# BLOW-UP ESTIMATES FOR HIGHER-ORDER SEMILINEAR PARABOLIC EQUATIONS

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ABSTRACT. We prove $L^\infty$ estimates on the blow-up behaviour of solutions of a $2m$-th order semilinear parabolic equation

$$u_t = -(-\Delta)^m u + q(u), \quad x \in \mathbf{R}^N, \ t > 0; \quad m > 1,$$

with a general even function $q(u) \geq 0$ with a superlinear growth for $|u| \gg 1$. Our comparison approach and estimates apply to general integral evolution equations. We also study the following problem: find a continuous function $q(u)$ with a superlinear growth as $u \to \infty$ such that the parabolic equation exhibits regional blow-up in a domain of finite non-zero measure. We show that such a regional blow-up can occur for $q(u) = u|\ln|u||^{2m}$. We present a formal asymptotic theory explaining that the stable (generic) blow-up behaviour as $t \to T^-$ is described by the self-similar solution

$$U_*(x,t) = \exp\{(T-t)^{-1/(2m-1)}\theta(x)\}, \quad \theta : \mathbf{R}^N \to \mathbb{C},$$

of the complex Hamilton-Jacobi equation

$$U_t = (-1)^{m+1} U^{1-2m} (\nabla U \cdot \nabla U)^m + U|\ln|U||^{2m}.$$

## 1. Introduction: estimates and regional blow-up

We consider the Cauchy problem for the $2m$-th order semilinear parabolic equation

$$u_t = \mathbf{A}u + q(u) \quad \text{in } Q = \mathbf{R}^N \times \mathbf{R}_+, \tag{1.1}$$

where

$$\mathbf{A} = -(-\Delta)^m,$$

with initial data $u_0 \in X = L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Higher-order semilinear and quasilinear diffusion operators occur in several applications including thin film theory, flame and wave propagation, phase transition at critical Lifschitz points and bi-stable systems (e.g., the Kuramoto-Sivashinskii equation and the extended Fisher-Kolmogorov equation). See references in the book [23].

Formation of singularities in higher-order heat equations, which is well understood for the second order reaction-diffusion equations, is an actual problem of the general theory of higher-order parabolic equations.

Let us describe a type of singularities we are going to study. We assume that a sufficiently smooth function $q(u)$ is superlinear for $|u| \gg 1$, so that we consider solutions

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$u(x,t)$ which *blow-up* in finite time $T = T(u_0) < \infty$. This means that $u$ is a bounded classical one in $Q_\tau = \mathbf{R}^N \times (0,\tau]$ for any $\tau \in (0,T)$ and the solution cannot be extended as a bounded one beyond $t = T$. By the classical parabolic regularity theory [5], [8], this means that

$$\sup_x |u(x,t)| \to \infty \quad \text{as } \ t \to T^-.$$

The limit is understood in the usual sense (one can see that if the limit $\liminf_{t\to T^-}$ would be finite, then the solution could be extended for all $t-T>0$ small as a classical bounded solution). General results on global nonexistence for higher-order parabolic equations are well known from 70's, see a survey paper [20] and references to Chapt. 4 in [24]. In particular, the critical Fujita exponent for the semilinear equation

$$u_t = -(-\Delta)^m u + |u|^p \quad \text{in } \ \mathbf{R}^N \times \mathbf{R}_+, \quad p > 1, \tag{1.2}$$

was proved in [4] to be $p_F = 1 + 2m/N$ (for $m = 1$ we arrive at Fujita's result [7]), so that for $p \in (1, p_F]$ any solution $u \not\equiv 0$ with arbitrarily small initial data $u_0$ with nonnegative first Fourier coefficient,

$$\int_{\mathbf{R}^N} u_0(x)dx \geq 0, \tag{1.3}$$

cannot be global in time, and by the interior parabolic regularity results, it blows up in finite time. For $p > p_F$ there exists a class of small global solutions decaying as $t \to \infty$ [13].

We consider the following three question concerning blow-up behaviour for $2m$-th order semilinear parabolic equations.

**1. Blow-up.** In Section 2 we present a simple proof of blow-up based on a classical analysis of the time evolution of the first Fourier coefficient of the solution. This is essentially an ODE analysis.

**2. Rate of blow-up.** The first result proved in Section 3 is devoted to a sharp estimate on blow-up rate of the solutions for rather arbitrary source-like terms $q(u)$. The asymptotic theory of blow-up is well developed in the case of semilinear and quasilinear second-order parabolic equations, $m = 1$, where the Maximum Principle applies and therefore several delicate asymptotic results have been proved. We refer to the book [24] (quasilinear equations) and to papers [6], [17], [21], [26] (semilinear equations with $q(u) = u^p$ and $q(u) = e^u$), see also references in [24]. In the second-order case corresponding to a local in time order-preserving flow and the solutions obey the Maximum Principle (a standard comparison), the blow-up analysis is essentially based on different types of comparison. In particular, a simple lower estimate follows by a straightforward comparison with a flat solution $U(t)$ independent of $x$ and satisfying the ODE

$$U' = q(U) \quad \text{for } \ t \in (0,T), \quad U(T) = \infty,$$

so that we compare solution $u(x,t)$ and $U(t)$ having the same blow-up time. Then they must intersect each other whence an estimate from below: $\sup_x u(x,t) > U(t)$ for any $t \in (0,T)$.

In the higher-order case such a comparison is not possible since the semigroup is not order-preserving. Nevertheless, we establish that a lower $L^\infty$-estimate on the blow-up solutions of the general semilinear equation (1.1) can be proved by using comparison with similar exact solutions of the associated ODE derived from the *majorizing order-preserving equation* introduced in [13]. In Section 4 such a comparison is extended to more general evolutions equations governed by pseudodifferential (convolution) operators.

