Stability of Infinite-Dimensional Sampled-Data Systems*

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Abstract: Suppose that a static state feedback stabilizes a continuous-time linear infinite-dimensional control system. We consider the following question: if we construct a sampled-data controller by applying an idealized sample-and-hold process to a continuous-time stabilizing feedback, will this sampled-data controller stabilize the system for all sufficiently small sampling times? Here the state space X and the control space U are Hilbert spaces, the system is of the form $\dot{x}(t) = Ax(t)+Bu(t)$, where A is the generator of a strongly continuous semigroup on X, and the continuous time feedback is $u(t) = Fx(t)$. The answer to the above question is known to be "yes" if X and U are finite dimensional spaces. In the infinite dimensional case, if F is not compact, then it is easy to find counter-examples. Therefore we restrict attention to compact feedback. We show that the answer to the above question is "yes", if B is a bounded operator from U into X. Moreover, if B is unbounded, we show that the answer "yes" remains correct, provided that the semigroup generated by A is analytic. We use the theory developed for static state feedback to obtain analogous results for dynamic output feedback control.

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1. Introduction

Consider the following system with state space X and input space U (both Hilbert spaces)

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = x^0 \in X, \tag{1.1}$$

where A is the generator of a strongly continuous semigroup $T(t)$ on X, B is a linear bounded operator from U into X_{-1}, where X_{-1} is a certain extrapolation space with $X \hookrightarrow X_{-1}$. If the input (or control) operator B maps boundedly into the state space X, then B is called bounded, otherwise B is called "unbounded" (with respect to the state space X). We give precise details of the set-up in Section 2. Suppose that the feedback control $u(t) = Fx(t)$, where F is a bounded linear operator from X into U, is an exponentially stabilizing state feedback control for (1.1) in the sense that $A+BF$ generates an exponentially stable strongly continuous semigroup on X. A natural implementation of this continuous-time control $u(t) = Fx(t)$ is to use sample and hold, so that u is given instead by

$$u(t) = Fx(k\tau)\,, \quad t \in [k\tau, (k+1)\tau)\,, \tag{1.2}$$

where $\tau > 0$ is the sampling time. The control (1.2) is called a sampled-data feedback control and the overall system given by (1.1) and (1.2) is called a sampled-data feedback system. Intuitively, we would expect that for all sufficiently small $\tau > 0$, (1.2) produces a stabilizing control for (1.1) in the sense that there exists $N \geq 1$ and $\nu > 0$ so that

$$\|x(t)\| \leq Ne^{-\nu t}\|x^0\|\,, \quad \forall\, x^0 \in X, \forall\, t \geq 0\,. \tag{1.3}$$

Integrating (1.1) and (1.2) over one sampling interval $[k\tau, (k+1)\tau)$, and setting $x_k := x(k\tau)$ yields the following discrete-time system

$$x_{k+1} = \Delta_\tau x_k\,, \quad \text{where} \quad \Delta_\tau := T(\tau) + \int_0^\tau T(s)BF\,ds\,.$$

It is straightforward to show that Δ_τ is a bounded operator from X to X and that (1.3) holds if and only if Δ_τ is power stable, i.e. $\|\Delta_\tau^k\| \to 0$ as $k \to \infty$ (see Section 2). By way of motivation, let us briefly consider the finite-dimensional case, i.e. $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, A, B and F are real matrices of compatible formats and $T(t) = e^{At}$. It is instructive to sketch a simple argument showing that in this case the sampled-data feedback (1.2) stabilizes (1.1), provided that τ is sufficiently small. Since $u(t) = Fx(t)$ is an exponentially stabilizing static state feedback control for (1.1), the eigenvalues of $A + BF$ have negative real parts, or, equivalently, there exists a positive definite matrix P satisfying

$$(A + BF)^T P + P(A + BF) = -I. \tag{1.4}$$

Next we observe, using the power series expansion of the matrix exponential e^{At}, that there exists $M_1 \geq 0$ so that

$$\|\Delta_\tau - I - \tau(A + BF)\| \leq M_1 \tau^2 \,, \quad \forall\, \tau \in (0, 1] \,.$$

Then, invoking (1.4), it is easy to see that there exists $M_2 > 0$ and $\tau^* > 0$

$$x^T \Delta_\tau^T P \Delta_\tau x - x^T P x \leq 2\tau x^T (A + BF) P x + M_2 \tau^2 \|x\|^2 \leq -\frac{1}{2}\tau \|x\|^2 \,, \quad \forall\, \tau \in (0, \tau^*) \,.$$

Therefore, for every $\tau \in (0, \tau^*)$, $\Delta_\tau^T P \Delta_\tau - P < 0$, or, equivalently, Δ_τ is power stable, showing that the sampled-data feedback (1.2) is stabilizing for all sufficiently small $\tau > 0$. A different proof of this result can be found in Chen and Francis [1], where, using the power series expansion of the matrix exponential and the continuous dependence of the eigenvalues of Δ_τ on τ, it is shown that the spectral radius of Δ_τ is less than one for all suffciently small $\tau > 0$ (which implies power-stability of Δ_τ for all suffciently small $\tau > 0$).

Whilst these simple observations are easy to prove and well-known in the finite-dimensional control literature, they do serve to indicate that extending these results to meaningful infinite-dimensional situations is far from trivial. Indeed, neither the proof sketched above nor the proof given in [1] would generalise well to the infinite-dimensional case because the exponential power series for semigroups (converging in the operator-norm topology) is only valid for semigroups generated by bounded operators (i.e. uniformly continuous semigroups), but not for strongly continuous semigroups.

It might appear that the case considered above of sampling a static state or output feedback control is too restrictive. However, results for the more practically relevant case of sampling a dynamic output feedback control can be obtained from the state feedback case quite easily as is shown in Section 5 (in an infinite-dimensional context). Such sampling of dynamic controllers forms the basis for an indirect sampled-data design methodology, where a continuous-time (analog) controller is designed on the basis of continuous-time performance criteria, but the controller is then implemented digitally using sampling and hold operations, yielding a discrete-time control for a continuous-time system (see Chen and Francis [2] and

Franklin, Powell and Workman [5] for more details in the finite-dimensional case). It is obvious that the question of whether stability of the continuous-time feedback system is inherited by the resulting sampled-data feedback system (a hybrid continuous/discrete-time system) for all small enough sampling times is of considerable importance in this approach. Of course, there are alternatives to the indirect design approach. At the core of these so-called direct approaches is the direct design of discrete-time controllers for the continuous-time system without discretizing a continuous-time control law by means of hold and sampling operations (see [2], [5] and Yamamoto [17]). In these direct approaches the question of whether stability is inherited from the continuous-time design is not an issue. Rather, a relevant issue in the context of the direct approach is whether properties of the continuous-time system such as controllability/stabilizability and observability/detectability are retained under sampling. Again, for finite-dimensional systems, the conditions under which sampling retains such properties are well-known. For example, retention of stabilizability/detectability under sampling is guaranteed by the Kalman-Ho-Narendra conditions, see [6]. Whilst the direct and indirect approaches to sampled-data control each have their merit, we feel the latter be most suited for infinite-dimensional systems for the following two reasons: (i) any physical meaning of the underlying continuous-time system equations might be difficult to exploit in a direct design which is based on a discretization of the system to be controlled; (ii) the indirect approach is based on a contiuous-time design and hence can make use of the vast literature on stabilization of infinite-dimensional systems by continuous-time feedback controls.

Recently there has been increasing interest in sampled-data control of infinite-dimensional systems. In the context of sampled-data stabilization with generalised hold, Tarn et al. [15] presented sufficient conditions, whilst Rebarber and Townley [10] obtained necessary conditions. For finite-dimensional systems, the necessary conditions and the sufficient conditions are equivalent. However, for infinite-dimensional systems there is a gap between these sets of conditions when the semigroup is not analytic: this is shown, in Rebarber and Townley [11], by way of a simple example. Such differences between results for finite-dimensional systems and those available for infinite-dimensional systems are common in sampled-data control. Rosen and Wang [13] have shown that generalizing the Kalman-Ho-Narendra condition [6] is very delicate and linked to loss of information when sampling high frequency modes. In Rebarber and Townley [10], it is shown that sampled-data stabilization of infinte-dimensional systems with compact input operators is possible only if certain restrictive conditions are satisfied. From most of these papers a trend emerges. If the underlying semigroup system is analytic, or the input operator is bounded, then typically the finite-dimensional result generalises. However, if the input operator is unbounded and the underlying semigroup is not analytic, then the generlization fails. It is not surprising that analyticity plays such a key role: in some sense, analyticity constrains the spectrum of the generator of the underlying semigroup so as to avoid any losses of high-frequency information when sampling.

We give a brief description of the results contained in this paper. In Section 2 we introduce the class of infinite-dimensional systems to which our results apply and we present some preliminary basic technical results to be used in the rest of the paper. In Section 3 we consider the case in which the input operator B is bounded, but the semigroup generated by A is not assumed to be analytic. We show that if the feedback operator F is compact and the semigroup generated by $A+BF$ is exponentially stable, then, for sufficiently small $\tau > 0$, the sampled-data feedback (1.2) is stabilizing. A key element of the proof is the introduction of a new, equivalent norm on the state space with respect to which the operator Δ_τ is a strict contraction provided that τ is sufficiently small. We also give a simple example which shows that the result is not true if the compactness assumption on F is removed. In Section 4 we consider the case in which the input operator B is unbounded and the semigroup generated by A is analytic. We show that for any compact feedback operator F, the operator $A+BF$ (with suitable domain) generates an analytic semigroup on the state-space X, and if this semigroup is exponentially stable, then the sampled-data feedback (1.2) is stabilizing for all sufficiently small $\tau > 0$. The proof of this result is quite involved and differs considerably from the argument leading to the result for bounded B. Instead of trying to estimate the spectral radius of Δ_τ directly, we invoke frequency-domain stability criteria for the power stability of Δ_τ, making repeated use of the analyticity of the underlying semigroup. Finally, in Section 5 we show how to use the static state feedback results of Sections 3 and 4 in the practically more relevant situation of sampling a dynamic output feedback controller.

Notation: For $\alpha \in \mathbb{R}$, $r > 0$ and $z \in \mathbb{C}$, we define

$$\mathbb{C}_\alpha := \{s \in \mathbb{C} \,|\, \operatorname{Re} s > \alpha\}\,, \quad \mathbb{B}(z,r) := \{s \in \mathbb{C} \,|\, |s-z| < r\}\,.$$

The exterior of the unit disc is denoted by $\mathbb{E}$, i.e.

$$\mathbb{E} := \{s \in \mathbb{C} \,|\, |s| > 1\}\,.$$

Let X and Y be Banach spaces. The space of all linear bounded operators from X to Y is denoted by $\mathcal{B}(X,Y)$; we set $\mathcal{B}(X) := \mathcal{B}(X,X)$. Let A be an operator with domain and range in X. Then $D(A)$ denotes the domain of A and $\varrho(A)$ denotes the resolvent set of A, i.e. the set of all $s \in \mathbb{C}$ such that $sI - A : D(A) \to X$ is bijective and $(sI-A)^{-1} \in \mathcal{B}(X)$. The spectrum of A, which is the complement of $\varrho(A)$, is denoted by $\sigma(A)$. Let $\sigma_p(A) \subset \sigma(A)$ denote the point spectrum (i.e. the set of all eigenvalues) of A. If $S \subset X$ and $A : D(A) \subset X \to Y$ is an operator, then $A|_S$ denotes the restriction of A to S, i.e. the operator defined by $A|_S\, x = Ax$ for all $x \in D(A|_S) := D(A) \cap S$. For an open set $\Omega \subset \mathbb{C}$, $H^\infty(\Omega, \mathcal{B}(X,Y))$ denotes the set of all bounded analytic $\mathcal{B}(X,Y)$-valued functions f defined on an open set $\Omega_f \subset \Omega$ (depending on f) such that $\Omega \setminus \Omega_f$ is a discrete set.[†] Clearly, each function $f \in H^\infty(\Omega, \mathcal{B}(X,Y))$ admits an extension $\tilde{f} \in H^\infty(\Omega, \mathcal{B}(X,Y))$ defined on Ω. For an arbitary set $\Lambda \subset \mathbb{C}$, we define

$$H^\infty(\Lambda, \mathcal{B}(X,Y)) := \bigcup_{\Omega \supset \Lambda,\ \Omega \text{ open}} H^\infty(\Omega, \mathcal{B}(X,Y))\,.$$

[†] This means that either $\Omega_f = \Omega$ or, if $\Omega_f \neq \Omega$, then for each $z \in \Omega \setminus \Omega_f$ there exists $\varepsilon > 0$ such that $\mathbb{B}(z,\varepsilon) \cap (\Omega \setminus \Omega_f) = \{z\}$.

