha + 2\beta = 1, \tag{1.10}$$

where the exponent $\beta > 0$ is determined as in [8], provides the leading order inner solution and, in view of the more complicated asymptotic analysis of the critical case that follows, it is worth outlining here the outer solution and the matching as well. The inner region is thus $|x| = O((T-t)^\beta)$; the outer one is $|x| = O(1)$ with

$$u(x,t) \sim (T-t)^{(m\alpha + (N-2)\beta)/m} \Phi(x), \tag{1.11}$$

where, since $F(\eta) \sim A\eta^{(N-2)/m}$ as $\eta \to \infty$ for some constant $A > 0$ that depends on the initial data, matching with (1.10) implies that Φ is the Green's function satisfying

$$\Delta\Phi = -(N-2)\omega_N A^m \delta(x) \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \partial\Omega, \tag{1.12}$$

where ω_N is the surface area of the unit ball in $\mathbf{R}^N$. Note that this asymptotic structure readily generalises to arbitrary domains Ω.

1.3. **Main results: the critical Sobolev case.** In the present paper we concentrate on the asymptotic extinction behaviour in the *critical* case

$$m = (N-2)/(N+2), \;\; N \ge 3 \quad (p = p_s). \tag{1.13}$$

It turns out that this case produces the most delicate and interesting asymptotic behaviour, which is constructed by a matching procedure. In Sections 2 and 3 we derive separately the asymptotic behaviour in the *inner* and the *boundary* (outer) regions respectively. Observe that in terms of the standard self-similar rescaled variable (1.8) we establish that the limit function S satisfies

$$S^m = 0 \quad \text{in } \mathcal{D}'(\Omega) \text{ and in } L^2(\Omega), \quad \text{and } S^m \ne 0 \text{ in } H_0^1(\Omega), \tag{1.14}$$

where $\mathcal{D}'(\Omega)$ denotes the linear space of distributions. Actually, we prove that in the limit sense $|\nabla S^m|^2(x) = c_2\delta(x)$, $c_2(N) > 0$ is a constant.

We show that in the inner region close to the origin, the rescaled orbit given in (1.8) evolves via a *slow motion* over the one-parameter family of positive stationary solutions, $\{S_\lambda(|x|),\, \lambda > 0\}$:

$$\Delta S^{(N-2)/(N+2)} + (N+2)S/4 = 0 \quad \text{in } \mathbf{R}^N, \tag{1.15}$$

which are Loewner-Nirenberg explicit solutions [11]

$$S_\lambda(r) = \lambda \left[\frac{2N(N-2)}{2N(N-2) + (N+2)\lambda^{4/(N+2)} r^2} \right]^{(N+2)/2} \equiv \lambda S_1(r\lambda^{2/(N+2)}), \tag{1.16}$$

with the parametrization $S_\lambda(0) = \lambda > 0$.

This kind of behaviour exhibits some common features of the stabilization to a singular stationary solution of semilinear and quasilinear parabolic equations studied in [3], where the Frank-Kamenetskii equation with the Dirichlet boundary condition

$$u_t = \Delta u + e^u \ \text{ in } B \times \mathbf{R}_+, \ B = \{|x| < \sqrt{2(N-2)}\}; \quad u = 0 \ \text{ on } \partial B \times \mathbf{R}_+, \tag{1.17}$$

is considered in dimension $N > 10$. The only stationary solution of this semilinear heat equation is the *singular* one

$$S_\infty(x) = \ln\left(2(N-2)/|x|^2\right), \quad S_\infty(0) = \infty,$$

satisfying the zero Dirichlet boundary condition. There exists a family $\{S_\lambda(x)\}$, with the same parametrization $S_\lambda(0) = \lambda \in \mathbf{R}$, of bounded (at the origin) stationary profiles such that $S_\lambda(x) \to S_\infty(x) - 0$, $\lambda \to \infty$, uniformly on any compact subsets bounded away from zero. In the problem (1.17), the rate of stabilization from below of a *bounded orbit* to the *singular stationary solution* is determined by matching of the outer (boundary) expansion and the inner one, where the solutions evolve on a family of stationary profiles. For (1.17) the linearized operator in the boundary domain does not have a centre subspace, so that the outer behaviour corresponds to the stable invariant subspace. The stable manifold behaviour in the boundary domain can be obtained independently of the inner centre manifold one. The boundary-inner matching consists in extending the known boundary behaviour into the the inner region (a *stable-centre* manifolds matching).

There exists an essential difference between these two problems. In the present problem (1.1), (1.2), (1.13), both inner and boundary domains exhibit asymptotic behaviour on non-trivial extended centre manifolds and the matching procedure becomes more interesting. The first term of the asymptotic expansion of the infinite-dimensional parameter describing the slow motion, a monotone increasing function $\alpha(t) \equiv \theta(0,t) \to \infty$, $t \to T^-$, is determined by matching of two extended centre manifolds (a *centre-centre* manifolds matching).

Section 4 is devoted to the *intermediate* region, where we match the inner and the boundary expansions and prove the following result.

Theorem 1.1. *Let $u_0 = u_0(r)$, $r = |x|$, be a symmetric function. In the critical case (1.13) the extinction rate as $t \to T^-$ is*

$$\|u(\cdot,t)\|_\infty \equiv u(0,t) = \gamma_0 (T-t)^{(N+2)/4} |\ln(T-t)|^{(N+2)/2(N-2)}(1 + o(1)), \tag{1.18}$$

where γ_0 is a constant depending on dimension N only.

The right-hand side in (1.18) represents an unbounded logarithmic perturbation of the self-similar rate $(T-t)^{(N+2)/4}$ corresponding to (1.9). Moreover, in the inner region near the origin, the asymptotic behaviour is given by a logarithmically perturbed self-similar stationary structure

$$u(x,t) = (T-t)^{(N+2)/4}\alpha(t)S_1(|x|\alpha^{2/(N+2)}(t))(1+o(1)), \tag{1.19}$$

with the slowly growing factor

$$\alpha(t) = \gamma_0|\ln(T-t)|^{(N+2)/2(N-2)}.$$

1.4. **Extensions: similar types of asymptotic pattern formation.** The type of the asymptotic behaviour studied in the paper is not a special or an exceptional one for classes of nonlinear parabolic equations of reaction-diffusion type. Let us mention other applications of such an asymptotic analysis. It has been well known for a long period that the semilinear (or quasilinear) heat equations with the critical Sobolev exponent

$$u_t = \Delta u + u^{(N+2)/(N-2)}, \quad x \in \Omega, \;\; t > 0, \tag{1.20}$$

with $u = 0$ on the boundary, admits, for a special choice of initial data $u_0(x) > 0$, global L^1-solutions $u(x,t)$, which are uniformly bounded in $L^1(B_1)$: $\|u(t)\|_1 \leq C$ for all $t > 0$, but not in $L^\infty(\Omega)$ [12]. Later on it was shown that such special solutions are classical ones, $u(t) \in L^\infty(B_1)$ for all $t > 0$, and $\|u(t)\|_\infty \to \infty$ as $t \to \infty$ [7]. The actual space-time structure of these special solutions as $t \to \infty$ remained open.