**3. Equation with regional blow-up.** Regional blow-up, i.e., blow-up of solutions in domains of non-zero measure, is well-known from 70's for essentially quasilinear second-order heat equations

$$u_t = \Delta u^{1+\sigma} + u^{1+\sigma}, \quad \sigma > 0, \tag{1.4}$$

describing processes with finite speed of propagation, see references to Chapt. 4 in [24]. Indeed, due to the same algebraic homogeneity of both operators in the right-hand side, equation (1.4) admits blow-up self-similar solutions in separate variables

$$u_*(x,t) = (T-t)^{-1/\sigma}\theta(x),$$

where $\theta \geq 0$ is a weak solution of the semilinear elliptic equation

$$\Delta\theta^{1+\sigma} + \theta^{1+\sigma} - \frac{1}{\sigma}\theta = 0 \quad \text{in} \;\; \mathbf{R}^N.$$

It always admit a compactly supported radial solution, and therefore $u_*(x,t)$ blows up as $t \to T^-$ in a bounded domain $\operatorname{supp}\theta$ and stays identically zero outside (a striking property of *blow-up localization* in a combustion medium). Such regional blow-up localization properties can be extended to more general solutions, see proofs in Chap. 4 in [24].

For the semilinear heat equations of any order with infinite speed of propagation, the question on regional blow-up and localization are more delicate. The localization phenomenon should be understood as an *effective* one in the sense that the blow-up set is a bounded subset of $\mathbf{R}^N$.

We show that, in a class of the semilinear equation (1.1), the *regional blow-up* (or the blow-up *S-regime* of *Standing* wave behaviour) occurs for the special equation

$$u_t = -(-\Delta)^m u + u|\ln|u||^{2m}. \tag{1.5}$$

For $m = 1$, this equation

$$u_t = \Delta u + u\ln^2 u \quad (u \geq 1) \tag{1.6}$$

exhibiting a striking regional blow-up property in domains with radius $L_1 = \pi$, was introduced more than 20 years ago, see [11], [9] and a list of references to § 3, Chapt. 4 in [24]. Different families of the so-called "weakly" quasilinear second-order parabolic equations with blow-up or global behaviour described by the quadratic Hamilton-Jacobi equations were studied in [9] and [12]. The Hamilton-Jacobi blow-up structure has been rigorously established in [14] (the semilinear equation in one dimension) and in [15] (radial solutions of some quasilinear equations).

We show that the asymptotic behaviour of blow-up solutions of (1.5) is described by a similarity solution of the following complex Hamilton-Jacobi equation:

$$U_t = (-1)^{m+1} \frac{(\nabla U \cdot \nabla U)^m}{U^{2m-1}} + U|\ln|U||^{2m}. \tag{1.7}$$

This similarity solution has a natural separable form

$$U_*(x,t) = \exp\{(T-t)^{-1/(2m-1)}\theta(x)\}, \tag{1.8}$$

and the support of the real part of the (radially symmetric) profile $\theta(x) : \mathbf{R}^N \to \mathbb{C}$ gives the actual *localization domain* of the solutions of both the Hamilton-Jacobi equation (1.7) and of the parabolic one (1.5). We show that the typical localization radius can be estimated from above by

$$L_m = \frac{\pi}{(2m-1)\sin(\pi/2m)} \quad (\Rightarrow\ L_1 = \pi). \tag{1.9}$$

In Section 5 we study self-similar solutions of the Hamilton-Jacobi equation

$$v_t = (-1)^{m+1}(\nabla v \cdot \nabla v)^m + |\mathrm{Re}\, v|^{2m}, \quad v : Q_T \to \mathbb{C}. \tag{1.10}$$

In Section 6 we present a formal asymptotic analysis of the rate of convergence of $u(x,t)$ to the self-similar solution of the Hamilton-Jacobi equation (1.9) and describe the first (generic) blow-up pattern for the equation under consideration.

In Section 7 we discuss three types of blow-up behaviour governed by similarity solutions of the Hamilton-Jacobi equation, for the semilinear equation

$$u_t = -(-\Delta)^m u + u|\ln|u||^{\gamma}. \tag{1.11}$$

We show that it exhibits three types of blow-up: (i) $\gamma > 2m$: single point blow-up (the LS-regime), (ii) $\gamma = 2m$: regional blow-up (the S-regime), and (iii) $1 < \gamma < 2m$: global blow-up (the HS-regime). Such a classification is quite similar to that for the second-order heat equation with $m = 1$, see p. 294 in [24].

It is worth mentioning that such a variety of different blow-up patterns of equation (1.11) is not available for a standard $2m$-th order semilinear parabolic equation (1.2). For any $p > 1$, similar to the case $m = 1$, it admits a countable subset of different blow-up patterns of single-point blow-up, which is constructed by means of a spectral analysis of the linearized rescaled non self-adjoint operator with a discrete real point spectrum, see [10] and a spectral analysis presented in Section 6.

## 2. **Proof of blow-up by ODE analysis**

Consider equation (1.1) with an even lower-order term satisfying

$$q = q(|u|) \ \text{in} \ \mathbf{R}, \quad q(u) > 0 \quad \text{and} \quad q(u) \ \text{is convex for} \ u > 0. \tag{2.1}$$

Using the same construction as in [13], we show that the blow-up theorem for higher-order equations can be proved by a special modification of the well-known method of Fourier coefficients (the eigenfunctions method) introduced in [18] for the second-order equations. Different extensions of this Kaplan's idea are discussed in Levine's survey paper [20]. Moreover, it is sufficient to control the first positive coefficient, and the result is based on blow-up properties of a single *ordinary differential inequality* (ODI). Without loss of generality, we consider initial data $u_0 \in X$ concentrated around the origin, $u_0(x) \to 0$ as $|x| \to \infty$ and $u(x,t)$ and its spatial derivatives vanish as $|x| \to \infty$ for $t > 0$. For instance, this is true for compactly supported data $u_0 \in X$, and the general case is then treated by approximation.