2. The sampled-data system

We start with a simple lemma which will be useful in the subsequent developments.

Lemma 2.1 *Let X, Y and Z be Hilbert spaces and let $L : [0,1] \to \mathcal{B}(X,Y)$ be given. If $\lim_{t\to 0} L(t)x = 0$ for all $x \in X$. and if $K \in \mathcal{B}(Z,X)$ is compact, then*

$$\lim_{t\to 0} \|L(t)K\|_{\mathcal{B}(Z,Y)} = 0\,. \tag{2.1}$$

Moreover, if $\lim_{t\to 0} L^*(t)y = 0$ for all $y \in Y$, and if $H \in \mathcal{B}(Y,Z)$ is compact, then

$$\lim_{t\to 0} \|HL(t)\|_{\mathcal{B}(X,Z)} = 0\,. \tag{2.2}$$

Proof: Assume that $\lim_{t\to 0} L(t)x = 0$ for all $x \in X$. If $K \in \mathcal{B}(Z,X)$ has finite rank, then it is easy to prove that (2.1) holds. In a Hilbert space a compact operator can be uniformly approximated by finite rank operators, and a standard argument shows that (2.1) is true for compact $K \in \mathcal{B}(Z,X)$.

Assuming that $\lim_{t\to 0} L^*(t)x = 0$ for all $x \in X$ and that $H \in \mathcal{B}(Y,Z)$ is compact, (2.2) can be obtained by an application of (2.1) to $L^*(t)$ and H^*, □

Throughout the paper, X and U denote Hilbert spaces. Let us consider the following sampled-data system with state space X and input space U

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0\,; \quad x(0) = x^0 \in X \tag{2.3a}$$
$$u(t) = Fx(k\tau), \qquad k\tau \leq t < (k+1)\tau, \quad k \in \mathbb{N}_0\,, \tag{2.3b}$$

where $x(t) \in X$, $u(t) \in U$, $\tau > 0$ is the sampling time, A is the generator of a strongly continuous semigroup $T(t)$ on X, $F \in \mathcal{B}(X,U)$ and $B \in \mathcal{B}(U,X_{-1})$, where X_{-1} denotes the closure of X in the norm $\|x\|_{-1} = \|(\lambda I - A)^{-1}x\|$ (here $\|\cdot\|$ denotes the norm of X and λ is any element in $\varrho(A)$). Clearly, $X \hookrightarrow X_{-1}$ and hence $\mathcal{B}(U,X) \subset \mathcal{B}(U,X_{-1})$. We say that B is *bounded* if $B \in \mathcal{B}(U,X)$, otherwise we say that B is *unbounded*. The semigroup $T(t)$ extends to a strongly continuous semigroup on X_{-1}. The generator of $T(t)$ on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbol $T(t)$ (respectively, A) for the original semigroup (respectively, its generator) and the associated extensions. With this convention, we may write $A \in \mathcal{B}(X,X_{-1})$. Considered as a generator on X_{-1}, the domain of A is X. The spectrum of A considered as an operator on X coincides with the spectrum of A considered as an operator on X_{-1}; moreover, the point spectra of A and its extension coincide including algebraic multiplicities of isolated eigenvalues, We refer the reader to pp. 123 in the book [4] by Engel and Nagel for more details on the extrapolation space X_{-1}.

The derivative on the left-hand side of (2.3a) has to be interpreted in the space X_{-1}. To solve the initial-value problem (2.3), we define a function x recursively by

$$\begin{aligned} x(0) &= x^0 && (2.4a) \\ x(k\tau + t) &= T(t)x(k\tau) + \int_0^t T(s)BFx(k\tau)\,ds\,,\ \forall\, t \in (0,\tau],\ \forall\, k \in \mathbb{N}_0\,. && (2.4b) \end{aligned}$$

It follows from standard results in the theory of strongly semigroups (applied to $T(t)$ considered as a semigroup on X_{-1}) that

$$x \in C(\mathbb{R}_+, X) \qquad \text{and} \qquad x|_{[k\tau,(k+1)\tau]} \in C^1([k\tau,(k+1)\tau], X_{-1})\,,\ \forall\, k \in \mathbb{N}_0\,.$$

Moreover, x satisfies the following differential equations in X_{-1}:

$$\dot{x}(t) = Ax(t) + BFx(k\tau)\,, \quad \forall\, t \in (k\tau,(k+1)\tau),\ \forall\, k \in \mathbb{N}_0\,.$$

It is also clear that x given by (2.4) is the only function with these properties. In this sense, the function x defined by (2.4) is the unique solution of (2.3).

We say that (2.3) is *exponentially stable* if there exists $N \geq 1$ and $\nu > 0$ such that $\|x(t)\| \leq Ne^{-\nu t}\|x^0\|$ for all initial values $x^0 \in X$ and all $t \geq 0$. To study the stability properties of the sampled-data system (2.3) we consider a related discrete time system. Let x be the solution of (2.3) given by (2.4) and set

$$x_k := x(k\tau).$$

Then, by (2.4b)

$$x_{k+1} = T(\tau)x_k + \int_0^\tau T(s)BFx_k\,ds = (T(\tau) + S_\tau F)x_k, \quad \forall\, k \in \mathbb{N}_0\,, \tag{2.5}$$

where S_τ is an operator defined on U given by

$$S_\tau u := \int_0^\tau T(s)Bu\,ds\,, \quad \forall\, u \in U\,. \tag{2.6}$$

Lemma 2.2 *For any $\tau \geq 0$, $S_\tau \in \mathcal{B}(U,X)$ and for any $\theta > 0$*

$$\sup_{0\leq\tau\leq\theta} \|S_\tau\|_{\mathcal{B}(U,X)} < \infty. \tag{2.7}$$

Moreover, if $F \in \mathcal{B}(X,U)$ is compact, then

$$\lim_{\tau\to 0} \|S_\tau F\|_{\mathcal{B}(X)} = 0\,. \tag{2.8}$$

Proof: By standard results from semigroup theory applied to $T(t)$ considered as a semigroup on X_{-1}, we know that $S_\tau U \subset X$ and

$$AS_\tau u = (T(\tau) - I)Bu\,, \quad \forall\, u \in U,\ \forall\, \tau \in (0,\infty)\,.$$

Hence, for $\lambda \in \varrho(A)$, we obtain

$$S_\tau u = (I - T(\tau))(\lambda I - A)^{-1}Bu + \lambda \int_0^\tau T(s)(\lambda I - A)^{-1}Bu\,ds\,, \quad \forall\, u \in U,\ \forall\, \tau \in [0,\infty)\,, \tag{2.9}$$

showing that $S_\tau \in \mathcal{B}(U,X)$ and that (2.7) holds. Moreover, let $F \in \mathcal{B}(X,U)$ be compact. We see from (2.9) that $\lim_{\tau\to 0}\|S_\tau u\| = 0$ for all $u \in U$, and thus (2.8) follows from an application of Lemma 2.1. □

Introducing the operator

$$\Delta_\tau := T(\tau) + S_\tau F\,, \tag{2.10}$$

(2.5) can be written in the form

$$x_{k+1} = \Delta_\tau x_k, \quad \forall\, k \in \mathbb{N}. \tag{2.11}$$

Note that by Lemma 2.2, $\Delta_\tau \in \mathcal{B}(X)$. Recall that an operator $L \in \mathcal{B}(X)$ is said to be *power stable* if $\|L^k\| \to 0$ as $k \to \infty$.

The following simple result can be obtained by a slight modification of the proof of Proposition 2.1 in Rebarber and Townley [11].

Lemma 2.3 *For any $\tau > 0$, (2.3) is exponentially stable if and only if Δ_τ is power stable.*

Hence, for the remainder of this paper, we study the power stability of Δ_τ in order to study the exponential stability of (2.3).

3. Bounded control

In this section we assume that the control operator B is bounded. We show that if the feedback operator F is compact, then exponential stability of the continuous-time semigroup generated by $A+BF$ implies exponential stability of the sampled-data system (2.3) provided the sampling time τ is small enough. More precisely, we prove the following result.

Theorem 3.1 *Assume that A generates a strongly continuous semigroup $T(t)$ on X, $B \in \mathcal{B}(U,X)$, $F \in \mathcal{B}(X,U)$ is compact and the semigroup generated by $A + BF$ is exponentially stable. Then there exists $\tau^* > 0$ such that for every $\tau \in (0,\tau^*)$, there exist $N \geq 1$ and $\nu > 0$ such that all solutions of (2.3) satisfy $\|x(t)\| \leq Ne^{-\nu t}\|x^0\|$ for all $x^0 \in X$ and all $t \geq 0$.*

Proof: Denote the semigroup generated by $A + BF$ by $T_{BF}(t)$. We start by writing Δ_τ as a perturbation of $T_{BF}(\tau)$. By the variation of parameters formula from the perturbation theory of semigroups (see, for example, [9], p. 79), we have

$$T_{BF}(\tau)x = T(\tau)x + \int_0^\tau T(\tau - s)BFT_{BF}(s)x\, ds\,, \quad \forall\, x \in X\,.$$

Using (2.10) and defining $P_\tau \in \mathcal{B}(X)$ by

$$P_\tau x := \int_0^\tau T(\tau - s)BF(I - T_{BF}(s))x\, ds\,, \quad \forall\, x \in X\,,$$

we see that

$$\Delta_\tau = T_{BF}(\tau) + P_\tau\,. \tag{3.1}$$

By the hypothesis that T_{BF} is exponentially stable, there exist $M \geq 1$ and $\omega > 0$ such that

$$\|T_{BF}(t)x\| \leq Me^{-\omega t}\|x\|\,, \qquad \forall\, x \in X,\ \forall\, t \in \mathbb{R}_+.$$

Adopting a standard technique from semigroup theory (see, for example, the proof of Theorem 5.2 on p. 19 in Pazy [9]), we introduce a new norm $|\cdot|$ on X by setting

$$|x| := \sup_{t\geq 0} \|e^{\omega t}T_{BF}(t)x\|\,, \quad \forall\, x \in X\,.$$

Clearly,

$$\|x\| \leq |x| \leq M\|x\|\,, \quad \forall\, x \in X\,, \tag{3.2}$$

and, moreover,

$$\begin{aligned} |T_{BF}(t)x| &= \sup_{s\geq 0} \|e^{\omega s}T_{BF}(s)T_{BF}(t)x\| \leq e^{-\omega t} \sup_{s\geq 0} \|e^{\omega s}T_{BF}(s)x\| \\ &= e^{-\omega t}|x|\,, \quad \forall\, x \in X,\ \forall\, t \geq 0\,. \end{aligned} \tag{3.3}$$

Since F is compact and $\lim_{s\to 0}(I - T_{BF}(s))x = 0$ for all $x \in X$, an application of Lemma 2.1 yields

$$\lim_{s\to 0} \|F(I - T_{BF}(s))\|_{\mathcal{B}(X,U)} = 0. \tag{3.4}$$

For $L \in \mathcal{B}(X)$ we denote the operator norm $\sup_{x\in X}(|Lx|/|x|)$ by $|L|$. Using (3.1)-(3.3) and invoking the inequality

$$e^{-\omega\tau} \leq 1 - \omega\tau e^{-\omega\tau}\,, \quad \forall\, \tau \in \mathbb{R}_+\,,$$

we obtain

$$\begin{aligned} |\Delta_\tau| &\leq e^{-\omega\tau} + |P_\tau| \leq e^{-\omega\tau} + M\|P_\tau\| \\ &\leq 1 - \omega\tau + [\omega(1 - e^{-\omega\tau}) + h(\tau)]\tau\,, \qquad \forall\, \tau \in \mathbb{R}_+, \end{aligned} \tag{3.5}$$

where

$$h(\tau) := M \sup_{0\le s\le\tau} \|T(\tau - s)BF(I - T_{BF}(s))\|_{\mathcal{B}(X)}.$$

By (3.4),

$$\lim_{\tau\to 0^+} h(\tau) = 0,$$

and hence it follows from (3.5) that for fixed but arbitrary $\varepsilon \in (0, \omega)$ there exists $\tau_\varepsilon > 0$ such that

$$|\Delta_\tau| < 1 - (\omega - \varepsilon)\tau < 1\,, \qquad \forall\, \tau \in (0, \tau_\varepsilon). \tag{3.6}$$

Applying (3.2), we see that for any $\tau \in (0, \tau_\varepsilon)$

$$\|\Delta_\tau^k x\| \le |\Delta_\tau^k x| \le |\Delta_\tau^k||x| \le M|\Delta_\tau|^k\|x\|\,, \quad \forall\, x \in X, \forall\, k \in \mathbb{N}.$$

Combining this with (3.6) shows that Δ_τ is power stable for all $\tau \in (0, \tau_\varepsilon)$. □

Remark 3.2 (1) Examining the proof of Theorem 3.1, we see that for every $\varepsilon \in (0, \omega)$, there exists $\tau_\varepsilon > 0$ such that

$$\|\Delta_\tau^k\| \le M(1 - (\omega - \varepsilon)\tau)^k, \qquad \forall\, k \in \mathbb{N}, \forall\, \tau \in (0, \tau_\varepsilon)\,.$$

(2) A key element in the proof of Theorem 3.1 is that $\lim_{\tau\to 0^+} h(\tau) = 0$ which implies that

$$\lim_{\tau\to 0^+} \frac{\|P_\tau\|}{\tau} = 0\,.$$

In the case of unbounded B, this does not seem to be true, even if B is an admissible control operator for $T(t)$ (in the sense of Weiss [16]). ◇

The next remark shows that the compactness assumption on F imposed in Theorem 3.1 cannot be relaxed.