The asymptotic behaviour of such solutions can be constructed by the same centre-centre manifolds matching technique described above and exhibits a similar structure. In order to exhibit such a strong asymptotic similarity of these apparently quite different nonlinear phenomena, we note that the function $v = \theta^{(N-2)/(N+2)}$, where θ is the rescaled function (1.8), solves the following quasilinear heat equation:

$$b_0 v^{4/(N-2)} v_\tau = \Delta v + b_1 v^{(N+2)/(N-2)}, \quad b_0 = p_s, \;\; b_1 = (N+2)/4, \tag{1.21}$$

where, as usual, $\tau = -\ln(T-t) \to \infty$ as $t \to T^-$. The stationary elliptic operators on the right-hand sides of (1.20) and (1.21) coincide (up to scaling) and both admit a of exact solutions of the form (1.16) as the manifold of the asymptotic slow evolution.

The same effects are available for quasilinear heat equations with the porous medium operator

$$u_t = \Delta u^m + u^p, \quad p = p_s = m(N+2)/(N-2), \tag{1.22}$$

($m > (N-2)/N$, $N > 2$) and with the p-Laplace one

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^p, \quad p = p_s = [N(\sigma+1)+\sigma+2]/[N-(\sigma+2)]$$

($\sigma > -2/(N+1)$, $N > \sigma+2$). These results will be published in a forthcoming paper [4]. Such a behaviour can also be observed for fully nonlinear equations like the dual porous medium equation with a source $u_t = |\Delta u|^{m-1}\Delta u + u^p$, with $m > 1$, where the critical Sobolev exponent $p = p_s$ is the same as in (1.22). A critical behaviour is expected to occur for the higher-order semilinear parabolic equation $u_t = -(-\Delta)^m + |u|^{p-1}u$ with $m > 1$ and $p = p_s = (N+2m)/(N-2m)$, $N > 2m$.

2. **Inner expansion**

Using the Strong Maximum Principle and natural eventual monotonicity results for solutions of autonomous quasilinear heat equations in bounded domains, we may assume that the symmetric solution $u = u(r,t)$ is strictly monotone decreasing in r:

$$u_r < 0 \quad \text{in } \ (0,1) \times [0,T),$$

so that

$$\|u(\cdot,t)\|_\infty = u(0,t), \quad t \in [0,T).$$

2.1. **Properties of separable solutions in $\mathbf{R}^N$.** In the critical case (1.13), equation (1.1) admits a family of strictly positive separable solutions in $\mathbf{R}^N \times (0,T)$

$$u_*(x,t) = (T-t)^{(N+2)/4} S_\lambda(x). \tag{2.1}$$

We need some straightforward asymptotic properties of the family $\{S_\lambda\}$ given in (1.16). First, for $r \gg 1$

$$S_1(r) = c_1 r^{-(N+2)} - c_2 r^{-(N+4)} + ..., \tag{2.2}$$

where $c_1 = [2N(N-2)/(N+2)]^{(N+2)/2}$, $c_2 = N(N-2)c_1$. Second, we prescribe the behaviour of the family as $\lambda \to \infty$ and $\lambda \to 0$.

Proposition 2.1. *As $\lambda \to \infty$,*

$$S_\lambda(r) \to 0 \quad \textit{uniformly on any subset } [\delta, \infty),\ \delta > 0, \tag{2.3}$$

$$S'_\lambda(r) \equiv \frac{d}{dr} S_\lambda(r) \to -\infty \quad \textit{on any level set } \{S_\lambda = \varepsilon\},\ \varepsilon > 0, \tag{2.4}$$

$$S_\lambda(r) \to \infty \quad \textit{uniformly on any subset } \{r = o(\lambda^{-1/(N+2)})\}. \tag{2.5}$$

In the class of radially symmetric functions, we identify $L^2(\Omega)$ with the weighted space $L^2_\rho((0,1))$, $\rho(r) = r^{N-1}$, with the inner product

$$(u,v)_\rho = \int_0^1 r^{N-1} u(r) v(r) dr.$$

Proposition 2.2. *(i) As $\lambda \to \infty$,*

$$(S_\lambda)^\alpha \to 0 \quad \textit{in } \ D'(\Omega) \ \textit{ if } \ n/(N+2) < \alpha < 2N/(N+2), \tag{2.6}$$

$$(S_\lambda)^{2N/(N+2)} \to d_1 \delta(x) \quad \textit{in } \ \mathcal{D}'(\Omega), \tag{2.7}$$

$$\big([(S_\lambda)^{(N-2)/(N+2)}]'\big)^2 \to d_2 \delta(x) \quad \textit{in } \ \mathcal{D}'(\Omega), \tag{2.8}$$

where c_1 and c_2 are positive constants depending on N:

$$d_1 = \int_0^\infty \xi^{N-1} S_1^{2N/(N+2)}(\xi) d\xi, \quad d_2 = [(N+2)^2/4N^2] \int_0^\infty \xi^{N+1} S_1^{2N/(N+2)}(\xi) d\xi.$$

(ii) As $\lambda \to 0$,

$$S_\lambda, \;\; S'_\lambda \to 0 \quad uniformly \; in \; \Omega. \tag{2.9}$$

2.2. **First estimates on rescaled orbits.** The rescaled solution (1.8), $\theta = \theta(x, \tau)$, with the new time variable $\tau = -\ln(T-t) \to \infty$ as $t \to T^-$, solves the quasilinear parabolic equation

$$\theta_\tau = \Delta \theta^m + (N+2)\theta/4 \quad \text{in } \; \Omega \times (\tau_0, \infty), \;\; \tau_0 = -\ln T, \tag{2.10}$$

with the conditions

$$\theta = 0 \quad \text{on } \partial\Omega \times (\tau_0, \infty), \quad \theta(x, 0) = \theta_0(x) \equiv T^{-(N+2)/4} u_0(x) \;\; \text{in } \; \Omega. \tag{2.11}$$

For a fixed $t \in (0, T)$, denote by $J(t, \lambda)$ the number of intersections in $r \in (0, 1)$ of the spatial profiles $\theta(r, t)$ and $S_\lambda(r)$, i.e., the number of sign changes of the difference $w(r, t) = \theta(r, t) - S_\lambda(r)$ for $r \in (0, 1)$.

Proposition 2.3. *We have*

$$J(t, \lambda) > 0 \quad for \; any \; t \in [0, T) \; and \; \lambda > 0. \tag{2.12}$$

Proof. One can see that if $J(t_0, \lambda_0) = 0$ for some $t_0 \in [0, T)$ and a $\lambda_0 > 0$, then the solutions $u_*(x, t)$ and $u(x, t)$, which satisfy by the Strong Maximum Principle the strict inequality $u_*(x, t_0 + \varepsilon) > u(x, t_0 + \varepsilon)$ in Ω, $\varepsilon > 0$ small, cannot have the same extinction time T, contradicting to assumption. □

Denote

$$\alpha(\tau) = \theta(0, \tau) \equiv \|\theta(\cdot, \tau)\|_\infty, \quad \tau > \tau_0. \tag{2.13}$$

The function $\alpha(\tau)$ is sufficiently smooth and positive on $(0, \infty)$.

Proposition 2.4. *$\alpha(\tau)$ is bounded away from zero: there exists a constant $\lambda_0 > 0$ such that*

$$\alpha(\tau) \geq \lambda_0 > 0 \quad for \; any \; \tau \geq \tau_0. \tag{2.14}$$

Proof. By the property (2.9) and the regularity of $u_0(r)$, there exists a small $\lambda_0 > 0$ such that

$$J(0, \lambda_0) = 1. \tag{2.15}$$

By the Sturmian argument (see e.g. references in [5], [7] and Chapter 4 of the book [15]) the number of intersections $J(t, \lambda)$ does not increase with time, so that (2.15) implies that

$$J(t, \lambda_0) = 1 \quad \text{for all } \; t \in (0, T). \tag{2.16}$$

In the rescaled variables, this implies (2.14). □

We now use the fact that the problem (2.10), (2.11) admits a standard Lyapunov function of the form

$$V(\theta)(\tau) = \frac{1}{2}\|\nabla\theta(\tau)\|_2^2 - \frac{N^2-4}{8N}\,\|\theta(\tau)\|_{m+1}^{m+1}, \tag{2.17}$$

satisfying the monotonicity condition

$$\frac{d}{d\tau}\,V(\theta)(\tau) = -\frac{N^2-4}{N^2}\,\|(\theta^{N/(N+2)})_\tau(\tau)\|_2^2 \le 0. \tag{2.18}$$

From this, we easily proof the following result.