We first construct a nonnegative smooth radial function $\psi_1 = \psi_1(|x|) > 0$ satisfying $\psi_1(0) = 1;\ \ 0 < \psi(r) \le 1,\ \ r > 0$, $\psi_1(r)$ is decreasing and $\psi_1(r) \to 0$ as $r \to \infty$ sufficiently fast. The analysis is essentially based on a special choice of this test function $\psi_1$ associated with an "eigenvalue" inequality for the elliptic operator $\Delta^m$. We will use the following estimate on $\psi_1$: there exists a constant $\lambda_1 > 0$ such that

$$|\Delta^m \psi_1| \le \lambda_1 \psi_1 \quad \text{in } \mathbf{R}^N. \tag{2.2}$$

One can see that in the present geometry such an "eigenfunction" $\psi_1$ always exists. Actually, we need to check if (2.2) holds for $r \gg 1$ where we put $\psi_1(r) = e^{-r^\beta}$ with an exponent $\beta \in (0,1]$. Then $\psi_1$ is extended for all $r \in [0,\infty)$ by a smooth approximation. We next normalize $\psi_1$ so that

$$\|\psi_1\|_1 \equiv \int_{\mathbf{R}^N} \psi(x)dx = 1.$$

Multiplying equation (1.1) by $\psi_1$ and integrating by parts in $\mathbf{R}^N$, we obtain that the first Fourier coefficient of the solution

$$E(t) = (u(t), \psi_1),$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbf{R}^N)$, satisfies

$$E'(t) = (u(t), (-1)^{m+1}\Delta^m \psi_1) + (q(u)(t), \psi_1), \quad t > 0. \tag{2.3}$$

By (2.1) and Jensen's inequality for convex functions

$$(q(u), \psi_1) \ge q((|u|, \psi_1)) \equiv q(\bar{E}),$$

where we introduce the notation

$$\bar{E}(t) = (|u(t)|, \psi_1) \ge |E(t)|. \tag{2.4}$$

Using (2.2), we deduce that

$$(u(t), (-1)^{m+1}\Delta^m \psi_1) \ge -(|u(t)|, |\Delta^m \psi_1|) \ge -\lambda_1(|u(t)|, \psi_1) = -\lambda_1 \bar{E}(t).$$

Substituting these two estimates into (2.3), we arrive at the ODI

$$E' \ge -\lambda_1 \bar{E} + q(\bar{E}) \equiv H(\bar{E}), \quad t > 0. \tag{2.5}$$

Let $s_0$ be the largest positive root of $H(s)$ and let $q(u)$ satisfy the condition

$$H(s) > 0,\ \ H'(s) > 0 \quad \text{for } \ s > s_0. \tag{2.6}$$

We assume that the initial function $u_0$ is such that

$$E(0) \equiv (u_0, \psi_1) > s_0. \tag{2.7}$$

Then since by definitions $\bar{E}(t) \geq E(t)$, and hence $\bar{E}(0) > s_0$, obviously, (2.5) implies that

$$\bar{E}(t) \geq E(t) \geq E(0) > s_0 \quad \text{for all } \ t > 0.$$

By the monotonicity assumption in (2.6) we then conclude that $E(t)$ satisfies the ODI

$$E' \geq H(\bar{E}) \geq H(E) \quad \text{for } \ t > 0, \quad E(0) > s_0. \tag{2.8}$$

Integrating gives finite time blow-up of the first Fourier coefficient $E(t)$ provided that *Osgood's criterion* holds:

$$\int^{\infty} \frac{ds}{q(s) - \lambda_1 s} < \infty. \tag{2.9}$$

It then follows that $E(t)$ blows up in finite time

$$T \leq \int_{E(0)}^{\infty} \frac{ds}{q(s) - \lambda_1 s},$$

and since $E(t) \leq \|u(t)\|_\infty$, this means blow-up of the solution in the $L^\infty$-norm. Let us summarize these computations as follows.

**Theorem 2.1.** *Let $q$ satisfy* (2.1), (2.6) *and* (2.9)*. If $u_0$ satisfies* (2.7)*, then the solution blows up in finite time.*

A further analysis of a family of the ODIs obtained by multiplication by the rescaled "eigenfunction" $\psi_R(x) = \psi_1(x/R)$ shows that in the case $q(u) = |u|^p$ blow-up occurs for arbitrarily small initial data satisfying (1.3), provided that $1 < p \leq p_F = 1 + 2m/N$ [13], thus introducing the critical Fujita exponent obtained in [4] by a different technique. Such a blow-up result can be extended to any even superlinear function $q(u) > 0$ for $u > 0$ satisfying

$$q(|u|) \geq c\,|u|^p \quad \text{for small } \ u > 0; \quad p \leq p_F, \ \ c > 0.$$

## 3. Lower blow-up estimates via majorizing order-preserving equation

3.1. **Nonlinear integral equation and oscillatory properties.** We consider the general equation (1.1) with a smooth function $q(u)$ satisfying $q(u) \neq 0$ for $u \neq 0$ and $q(0) = 0$. In this section we are going to use the integral representation of the parabolic equation:

$$u(t) = \mathbf{M}_{2m}(u) \equiv b(t) * u_0 + \int_0^t b(t-s) * q(u(s))ds, \quad u_0 \in X, \tag{3.1}$$

where $b(t) * u_0 \equiv e^{\mathbf{A}t} u_0$ is the convolution representation of the continuous semigroup $e^{\mathbf{A}t}$ with the infinitesimal generator $\mathbf{A}$, i.e., $u(t)$ is a fixed point of operator $\mathbf{M}_{2m}$ for any $t > 0$. Then

$$b(x,t) = t^{-N/2m} f(\xi), \quad \xi = x/t^{1/2m}, \tag{3.2}$$

is the fundamental solution (the self-similar kernel) of the linear operator $\partial/\partial t - \mathbf{A}$. The function $f$ is the unique radial solution of the elliptic equation

$$\mathbf{C} f \equiv \mathbf{A} f + \frac{1}{2m} \xi \cdot \nabla f + \frac{N}{2m} f = 0 \quad \text{in } \mathbf{R}^N, \quad \int_{\mathbf{R}^N} f(\xi) d\xi = 1. \tag{3.3}$$

On the interval $(0,T)$ of the classical solvability, both integral and differential equations give the same solution $u(x,t)$.