Remark 3.3 It is easy to find counterexamples which show that in general Theorem 3.1 is not true for non-compact feedback operators F. Indeed, consider the case where $X = l^2(\mathbb{Z})$, $A = \mathrm{diag}_{k\in\mathbb{Z}}\,(1 + ki)$, $B = I$ and $F = -2I$. Then F is a continuous-time exponentially stabilizing feedback since $A + BF$ generates the exponentially stable semigroup

$$T_{BF}(t) = \mathrm{diag}_{k\in\mathbb{Z}}(e^{(-1+ki)t}).$$

However, applying the sampled-data feedback given by (2.3b) results in the discrete-time operator

$$\Delta_\tau = \mathrm{diag}_{k\in\mathbb{Z}}(e^{(1+ki)\tau}) - 2\mathrm{diag}_{k\in\mathbb{Z}}\left(\frac{e^{(1+ki)\tau} - 1}{1 + ki}\right).$$

Since

$$\left| e^{(1+ki)\tau} - 2\left(\frac{e^{(1+ki)\tau}-1}{1+ki}\right)\right| \geq e^{\tau} - \frac{2}{|1+ik|}(e^{\tau}+1),$$

we see that for any $\tau > 0$, the eigenvalues λ_k^τ of Δ_τ satisfy

$$\liminf_{|k|\to\infty} |\lambda_k^\tau| \geq e^{\tau} > 1\,,$$

showing that Δ_τ is not power stable for any $\tau > 0$.

Furthermore, it is possible to find counterexamples which show that in general Theorem 3.1 is not true when the B is not bounded. In particular, in [12] an example is given for which U is one-dimensional, F is bounded, B is not bounded, $A + BF$ generates an exponentially stable semigroup, $A^{-\varepsilon}B \in \mathcal{B}(U, X)$ for any $\varepsilon > 0$ (so B is "barely unbounded" by a common measure of unboundedness), and there exists a sequence of positive sampling times (τ_n) such that $\tau_n \to 0$ and Δ_{τ_n} is not power stable.

◇

4. Unbounded control and analytic semigroup

In order to formulate and prove the main result of this section a number of preparations are required. Throughout this section, we assume that A generates a strongly continuous semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X_{-1})$ and $F \in \mathcal{B}(X, U)$. Further assumptions on (A, B, F) will be introduced when needed. We define an operator A_{BF} on X by setting

$$A_{BF}x = (A + BF)x\,, \quad \forall\, x \in D(A_{BF}) := \{x \in X \mid (A + BF)x \in X\}\,. \tag{4.1}$$

It is easy to show that A_{BF} is closed. We carefully distinguish between the operators A_{BF} and $A+BF$, the latter being an unbounded operator on X_{-1} with domain $D(A+BF) = X$. Clearly, $A + BF \in \mathcal{B}(X, X_{-1})$. Note that A_{BF} is the part of $A + BF$ in X. The transfer function of the continuous time system given by (A, B, F) is defined as follows:

$$\mathbf{G} : \varrho(A) \to \mathcal{B}(U), \quad s \mapsto F(sI - A)^{-1}B\,. \tag{4.2}$$

We introduce the space $X_1 := D(A^*)$, endowed with the norm

$$\|x\|_1 := \|(\bar{\lambda}I - A^*)x\|\,,$$

where λ is any element of $\varrho(A)$. Clearly, $X_1 \hookrightarrow X = X^*$. It is well-known that X_{-1} and X_1^* are isometrically isomorphic. Therefore, the triple $X_1 \hookrightarrow X = X^* \hookrightarrow X_{-1}$ is a so-called Gelfand triple. Moreover, if $B \in \mathcal{B}(U, X_{-1})$, then $B^* \in \mathcal{B}(X_1, U)$.

Lemma 4.1 *Under the above assumptions on A, B and F, the following statements hold:*
(1) $\varrho(A+BF)\cap\varrho(A)=\{s\in\varrho(A)\,|\,1\in\varrho(\mathbf{G}(s))\}$;
(2) if $\varrho(A+BF)\neq\emptyset$, *then* $\varrho(A_{BF})=\varrho(A+BF)$ *and*

$$(sI-A_{BF})^{-1}=(sI-A-BF)^{-1}|_X\,,\quad \forall\, s\in\varrho(A_{BF})=\varrho(A+BF)\,;$$

(3) if F is compact, then

$$\varrho(A_{BF})\cap\varrho(A)\subset\varrho(A+BF)\,;$$

(4) if F is compact, then $A^*_{BF}=A^*+F^*B^*$ *with* $D(A^*_{BF})=D(A^*)=X_1$.

Proof: To prove statements (1) and (3), let $s\in\varrho(A)$, and write

$$sI-A-BF=(sI-A)[I-(sI-A)^{-1}BF]\,. \tag{4.3}$$

It follows that

$$s\in\varrho(A+BF)\;\Leftrightarrow\;1\in\varrho((sI-A)^{-1}BF)\;\Leftrightarrow\;1\in\varrho(F(sI-A)^{-1}B)\,,$$

yielding statement (1). Assume that F is compact. Then, by (4.3),

$$s\in\sigma(A+BF)\;\Leftrightarrow\;1\in\sigma_p((sI-A)^{-1}BF)\,.$$

Therefore, if $s\in\sigma(A+BF)$, there exists $x\in X$, $x\neq0$, such that $(sI-A)^{-1}BFx=x$. Hence, $(A+BF)x=sx$, showing that $s\in\sigma_p(A_{BF})$. We may conclude that $\sigma(A+BF)\cap\varrho(A)\subset\sigma(A_{BF})$, proving statement (3).

To prove statement (2), let $\lambda\in\varrho(A+BF)$ and note that, by Proposition 2.17 on p. 261 in [4], we only need to show that the topology on X given by the norm

$$x\mapsto|x|=\|(\lambda I-A-BF)x\|_{-1}\,,$$

is stronger than the original norm topology. But this follows immediately, since with

$$k:=\|(\lambda I-A-BF)^{-1}\|_{\mathcal{B}(X_{-1},X)}\,,\quad l:=\|\lambda I-A-BF\|_{\mathcal{B}(X,X_{-1})}\,.$$

we have

$$\|x\|\le k\|(\lambda I-A-BF)x\|_{-1}=k|x|\,,\;\;|x|\le l\|x\|\,;\quad\forall\,x\in X\,,$$

i.e. the two norms are even equivalent.

To prove statement (4), consider the operator $A^*+F^*B^*$ with $D(A^*+F^*B^*)=D(A^*)=X_1$. The compactness of F together with the fact that $B^*\in\mathcal{B}(X_1,U)$ yields that F^*B^* is relatively compact with respect to A^*. Therefore, since A^* is closed, $A^*+F^*B^*$ is closed (see Theorem 1.1 on p. 194 in [7]) and so $(A^*+F^*B^*)^{**}=A^*+F^*B^*$ (see Theorem 5.29 on p. 168 in [7]). As a consequence, statement (4) will follow if we can show that $(A^*+F^*B^*)^*=A_{BF}$.

To this end assume that $x \in D((A^* + F^*B^*)^*)$. Setting $x' := (A^* + F^*B^*)^*x \in X$ we have that

$$\langle x, (A^* + F^*B^*)z\rangle = \langle x', z\rangle\,, \quad \forall\, z \in X_1\,, \tag{4.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in X. The duality pairing on $X_{-1} \times X_1$, denoted by $[\cdot,\cdot]$, is given by

$$[y, z] := \langle(\lambda I - A)^{-1}y, (\bar{\lambda} I - A^*)z\rangle\,, \quad \forall\,(y, z) \in X_{-1} \times X_1\,,$$

where $\lambda \in \varrho(A)$. It is straightforward to show that

$$[y, z] = \langle y, z\rangle\,, \quad [Ay, z] = \langle y, A^*z\rangle\,, \quad [BFy, z] = \langle y, F^*B^*z\rangle\,; \quad \forall\,(y, z) \in X \times X_1\,. \tag{4.5}$$

Thus, by (4.4),

$$[x' - (A + BF)x, z] = 0\,, \quad \forall z \in X_1\,.$$

and so $(A + BF)x = x' \in X$, proving that $x \in D(A_{BF})$ and $A_{BF}x = (A^* + F^*B^*)^*x$. This shows that A_{BF} is an extension of $(A^* + F^*B^*)^*$. It remains to prove that $D(A_{BF}) \subset D((A^* + F^*B^*)^*)$. Let $x \in D(A_{BF})$. Invoking (4.5), we obtain for all $z \in X_1$

$$\langle x, (A^* + F^*B^*)z\rangle = [(A + BF)x, z] = \langle A_{BF}x, z\rangle\,,$$

showing that $x \in D((A^* + F^*B^*)^*)$. □

Remark 4.2 Statements (2) and (3) of Lemma 4.1 are wrong if the operator A_{BF} is replaced by a proper restriction of A_{BF}. More precisely, consider an operator $\tilde{A}_{BF}$ on X given by

$$\tilde{A}_{BF}x = (A + BF)x\,, \quad \forall\, x \in D(\tilde{A}_{BF}) \subset D(A_{BF})\,.$$

If $D(\tilde{A}_{BF}) \neq D(A_{BF})$, then

$$\varrho(\tilde{A}_{BF}) \subset \sigma_p(A_{BF}) \subset \sigma_p(A + BF)\,. \tag{4.6}$$

The second inclusion holds trivially. To prove the first inclusion, let $\lambda \in \varrho(\tilde{A}_{BF})$. Then, $\lambda I - \tilde{A}_{BF} : D(\tilde{A}_{BF}) \to X$ is bijective and hence, for $y \in D(A_{BF}) \setminus D(\tilde{A}_{BF})$, there exists $x \in D(\tilde{A}_{BF})$ such that $(\lambda I - \tilde{A}_{BF})x = (\lambda I - A_{BF})y$. It follows that $(\lambda I - A_{BF})(y - x) = 0$, and since $y - x \neq 0$, we see that $\lambda \in \sigma_p(A_{BF})$.