Proposition 2.5. *$\alpha(\tau)$ is unbounded as $\tau \to \infty$.*

Proof. Indeed, if $\alpha(\tau)$ were uniformly bounded, this would mean that the regular orbits $\{\theta(x,\tau),\ \tau > \tau_0\}$ would stabilize as $\tau \to \infty$ to a classical stationary solution $S(x)$ of the elliptic problem (1.15). Then S is a non-trivial solution by Proposition 2.4. This contradicts the non-existence result for the critical case. □

Proposition 2.6. *There holds*

$$\alpha'(\tau) > 0 \quad for\ \ \tau \gg 1. \tag{2.19}$$

Proof. It follows from the properties (2.3)-(2.5) that

$$J(0,\lambda) = 2 \quad \text{for all}\ \ \lambda \gg 1.$$

Fix a $\lambda \gg 1$. By continuity, there exists a $\tau_\lambda > \tau_0$ such that $\alpha(\tau_\lambda) = \lambda$. Therefore, by the Strong Maximum Principle, and using a standard geometric argument, at $\tau = \tau_\lambda$ the minimal intersection disappears, so that $J(\tau_\lambda,\lambda) = 1$ and $\theta(r,\tau_\lambda) > S_\lambda(r)$ for small $r > 0$, whence $\alpha'(\tau_\lambda) \ge 0$ by the standard local comparison with the stationary solution S_λ. Since $\lambda \gg 1$ is otherwise arbitrary, we conclude that $\alpha'(\tau) \ge 0$ for all $\tau \gg 1$.

Let us now prove the strict inequality (2.19). The derivative $z = \theta_\tau$ satisfies a linear non-degenerate parabolic equation

$$z_\tau = \Delta(m\theta^{m-1}z) + (N+2)z/4. \tag{2.20}$$

Moreover, in a neighbourhood of the origin, equations (2.10) and (2.20) have analytic coefficients, so that their solutions are analytic in x. Then the number of zeros of $z(r,\tau)$ is finite for any $\tau > \tau_0$ (in fact, the analyticity assumption is not necessary, see references in Chapter 4 of [15]). By the Strong Maximum Principle this implies that $z > 0$ for small $r \ge 0$. □

From Propositions 2.5 and 2.6 we deduce that

$$\alpha(\tau) \to \infty \quad \text{as}\ \ \tau \to \infty. \tag{2.21}$$

2.3. **Further rescaling.** In view of (2.21), using the invariance properties of equation (2.10), we introduce a new rescaled variable $f(\xi, \tau)$ defined by

$$\theta(r, \tau) = \alpha(\tau) f(\xi, \tau), \quad \xi = r\, \alpha^{2/(N+2)}(\tau), \tag{2.22}$$

where the asymptotic behaviour of the monotone increasing smooth function $\alpha(\tau)$ remains to be determined. The function f solves the non-autonomous quasilinear equation

$$f_\tau = \mathbf{B}(f) + g(\tau)\, \mathbf{C} f, \quad g(\tau) = \alpha'(\tau)/\alpha(\tau), \tag{2.23}$$

where $\mathbf{B}$ is a nonlinear operator

$$\mathbf{B}(f) = \Delta f^m + (N+2) f/4, \tag{2.24}$$

and $\mathbf{C}$ is a linear first-order operator

$$\mathbf{C} f = -2 f_\xi \xi/(N+2) - f. \tag{2.25}$$

The boundary condition is transformed into

$$f = 0 \quad \text{at } \xi = \xi_*(\tau) \equiv \alpha^{2/(N+2)}(\tau), \;\; \tau > \tau_0. \tag{2.26}$$

By the definition of the function $\alpha(\tau)$, (2.13), there holds

$$f(0, \tau) \equiv 1, \quad \tau > \tau_0. \tag{2.27}$$

The stationary profiles are invariant under scaling (2.22), so that Proposition 2.6 implies the following condition.

Proposition 2.7. *For any fixed $\tau \gg 1$, the profile $g(\xi, \tau)$ intersect $S_1(\xi)$ exactly once and $f(\xi, \tau) > S_1(\xi)$ for small $\xi > 0$, $f(\xi, \tau) < S_1(\xi)$ for $\xi \gg 1$.*

Equation (2.23) is a perturbed (by $O(g(\tau))$) autonomous equation

$$f_s = \mathbf{B}(f) \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+, \;\; f(0, s) \equiv 1. \tag{2.28}$$

Proposition 2.8. *The perturbation in (2.23) is uniformly bounded:*

$$0 < g(\tau) \le (N+2)/4, \quad \tau > \tau_0. \tag{2.29}$$

Proof. Setting $x = 0$ in (2.10) and using that $\Delta \theta^m \le 0$ at any maximum point in ξ, we get $\theta_\tau(0, \tau) \equiv \alpha'(\tau) \le (N+2)\alpha(\tau)/4$, whence follows the result. □

Differentiating equation (2.10) and using Bernstein estimates on θ and on the derivative θ_τ, we have

$$|\alpha''(\tau)| \le \text{const}[\alpha(\tau) + \alpha'(\tau)] \le \text{const}\, \alpha(\tau).$$

If particular, this means that for $\tau \gg 1$

$$|g'(\tau)| \equiv |\alpha''/\alpha - (\alpha'/\alpha)^2\,| \le \text{const}. \tag{2.30}$$

Proposition 2.9. *The perturbation is asymptotically small:*

$$g(\tau) \to 0 \quad \textit{as } \tau \to \infty. \tag{2.31}$$

This means that as $\tau \to \infty$, $\alpha(\tau)$ is a slowly varying function relative to any exponential one $e^{\mu\tau}$, $\mu \neq 0$. By the definition, smooth exponential-like functions $\alpha(\tau) = e^{\mu\tau}$ satisfy $\alpha'(\tau)/\alpha(\tau) \to \mu \neq 0$, $\tau \to \infty$ instead of (2.31). We always assume that $g \notin L^1((1,\infty))$ is a non-integrable slowly (non-exponentially fast) decaying function as $\tau \to \infty$. Actually, in Section 4 we shall prove that

$$g(\tau) \sim 1/\tau \quad \text{for } \tau \gg 1,$$

a typical order for perturbations in perturbed dynamical systems arising in applications to blow-up and extinction phenomena for quasilinear reaction-diffusion equations, see e.g. [6]. It satisfies the above hypotheses.