Oscillatory properties of fundamental solutions of higher-order parabolic operators as well as of solutions of higher-order ODEs are well known, see a survey paper [19]. We observe that for odd $m$'s the ODE (3.3) is not oscillatory since a standard asymptotic analysis shows that it admits a monotone exponentially decaying solution, but such a mode is not a leading one in the fundamental rescaled profile $f$. For convenience, we state a result on oscillatory properties of solutions of $2m$-th order parabolic equations.

**Proposition 3.1.** *Let $m>1$ and $f=f(\eta,\tau)$, $\eta=|\xi|\geq 0$, $\tau>0$, be a smooth radial solution of the inhomogeneous parabolic equation*

$$f_\tau = \mathbf{C}f. \tag{3.4}$$

*Assume that $f$ vanishes at $\eta=\infty$ with its derivatives $f_\eta, ..., f_\eta^{(2m)}$, and $f_\tau \geq 0$. If $f\geq 0$ for $\eta \gg 1$, then $f\equiv 0$.*

*Proof.* We argue by contradiction. Assume that $f\geq 0$, $f\not\equiv 0$ for $\eta\gg 1$, is a solution of the ODI $\mathbf{C}f\geq 0$. Denoting $Z=\Delta^{m-1}f$, we integrate the ODI in the divergent form

$$(-1)^{m+1}(\eta^{N-1}Z')' + \frac{1}{2m}(\eta^N f)' \geq 0$$

over the interval $(\eta,\infty)$ and obtain that

$$-(-1)^{m+1}\eta^{N-1}Z' - \frac{1}{2m}\eta^N f \geq 0,$$

so that $(-1)^m Z'>0$ on a subset where $f>0$. Integrating again, we get $(-1)^{m-1}Z>0$ there, and we conclude that

$$f>0 \implies (-\Delta)^{m-1}f>0 \implies -(-\Delta)^{m-2}f>0 \implies \dots .$$

Finally, we obtain $-\Delta f>0$ if $m$ is even, and $\Delta f>0$ if $m$ is odd. In the first case, integrating $\Delta f<0$ twice yields $f_\eta>0$ in the positivity domain of $f$, a contradiction. In the second case, we obtain $f_\eta<0$. Since by construction, this inequality is valid if $f>0$, there exists the positive extension of $f(\eta,\tau)$ up to $\eta=0$, which does not satisfy the symmetry condition at the origin. □

As a consequence, we deduce that any suitable symmetric solution $f(\eta)$, $\eta\geq 0$, of the stationary ODE (3.3) is oscillatory at $\eta=\infty$.

3.2. **Majorizing equation.** Using an idea introduced in [13], given the integral (evolution) equation (3.1), we construct the corresponding *order-preserving majorizing equation.* The following estimate of the rescaled kernel $f$ follows from the ODE (the radial restriction of the elliptic equation (3.3)), see more general results in [5], p. 48.

**Proposition 3.2.** *Let $m>1$. There exist constants $D>1$ and $d>0$ depending on $m$ and $N$ such that*

$$|f(\eta)| < D\,F(\eta) \equiv D\omega_1 e^{-d|\eta|^\alpha} \quad in\ \mathbf{R}^N, \tag{3.5}$$

*where*

$$\alpha = \frac{2m}{2m-1} \in (1,2) \quad and \quad \omega_1 = \left( \int_{\mathbf{R}^N} e^{-d|\eta|^\alpha}\, d\eta \right)^{-1}.$$

We next introduce the *majorizing kernel*

$$\bar{b}(x,t) = t^{-N/2m} F(\eta), \quad \eta = x/t^{1/2m}, \tag{3.6}$$

which is *strictly positive* for $t > 0$. Therefore, if $q(u)$ is increasing in $\mathbf{R}^N$, the corresponding majorizing integral equation with nonlinear Hammerstein's integral operator ($t$ plays a role of a parameter)

$$v(t) = \bar{b}(t) * v_0 + \int_0^t \bar{b}(t-s) * (Dq(v(s)))ds, \quad t > 0, \tag{3.7}$$

describes the *order-preserving* evolution in $X$ with the usual partial order: given two solutions $v(t)$ and $\tilde{v}(t)$ of (3.7),

$$v_0 \le \tilde{v}_0 \implies v(t) \le \tilde{v}(t) \ \text{ for } \ t > 0.$$

The nonlinear term is a locally Lipschitz map and local existence and uniqueness are straightforward, see [25], Chapt. 15. In particular, we have the *positivity property*:

$$v_0 \ge 0,\ v_0 \not\equiv 0 \implies v(t) > 0 \ \text{ for } \ t > 0. \tag{3.8}$$

By comparing (3.1) and (3.7), we deduce the following result on the Maximum Principle (comparison) duality between the semilinear parabolic equation and the integral majorizing one, cf. [13].