Moreover, combining statement (1) of Lemma 4.1 and (4.6), we see that if $\tilde{A}_{BF}$ is a proper restriction of A_{BF}, then $I - \mathbf{G}(s)$ is not invertible for any $s \in \varrho(\tilde{A}_{BF}) \cap \varrho(A)$. ◇

Lemma 4.3 *Assume that $F \in \mathcal{B}(X, U)$ is compact and that the following two spectral assumptions hold*

(S1) $\bar{\mathbb{C}}_0 \subset \varrho(A_{BF})$*;*

(S2) *there exists* $\varepsilon > 0$ *such that* $\sigma(A) \cap \mathbb{C}_{-\varepsilon}$ *is bounded.*

Then there exists $\alpha < 0$ *such that* $\sigma(A) \cap \mathbb{C}_\alpha$ *consists of finitely many eigenvalues of* A *with finite algebraic multiplicities.*

Proof: Since F is compact, it follows from statement (3) of Lemma 4.1 and assumption (S1) that $\varrho(A + BF) \neq \emptyset$. Hence, combining statement (2) of Lemma 4.1, assumption (S1) and the fact that $\varrho(A_{BF})$ is open shows that there exists an open set Ω such that

$$\varrho(A + BF) = \varrho(A_{BF}) \supset \Omega \supset \bar{\mathbb{C}}_0 \,. \tag{4.7}$$

Recall that the spectrum of A considered as an operator on X coincides with the spectrum of A considered as an operator on X_{-1} (see [4], p. 261, Proposition 2.17). Moreover, the point spectra of A and its extension are identical including algebraic multiplicities of isolated eigenvalues, For the rest of this proof we will consider A as operator on X_{-1} with domain X. Clearly,

$$sI - A = [I + BF(sI - A - BF)^{-1}](sI - A - BF)\,, \quad \forall\, s \in \Omega\,, \tag{4.8}$$

and therefore, by the compactness of F

$$s \in \sigma(A) \;\Leftrightarrow\; -1 \in \sigma_p(BF(sI - A - BF)^{-1})\,, \quad \forall\, s \in \Omega\,. \tag{4.9}$$

Let $s \in \sigma(A) \cap \Omega$. Then, by (4.8) and (4.9), there exists $x \in X_{-1}$, $x \neq 0$, such that

$$(sI - A)(sI - A - BF)^{-1}x = 0\,,$$

showing that s is an eigenvalue of A, and so

$$\sigma(A) \cap \Omega = \sigma_p(A) \cap \Omega\,. \tag{4.10}$$

Combining (4.9), (4.10) and the compactness of F with Theorem 1.9 on p. 370 in Kato [7] shows that for any compact set $K \subset \Omega$, the intersection $\sigma_p(A) \cap K$ is finite. It now follows from assumption (S2) that there exists $\alpha < 0$ such that $\sigma_p(A) \cap \mathbb{C}_\alpha$ is finite and $\sigma_p(A) \cap \mathbb{C}_\alpha \subset \Omega$. It remains to show that the each element in $\sigma_p(A) \cap \mathbb{C}_\alpha$ has finite algebraic multiplicity. Seeking a contradiction, let $\lambda \in \sigma_p(A) \cap \mathbb{C}_\alpha \subset \Omega$ and suppose that λ has infinite algebraic multiplicity. Then, by Theorem 5.28 on p. 239 in [7], λ belongs to the essential spectrum of A (as defined on p. 243 in [7]). As an unbounded perturbation on X_{-1}, the operator BF is relatively compact with respect to A (considered on X_{-1}) due to the compactness of F. By Theorem 5.35 on p. 244 in [7], the essential spectrum of an operator remains fixed under relatively compact perturbations, and thus, $\lambda \in \sigma(A + BF) \cap \Omega$, contradicting (4.7). $\square$

Assume that $F \in \mathcal{B}(X,U)$ is compact and the assumptions (S1) and (S2) hold. Then, by Lemma 4.3, there exists $\alpha < 0$ such that $\sigma(A) \cap \mathbb{C}_\alpha$ consists of finitely many eigenvalues of A with finite algebraic multiplicities. Hence, there exists a simple closed curve γ in the half-plane $\mathbb{C}_\alpha$ not intersecting $\sigma(A)$ and enclosing $\sigma(A) \cap \bar{\mathbb{C}}_0$ in its interior. The operator

$$\Pi := \frac{1}{2\pi i} \int_\gamma (sI - A)^{-1} \, ds \tag{4.11}$$

is a projection operator and we have

$$X = X_1 \oplus X_2 \,, \qquad \text{where} \quad X_1 := \Pi X \,, \quad X_2 := (I - \Pi)X \,. \tag{4.12}$$

By a standard result (see, for example, Lemma 2.5.7 in Curtain and Zwart [3]), the above decomposition has the following properties

(D1) $\dim X_1 < \infty$ and $X_1 \subset D(A)$;

(D2) X_1 and X_2 are $T(t)$-invariant for all $t \geq 0$;

(D3) $\sigma(A|_{X_1}) = \sigma(A) \cap \bar{\mathbb{C}}_0$ and $\sigma(A|_{X_2}) = \sigma(A) \cap (\mathbb{C} \setminus \bar{\mathbb{C}}_0)$.

It is useful to introduce the following notation

$$A_j := A|_{X_j} \,, \quad T_j(t) := T(t)|_{X_j} \qquad j = 1, 2 \,. \tag{4.13}$$

Clearly, by (D1) and (D2), $T_1(t)$ is a semigroup on the finite-dimensional space X_1 with generator A_1, i.e. $T_1(t) = e^{A_1 t}$, and $T_2(t)$ is a strongly continuous semigroup on X_2 with generator A_2. Since the spectrum of A considered as an operator on X coincides with the spectrum of A considered as an operator on X_{-1}, the projection operator Π on X defined in (4.11) extends to a projection on X_{-1}. We will use the same symbol Π for the original projection and its associated extension. The decomposition (4.12) induces decompositions of the control operator $B \in \mathcal{B}(U, X_{-1})$ and the feedback operator $F \in \mathcal{B}(X, U)$:

$$B_1 := \Pi B \,, \quad B_2 := (I - \Pi)B \,, \quad F_1 := F|_{X_1} \,, \quad F_2 := F|_{X_2} \,. \tag{4.14}$$

Lemma 4.4 *Assume that $F \in \mathcal{B}(X,U)$ is compact and that assumptions (S1) and (S2) hold. Then the pair (A_1, B_1) is controllable and the pair (A_1, F_1) is observable.*

Proof: Seeking a contradiction suppose that (A_1, B_1) is not controllable. Then, by the Kalman controllability decomposition lemma, we may assume without loss of generality that A_1 and B_1 can be written in the form

$$A_1 = \begin{pmatrix} A_{11} & * \\ 0 & A_{12} \end{pmatrix} , \quad B_1 = \begin{pmatrix} B_{11} \\ 0 \end{pmatrix} . \tag{4.15}$$

A straightforward calculation combined with an application of statement (2) of Lemma 4.1 shows that

$$(sI - A_{BF})^{-1} = (sI - A)^{-1} + (sI - A)^{-1}BF(sI - A_{BF})^{-1}\,, \quad \forall\, s \in \varrho(A) \cap \bar{\mathbb{C}}_0\,. \tag{4.16}$$

Using the decomposition of A and B and (4.15), we see that $(sI - A)^{-1}BF$ may be written in the form

$$(sI - A)^{-1}BF = \begin{pmatrix} (sI - A_{11})^{-1} & * & 0 \\ 0 & (sI - A_{12})^{-1} & 0 \\ 0 & 0 & (sI - A_2)^{-1} \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix}. \tag{4.17}$$

Let $\lambda \in \sigma(A_{12})$ and let y be a corresponding eigenvector of A_{12}. Note that, by (D3), $\lambda \in \bar{\mathbb{C}}_0$. Setting $x := \mathrm{col}\,(0, y, 0) \in X$, we may conclude from (4.16) and (4.17) that

$$(sI - A_{BF})^{-1}x = \begin{pmatrix} * \\ (sI - A_{12})^{-1}y \\ 0 \end{pmatrix} + \begin{pmatrix} * \\ 0 \\ * \end{pmatrix} = \begin{pmatrix} * \\ (s - \lambda)^{-1}y \\ * \end{pmatrix}.$$

This shows that $\|(sI - A_{BF})^{-1}x\| \to \infty$ as $s \to \lambda$ in $\varrho(A) \cap \bar{\mathbb{C}}_0$, leading to a contradiction, since by (S1), $\lambda \in \varrho(A_{BF})$.

To show the observability of (A_1, F_1), note that

$$(sI - A_{BF})^{-1} = (sI - A)^{-1} + (sI - A - BF)^{-1}BF(sI - A)^{-1}\,, \quad \forall\, s \in \varrho(A) \cap \bar{\mathbb{C}}_0\,, \tag{4.18}$$

which can be obtained from a straightforward calculation combined with an application of statement (2) of Lemma 4.1. Based on the Kalman observability decomposition lemma and (4.18), an argument similar to that establishing controllability of (A_1, B_1) can be used to prove observability of (A_1, F_1). We omit the details for the sake of brevity. □

The next result shows that if A generates an analytic semigroup and if F is compact, then A_{BF} generates an analytic semigroup.

Proposition 4.5 *Assume that A generates an analytic semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X_{-1})$ and $F \in \mathcal{B}(X, U)$ is compact. Then A_{BF} generates an analytic semigroup $T_{BF}(t)$ on X. Moreover, if $T_{BF}(t)$ is exponentially stable, then $(I - \mathbf{G})^{-1} \in H^\infty(\mathbb{C}_0, \mathcal{B}(U))$, where $\mathbf{G}$ is given by (4.2).*

Proof: By statement (4) of Lemma 4.1, $A^*_{BF} = A^* + F^*B^*$ with $D(A^*_{BF}) = D(A^*) = X_1$. Clearly, since A generates an analytic semigroup, so does A^*. Moreover, by the assumptions on B and F, $B^* \in \mathcal{B}(X_1, U)$ and F^* is compact, and so F^*B^* is relatively compact with respect to A^*. By a well-known result from the perturbation theory of analytic semigroups (see Corollary 2.17 on p. 180 in [4] or Proposition 1 in Zabczyk [18]), it follows that $A^*_{BF} =$

$A^* + F^*B^*$ generates an analytic semigroup, and hence, so does A_{BF}^{**}. By the closedness of A_{BF}, we have that $A_{BF}^{**} = A_{BF}$, and thus we may conclude that A_{BF} generates an analytic semigroup.

Assume that $T_{BF}(t)$ exponentially stable. Then, by Lemma 4.1,

$$\varrho(A_{BF}) = \varrho(A + BF) \supset \bar{\mathbb{C}}_0\,,$$

and for all $s \in \bar{\mathbb{C}}_0$ and for all $x \in X_{-1}$ we have that

$$(sI - A - BF)^{-1}x = (A + BF)(sI - A - BF)^{-1}(A + BF)^{-1}x\,.$$

Since $(A + BF)^{-1}x \in X$, we may conclude using Lemma 4.1 that for all $s \in \bar{\mathbb{C}}_0$ and for all $x \in X_{-1}$

$$\begin{aligned} (sI - A - BF)^{-1}x &= A_{BF}(sI - A_{BF})^{-1}(A + BF)^{-1}x \\ &= (s(sI - A_{BF})^{-1} - I)(A + BF)^{-1}x\,. \end{aligned} \tag{4.19}$$

A straightforward calculation shows that

$$(I - \mathbf{G}(s))^{-1} = (I - \mathbf{G}(s))^{-1}\mathbf{G}(s) + I = F(sI - A - BF)^{-1}B + I\,,$$

and so, by (4.19),

$$(I - \mathbf{G}(s))^{-1} = F(s(sI - A_{BF})^{-1} - I)(A + BF)^{-1}B + I\,. \tag{4.20}$$

Since $T_{BF}(t)$ is an exponentially stable analytic semigroup and since $(A + BF)^{-1}B \in \mathcal{B}(U, X)$, the right-hand side of (4.20) is uniformly bounded for $s \in \mathbb{C}_0$. Therefore, $(I - \mathbf{G})^{-1} \in H^\infty(\mathbb{C}_0, \mathcal{B}(U))$. □

Corollary 4.6 *Assume that A generates an analytic semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X_{-1})$ and $F \in \mathcal{B}(X, U)$ is compact. If the semigroup generated by A_{BF} is exponentially stable, then there exists a decomposition $X = X_1 \oplus X_2$ such that (D1)-(D3) are satisfied and, morover, the following property holds*

(D4) *$T_2(t)$ is an exponentially stable analytic semigroup on X_2 with generator A_2,*

where $T_2(t)$ and A_2 are given by (4.13).

Proof: By the exponential stability of the semigroup generated by A_{BF}, assumption (S1) in Lemma 4.3 holds. Since A generates an analytic semigroup, for any $\beta \in \mathbb{R}$, the intersection $\sigma(A) \cap \mathbb{C}_\beta$ is bounded, and so assumption (S2) in Lemma 4.3 is satisfied. Therefore, by Lemma 4.3, properties (D1) - (D3) hold. Moreover, it is clear that the semigroup $T_2(t)$ is analytic, and therefore satisfies the spectrum determined growth assumption. By (D3) and

Lemma 4.3, the generator A_2 of $T_2(t)$ satisfies $\sigma(A_2) \subset \mathbb{C} \setminus \mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$. Therefore, we may conclude that $T_2(t)$ is exponentially stable. $\square$

Consider the discrete-time system $(T(\tau), S_\tau, F)$, where S_τ is given by (2.6), and denote the transfer function of this system by $\mathbf{H}_\tau$, i.e.

$$\mathbf{H}_\tau(z) = F(zI - T(\tau))^{-1} S_\tau. \tag{4.21}$$

Then, under the assumptions of Lemma 4.3 (i.e. F is compact and (S1) and (S2) hold), we may write

$$\mathbf{H}_\tau = \mathbf{H}_\tau^1 + \mathbf{H}_\tau^2\,, \tag{4.22}$$

where, using the notation of (4.13) and (4.14),

$$\mathbf{H}_\tau^j(z) := F_j(zI - T_j(\tau))^{-1} \int_0^\tau T_j(s) B_j \, ds\,, \quad j = 1, 2. \tag{4.23}$$

In light of Lemma 2.3, in order to prove that the sampled-data system (2.3) is exponentially stable for a given sampling time $\tau > 0$, we need to show that Δ_τ given by (2.10) is power stable. The following lemma provides a sufficient condition for the power-stability of Δ_τ in terms of τ and $\mathbf{H}_\tau$.