Proof. Assume that (2.31) is not true and that there exists a monotone sequence $\{\tau_k\} \to \infty$ such that $g(\tau_k) \to \varepsilon > 0$. Using a standard interior regularity for uniformly parabolic equations and the estimate (2.30), we conclude that $f(\xi, \tau_k + s) \to h(\xi, s)$ in $L^\infty_{\rm loc}(\mathbf{R}_+; H^2_{\rm loc}(\mathbf{R}_+))$, where $h \geq 0$ solves the equation

$$h_s = \mathbf{B}(h) + \varepsilon \mathbf{C} h \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+ \tag{2.32}$$

and

$$h(0, s) \equiv 1, \quad s \geq 0. \tag{2.33}$$

By the Strong Maximum Principle for autonomous uniformly parabolic equations, (2.33) implies that h is a stationary solution with $h(\xi) \geq 0$ and

$$\mathbf{B}(h) + \varepsilon \mathbf{C} h = 0, \quad \xi > 0; \quad h(0) = 1; \tag{2.34}$$

see Section 3.2 in [3]. Let us rewrite (2.34) in the form

$$\Delta h^m - 2\varepsilon h'\xi/(N+2) + [-\varepsilon + 4/(N+2)]h = 0, \quad \xi > 0, \quad h(0) = 1, \;\; h'(0) = 0. \tag{2.35}$$

We thus have a contradiction with the non-existence of such a profile $h \geq 0$ in $\mathbf{R}_+$. □

Proposition 2.10. *For any $\varepsilon > 0$, the solution $h(\xi)$ of (2.35) vanishes at a finite point and cannot be extended to $\mathbf{R}_+$.*

Proof. Nonexistence follows from Pohozaev's identity for the elliptic operator $\Delta V + cV^{1/m}$ in $\mathbf{R}^N$. Observe that the additional first-order term $-2\varepsilon h'\xi/(N+2)$ in (2.35) is strictly positive for nontrivial monotone solutions, so that the same technique applies, see e.g. p. 266 in [15].

□

We thus obtain the first asymptotic result in the inner region.

Theorem 2.1. *There holds*

$$f(\xi, \tau) \to S_1(\xi) \quad \textit{as } \tau \to \infty \textit{ uniformly.} \tag{2.36}$$

Next, we conclude that

$$g'(\tau) \to 0, \quad \tau \to \infty. \tag{2.37}$$

Indeed, by standard interior regularity results for uniformly parabolic equations, convergence (2.36) implies that the time derivatives satisfy

$$f_\tau,\ f_{\xi\tau},\ f_{\tau\tau}, ... \to 0, \quad \tau \to \infty.$$

Since $z = f_\tau$ solves the parabolic equation

$$z_\tau = m\Delta(f^{m-1}z) + (N+2)z/4 + g(\tau)\mathbf{C}z + g'(\tau)\mathbf{C}f$$

and the solution $f(\cdot,\tau) \to S_1(\cdot)$, together with its spatial derivatives, so passing to the limit $\tau \to \infty$ in the above equation one deduces that

$$g'(\tau)\mathbf{C}S_1 = o(1),$$

whence (2.37) follows since $\mathbf{C}S_1 \not\equiv 0$.

2.4. **Linearized rescaled equation.** In order to apply our matching procedure, we need to derive the rate of convergence in (2.36). Setting in (2.23)

$$f(\xi,\tau) = S_1(\xi) + S_1^{1-m}(\xi)Y(\xi,\tau) \quad (Y(0,\tau) \equiv 0), \tag{2.38}$$

we obtain the equation

$$Y_\tau = -\mathbf{A}Y + g(\tau)p + g(\tau)\tilde{\mathbf{C}}Y + \mathbf{D}(Y), \tag{2.39}$$

where $\mathbf{A}$ and $\tilde{\mathbf{C}}$ are the linear operators:

$$\mathbf{A}Y = -mS_1^{m-1}\Delta Y - (N+2)Y/4, \quad \tilde{\mathbf{C}}Y = S_1^{m-1}\mathbf{C}(S_1^{1-m}Y), \tag{2.40}$$

$p(\xi)$ is the function

$$p = S_1^{m-1}\mathbf{C}S_1 \sim \xi^{-(N-2)}, \quad \xi \to \infty,$$

and $\mathbf{D}$ is a nonlinear one

$$\mathbf{D}(Y) = S_1^{m-1}[\Delta(S_1 + S_1^{1-m}Y)^m - \Delta S_1^m - m\Delta Y]. \tag{2.41}$$

It is a quadratic perturbation (locally in ξ) on smooth small symmetric functions $Y \in C^2$:

$$\mathbf{D}(Y) = \frac{m(m-1)}{2}S_1^{m-1}\Delta(S_1^{-m}Y^2) + O(\|Y\|^3_{H^2_\rho}) \quad \text{as } Y \to 0. \tag{2.42}$$

By the intersection comparison analysis (see the proof of Proposition 2.6), we have that for $\tau \gg 1$ the function $Y(\xi,\tau)$ has a unique zero for $\xi > 0$ and $Y(\xi,\tau) > 0$ for sufficiently small $\xi > 0$ and $Y(\xi,\tau) < 0$ near the boundary point $\xi = \xi_*(\tau) \equiv \alpha^{2/(N+2)}(\tau)$.

2.5. **Properties of the linear operator.** The linear operator $\mathbf{B} = \mathbf{A} + \frac{N+2}{4}$ is symmetric and semibounded, $\mathbf{B} \geq 0$. Therefore there exists a unique Friedrichs extension [2] denoted again by $\mathbf{B}$ which is a self-adjoint semibounded operator in the weighted space $L^2_\rho(\mathbf{R}_+)$ of symmetric functions with the inner product $(\cdot,\cdot)_\rho$ and the weight function

$$\rho(\xi) = m^{-1}\xi^{N-1}S_1^{1-m}(\xi) > 0, \quad \xi > 0.$$

The domain of $\mathcal{D}(\mathbf{B})$ consists of symmetric functions $w \in L^2_\rho$ satisfying $\mathbf{B}w \in L^2_\rho$, $w'(0) = 0$. $\mathbf{B}$ satisfies the same bound $(\mathbf{B}Y, Y)_\rho \geq 0$ in $\mathcal{D}(\mathbf{B})$. Consider the singular point $\xi = \infty$ of the Sturm-Liouville operator $\mathbf{B}$.

Proposition 2.11. *For $N \geq 4$, the point $\xi = \infty$ is in the limit-point case for* $\mathbf{B}$.

Proof. We have
$$S_1^{1-m}(\xi) \sim \xi^{-4}, \quad \xi \gg 1.$$
Hence,
$$\rho(\xi) \sim \xi^{N-5} \quad \text{as } \xi \to \infty,$$
so that for $\xi \gg 1$ the operator $\mathbf{B}$ is asymptotically equivalent to
$$\mathbf{B}Y \sim -\xi^{5-N}(\xi^{N-1}Y')'.$$
Solving the homogeneous equation $\mathbf{B}Y = 0$ as $\xi \to \infty$, we obtain two linearly independent solutions $Y_1 \sim 1, \quad Y_2 \sim \xi^{2-N}$. Since for $N \geq 4$, $Y_1 \notin L^2_\rho((1,\infty))$ and $Y_2 \in L^2_\rho((1,\infty))$, we get the result, see [10].

□

Therefore, for $N \geq 4$, $\mathbf{A}$ has a purely discrete spectrum consisting of an increasing sequence of simple eigenvalues $\sigma(\mathbf{A}) = \{\lambda_k,\ k = 0, 1, ...\}$. The corresponding eigenfunctions $\{\psi_k(\xi)\}$ form an orthonormal basis in $L^2_\rho(\mathbf{R}_+)$.

Proposition 2.12. *For $N \geq 4$ the spectrum $\sigma(\mathbf{A})$ in $L^2_\rho(\mathbf{R}_+)$ is discrete. If $N = 3$ then the same is true in the space of functions satisfying the additional condition*

$$Y(\xi) \to 0, \quad \xi \to \infty. \tag{2.43}$$

Proof. For $N = 3$ this extra condition excludes the behaviour $Y_1(\xi)$ as $\xi \to \infty$, thus making the spectrum discrete.