**Theorem 3.1.** *Let $q(u)$ satisfy the following properties:*

$$|q(u)| \le q(|u|) \ \text{ in } \mathbf{R}, \quad \text{and } q'(u) > 0 \ \text{ in } \mathbf{R}_+. \tag{3.9}$$

*Then the integral equation* (3.7) *is majorizing for the $2m$-th order parabolic equation* (1.1) *in the following sense:*

$$D|u_0(x)| \le v_0(x) \ \text{ in } \ \mathbf{R}^N \implies |u(x,t)| \le v(x,t) \ \ for \ \ x \in \mathbf{R}^N, \ \ t > 0. \tag{3.10}$$

*Proof.* We have that $v(t)$ solves the integral inequality

$$v(t) \ge \bar{b}(t) * (D|u_0|) + \int_0^t \bar{b}(t-s) * (Dq(v(s)))ds, \quad t > 0, \tag{3.11}$$

On the other hand, it follows from (3.1) and Proposition 3.2 that

$$|u(t)| \le |b(t)| * |u_0| + \int_0^t |b(t-s)| * |q(u(s))|ds$$

$$\le |D\bar{b}(t)| * |u_0| + \int_0^t |D\bar{b}(t-s)| * q(|u(s)|)ds. \tag{3.12}$$

Using (3.9), we conclude that the difference $w = v - |u|$ satisfies the linear integral inequality

$$(3.13) \qquad w(t) \ge \int_0^t D\bar{b}(t-s) * (q(v(s)) - q(|u(s)|)ds \equiv \int_0^t K(t)w(s)ds, \;\; t > 0,$$

with the positive integral operator

$$K(t)w(s) = D\bar{b}(t-s) * [q'(\xi(s))w(s)],$$

with strictly positive kernel. Here $\xi(s) \in (|u(s)|, v(s))$ denotes intermediate points obtained via Lagrange's formula of finite increments. Since $D > 1$, we have $w(0) = v_0 - |u_0| \ge v_0 - D|u_0| \ge 0$. If $w(0) > \varepsilon > 0$ in $\mathbf{R}^N$, then the integral inequality with positive operator implies that $w(t) \ge 0$ for all $t > 0$, see below. If $w(0)$ is not uniformly positive and, say, both $u_0$ and $v_0$ are compactly supported, we then use a standard approximation of the solution on any compact subset by constructing $v(t)$ as the limit of a monotone decreasing sequence $\{v_\varepsilon(t)\}$ of strictly solutions $v_\varepsilon \ge \varepsilon$ in $\mathbf{R}^N$ of the integral equation (3.7) (such a sequence is monotone by the order-preserving property of the majorizing equation and the limit solution does not depend on approximation by uniqueness). In both cases, we have by continuity that (we set $w = w_\varepsilon$ if necessary)

$$\tau = \sup\{t \ge 0 : \;\; w(t) > 0\} > 0.$$

It follows from (3.13) that, for all $t \ge \tau$, $w = w(x,t)$ cannot vanish a any point $x \in \mathbf{R}^N$, which completes the proof. $\square$

One can see that, actually, we have the strong comparison property and the inequality in (3.10) is strict: $|u(x,t)| < v(x,t)$ for $t > 0$.

For $m > 1$, the majorizing integral equation (3.7) is not generated by a semigroup unlike the second-order case $m = 1$, where $b(t) * u_0 = e^{\Delta t}u_0 > 0$ and the semigroup is order-preserving since $b(t) > 0$ for $t > 0$. Therefore, solutions $v(x,t)$ are not time-translational invariant. Nevertheless, the spatially homogeneous solutions $V = V(t)$ satisfying the integral equation (we recall that $\int \bar{b}(t) \equiv 1$)

$$V(t) = V(0) + \int_0^t Dq(V(s))ds, \;\; t > 0,$$

and hence the ODE

$$(3.14) \qquad V' = Dq(V), \quad t > 0; \quad V(0) > 0,$$

admit translation in time, so that $V(t+\tau)$ is a solution of the majorizing equation for any constant time-shift $\tau$.

In view of the order-preserving majorizing evolution, we now compare $u(x,t)$ with such spatially homogeneous solutions $V = V(t)$ and obtain the following simple estimate from above of the solutions and hence a lower estimate of the blow-up time.

**Corollary 3.1.** *Let $f$ satisfy* (3.9). *Denote $m_0 = \sup|u_0(x)| > 0$. Then*

$$(3.15) \qquad |u(x,t)| \le V(t) < \infty, \quad 0 < t < t_0 = \frac{1}{D}\int_{m_0 D}^{\infty} \frac{dz}{q(z)},$$

*where $V(t) > 0$ is the solutions of the ODE* (3.14) *with initial data* $V(0) = Dm_0$.

Finally, we establish a more delicate *intersection-like* property of the solutions $u(x,t)$ and $V_T(t)$, $V_T(T) = \infty$, having the same blow-up time $T$. Estimates based on such intersection properties play a fundamental role for the blow-up analysis in the second-order case $m = 1$, see Chapt. 4 in [24] and references therein.

**Theorem 3.2.** *Let $u(x,t)$ and $V = V_T(t)$ solving* (3.14) *have the same finite blow-up time $T > 0$. Then, for any $t \in [0,T)$, the function $|u(\cdot,t)| \in X$ intersects $DV_T(t)$, so that*

$$\|u(t)\|_\infty \ge DV_T(t), \quad t \in [0,T). \tag{3.16}$$

*Proof.* Assume for contradiction that $|u(x,t_0)| < DV_T(t_0)$ in $\mathbf{R}^N$ for some $t_0 \in [0,T)$. By continuity, there exists a small $\varepsilon > 0$ such that

$$|u(x,t_0)| < DV_T(t_0 - \varepsilon) \equiv DV_{T+\varepsilon}(t_0), \quad x \in \mathbf{R}^N.$$

By the comparison Theorem 3.1 we then have that $|u(x,t)| < V_{T+\varepsilon}(t)$ for all $t \in [t_0,T)$. This means that at $t = T$, the solution satisfies $|u(x,T)| < V_{T+\varepsilon}(T) < \infty$ and hence is uniformly bounded at its blow-up time. This contradiction completes the proof. □

It follows that for the weakly nonlinear logarithmic function with the exponent $\gamma > 1$,

$$q(u) = u \ln^\gamma(1 + |u|) = u|\ln|u||^\gamma(1 + o(1)) \ \text{ as } \ u \to \infty, \tag{3.17}$$

the following lower $L^\infty$-estimate on blow-up solutions holds:

$$\|u(\cdot,t)\|_\infty \ge DV_T(t) = D\exp\{[(\gamma-1)D(T-t)]^{-1/(\gamma-1)}(1+o(1))\}, \quad t \to T^-. \tag{3.18}$$

This estimate well corresponds to the asymptotic analysis of blow-up patterns we are going to perform in the next sections. It is easy to see that the strict inequality holds in (3.16) and (3.18).