Lemma 4.7 *Assume that A generates an analytic semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X_{-1})$, $F \in \mathcal{B}(X, U)$ is compact and the semigroup generated by A_{BF} is exponentially stable. If $\tau > 0$ is such that*

$$\tau(\lambda_2 - \lambda_1) \neq 2k\pi i\,, \quad \forall\, k \in \mathbb{Z} \setminus \{0\},\ \forall\, \lambda_1, \lambda_2 \in \sigma(A) \cap \bar{\mathbb{C}}_0\,, \tag{4.24}$$

and if $(I - \mathbf{H}_\tau)^{-1} \in H^\infty(\mathbb{E}, \mathcal{B}(U))$, then Δ_τ is power-stable.

Note that under the assumptions of the above lemma, (4.24) is satisfied for all sufficiently small $\tau > 0$.

Proof of Lemma 4.7: By Corollary 4.6, there exists a decomposition $X = X_1 \oplus X_2$ such that (D1)-(D4) hold. In particular, using the notation of (4.13) and (4.14), we have $\sigma(A_1) = \sigma(A) \cap \bar{\mathbb{C}}_0$, and, moreover, setting $S_\tau^j := \int_0^\tau T_j(s) B_j \, ds$, $j = 1, 2$, we may write

$$T(\tau) = \begin{pmatrix} T_1(\tau) & 0 \\ 0 & T_2(\tau) \end{pmatrix}, \quad S_\tau = \begin{pmatrix} S_\tau^1 \\ S_\tau^2 \end{pmatrix}, \quad F = (F_1, F_2)\,. \tag{4.25}$$

Assume that τ satisfies (4.24). By Lemma 4.4, the finite-dimensional controlled system (A_1, B_1) is controllable, and so by a well-known result (see [6] or Theorem 4 on p. 102 in Sontag [14]), the discrete-time system $(T_1(\tau), S_\tau^1)$ is controllable as well. This implies in particular, that there exists $K_1 \in \mathcal{B}(X_1, U)$ such that the matrix $T_1(\tau) + S_\tau^1 K_1$ is power-stable. Now $T_2(\tau)$ is power-stable (since the semigroup $T_2(t)$ is exponentially stable by

(D4)) and, setting $K := (K_1, 0) \in \mathcal{B}(X, U)$, it follows from (4.25) that $T(\tau) + S_\tau K$ is power-stable, showing that the controlled discrete-time system $(T(\tau), S_\tau)$ is stabilizable. A similar argument shows that the observed discrete-time system $(T(\tau), F)$ is detectable. Hence, if $(I - \mathbf{H}_\tau)^{-1} \in H^\infty(\mathbb{E}, \mathcal{B}(U))$, we may conclude that Δ_τ is power-stable (see Theorem 2 in Logemann [8]). □

The following theorem is the main result of this section.

Theorem 4.8 *Assume that A generates an analytic semigroup $T(t)$ on X, $B \in \mathcal{B}(U, X_{-1})$, $F \in \mathcal{B}(X, U)$ is compact. If the semigroup generated by A_{BF} is exponentially stable, then there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, there exist $N \geq 1$ and $\nu > 0$ such that all solutions of (2.3) satisfy $\|x(t)\| \leq Ne^{-\nu t}\|x^0\|$ for all $x^0 \in X$ and all $t \geq 0$.*

The following lemma is the key tool for the proof of Theorem 4.8.

Lemma 4.9 *For $\eta > 0$, define $W(\eta) := \{z = e^s \mid s \in \mathbb{C}_0, |s| < \eta\}$. Under the assumptions of Theorem 4.8, there exist $\eta > 0$ and $\theta > 0$ such that*

$$(I - \mathbf{H}_\tau)^{-1} \in H^\infty(W(\eta), \mathcal{B}(U)) \quad \forall \tau \in (0, \theta), \tag{4.26}$$

where $\mathbf{H}_\tau$ is given by (4.21).

We defer the long and technical proof of Lemma 4.9 to the end of this section.

Proof of Theorem 4.8: By Lemma 2.3 and Lemma 4.7, it is sufficient to show that there exists $\tau^* > 0$ such that

$$(I - \mathbf{H}_\tau)^{-1} \in H^\infty(\mathbb{E}, \mathcal{B}(U)), \quad \forall \tau \in (0, \tau^*). \tag{4.27}$$

To this end, let $M \geq 1$ and $\mu \in \mathbb{R}$ be such that

$$\|T(t)\| \leq Me^{\mu t}, \quad \forall t \in \mathbb{R}_+. \tag{4.28}$$

We split $\mathbb{E}$ into three disjoint parts

$$\mathbb{E} = E_1(\delta) \cup E_2(\delta) \cup E_3, \tag{4.29}$$

where, for $\delta \in (0, 1)$,

$$\begin{aligned} E_1(\delta) &:= \{z \in \mathbb{C} \mid 1 < |z| < 4M, |z - 1| \leq \delta\}, \\ E_2(\delta) &:= \{z \in \mathbb{C} \mid 1 < |z| < 4M, |z - 1| > \delta\}. \end{aligned}$$

and $E_3 := \{z \in \mathbb{C} \mid |z| \geq 4M\}$. We proceed in several steps.

Step 1: We claim that there exist $\delta > 0$ and $\tau_1 > 0$ such that

$$(I - \mathbf{H}_\tau)^{-1} \in H^\infty(E_1(\delta), \mathcal{B}(U)), \quad \forall \tau \in (0, \tau_1). \tag{4.30}$$

By Lemma 4.9 there exists $\eta > 0$ and $\theta > 0$ such that (4.26) holds. Choosing $\delta \in (0,1)$ such that $E_1(\delta) \subset W(\eta)$, we see that (4.30) follows with $\tau_1 = \theta$.

Step 2: Let $\delta \in (0,1)$ be as in Step 1. We claim that there exists $\tau_2 > 0$ such that

$$(I - \mathbf{H}_\tau)^{-1} \in H^\infty(E_2(\delta), \mathcal{B}(U)), \quad \forall \tau \in (0, \tau_2). \tag{4.31}$$

To this end first note that by (4.23)

$$\mathbf{H}_\tau^1(z) = F_1((z-1)I - (T_1(\tau) - I))^{-1} \int_0^\tau T_1(s) B_1 \, ds .$$

Since $T_1(t)$ is a semigroup on the finite-dimensional space X_1, we have $\lim_{\tau \to 0} \|T_1(\tau) - I\| = 0$. Moreover,

$$\lim_{\tau \to 0} \| \int_0^\tau T_1(s) B_1 \, ds \| = 0 .$$

Since $|z - 1| > \delta > 0$ for all $z \in E_2(\delta)$, a routine argument involving the application of the Neumann series shows that there there exists $\theta^* > 0$ such that

$$\|\mathbf{H}_\tau^1(z))\| \leq 1/4 , \quad \forall z \in E_2(\delta), \forall \tau \in (0, \theta^*). \tag{4.32}$$

Now we analyze $\mathbf{H}_\tau^2(z)$ on $E_2^{cl}(\delta)$, the closure of $E_2(\delta)$. Using analyticity and boundedness of the semigroup $T_2(t)$ (guaranteed by (D4)), an application of Theorem 5.6 (b) on p. 66 in Pazy [9] shows that for each $\lambda \in E_2^{cl}(\delta)$ there exist $\theta_\lambda > 0$ and $k_\lambda > 0$ such that

$$\|(\lambda I - T_2(\tau))^{-1}\| \leq k_\lambda , \quad \forall \tau \in (0, \theta_\lambda). \tag{4.33}$$

For $z, \lambda \in E_2^{cl}(\delta)$ we have

$$\|(zI - T_2(\tau))^{-1}\| \leq \|(T_2(\tau) - \lambda I)^{-1}\| \|(I - (z - \lambda)(T_2(\tau) - \lambda I)^{-1})^{-1}\| .$$

Combining this with (4.33), an application of the Neumann series yields that for each $\lambda \in E_2^{cl}(\delta)$

$$\|(zI - T_2(\tau))^{-1}\| \leq 2k_\lambda , \quad \forall z \in \mathbb{B}(\lambda, 1/(2k_\lambda)), \forall \tau \in (0, \theta_\lambda).$$

Since $E_2^{cl}(\delta)$ is compact we may conclude using a standard compactness argument that there exist $k > 0$ and $\tilde{\theta} > 0$ such that

$$\|(zI - T_2(\tau))^{-1}\| \leq k , \quad \forall z \in E_2^{cl}(\delta), \tau \in (0, \tilde{\theta}). \tag{4.34}$$

By (4.23),

$$\mathbf{H}_\tau^2(z) = F_2(T_2(\tau) - I)(zI - T_2(\tau))^{-1} A_2^{-1} B_2 ,$$

and since F_2 is compact, it follows from Lemma 2.1 and (4.34) that there exists $\hat{\theta} \in (0, \tilde{\theta})$ such that

$$\|\mathbf{H}_\tau^2(z)\| \le 1/4\,, \quad \forall\, z \in E_2(\delta),\ \tau \in (0, \hat{\theta})\,.$$

Combining this with (4.32) and (4.22) shows that

$$\|\mathbf{H}_\tau(z)\| \le 1/2\,, \quad \forall\, z \in E_2(\delta),\ \tau \in (0, \tau_2)\,,$$

where $\tau_2 := \min(\theta^*, \hat{\theta})$. Yet another application of the Neumann series yields (4.31).

Step 3: We claim that there exists $\tau_3 > 0$ such that

$$(I - \mathbf{H}_\tau)^{-1} \in H^\infty(E_3, \mathcal{B}(U))\,, \quad \forall\, \tau \in (0, \tau_3)\,. \tag{4.35}$$

By (2.10) and (4.28), together with Lemma 2.2, there exists $\tau_3 > 0$ such that

$$\|\Delta_\tau\| \le 2M\,, \quad \forall\, \tau \in (0, \tau_3)\,.$$

Defining the open set

$$\tilde{E}_3 := \{z \in \mathbb{C} \,|\, |z| > 3M\} \supset E_3\,,$$

a routine argument involving the Neumann series then shows that

$$\|(zI - \Delta_\tau)^{-1}\| = |z^{-1}| \|(I - z^{-1}\Delta_\tau)^{-1}\| \le 1/M\,, \quad \forall\, z \in \tilde{E}_3, \forall\, \tau \in (0, \tau_3)\,. \tag{4.36}$$

Now

$$(I - \mathbf{H}_\tau(z))^{-1} = I + F(zI - \Delta_\tau)^{-1} S_\tau\,, \tag{4.37}$$

and so (4.35) follows from (4.36), (4.37) and Lemma 2.2.

Step 4: Finally, setting $\tau^* = \min(\tau_1, \tau_2, \tau_3)$, a combination of (4.29)-(4.31) and (4.35) yields (4.27). □

It remains to prove the crucial Lemma 4.9.

Proof of Lemma 4.9: Recall the definition of $\mathbf{G}$, see (4.2). Appealing to (D1)-(D4), (4.13) and (4.14), we may write

$$\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2\,, \quad \text{where} \quad \mathbf{G}_j(s) = F_j(sI - A_j)^{-1} B_j\,, \ \ j = 1, 2\,. \tag{4.38}$$

It is convenient to set

$$\mathbb{C}_0^* := \mathbb{C}_0 \setminus \sigma(A) = \mathbb{C}_0 \setminus \sigma(A_1)\,.$$

For $\eta > 0$ and $\theta > 0$, we define

$$\begin{aligned} V(\theta, \eta) &:= \{(\tau, s) \in (0, \infty) \times \mathbb{C}_0 \mid \tau \in (0, \theta),\ \tau|s| < \eta\}\,, \\ V^*(\theta, \eta) &:= \{(\tau, s) \in (0, \infty) \times \mathbb{C}_0^* \mid \tau \in (0, \theta),\ \tau|s| < \eta\}\,. \end{aligned}$$

The idea of the proof is to compare $\mathbf{H}_\tau(e^{\tau s})$ to $\mathbf{G}(s)$ for all $(\tau, s) \in V^*(\theta, \eta)$, where $\eta > 0$ and $\theta > 0$ are sufficiently small. We proceed in several steps.