□

Proposition 2.13. *There holds*

$$\lambda_0 > \lambda_* = (N+2)/4. \tag{2.44}$$

Proof. Obviously, λ_* is not an eigenvalue: $\mathbf{A}Y = -\lambda_* Y$ implies $Y(\xi) = 1 \notin L^2_\rho(\mathbf{R}_+)$. The function $Y \equiv 1$ does not change sign in $\mathbf{R}_+$, whence $\lambda_0 > \lambda_*$ by the Sturm Theorem.

□

2.6. **The asymptotic behaviour.** We first extend our rescaled initial-boundary value problem to the Cauchy problem for equation (2.10) in $\mathbf{R}_+ \times \mathbf{R}_+$. Consider the equivalent (parabolic) equation extended to the negative values $\theta < 0$ by replacing the fast diffusion operator by $\Delta(|\theta|^{m-1}\theta)$. We then define the solution for $r \in [1, 3/2]$ by the smooth odd reflection with respect to $r = 1$: $\theta(r,\tau) = -\theta(2-r,\tau)$. We put $\theta(r,\tau) = -\theta(1/2,\tau) \to 0$, $\tau \to \infty$ for $r \in (2,3)$ and perform a uniformly smooth transition on $(3/2, 2)$. Finally, we put $\theta(\tau, r) = -\theta(1/2,\tau)v(r)$ for $r \geq 3$ where $v(r) > 0$, $v(3) = 1$, is a smooth monotone decreasing function with exponential decay at infinity. As a result, after rescaling, we obtain a global smooth solution f of the Cauchy problem in $\mathbf{R}_+ \times \mathbf{R}_+$ for an equation which coincides with the original one (2.23) in the domain $\{\xi < \xi_*(\tau) = \alpha^{2/(N+2)}(\tau),\ \tau > 0\}$. In the complement, we put a perturbation into the equation which is uniformly bounded and moreover uniformly vanishing on any domain with $\xi \gg 1$, $\tau \to \infty$. This extension does not affect the asymptotic analysis we present below and we omit this negligible perturbation term in such remote domains of $\mathbf{R}_+ \times \mathbf{R}_+$.

By construction of the extended solution f, we have that

$$|Y| \le \mathrm{const}\, \xi^{2-N} \quad \text{for } \xi \gg \xi_*(\tau).$$

By Theorem 2.1, it follows from (2.38) that, uniformly on $(0, \xi_*(\tau))$,

$$|Y| \le M(\tau)(1+\xi)^4, \quad M(\tau) \to 0 \quad \text{as } \tau \to \infty.$$

Similar estimates are available for the derivatives Y_ξ, $Y_{\xi\xi}$. Consider local properties of the solution. For $\tau \gg 1$, equation (2.39) is uniformly parabolic on any compact subset $\xi \in K = [0, \xi_K]$. By the standard regularity theory for uniformly parabolic equations, we have that for uniformly small solutions $Y(\tau)$ with $\tau \gg 1$ on K

$$|Y_\xi(\tau)|, \ |Y_{\xi\xi}(\tau)| \le \mathrm{const}\, \|Y(\tau)\|_\infty,$$

so that the linear perturbation term and the quadratic one in equation (2.39) satisfy on K

$$\tilde{\mathbf{C}} Y(\tau) = O(\|Y(\tau)\|_\infty), \quad \mathbf{D}(Y(\tau)) = O(\|Y(\tau)\|_\infty^2).$$

Then $Y(\tau) \to 0$ as $\tau \to \infty$ satisfies

$$Y_\tau = -\mathbf{A}Y + g(\tau)p + O(g(\tau)\|Y(\tau)\|_\infty + \|Y(\tau)\|_\infty^2). \tag{2.45}$$

We first consider the *linearized non-local problem.* In view of an extra boundary condition at the origin: $Y = 0$ at $\xi = 0$ (and $Y_\xi = 0$ for smooth symmetric functions), the corresponding linearized inhomogeneous equation (2.39) without the higher-order linear and quadratic terms is a differential-functional equation with a nonlocal term defined via the localized measure $\delta(\xi)$. Namely, setting

$$L(\tau)Y = -\mathbf{A}Y + g(\tau)p,$$

one may write it down as follows

$$Y_\tau = L(\tau)Y - L(\tau)Y|_{\xi=0}, \quad \tau > 0, \tag{2.46}$$

so that $Y_\tau \equiv 0$ at $\xi = 0$ and $Y(\tau)|_{\xi=0} \equiv Y(0)|_{\xi=0} = 0$. Equation (2.46) is a linear inhomogeneous parabolic equation $Y_\tau = L(\tau)Y$ restricted to the subspace

$$W_0 = H^2_\rho(\mathbf{R}_+) \cap \{Y|_{\xi=0} = 0\}.$$

Let $p = \sum_{(k)} \gamma_k \psi_k$ be the eigenfunctions expansion of the function $p \in L^2_\rho(\mathbf{R}_+)$. The eigenfunctions expansion of the solution on W_0,

$$Y(\tau) = \sum_{(k)} a_k(\tau)\psi_k,$$

leads to the following non-diagonal system governing the expansion coefficients:

$$a_k'(\tau) = \sum_{(j)} [\beta_j \psi_j(0) - \delta_{jk}]\lambda_j a_j(\tau) + g(\tau)[\gamma_k - \beta_k p(0)], \ \ k = 0, 1, 2, \dots, \tag{2.47}$$

where we denote $\beta_j = (1, \psi_j)_\rho$. Observe that $\psi_j(\xi) \sim \xi^{2-N}$, $\xi \to \infty$, so that $\psi_j \in L^1_\rho(\mathbf{R}_+)$. By construction, the system admits the first integral

$$\sum a_k(\tau)\psi_k(0) = 0, \quad \tau > 0.$$

This system does not have a typical diagonal structure and the analysis becomes more delicate than for the standard systems occurring for local self-adjoint differential operators. Nevertheless, it can be shown that the slow (non-exponential) term of the order $g(\tau)$ in (2.47) becomes dominant for $\tau \gg 1$. We prove that for the nonlinear problem.

Consider equation (2.45). We have $p \in L^2_\rho(\mathbf{R}_+)$ and the main perturbation has the order $g(\tau)$, which is slowly decaying and non-integrable as $\tau \to \infty$. Estimating the perturbations in $L^2_\rho(\mathbf{R}_+)$ or $H^2_\rho(\mathbf{R}_+)$ and multiplying equation by Y in $L^2_\rho(\mathbf{R}_+)$, we obtain that

$$\frac{d}{d\tau}\|Y(\tau)\|_{2,\rho} = O(g(\tau)), \;\; \tau \gg 1.$$

Therefore, $Y(\tau) = O(g(\tau))$ for $\tau \gg 1$ in $L^2_\rho(\mathbf{R}_+)$ and hence in $H^2_\rho(\mathbf{R}_+)$ and locally pointwise. This suggests the change of the dependent variable

$$Y(\xi,\tau) = g(\tau)W(\xi,\tau),$$

where W is uniformly bounded on compact subsets in ξ. For $\tau \gg 1$, the equation for W takes the asymptotic form

$$W_\tau = -\mathbf{A}W + p + q(\tau)W + g(\tau)O(\|W\|_\infty + \|W\|^2_\infty), \tag{2.48}$$

with the perturbation being of the order

$$q(\tau) = g'(\tau)/g(\tau) \equiv (\alpha''/\alpha' - \alpha'/\alpha)(\tau),$$

and with $W = W_\xi \equiv 0$ at the origin $\xi = 0$.