## 4. Blow-up estimates for nonlinear pseudodifferential evolution

Consider a general class of nonlinear integral equations (3.1) with the pseudodifferential (convolution) operator $\mathbf{B}\phi = b(t)*\phi$, $\phi \in X$, where the kernel $b(t)$ has the scaling structure as in (3.2) with a smooth $L^1$ rescaled kernel $f$. Now it is not generated by a linear parabolic differential operator. Then (3.1), where $t$ is a parameter, is to be considered as an evolution equation with evolution governed by a pseudodifferential operator $\mathbf{B}$. This equation includes all the higher-order parabolic equations, and other integro-differential equations which do not admits a local pointwise PDEs formulation. If the symbol of $\mathbf{B}$, i.e., the Fourier transform $W(t) = \mathcal{F}(b(t))$, changes sign for $t > 0$, then $\mathbf{B}$ is not invertible in $L^1(\mathbf{R}^N)$ and is not a Fredholm operator, see Chap. 12 in [16].

We now apply the same idea of the majorizing operator. We assume that the rescaled kernel $f$ has an $L^1$ positive majorizing kernel and the following hypothesis holds.

**(H)** There exists a continuous function $F(\eta) > 0$ and a constant $D \ge 1$ such that

$$|f(\eta)| \le DF(\eta) \quad \text{in } \mathbf{R}^N, \quad \int_{\mathbf{R}^N} F(\eta)\,d\eta = 1. \tag{4.1}$$

One can see that the *order deficiency* $D = D(f) = 1$ if $f \ge 0$, i.e., if the pseudodifferential evolution is order-preserving. Otherwise, if $f$ changes sign, we always have that $D > 1$. Actually, the optimal deficiency $D$ corresponds to the majorizing kernel

$$F(\eta) = \frac{1}{D}\,|f(\eta)| \implies D = \left( \int_{\mathbf{R}^N} |f(\eta)| d\eta \right)^{-1}.$$

Then $D(f) = 1$ if and only if $f \ge 0$.

We next introduce the majorizing kernel (3.6) and the corresponding majorizing pseudodifferential evolution equation (3.7) with Hammerstein's operator. A local in time solution exists and is unique [25]. Obviously, if a smooth solution cannot be extended globally in time, then its $L^\infty$-norm blows up in finite time. The solutions of (3.7) satisfy the positivity property (3.8). Moreover, the comparison Theorem 3.1 holds. We then arrive at the following estimate of blow-up solutions of the pseudodifferential evolution equation (3.1).

**Theorem 4.1.** *Let* (H) *and* (3.9) *hold. Let* $u(x,t)$ *satisfying* (3.1) *and* $V = V_T(t)$ *solving* (3.14) *have the same finite blow-up time* $T > 0$. *Then*

$$\|u(t)\|_\infty \ge D V_T(t), \quad t \in [0,T). \tag{4.2}$$

It is important that the only influence of the type of a pseudodifferential evolution is in the order deficiency $D$ depending on the rescaled kernel $f$. We deduce that for the weakly superlinear function (3.17), the corresponding estimate of the blow-up rate from below takes the same form (3.18) as for the semilinear $2m$-th order parabolic equation.

## 5. **Transition to the Hamilton-Jacobi equation**

5.1. **Nonlinear scaling.** The idea of the asymptotic transition to the Hamilton-Jacobi equation consists in the nonlinear scaling by setting

$$u = e^v. \tag{5.1}$$

For any $m > 1$, the kernel of the parabolic operator $\partial/\partial t + (-\Delta)^m$ is oscillatory (the rescaled kernel $f(\xi)$ in (3.2) is oscillatory as $\xi \to \infty$), and the corresponding semigroup $e^{\mathbf{A}t}$ is not order-preserving, see [5]. Therefore, if blow-up occurs as $t \to T^-$ on a blow-up set of a non-zero measure, the solution $u(x,t)$ can take arbitrarily large positive and negative values. In general, we then obtain a function $v = \ln u : Q_T \to \mathbb{C}$, and in the case $m = 1$ only, when the linear semigroup is order-preserving, we arrive at a real-valued function $v$ if $u > 0$. Therefore, performing transformation (5.1), we treat (1.5) as a *complex parabolic equation* for $u : Q_T \to \mathbb{C}$.

The function $v$ satisfies the following semilinear parabolic equation:

$$v_t = \mathbf{H}_{2m}(v) + \mathbf{P}_{2m}(v), \tag{5.2}$$

where

$$\mathbf{H}_{2m}(v) = (-1)^{m+1}(\nabla v \cdot \nabla v)^m + |\mathrm{Re}\, v|^{2m} \tag{5.3}$$

is the first-order Hamilton-Jacobi operator of the algebraic homogeneity $2m$, and $\mathbf{P}_{2m}$ is a $2m$-th order semilinear elliptic operator of the form

$$\mathbf{P}_{2m}(v) = -(-\Delta)^m v + \mathbf{Q}_{2m-1}(v), \tag{5.4}$$

where $\mathbf{Q}_{2m-1}$ is a lower $(2m-1)$-th order polynomial operator. For instance, $\mathbf{P}_2(v) = \Delta v$ ($\mathbf{Q}_1 = 0$) and

$$\mathbf{P}_4(v) = -\Delta^2 v + \mathbf{Q}_3(v) \equiv -\Delta^2 v$$
$$-\Delta(\nabla v \cdot \nabla v) - 2\nabla v \cdot \nabla \Delta v - 2(\nabla v \cdot \nabla v)\Delta v - 2\nabla v \cdot \nabla(\nabla v \cdot \nabla v) - (\Delta v)^2.$$