Step 1: In this step we analyze the term $\mathbf{H}^1_\tau(e^{\tau s}) - \mathbf{G}_1(s)$. We claim that

$$\mathbf{G}_1(s) - \mathbf{H}^1_\tau(e^{\tau s}) = F_1(sI - A_1)^{-1}Q(\tau, s)B_1 , \quad \forall\, (\tau, s) \in (0, \infty) \times \mathbb{C}^*_0 , \tag{4.39}$$

where Q is a matrix-valued function defined on $(0, \infty) \times \mathbb{C}^*_0$ such that

$$\lim_{(\theta,\eta)\to 0} (\sup\{\|Q(\tau, s)\| \mid (\tau, s) \in V^*(\theta, \eta)\}) = 0 . \tag{4.40}$$

Using the finite-dimensionality of X_1, we obtain using Taylor expansions

$$\int_0^\tau T_1(t)B_1 \, dt = \tau(I + Q_1(\tau))B_1 , \quad I - e^{-\tau s}T_1(\tau) = \tau(I + Q_2(\tau, s))(sI - A_1) , \tag{4.41}$$

where

$$Q_1(\tau) := \sum_{k=2}^{\infty} \frac{\tau^{k-1}}{k!} A_1^{k-1} , \quad Q_2(\tau, s) := \sum_{k=2}^{\infty} \frac{\tau^{k-1}}{k!} (A_1 - sI)^{k-1} .$$

Note that Q_1 and Q_2 satisfy

$$\lim_{\tau\to 0} Q_1(\tau) = 0 , \quad \lim_{(\tau,\tau s)\to 0} Q_2(\tau, s) = 0 . \tag{4.42}$$

The first limit follows immediately from the definition of Q_1, whilst the second limit follows from the inequality

$$\|Q_2(\tau, s)\| \le e^{\tau(|s|+\|A_1\|)} - 1 ,$$

which is the result of a routine estimate. Using (4.41), a straightforward calculation shows that (4.39) holds with

$$Q(\tau, s) := \tau[(1 - e^{-\tau s})I - e^{-\tau s}Q_1(\tau) + Q_2(\tau, s)](sI - A_1)(I - e^{-\tau s}T_1(\tau))^{-1} . \tag{4.43}$$

Note that, by the spectral mapping theorem, $I - e^{-\tau s}T_1(\tau)$ is invertible for all $(\tau, s) \in (0, \infty) \times \mathbb{C}^*_0$, and so, $Q(\tau, s)$ is well-defined for all $(\tau, s) \in (0, \infty) \times \mathbb{C}^*_0$. From (4.41) we obtain

$$Q(\tau, s) = [(1 - e^{-\tau s})I - e^{-\tau s}Q_1(\tau) + Q_2(\tau, s)](I + Q_2(\tau, s))^{-1} ,$$

and it follows from (4.42) that Q satisfies (4.40).

Step 2: We show that

$$\lim_{(\theta,\eta)\to 0} (\sup\{\|\mathbf{H}^2_\tau(e^{\tau s}) - \mathbf{G}_2(s)\| \mid (\tau, s) \in V(\theta, \eta)\}) = 0 . \tag{4.44}$$

In order to estimate the difference $\mathbf{H}^2_\tau(e^{\tau s}) - \mathbf{G}_2(s)$ note that

$$\mathbf{H}^2_\tau(e^{\tau s}) - \mathbf{G}_2(s) = F_2 \left[(e^{\tau s}I - T_2(\tau))^{-1}(T_2(\tau) - I)A_2^{-1} - (sI - A_2)^{-1}\right] B_2$$

$$= F_2 \left[(T_2(\tau) - I)A_2^{-1}(sI - A_2) - (e^{\tau s}I - T_2(\tau)) \right] (e^{\tau s}I - T_2(\tau))^{-1}(sI - A_2)^{-1}B_2 \,.$$

Setting $R_2(s) := (sI - A_2)^{-1}$, routine algebraic manipulations show that

$$\mathbf{H}_\tau^2(e^{\tau s}) - \mathbf{G}_2(s) = P_1(\tau, s)P_2(\tau, s)\,, \tag{4.45}$$

where

$$\begin{aligned} P_1(\tau, s) &:= F_2 \left(e^{-\tau s}\frac{T_2(\tau) - I}{\tau}A_2^{-1} - \frac{1 - e^{-\tau s}}{\tau s}I \right), \\ P_2(\tau, s) &:= \left(\frac{I - e^{-\tau s}T_2(\tau)}{\tau s} \right)^{-1} R_2(s)B_2 \,. \end{aligned}$$

We first analyze $P_1(\tau, s)$. Since the term

$$e^{-\tau s}\frac{T_2(\tau) - I}{\tau}A_2^{-1} - \frac{1 - e^{-\tau s}}{\tau s}I$$

converges strongly to 0 as $(\tau, \tau s) \to 0$, the compactness of F_2 together with Lemma 2.1 shows that

$$\lim_{(\tau,\tau s)\to 0} \|P_1(\tau, s)\| = 0\,. \tag{4.46}$$

Estimating $P_2(\tau, s)$ is considerably more difficult. In view of (4.45) and (4.46), to establish (4.44), it is sufficient to prove the following claim.

Claim: For any $p \in (0, \pi)$ there exist $\delta > 0$ and $k > 0$ such that

$$\|P_2(\tau, s)\| \leq k < \infty\,, \quad \forall\, (\tau, s) \in V(\delta, p)\,. \tag{4.47}$$

Proof: Since the semigroup $T_2(t)$ is exponentially stable, for fixed $s \in \mathbb{C}_0$ and $\tau > 0$, the spectral radius of $e^{-\tau s}T_2(\tau)$ is smaller than 1, and the Neumann series gives

$$(I - e^{-\tau s}T_2(\tau))^{-1} = \sum_{k=0}^{\infty} e^{-k\tau s}T_2(k\tau)\,. \tag{4.48}$$

Since $T_2(t)$ is an exponentially stable analytic semigroup there exist $\beta \in (0, \pi/2)$, $\omega > 0$ and $M > 0$ such that

$$\varrho(A_2) \supset \Xi := \{\xi \in \mathbb{C} \mid |\arg(\xi + \omega)| < \pi/2 + \beta\}\,,$$

and

$$\|R_2(\xi)\| \leq M/|\xi|\,, \quad \forall\, \xi \in \Xi \setminus \{0\}\,,. \tag{4.49}$$

Upon noting that $R_2(\xi)B_2 = (\xi R_2(\xi) - I)A_2^{-1}B_2$, (4.49) shows that for any $\varepsilon > 0$ there exists $N > 0$ such that

$$\|R_2(\xi)B_2\|_{\mathcal{B}(U,X)} \leq N\,, \quad \forall\, \xi \in \Xi\,. \tag{4.50}$$

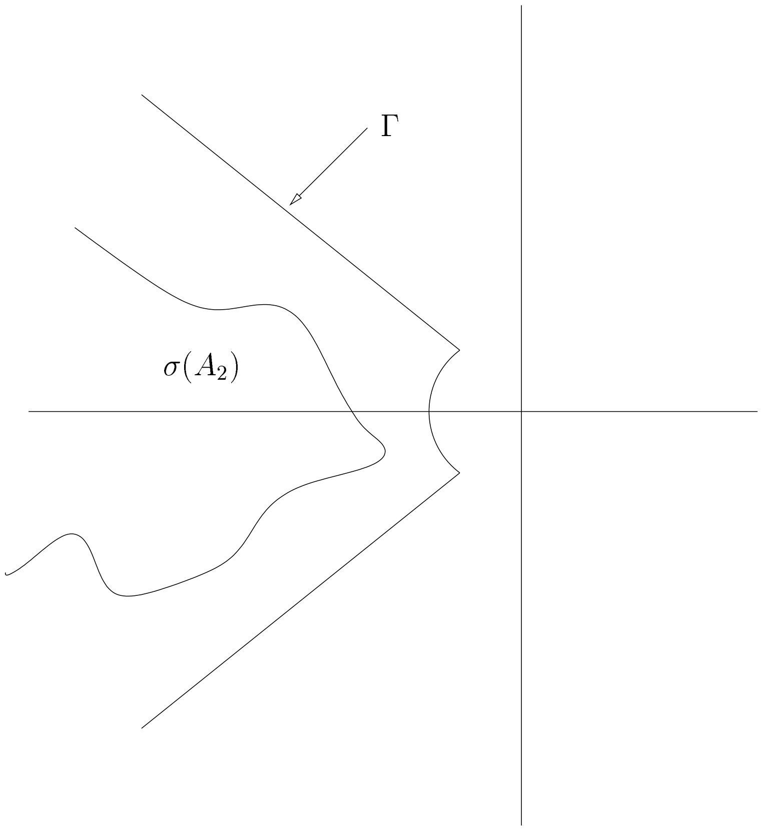

Figure 4.1: *The contour* Γ *with* $\sigma(A_2)$ *to the left of* Γ.

Let $\psi \in (\pi/2, \pi/2+\beta)$ and $r_0 \in (0,\omega)$. Define the contour

$$\Gamma = \{re^{-i\psi} \mid r > r_0\} \cup \{r_0 e^{i\phi} \mid \phi \in [\psi, 2\pi - \psi]\} \cup \{re^{i\psi} \mid r > r_0\},$$

oriented bottom to top. Note that by construction $\Gamma \subset \Xi$ and $\sigma(A_2)$ is to the left of Γ, see Figure 4.1.

Invoking (4.48) and using a standard relationship between $T_2(t)$ and $R_2(s)$ (see [9], Theorem 7.7, p. 30), we have

$$(I - e^{-\tau s} T_2(\tau))^{-1} = I + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \int_\Gamma e^{-k\tau s} e^{k\tau\lambda} R_2(\lambda)\, d\lambda. \tag{4.51}$$

By (4.49)

$$\|e^{-k\tau s} e^{k\tau\lambda} R_2(\lambda)\| \leq \frac{M}{|\lambda|} |e^{k\tau(\lambda - s)}|, \quad \forall\, \lambda \in \Gamma,\, \forall\, s \in \mathbb{C}_0,\, \forall\, \tau > 0\,.$$

Hence we can use Lebesgue's dominated convergence theorem to interchange the order of summation and integration in (4.51). Evaluating the resulting infinite sum we obtain

$$P_2(\tau, s) = \tau s R_2(s) B_2 + \frac{1}{2\pi i} \int_\Gamma \frac{\tau s e^{\tau(\lambda - s)}}{1 - e^{\tau(\lambda - s)}} R_2(\lambda) R_2(s) B_2\, d\lambda. \tag{4.52}$$

Setting

$$f(\tau, \lambda, s) := \frac{\tau s e^{\tau(\lambda - s)}}{1 - e^{\tau(\lambda - s)}} + \frac{s}{\lambda - s} = \frac{\tau s}{e^{\tau(s-\lambda)} - 1} - \frac{s}{s - \lambda}\,, \tag{4.53}$$

and letting $p \in (0,\pi)$ be fixed, but arbitary, an elementary analysis (see Appendix) shows that

$$\sup\{|\frac{f(\tau,\lambda,s)}{\tau(s-\lambda)}| \mid (\tau,\lambda,s) \in (0,\infty)\times\Gamma\times\mathbb{C}_0 \quad \text{s.t. } \tau|\lambda| \le p,\ \tau|s| \le p\} < \infty \tag{4.54}$$

Define $\delta := p/r_0$ and for $\tau \in (0,\delta)$ introduce

$$\Gamma_\tau := \{\lambda \in \Gamma \mid \tau|\lambda| \le p\}\,.$$

Clearly, $\Gamma_\tau \neq \emptyset$ for all $\tau \in (0,\delta)$. We decompose the integral on the right-hand side of (4.52) as follows

$$I(\tau,s) := \int_\Gamma \frac{\tau s e^{\tau(\lambda-s)}}{1-e^{\tau(\lambda-s)}} R_2(\lambda)R_2(s)B_2\,d\lambda = I_1(\tau,s) + I_2(\tau,s)R_2(s)B_2\,, \tag{4.55}$$

where

$$I_1(\tau,s) := \int_{\Gamma_\tau} \frac{\tau s e^{\tau(\lambda-s)}}{1-e^{\tau(\lambda-s)}} R_2(\lambda)R_2(s)B_2\,d\lambda\,, \quad I_2(\tau,s) := \int_{\Gamma\setminus\Gamma_\tau} \frac{\tau s e^{\tau(\lambda-s)}}{1-e^{\tau(\lambda-s)}} R_2(\lambda)\,d\lambda\,.$$

We first analyze $I_1(\tau,s)$. Using the resolvent identity

$$(s-\lambda)R_2(\lambda)R_2(s) = R_2(\lambda) - R_2(s)\,,$$

and invoking (4.53), we obtain

$$I_1(\tau,s) = \int_{\Gamma_\tau} f(\tau,\lambda,s)\frac{R_2(\lambda)-R_2(s)}{s-\lambda}B_2\,d\lambda + \int_{\Gamma_\tau}\frac{s}{s-\lambda}R_2(\lambda)R_2(s)B_2\,d\lambda\,. \tag{4.56}$$