Let $\omega(W_0)$ be the omega-limit set, defined in the uniform metric on compact subsets, of a bounded orbit $\{W(\tau),\, \tau > \tau_0\}$ with initial datum W_0. Passing to the limit in (2.48) along a sequence $\{\tau_k\} \to \infty$ such that $W(\tau_k) \to f \in \omega(W_0)$ and assuming that

$$q(\tau) \to 0, \quad \tau \to \infty$$

(this is true for slowly decaying functions as in Proposition 2.9 but is a new assumption to be confirmed later on), we have that $W(\tau_k + s) \to h(s)$ in $L^\infty_{\rm loc}(\mathbf{R}_+; H^2_{\rm loc}(\mathbf{R}))$, where $h(s)$ is a smooth solution of the linear non-homogeneous parabolic equation

$$h_s = -\mathbf{A}h + p, \quad \xi > 0, \; s > 0,$$

with the initial datum $h_0 \in \omega(W_0)$. Since by construction

$$h = h_\xi = 0 \;\text{ at } \xi = 0 \quad \text{for any } s \ge 0,$$

by the Sturmian argument for uniformly parabolic equations (see a similar analysis on p. 673 in [3]), we conclude that $h(s)$ is a stationary solution:

$$h(\xi,s) \equiv R(\xi): \quad -\mathbf{A}R + p = 0, \;\; \xi > 0, \quad R(0) = R'(0) = 0. \tag{2.49}$$

Indeed, if on the other hand $h(\xi, s_1) \not\equiv R(\xi)$ for some $s_1 > 0$, then by the Strong Maximum Principle $h(\xi, s_1) \ne R(\xi)$ a.e. near the origin $\xi = 0$ and hence $h(0,s) \ne 0 = R(0)$ a.e. in

$s > s_1$ contradicting the definition of h. It is easy to see (and this also follows from the intersection comparison results given above) that $R(\xi)$ has a unique zero for $\xi > 0$, $R(\xi)$ is positive for small $\xi > 0$ and is negative for $\xi \gg 1$.

We thus obtain the following asymptotic behaviour:

$$Y(\xi,\tau) = g(\tau)(R(\xi) + o(1)), \quad \tau \to \infty. \tag{2.50}$$

Since $\lambda = 0 \notin \sigma(\mathbf{A})$, the stationary equation

$$\mathbf{A}R = p \equiv \sum \gamma_k \psi_k \in L^2_\rho(\mathbf{R}_+)$$

admits a unique solution

$$R_* = \sum (\gamma_k/\lambda_k)\psi_k \in L^2_\rho(\mathbf{R}_+),$$

and by the above asymptotic results $R_*(\xi) \sim \xi^{2-N}, \quad \xi \to \infty$. We then decompose R as follows

$$R = R_* + R_0, \tag{2.51}$$

where $R_* \in L^2_\rho(\mathbf{R}_+)$ is the solution of the inhomogeneous equation and R_0 is a solution of the homogeneous equation

$$\mathbf{A}R_0 = 0, \quad R_0(0) = -R_*(0) \neq 0, \;\; R_0'(0) = 0.$$

Obviously, $R_0 \notin L^2_\rho(\mathbf{R}_+)$ and by the properties of the singular point $\xi = \infty$

$$R_0(\xi) = -c_*(1 + o(1)), \quad \xi \to \infty \quad (c_* > 0), \tag{2.52}$$

so that R_0 belongs to the *extended* ($\not\subset H^2_\rho(\mathbf{R}_+)$) *centre subspace* of the linear operator $\mathbf{A}$ (the extra term R_* in the decomposition (2.51) means that the invariant subspace is affine one since the main perturbation in the equation is non-autonomous). Let us note that in the extended weighted space, $\lambda = 0$ belongs to the spectrum of $\mathbf{A}$ but, indeed, the spectrum is continuous ($\lambda \in \mathbf{R}$ is arbitrary). Such an asymptotic behaviour is a stable slow motion on the extended affine centre manifold of the nonlinear operator in equation (2.39).

Let us present the final inner expansion of the solution. It follows from (2.2), (2.50) and (2.38), (2.52) that for large ξ as $\tau \to \infty$

$$f(\xi,\tau) = c_1\xi^{-(N+2)} - c_2\xi^{-(N+4)} - c_3 g(\tau)\xi^{-4} + \dots, \tag{2.53}$$

where $c_1,\; c_2, \dots$ denote positive constants depending on dimension N only (c_1 and c_2 are prescribed in (2.2)). Using now the rescaled variables (1.8), (2.22), we derive the inner expansion

$$\begin{aligned} u(r,t) &= (T-t)^{(N+2)/4}\alpha(\tau)[S_1(r\alpha^{2/(N+2)}(\tau)) + g(\tau)(S_1^{1-m}R)(r\alpha^{2/(N+2)}(\tau)) + \dots] \\ &= (T-t)^{(N+2)/4}\alpha[c_4\alpha^{-2}r^{-(N+2)} - c_5\alpha^{-2(N+4)/(N+2)}r^{-(N+4)} - c_6 g\alpha^{-8/(N+2)}r^{-4} + \dots]. \end{aligned}$$

Since $g = \alpha'/\alpha$, we finally obtain that in the inner region

$$u(r,t) = (T-t)^{(N+2)/4}\left[\alpha^{-1}\frac{c_4}{r^{N+2}} - \alpha^{-(N+6)/(N+2)}\frac{c_5}{r^{N+4}} - \alpha^{-8/(N+2)}\alpha'\frac{c_6}{r^4} + \dots\right]. \tag{2.54}$$

3. **Outer expansion (boundary layer)**

The analysis in the outer (boundary) domain is easier and on the other hand exhibits certain similarities. In the domain away from the origin $r = |x| = 0$, we observe a standard asymptotic behaviour. The rescaled solution satisfies uniformly

$$u(r,t) \to 0, \quad t \to T^-. \tag{3.1}$$

In order to get a further information about the asymptotic behaviour, we perform a natural rescaling of the separable type

$$u(r,t) = A(t)v(r,t),$$

where the function $A(t) \to 0$, as $t \to T^-$, and is chosen so that v is uniformly bounded on intervals $r \in [\delta, 1]$, $\delta > 0$ small, and is uniformly bounded away from zero on $[\delta, 1-\delta]$. It is convenient to choose $A(t)$ as follows

$$A^m(t) = -(u^m)_r(1,t) > 0. \tag{3.2}$$

Then $v(r,t)$ satisfies the boundary conditions

$$v = 0, \;\; (v^m)_r = -1 \quad \text{at } \; r = 1, \; t \in (0,T). \tag{3.3}$$

Substituting into the original equation, we obtain

$$v_s = \Delta v^m + p(s)v,$$

where the order of the linear perturbation is given by

$$p(s) = -A^{-m}(t)A'(t) > 0,$$

and s is the new time variable determined from the ODE $ds = A^{m-1}(t)dt$. We suppose that $s \to \infty$ as $t \to T^-$ (a condition to be checked). Assuming that the perturbation is asymptotically small: $p(s) \to 0$ as $s \to \infty$, and passing to the limit in the equation, one concludes that the limit-time profile,

$$v(r,s) \to \phi_0(r), \quad s \to \infty,$$

is indeed the stationary solution:

$$\Delta\phi_0^m \equiv \frac{1}{r^{N-1}}(r^{N-1}(\phi_0^m)')' = 0, \;\; r \in (0,1); \quad \phi_0(1) = 0, \;\; (\phi_0^m)'(1) = -1, \tag{3.4}$$

so that we put

$$\phi_0(r) = \big[(N-2)^{-1}\big(\, r^{2-N} - 1 \,\big)\big]^{(N+2)/(N-2)} = \frac{c_7}{r^{N+2}} - \frac{c_8}{r^4} + ..., \;\; r \to 0. \tag{3.5}$$