An important point is that, in comparison with the Hamilton-Jacobi operator $\mathbf{H}_{2m}$, the higher-order differential operator $\mathbf{P}_{2m}$ on the right-hand side of (5.2) is a lower-order one relative to its algebraic homogeneity in $u$. Indeed, given a smooth non-trivial function $\varphi$, as $\lambda \to \infty$,

$$|\mathbf{P}_{2m}(\lambda\varphi)| = O(\lambda^{2m-1}) \ll |\mathbf{H}_{2m}(\lambda\varphi)| = O(\lambda^{2m}).$$

5.2. **Hamilton-Jacobi profiles.** Keeping in (5.2) the leading operators of the homogeneity $2m$, we arrive at the Hamilton-Jacobi equation

$$V_t = \mathbf{H}_{2m}(V) \equiv (-1)^{m+1}(\nabla V \cdot \nabla V)^m + |\mathrm{Re}\, V|^{2m}. \tag{5.5}$$

The structure of blow-up patterns for (5.5) is described by the *Standing wave solutions* (the blow-up S-regime) in the separable form

$$V_*(x,t) = (T-t)^{-1/(2m-1)}\theta(x), \tag{5.6}$$

where $\theta : \mathbf{R}^N \to \mathbb{C}$ solves the following *complex stationary Hamilton-Jacobi equation*:

$$\bar{\mathbf{H}}_{2m}(\theta) \equiv (-1)^{m+1}(\nabla\theta \cdot \nabla\theta)^m + |\mathrm{Re}\,\theta|^{2m} - \frac{\theta}{2m-1} = 0. \tag{5.7}$$

Of course, it admits a trivial constant (real) solution:

$$\theta \equiv \theta_* = (2m-1)^{-1/(2m-1)},$$

describing flat global blow-up in $\mathbf{R}^N$ for both Hamilton-Jacobi equation and the parabolic one. We are looking for a smooth compactly supported solution $\theta : \mathbf{R}^N \to \mathbb{C}$, describing a non-trivial spatial shape of regional blow-up.

5.3. **Example: a real-valued profile for odd $m$'s.** In order understand a general structure of such solutions, we first consider the case of odd $m = 3, 5, ...$ (the order-preserving case $m = 1$ is also included). The equation admits a real-valued non-negative solution $\theta \geq 0$ so that it takes the form of a standard Hamilton-Jacobi equation

$$|\nabla\theta|^{2m} = \frac{\theta}{2m-1} - \theta^{2m}. \tag{5.8}$$

Actually, such real-valued profiles do not occur in the oscillatory blow-up behaviour for $m > 1$, where the complex Hamilton-Jacobi equation becomes a system of two first-order ODEs. Nevertheless, such real first-order equations play a key role in estimating of solutions of complex ones.

Setting

$$S(x) = \int_0^{\theta(x)} \left( \frac{z}{2m-1} - z^{2m} \right)^{-1/2m} dz, \quad 0 \le \theta \le \theta_*, \tag{5.9}$$

we obtain the classical *eikonal equation*:

$$|\nabla S|^2 = 1. \tag{5.10}$$

We are interested in bell-shaped profiles $\theta(x)$ concentrated around the origin so that we always assume that $\theta(0) = \theta_*$. This implies that

$$S(0) = S_* = \int_0^{\theta_*} (z/(2m-1) - z^{2m})^{-1/2m} dz = \frac{1}{2m-1} \int_0^1 \eta^{1/2m-1} (1-\eta)^{(2m-1)/2m-1} d\eta$$

$$= \frac{1}{2m-1} \Gamma(1/2m) \Gamma((2m-1)/2m) = \frac{\pi}{(2m-1)\sin(\pi/2m)} \equiv L_m, \tag{5.11}$$

where $\Gamma$ denotes Euler's Gamma function.

5.4. **The radial profile.** The radial compactly supported solution $\hat{S}$ depending on $r = |x|$ satisfies the ODE $\hat{S}'(r) = -1$ so that

$$\hat{S}(r) = [S_* - r]_+. \tag{5.12}$$

Observe that due to transformation (5.9), the Hamilton-Jacobi profile is a (classical) viscosity $C^1$-solution of the ODE (5.8). This gives the localization ball of the radius $S_* = L_m$:

$$\theta(L_m) = 0, \quad \theta(r) > 0 \text{ for any } r \in [0, L_m).$$

5.5. **General solution of the eikonal equation: nonexistence of non-symmetric patterns.** Let us show that the above eikonal equation has no other sufficiently smooth non-radial solutions so that the Hamilton-Jacobi equation (5.5) admits no self-similar non-symmetric blow-up patterns.

According to the transformation (5.9), we are looking for a solution $S$ which is a $C^1$-function a.e., except a unique point, similar to the conical point $r = 0$ in the radially symmetric profile (5.12). Let us note that by translation and a natural continuous constant extension, for any $a > 0$ the function

$$S^a(r) = \{ S_* \text{ for } r \le a, \quad S_* + a - r \text{ for } a < r \le a + S_*, \quad 0 \text{ for } r \ge a + S_* \}$$

also generates a viscosity $C^1$-solution $\theta^a$ of (5.8). Following the evolution stability theory developed for $m = 1$ in [14], [15], such profiles with nontrivial mesa-like parts should be excluded from the reduced $\omega$-limit set of the Hamilton-Jacobi equation (5.5), since they cannot be approached as the $\omega$-limits of the parabolic orbits. We expect that the same remains true for the higher-order equations.

More precisely, we will prove the following result.

**Lemma 5.1.** *The radial solution* (5.12) *is the unique solution of the eikonal equation* (5.10) *having a single singularity point.*

By the methods of characteristics for the first-order equations with smooth coefficients, such a unique singular point means that in the canonical form, all the characteristics intersect each other at this point only, and the solutions stays smooth away from it.