Setting

$$I_{11}(\tau,s) := \int_{\Gamma_\tau} f(\tau,\lambda,s)\frac{R_2(\lambda)-R_2(s)}{s-\lambda}B_2\,d\lambda\,,\ \ I_{12}(\tau,s) := \int_{\Gamma_\tau}\frac{s}{s-\lambda}R_2(\lambda)\,d\lambda\,,$$

we have

$$I_1(\tau,s) = I_{11}(\tau,s) + I_{12}R_2(s)B_2\,. \tag{4.57}$$

Invoking (4.50) and (4.54) shows that there exists $k_1 > 0$ such that for all $(\tau,s) \in V(\delta,p)$

$$\begin{aligned} \|I_{11}(\tau,s)\| &\le \int_{\Gamma_\tau} \frac{|f(\tau,\lambda,s)|}{|\lambda-s|}\,(\|R_2(\lambda)B_2\| + \|R_2(s)B_2\|)\ d|\lambda| \\ &\le k_1\tau\ \text{length}\,(\Gamma_\tau)\,. \end{aligned}$$

Combining this with

$$\text{length}\,(\Gamma_\tau) \le 2(\pi+1)p/\tau$$

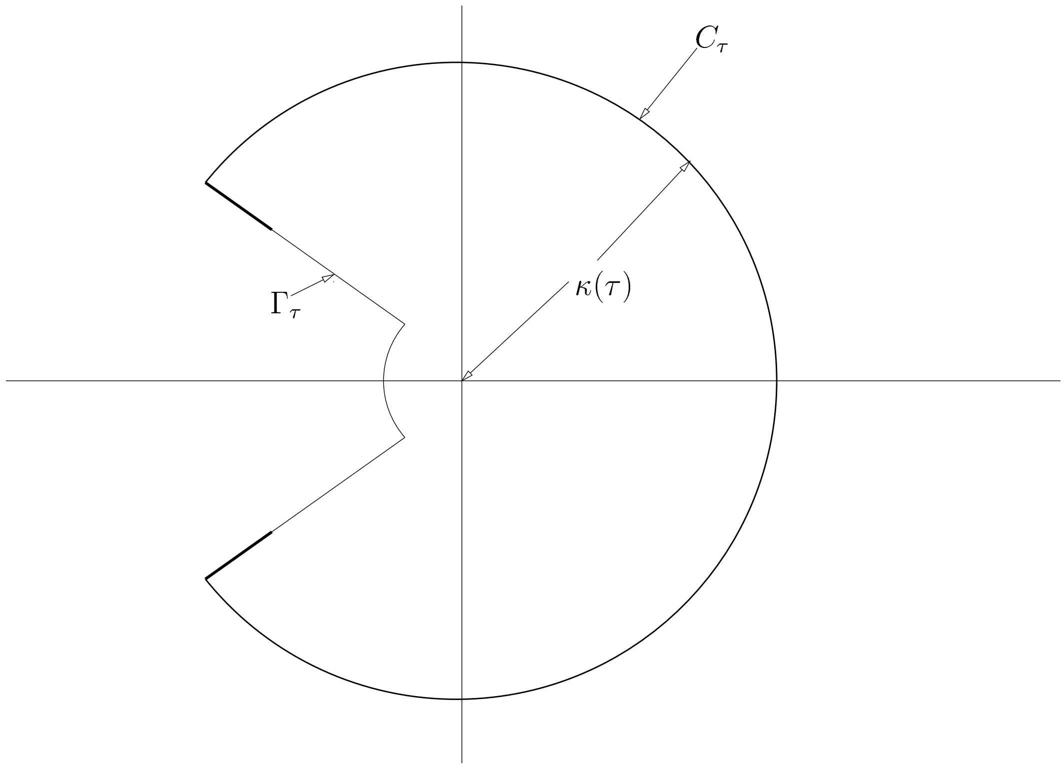

Figure 4.2: *The contour Γ_τ embedded in a closed contour C_τ.*

and setting $k_2 := 2k_1(\pi+1)p$ yields

$$\|I_{11}(\tau,s)\| \leq k_2\,, \quad \forall\,(\tau,s)\in V(\delta,p)\,. \tag{4.58}$$

To estimate $I_{12}(\tau,s)$, we define

$$\kappa(\tau) := 2p/\tau\,,$$

and we embed, for each $\tau\in(0,\delta)$, Γ_τ into a closed contour

$$\begin{aligned} C_\tau \;&:=\; \{re^{-i\psi}\mid r\in[r_0,\kappa(\tau)]\}\cup\{r_0e^{i\phi}\mid \phi\in[\psi,2\pi-\psi]\}\cup\{re^{i\psi}\mid r\in[r_0,\kappa(\tau)]\}\\ &\quad\;\cup\{\kappa(\tau)e^{i\phi}\mid\phi\in[-\psi,\psi]\}, \end{aligned}$$

oriented clockwise (see Figure 4.2). Then

$$\begin{aligned} I_{12}(\tau,s) \;=\;& \int_{C_\tau}\frac{s}{(s-\lambda)}R_2(\lambda)\,d\lambda - \int_{p/\tau}^{\kappa(\tau)}\frac{s}{s-re^{i\psi}}R_2(re^{i\psi})e^{i\psi}\,dr\\ &+\int_{p/\tau}^{\kappa(\tau)}\frac{s}{s-re^{-i\psi}}R_2(re^{-i\psi})e^{-i\psi}\,dr+\int_{-\psi}^{\psi}\frac{is\kappa(\tau)e^{i\phi}}{s-\kappa(\tau)e^{i\phi}}R_2(\kappa(\tau)e^{i\phi})d\phi. \end{aligned}\tag{4.59}$$

By construction, for a given $\tau\in(0,\delta)$, any $s\in\mathbb{C}_0\cap\mathbb{B}(0,p/\tau)$ is inside the contour C_τ and so Cauchy's integral formula combined with (4.49) shows that there exists $k_3>0$ such that

$$\|\int_{C_\tau}\frac{s}{(s-\lambda)}R_2(\lambda)\,d\lambda\| = 2\pi\|sR_2(s)\|\leq k_3\,,\quad\forall\,(\tau,s)\in V(\delta,p)\,. \tag{4.60}$$

Moreover, since $|s-re^{i\psi}| > r|\cos\psi|$ for $s \in \mathbb{C}_0$, it follows from (4.49) that there exists $k_4 > 0$ such that

$$\| \int_{p/\tau}^{\kappa(\tau)} \frac{s}{s-re^{i\psi}} R_2(re^{i\psi}) e^{i\psi} dr \| \leq \frac{k_4|s|}{|\cos\psi|} \int_{p/\tau}^{\infty} \frac{dr}{r^2} = \frac{k_4}{p|\cos\psi|}\tau|s|\,, \quad \forall\,(\tau,s) \in V(\delta,p)\,.$$

As an immediate consequence we see that

$$\| \int_{p/\tau}^{\kappa(\tau)} \frac{s}{s-re^{i\psi}} R_2(re^{i\psi}) e^{i\psi}\, dr \| \leq k_5\,, \quad \forall\,(\tau,s) \in V(\delta,p)\,, \tag{4.61}$$

for some suitable $k_5 > 0$. Similarly, there exists $k_6 > 0$ such that

$$\| \int_{p/\tau}^{\kappa(\tau)} \frac{s}{s-re^{-i\psi}} R_2(re^{-i\psi}) e^{-i\psi}\, dr \| \leq k_6\,, \quad \forall\,(\tau,s) \in V(\delta,p)\,. \tag{4.62}$$

Trivially, since $\kappa(\tau) = 2p/\tau$,

$$|s - \kappa(\tau)e^{i\phi}| \geq p/\tau\,, \quad \forall\,(\tau,s) \in V(\delta,p), \forall\,\phi \in [-\psi,\psi]\,,$$

and hence, using (4.49), we see that there exists $k_7 > 0$ such that

$$\begin{aligned} \| \int_{-\psi}^{\psi} \frac{is\kappa(\tau)e^{i\phi}}{s-\kappa(\tau)e^{i\phi}} R_2(\kappa(\tau)e^{i\phi}) d\phi \| &\leq \frac{k_7\tau|s|}{p} \int_{-\psi}^{\psi} d\phi = \frac{2k_7\psi\tau|s|}{p} \\ &\leq 2k_7\psi\,, \quad \forall\,(\tau,s) \in V(\delta,p)\,. \end{aligned} \tag{4.63}$$

Combining (4.59)-(4.63), shows that there exists $k_8 > 0$ such that

$$\|I_{12}(\tau,s))\| \leq k_8\,, \quad \forall\,(\tau,s) \in V(\delta,p)\,. \tag{4.64}$$

By (4.50) we may conclude that $I_{12}(\tau,s)R_2(s)B_2$ is uniformly bounded on $V(\delta,p)$, and therefore, by (4.57) and (4.58), we obtain that there exists $k_9 > 0$ such that

$$\|I_1(\tau,s))\| \leq k_9\,, \quad \forall\,(\tau,s) \in V(\delta,p)\,. \tag{4.65}$$

To estimate $I_2(\tau,s)$ first note that there exists $k_{10} > 0$ such that

$$|1 - e^{\tau(\lambda-s)}| \geq k_{10} > 0\,, \quad \forall\,(\tau,\lambda,s) \in (0,\infty)\times\Gamma\times\mathbb{C}_0 \ \text{ s.t. } \lambda \in \Gamma\setminus\Gamma_\tau\,.$$

Combining this with an application of (4.49), we obtain

$$\|I_2(\tau,s)\| \leq \frac{2k_{11}p}{k_{10}} \int_{p/\tau}^{\infty} \frac{e^{\tau r\cos\psi}}{r}\, dr\,, \quad \forall\,(\tau,s) \in V(\delta,p)\,, \tag{4.66}$$

for some suitable constant $k_{11} > 0$. Since $\cos\psi < 0$

$$\int_{p/\tau}^{\infty} \frac{e^{\tau r \cos\psi}}{r}\, dr \leq \frac{\tau}{p\tau|\cos\psi|} e^{p\cos\psi} \leq \frac{e^{p\cos\psi}}{p|\cos\psi|} < \infty\,.$$

Together with (4.66) this shows that there exists $k_{12} > 0$ such that

$$\|I_2(\tau, s)\| \leq k_{12}\,, \quad \forall\,(\tau, s) \in V(\delta, p)\,. \tag{4.67}$$

Combining (4.52), (4.55), (4.65) and (4.67) and invoking (4.50), shows that (4.47) holds, completing the proof of the claim.

Step 3: By assumption, the analytic semigroup generated by A_{BF} is exponentially stable. Therefore, by Proposition 4.5,

$$(I - \mathbf{G})^{-1} \in H^\infty(\mathbb{C}_0, \mathcal{B}(U))\,, \tag{4.68}$$

which in particular implies that $I - \mathbf{G}(s)$ is invertible for all $s \in \mathbb{C}_0^*$. Setting

$$D(\tau, s) := \mathbf{G}_2(s) - \mathbf{H}_\tau^2(e^{s\tau})\,, \quad \forall\,(\tau, s) \in (0, \infty) \times \mathbb{C}_0\,, \tag{4.69}$$

and

$$\tilde{Q}(\tau, s) := \begin{pmatrix} Q(\tau, s) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(X_{-1}, X)\,, \quad \forall\,(\tau, s) \in (0, \infty) \times \mathbb{C}_0^*\,, \tag{4.70}$$

with Q given by (4.43), we obtain from (4.38) and (4.39) for all $(\tau, s) \in (0, \infty) \times \mathbb{C}_0^*$

$$\begin{aligned} I - \mathbf{H}_\tau(e^{\tau s})) &= I - \mathbf{G}_1(s) - \mathbf{G}_2(s) + F_1(sI - A_1)^{-1}Q(\tau, s)B_1 + D(\tau, s) \\ &= (I - \mathbf{G}(s))[I + (I - \mathbf{G}(s))^{-1}F(sI - A)^{-1}\tilde{Q}(s, \tau)B \\ &\quad + (I - \mathbf{G}(s))^{-1}D(\tau, s)]. \end{aligned} \tag{4.71}$$

A straightforward calculation combined with an application of statements (2) and (3) of Lemma 4.1 shows that

$$\begin{aligned} (I - \mathbf{G}(s))^{-1}F(sI - A)^{-1}x &= F(sI - A - BF)^{-1}x \\ &= F(sI - A_{BF})^{-1}x\,, \quad \forall\, s \in \mathbb{C}_0^*,\ \forall\, x \in X\,. \end{aligned} \tag{4.72}$$