The rest of the outer asymptotic analysis is similar to that of the inner one. Setting

$$v = \phi_0 + \phi_0^{1-m}W,$$

we obtain the perturbed equation

$$W_s = -\mathbf{B}W + p(s)\phi_0^m + p(s)W + \mathbf{D}(W), \tag{3.6}$$

where $\mathbf{B} > 0$ is the Laplacian operator and in the space $L^2_\rho((0,1))$ of symmetric functions vanishing on the boundary $r = 1$, it takes the form:

$$\mathbf{B} = -\frac{1}{\rho}\frac{d}{dr}\left(r^{N-1}\frac{d}{dr}\right); \quad \rho = m^{-1}r^{N-1}\phi_0^{1-m}(r) \sim r^{N-5}, \;\; r \to 0.$$

The nonlinear perturbation is locally quadratic:

$$\mathbf{D}(W) = \frac{m(m-1)}{2}\phi_0^{m-1}\Delta(\phi_0^{-m}W^2) + O(\|W\|^3_{H^2_\rho}).$$

Recall that $W = W_r = 0$ at $r = 1$ by construction. In both inner and outer domains the flow is governed by the extended affine centre manifolds with the centre subspaces generated by the same linearized operators defined and self-adjoint in similar weighted spaces. Namely, $r = 0$ is in the limit-point case for $\mathbf{B}$ if $N \geq 5$. The same is always true for $r = 1$ ($\rho(1) = 0$). For $N = 3$ and 4 an extra condition at $r = 0$ is necessary (coming from the inner analysis) to guarantee that $\sigma(\mathbf{B})$ is discrete in $L^2_\rho((0,1))$. The analysis on the affine extended centre manifold is similar and gives the behaviour

$$W(r,s) = p(s)(R(r) + o(1)), \quad s \to \infty,$$

where R solves the rescaled stationary equation

$$-\mathbf{B}R + \phi_0^m = 0$$

with the conditions $R(1) = R'(1) = 0$. Observe that as $r \to 0$, $\phi_0^m(r) \sim r^{2-N} \notin L^2_\rho((0,1))$ so that $R(r) \sim r^{-N}$. The linearized inhomogeneous problem is again a non-local one equivalent to a linear parabolic flow on a subspace of co-dimension 1 in $H^2_\rho(\mathbf{R}_+)$.

Finally, we obtain that the boundary behaviour has the following form: as $s \to \infty$ uniformly on any interval $[\delta, 1]$ there holds

$$u(r,t) = A(t)[\phi_0(r) + \phi_0^{1-m}(r)p(s)(R(r) + o(1))].$$

Using the above asymptotic development of functions ϕ_0 and R, we deduce that for $r > 0$ sufficiently small, the outer solution is described by the expansion

$$u(r,t) = A(t)\left(\frac{c_7}{r^{N+2}} - \frac{c_8}{r^4} - p(s(t))\frac{c_9}{r^{N+4}} + \ldots\right). \tag{3.7}$$

4. **Matching analysis in the intermediate region**

4.1. **Extension to intermediate region.** We now extend the inner expansion (2.54) in order to match it with the outer one (3.7). In order to obtain the scaling transformation corresponding to the intermediate region, we formally assume that the second term in the inner expansion (2.53) is negligible in comparison with two others, so that

$$f(\xi,\tau) = c_1\xi^{-(N+2)} - c_3 g(\tau)\xi^{-4} + \ldots \equiv \xi^{-(N+2)}(c_1 - c_3 g(\tau)\xi^{N-2} + \ldots). \tag{4.1}$$

This suggests the change of the spatial variable

$$\zeta = \xi g^{1/(N-2)}(\tau), \tag{4.2}$$

and then the inner expansion (2.53) takes the form

$$f(\xi,\tau)=g^{(N+2)/(N-2)}\zeta^{-(N+2)}(c_1-c_2\zeta^{-2}g^{2/(N-2)}-c_3\zeta^{N-2}+...). \tag{4.3}$$

Observe that the second term is negligible if $\zeta^{-2}g^{2/(N-2)}(\tau)\ll 1$ or if $\zeta \gg g^{1/(N-2)}(\tau)$, $\tau\to\infty$. Thus

$$f(\xi,\tau)=g^{(N+2)/(N-2)}\zeta^{-(N+2)}(c_1-c_3\zeta^{N-2}+...) \quad \text{in } \{g^{1/(N-2)}\ll\zeta\ll 1\}. \tag{4.4}$$

According to this structure, we introduce the rescaled solution F given by

$$f(\xi,\tau)=g^{(N+2)/(N-2)}(\tau)F(\zeta,\tau). \tag{4.5}$$

This describes the asymptotic behaviour in the *intermediate region* determined in terms of sufficiently small compact subsets in ζ bounded away from zero, where F solves the fast diffusion equation with the three perturbation terms:

$$F_s=\Delta_\zeta F^m+\frac{N+2}{4}g^{2/(N-2)}F-(g^{2/(N-2)})'\frac{1}{2}[(N+2)F-F_\zeta\zeta]+g^{N/(N-2)}\mathbf{C}F, \tag{4.6}$$

where the new time $s(\tau)\to\infty$ as $\tau\to\infty$ is given by the ODE

$$g^{-2/(N-2)}(\tau)d\tau=ds. \tag{4.7}$$

The inner expansion then provides us with the following expansion near the origin:

$$F(\zeta,\tau)=\zeta^{-(N+2)}(c_1-c_3\zeta^{N-2}+...) \quad \text{in } \{g^{1/(N-2)}\ll\zeta\ll 1\}, \tag{4.8}$$

so that F is bounded for small ζ uniformly in $\tau\gg 1$. By monotonicity in ζ, the orbit $\{F(\cdot,\tau),\,\tau\gg 1\}$ is uniformly bounded on any compact subset $\{0<\delta_1\le\zeta\le\delta_2\}$.