Moreover, by (4.70),

$$(\tilde{Q}(\tau, s)B)U \subset X\,, \quad \forall\,(\tau, s) \in (0, \infty) \times \mathbb{C}_0^*\,. \tag{4.73}$$

Combining (4.71)-(4.73) shows that for all $(\tau, s) \in (0, \infty) \times \mathbb{C}_0^*$

$$I - \mathbf{H}_\tau(e^{\tau s}) = (I - \mathbf{G}(s))[I + F(sI - A_{BF})^{-1}\tilde{Q}(\tau, s)B + (I - \mathbf{G}(s))^{-1}D(\tau, s)]\,. \tag{4.74}$$

By assumption, the analytic semigroup generated by A_{BF} is exponentially stable. Consequently, the function $s \mapsto (sI - A_{BF})^{-1}$ is in $H^\infty(\mathbb{C}_0, \mathcal{B}(X))$, and thus we may conclude from (4.40) and (4.70) that

$$\lim_{(\theta,\eta)\to 0}(\sup\{\|F(sI - A_{BF})^{-1}\tilde{Q}(\tau,s)B\| \mid (\tau,s) \in V^*(\theta,\eta)\}) = 0\,. \tag{4.75}$$

From (4.44) and (4.69) combined with (4.68), we obtain

$$\lim_{(\theta,\eta)\to 0}(\sup\{\|(I - \mathbf{G}(s))^{-1}D(\tau,s)]\| \mid (\tau,s) \in V(\theta,\eta)\}) = 0\,. \tag{4.76}$$

Finally, combining (4.74)-(4.76) shows that that there exist $\theta > 0$ and $\eta > 0$ such that (4.26) holds. □

5. Dynamic sampled-data feedback

In this section we apply the results of the previous sections to dynamic sampled-data feedback. For simplicity and the sake of brevity, we restrict ourselves here to systems with bounded control and bounded observation. The system to be controlled, the plant, is formally given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0\,;\ x(0) = x^0 \in X\,, \tag{5.1a}$$
$$y(t) = Cx(t)\,, \tag{5.1b}$$

where $A : D(A) \subset X \to X$ generates a strongly continuous semigroup $T(t)$, $B \in \mathcal{B}(U,X)$, $C \in \mathcal{B}(X,Y)$ and X, U and Y are Hilbert spaces.

Consider a continuous-time controller of the form

$$\dot{z}(t) = A_c z(t) + B_c v(t), \quad t \geq 0\,;\ z(0) = z^0 \in X_c\,, \tag{5.2a}$$
$$w(t) = C_c z(t) + D_c v(t)\,, \tag{5.2b}$$

where $A_c : D(A_c) \subset X_c \to X_c$ generates a strongly continuous semigroup $T_c(t)$, $B_c \in \mathcal{B}(Y,X_c)$, $C_c \in \mathcal{B}(X_c,U)$, $D_c \in \mathcal{B}(Y,U)$ and X_c is a Hilbert spaces. Using the output (5.1b) of (5.1) as the input for (5.2) and the output (5.2b) of (5.2) as the input for (5.1), i.e.

$$v = y \quad \text{and} \quad u = w\,, \tag{5.3}$$

we obtain the feedback interconnection of (5.1) and (5.2). It is convenient to define

$$\tilde{A} := \begin{pmatrix} A & 0 \\ 0 & A_c \end{pmatrix},\ \tilde{B} := \begin{pmatrix} B & 0 \\ 0 & B_c \end{pmatrix},\ \tilde{C} := \begin{pmatrix} C & 0 \\ 0 & C_c \end{pmatrix},\ \tilde{K} := \begin{pmatrix} D_c & I_U \\ I_Y & 0 \end{pmatrix}.$$

Clearly, $\tilde{A}$ generates the strongly continuous semigroup

$$\tilde{T}(t) := \begin{pmatrix} T(t) & 0 \\ 0 & T_c(t) \end{pmatrix}.$$

We say that the continuous-time feedback system given by (5.1)-(5.3) is *exponentially stable* if the strongly continuous semigroup generated by

$$\tilde{A} + \tilde{B}\tilde{K}\tilde{C} = \begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix}$$

is exponentially stable.

Let $\tau > 0$ and consider the following discretization of (5.2) which is obtained by applying the standard hold and sampling operations to (5.2)

$$\begin{aligned} z_{k+1} &= T_c(\tau)z_k + \int_0^\tau T_c(s)B_cv_k\,ds\,, \quad k \in \mathbb{N}_0\,;\ z_0 = z^0 \in X_c\,, && \text{(5.4a)} \\ w_k &= C_cz_k + D_cv_k\,. && \text{(5.4b)} \end{aligned}$$

We use the discrete-time system (5.4) to control the continuous-time system (5.1) by sampled-data feedback, i.e. we consider the feedback interconnection of (5.1) and (5.4) given by

$$v_k = y(k\tau) \quad \text{and} \quad u(k\tau + t) = w_k\,,\ t \in [0,\tau);\ \ k \in \mathbb{N}_0\,.$$

Via an application of the variation of parameters formula to (5.1a), this leads to the following sampled-data feedback equations

$$\begin{aligned} x(k\tau + t) &= T(t)x(k\tau) + \int_0^t T(s)B(D_cCx(k\tau) + C_cz_k)\,ds\,,\ t \in [0,\tau)\,;\, x(0) = x^0, && \text{(5.5a)} \\ z_{k+1} &= T_c(\tau)z_k + \int_0^\tau T_c(s)B_cCx(k\tau)\,ds\,, \quad k \in \mathbb{N}_0\,;\ z_0 = z^0\,. && \text{(5.5b)} \end{aligned}$$

The sampled-data feedback system (5.5) is called *exponentially stable* if there exist $N \geq 1$ and $\nu > 0$ such that for all initial conditions $(x^0, z^0) \in X \times X_c$

$$\|(x(k\tau + t), z_k)\| \leq Ne^{-\nu(k\tau+t)}\|(x^0, z^0)\|\,, \quad \forall\, t \in [0,\tau),\ \forall\, k \in \mathbb{N}_0\,.$$

By a straightforward calculation it follows from (5.5) that

$$\begin{pmatrix} x((k+1)\tau) \\ z_{k+1} \end{pmatrix} = \tilde{\Delta}_\tau \begin{pmatrix} x(k\tau) \\ z_k \end{pmatrix}, \quad k \in \mathbb{N}_0\,,$$

where

$$\tilde{\Delta}_\tau : X \times X_c \to X \times X_c\,, \quad \tilde{x} \mapsto \tilde{T}(\tau)\tilde{x} + \int_0^\tau \tilde{T}(s)\tilde{B}\tilde{K}\tilde{C}\tilde{x}\,ds\,.$$

The proof of the following lemma is a routine exercise and is therefore left to the reader.

Lemma 5.1 *For any $\tau > 0$, the sampled-data feedback system (5.5) is exponentially stable if and only if $\tilde{\Delta}_\tau$ is power-stable.*

We are now in the position to formulate the main result of this section.

Theorem 5.2 *Assume that the operator $\tilde{K}\tilde{C}$ is compact. If the continuous-time feedback system (5.1)-(5.3) is exponentially stable (i.e. the strongly continuous semigroup generated by $\tilde{A} + \tilde{B}\tilde{K}\tilde{C}$ is exponentially stable), then there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$ the sampled-data feedback system (5.5) is exponentially stable.*

Proof: Introducing the operator

$$\tilde{S}_\tau : U \times Y \to X \times X_c\,, \quad \tilde{u} \mapsto \int_0^\tau \tilde{T}(s)\tilde{B}\tilde{u}\, ds\,.$$

we may write $\tilde{\Delta}_\tau = \tilde{T}(\tau) + \tilde{S}_\tau\tilde{K}\tilde{C}$, and we see that $\tilde{\Delta}$ is of the form (2.10). By the compactness of $\tilde{K}\tilde{C}$ and the exponential stability of the semigroup generated by $\tilde{A} + \tilde{B}\tilde{K}\tilde{C}$ it follows from Theorem 3.1 and Lemma 2.3 that there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$ the operator $\tilde{\Delta}_\tau$ is power-stable. An application of Lemma 5.1 yields the claim. □

Trivially, the compactness assumption on $\tilde{K}\tilde{C}$ is satisfied if the observation operators C and C_c are compact. In particular, compactness of $\tilde{K}\tilde{C}$ is guaranteed if U and Y are finite-dimensional.

Under suitable assumptions, Theorem 4.8 can be used to extend Theorem 5.2 to systems and controllers with unbounded control operators B and B_c, respectively, provided the semigroups $T(t)$ and $T_c(t)$ are analytic. For the sake of brevity we omit the details.

Appendix: Derivation of (4.54)

Let $p \in (0, \pi)$ and consider the meromorphic function g defined on $\mathbb{C}$ given by

$$g(\xi) := \frac{1 + \xi - e^\xi}{\xi(e^\xi - 1)}\,.$$

Clearly, for any $\varepsilon \in (0, 2p)$, the function g is bounded on $\mathbb{B}(0, 2p) \setminus \mathbb{B}(0, \varepsilon)$. Moreover, invoking the Taylor expansion of e^ξ or using L'Hôspital's rule we have

$$\lim_{\xi \to 0} g(\xi) = -1/2\,.$$

Therefore, we may conclude that

$$\sup\{|g(\xi)| \mid \xi \in \mathbb{B}(0, 2p),\ \xi \neq 0\} < \infty\,. \tag{A.1}$$

By a straightforward calculation it follows from (4.53) that

$$\frac{|f(\tau,\lambda,s)|}{|\tau(s-\lambda)|} = \frac{|s|}{|s-\lambda|}|g(\tau(s-\lambda))|\,.$$

Noting that

$$\sup\{|s|/|s-\lambda| \mid (\lambda,s)\in\Gamma\times\mathbb{C}_0\} < \infty\,,$$

it follows from (A.1) that

$$\sup\{|\frac{f(\tau,\lambda,s)}{\tau(s-\lambda)}| \mid (\tau,\lambda,s)\in(0,\infty)\times\Gamma\times\mathbb{C}_0 \quad \text{s.t. } \tau|\lambda|\le p,\, \tau|s|\le p\} < \infty\,,$$

which is (4.54).

References

[1] T. Chen and B. Francis, Input-output stability of sampled-data systems, *IEEE Trans. Auto. Cont.* **36** (1991), pp. 50-58.

[2] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*, Springer, London, 1995.

[3] R.F. Curtain and H.J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, New York, 1995.

[4] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.

[5] G.F. Franklin, J.D. Powell and M. Workman, *Digital Control of Dynamic Systems*, 3rd edition, Addison Wesley, Menlo Park, 1998.

[6] R. Kalman, B. Ho and K. Narendra, Controllability of linear dynamical systems, *Contributions to Differential Equations* **1** (1963), pp. 188-213.

[7] T. Kato, *Perturbation Theory for Linear Operators*, 2nd edition, Springer, Berlin, 1980.

[8] H. Logemann, Stability and stabilizability of linear infinite-dimensional discrete-time system, *IMA J. of Mathematical Control & Information* **9** (1992), pp. 255-263.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[10] R. Rebarber and S. Townley, Stabilization of distributed parameter systems by piecewise polynomial control, *IEEE Trans. Auto. Cont.* **42** (1997), pp. 1254-1257.

[11] R. Rebarber and S. Townley, Generalized sampled data feedback control of distributed parameter systems, *Systems & Control Letters* **34** (1998), pp. 229-240.

[12] R. Rebarber and S. Townley, Robustness of distributed parameter systems with respect to sample and hold - counterexamples, Proceedings of the 2000 Conference on Decision and Control, Sydney, Australia.

[13] I.G. Rosen and C. Wang, On stabilizability and sampling for infinite dimensional systems, *IEEE Trans. Auto. Control* **37** (1992), pp. 1653-1656.

[14] E.D. Sontag, *Mathematical Control Theory*, 2nd edition, Springer, New York, 1998.

[15] T. J. Tarn, J. R. Zavgren and X. M. Zeng, Stabilization of infinite-dimensional systems with periodic gains and sampled output, *Automatica*, **24** (1988), pp. 95-99.

[16] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control & Optim.* **27** (1989), pp. 527-545.

[17] Y. Yamamoto, A function space approach to sampled data control systems and tracking problems, *IEEE Trans. Auto. Control* **39** (1994), pp. 703-713.

[18] J. Zabczyk. On decompositions of generators, *SIAM J. Control & Optim.* **16** (1978), pp. 523-534 (erratum in *SIAM J. Control & Optim.* **18** (1980), p. 325).