The asymptotic analysis of this equation is similar to that in Section 2. We see that the second and the third perturbations on the right-hand side are asymptotically small by Theorem 2.1. On the other hand, the second perturbation of the order $(g^{2/(N-2)}(\tau))'$ is small in the weak sense. By the standard parabolic regularity theory, it follows from (4.6) that any bounded smooth orbit satisfies

$$F(\zeta,s_k+s)\to\bar F(\zeta,s) \quad \text{in } L^\infty_{\rm loc}(\mathbf{R}_+;\, C_{\rm loc}(\mathbf{R}_+)) \tag{4.9}$$

along a subsequence $\{s_k\}\to\infty$, where $\bar F$ is a weak and hence a classical solution of the unperturbed equation

$$\bar F_s=\Delta_\zeta\bar F^m, \tag{4.10}$$

with initial data $\bar F_0$ from the ω-limit set of the orbit $\{F(s),s>0\}$ satisfying the stationary condition near the origin:

$$\bar F_0(\zeta)=\zeta^{-(N+2)}(c_1-c_3\zeta^{N-2}+...). \tag{4.11}$$

Using a standard Lyapunov technique for equation (4.10), which is of divergence form, we conclude that as $s\to\infty$, $F(\zeta,s)\to F_0(\zeta)$, where F_0 is the unique stationary solution:

$$\Delta_\zeta F_0^m=0, \tag{4.12}$$

with the same behaviour as $\zeta \to 0$. Thus, solving (4.12), we obtain

$$F_0(\zeta) = (c_{10}\zeta^{2-N} - c_{11})^{(N+2)/(N-2)} = c_1\zeta^{-(N+2)} - c_3\zeta^{-4} + ... \tag{4.13}$$

for small $\zeta > 0$. Moreover, using standard contractivity properties of flows induced by the diffusion equation (4.10), we have that the ω-limit set of the problem (4.10), (4.11) is uniformly stable in the L^1-metric and hence the ω-limit set of the bounded orbit $\{F(\tau)\}$ consists of the unique profile F_0; see Theorem 3 in [6] where other applications of such stability techniques are presented. In order to derive the rate of convergence on subsets $\{\delta_1 \le \zeta \le \delta_2\}$, we set, as usual,

$$F = F_0 + F_0^{1-m} Z,$$

to get the perturbed equation for the function Z

$$Z_s = mF_0^{m-1}\Delta Z + g^{2/(N-2)}(\tau)(N+2)F_0^m/4 + ... , \tag{4.14}$$

which is similar to (3.6) in the boundary layer. We omit the higher-order terms and keep the first perturbation, which will be shown to be the dominant one. The operator $\mathbf{B} = mF_0^{m-1}\Delta$ is self-adjoint in $L^2_\rho((0,\zeta_*))$, where $\rho = \zeta^{N-1}F_0^{1-m}/m \sim \zeta^{N-5}$, $\zeta \to 0$, and ζ_* is the positive root of the equation $F_0(\zeta) = 0$. Observe that $\rho(\zeta) = 0$ at $\zeta = \zeta_*$ (it is in the limit-point case). The behaviour on the extended affine centre manifold is then given by the standard expression

$$Z(\zeta, s) = g^{2/(N-2)}(\tau)Q(\zeta) + ...,$$

where Q solves the stationary inhomogeneous equation

$$mF_0^{m-1}\Delta Q + (N+2)F_0^m/4 = 0.$$

We have $F_0^m(\zeta) \sim \zeta^{2-N} \notin L^2_\rho((0,\zeta_*))$ for small $\zeta > 0$ and also $Q(\zeta) \sim \zeta^{-N} \notin L^2_\rho((0,\zeta_*))$. Using the above expansions, we arrive at the following asymptotic behaviour: for sufficiently small $\zeta > 0$ as $s \to \infty$

$$F(\zeta, s) = F_0 + F_0^{1-m}g^{2/(N-2)}Q + ... = c_1\zeta^{-(N+2)} - c_3\zeta^{-4} - c_{12}g^{2/(N-2)}\zeta^{-(N+4)} + ...$$

We thus obtain that two terms from the inner expansion (2.54) extend into the intermediate region, whereby

$$f(\xi, \tau) = c_1\xi^{-(N+2)} - c_3 g(\tau)\xi^{-4} + ... \tag{4.15}$$

Therefore in the intermediate region

$$u(r,t) = (T-t)^{(N+2)/4}\left[\alpha^{-1}\frac{c_{13}}{r^{N+2}} - \alpha^{-8/(N+2)}\alpha'\frac{c_{14}}{r^4} + ...\right]. \tag{4.16}$$

The stationary profile $F_0(\zeta)$ vanishes at a finite $\zeta_* > 0$, and this describes the homogeneous Dirichlet boundary condition at $r = 1$ (it can be considered as an additional matching condition, see below).

4.2. **Matching the regions.** The matching procedure determines both expansions in the inner region and in the boundary layer. We assume that compact subsets $\{\delta_1 \le \zeta \le \delta_2\}$ and $\{\delta_3 \le r \le 1\}$ in the intermediate region and in the boundary one are overlapping for some choice of small δ_1, δ_3 and sufficiently large δ_2. This is the matching assumption to be checked later on. We choose $r = r_0 > 0$ small enough and compare the first terms in the boundary expansion (3.7) and in the intermediate one (4.16), which are now viewed as standard finite Taylor expansions in r of the same solution $u(r,t)$ at $r = r_0$. Taking into account that the two available coefficients of $r_0^{-(N+2)}$ and r_0^{-4} are given with the accuracy $o(1)$ as $t \to T^-$ ($\tau, s \to \infty$), we arrive at the system of two equations

$$A(t) = c_{15}(T-t)^{(N+2)/4}\alpha^{-1}(\tau)(1+o(1)) \quad (\tau = -\ln(T-t)),$$

$$A(t) = c_{16}(T-t)^{(N+2)/4}\alpha^{-8/(N+2)}(\tau)\alpha'(\tau)(1+o(1)), \quad t \to T^-,$$

with two unknown functions: $\alpha(\tau)$ from the inner expansion and $A(t)$ from the boundary layer analysis. One can study this system asymptotically as a standard system of the ODEs. We first derive a single ODE for the function α:

$$\alpha^{(N-6)/(N+2)}\alpha' = c_{18}(1+o(1)), \quad \tau \to \infty, \tag{4.17}$$

and hence

$$\alpha(\tau) = \gamma_0 \tau^{(N+2)/2(N-2)})(1+o(1)) \tag{4.18}$$

$$\equiv \gamma_0 |\ln(T-t)|^{(N+2)/2(N-2)}(1+o(1)), \quad t \to T^- \;\; (\gamma_0(N) > 0).$$

We thus obtain that $\alpha(\tau)$ is slow growing for $\tau \gg 1$ and all the above assumptions are valid for such a power-like function. In particular, the main perturbation order $g(\tau)$ in the inner region is estimates as follows:

$$g(\tau) = \frac{\alpha'(\tau)}{\alpha(\tau)} = c_{18}\alpha^{-2(N-2)/(N-6)}(\tau)(1+o(1)) = \frac{\gamma_0(N+2)}{2(N-2)}\,\frac{1}{\tau}(1+o(1)),$$

and is not integrable at $\tau = \infty$, as was expected.

The decay rate in the boundary layer as $t \to T^-$ is

$$u(r,t) = \gamma_1(T-t)^{(N+2)/4}|\ln(T-t)|^{-(N+2)/2(N-2)}\phi_0(r)(1+o(1)) \quad (\gamma_1(N) > 0)$$

uniformly on intervals $\{\delta \le r \le 1\}$.

This completes the asymptotic analysis and the proof of Theorem 1.1.

4.3. **Matching on the boundary.** The slow growing factor $\alpha(\tau)$ can be determined independently of the boundary expansion via a single ODE. It follows from the expansion for F that the boundary point $r = 1$ corresponds to $\zeta = \zeta_*(1+o(1))$ as $s \to \infty$, so that one obtains the equation

$$\zeta \equiv r\alpha^{2/(N+2)}(\tau)(\alpha'/\alpha)^{1/(N-2)}|_{r=1} = \zeta_*(1+o(1)), \quad \tau \to \infty,$$

which is indeed the asymptotic ODE (4.17). Therefore,

$$\gamma_0(N) = (2m\zeta_*^{N-2})^{1/2m} \quad (m = 1/p_s).$$

This clearly shows that with such a choice of $\alpha(\tau)$, there holds $\zeta = \zeta_* r(1 + o(1))$, so that compact subsets from the intermediate and boundary regions are actually overlapping.

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