For simplicity, we explain this fact in two-dimensions:

$$S_x^2 + S_y^2 = 1. \tag{5.13}$$

In order to see that the radial solution is the unique one which "collapses" at the origin only, we solve equation (5.13) with the data $S = 0$ prescribed on a smooth simple closed curve $\Gamma_0 \subset \mathbf{R}_2$ given by

$$x = x_0(s), \quad y = y_0(s), \tag{5.14}$$

where $s$ is a parameter ($\Gamma_0$ is the boundary of the support of $\theta$). We introduce the canonical variables $S_x = p$, $S_y = q$, and solve Charpit's equations:

$$\dot{x} = p, \ \dot{y} = q, \ \dot{S} = 1, \ \dot{p} = \dot{q} = 0, \tag{5.15}$$

where dot means $d/d\tau$, along the characteristics for the functions $p$ and $q$ that originate on the boundary given by (5.14), on which also

$$S = 0, \ p_0^2 + q_0^2 = 1, \ p_0 x_0' + q_0 y_0' = 0.$$

One can see that, given an arbitrary point $(x_0(s), y_0(s)) \in \Gamma$ and $\tau \geq 0$, we have that $p = p_0 = -\sin\alpha$, $q = q_0 = \cos\alpha$ ($\alpha = \alpha(s)$ is the angle of the boundary to the $x$-axis at the given point on the boundary), and $S = \tau$ at the point $x = x(s) - \tau \sin\alpha$, $y = y_0(s) + \tau\cos\alpha$, so that $(x - x_0(s))^2 + (y - y_0(s))^2 = \tau^2$. The envelope $\Gamma_\tau$ of all these spherical wavefronts gives the characteristic for $S$: $S = \tau$, and it is a level surface of $S$. This determines a parametric form of the solution $S$ for arbitrary curve $\Gamma_0$. This solution can be looked at as a set of wavefronts propagating from the initial surface, where $S = 0$, and is generated by a set of small spherical waves, one emanating from each point of the initial surface (Huyghens' Principle). The level surfaces $\Gamma_\tau$ of any solution form a set of parallel surfaces and it is obvious that the radial solution is the unique one with level surfaces collapsing at a single point (say, at the origin). The analysis is similar for $N > 2$.

5.6. **On solvability of the complex Hamilton-Jacobi equation.** We now consider complex-valued radial solutions $\theta = \theta(r) : \mathbf{R}_+ \to \mathbb{C}$ of the Hamilton Jacobi equation (5.7). It is an ODE with the real independent variable $r$ for a complex-valued function $\theta(r)$

$$(\theta')^{2m} = F(\theta) \equiv (-1)^{m+1}(\gamma\theta - |\mathrm{Re}\,\theta|^{2m}) \quad \text{for } r > 0, \ \theta'(0) = 0, \tag{5.16}$$

where $\gamma = 1/(2m-1)$. We will look for a solution

$$\theta(r) = f(r) + i\,g(r),$$

where $f$ and $g$ are smooth real-valued functions. Then the symmetry condition at the origin implies that $\theta(0) = \theta_* = \gamma^\gamma$ is real, so that we arrive at the conditions

$$f(0) = \gamma^\gamma, \quad f'(0) = g(0) = g'(0) = 0. \tag{5.17}$$

We first prove that the real part $f(r)$ vanishes at a finite point which makes it possible to consider compactly supported solutions of the complex Hamilton-Jacobi equation. The right-hand side $F$ in (5.16) is not analytic, and we cannot integrate (5.16) as a standard ODE. Substituting $\theta = f + ig$ and hence $(f' + ig')^{2m} = P_{2m}(f', g') + iQ_{2m}(f', g')$, we obtain a system of two first-order ODEs

$$P_{2m}(f', g') = (-1)^{m+1}(\gamma f - |f|^{2m}), \quad Q_{2m}(f', g') = (-1)^{m+1}\gamma g, \tag{5.18}$$

where $P_{2m}$ and $Q_{2m}$ are polynomials of the form

$$P_{2m}(p, q) = p^{2m} - \frac{2m(2m-1)}{2} p^{2m-2} q^2 + ... + \frac{2m(2m-1)}{2}(-1)^{m-1} p^2 q^{2m-2} + (-1)^m q^{2m},$$

$$Q_{2m}(p, q) = 2mp^{2m-1}q + ... + 2m(-1)^{m-1}pq^{2m-1}.$$

Setting for convenience

$$g'(r) = \beta f'(r) \quad (\beta = \beta(r)),$$

one can obtain the following representation of these polynomials:

$$P_{2m}(f', g') = (f')^{2m} P_{2m}(1, \beta), \quad Q_{2m}(f', g') = (f')^{2m} Q_{2m}(1, \beta),$$

where

$$P_{2m}(1, \beta) = (1 + \beta^2)^m \cos(2ms), \;\; Q_{2m} = (1 + \beta^2)^m \sin(2ms), \;\; s = \tan^{-1}\beta. \tag{5.19}$$

We begin with a local analysis of solvability of the system near the origin $r = 0$, which is a singular point $(f, f', g, g') = (\gamma^\gamma, 0, 0, 0) = P_0$. A careful checking of equations (5.18) shows that on the unstable manifold of $P_0$, the derivatives $f'$ and $g'$ are balanced in the sense that in a small neighbourhood of $P_0$ there holds

$$g' = \beta f' + ..., \quad \beta = \text{const} > 0.$$

It then follows from equations (5.18) that as $r \to 0$,

$$f(r) = \gamma^\gamma - Cr^\alpha + ..., \quad g(r) = -\beta C r^\alpha + ... ,$$

where by (5.19) the expansion coefficient $C = C(\beta)$ satisfies the following algebraic system:

$$C^{2m-1} = \frac{(-1)^{m+1}}{\alpha^{2m} P_{2m}(1, \beta)} = \frac{(-1)^m \gamma\beta}{\alpha^{2m} Q_{2m}(1, \beta)} > 0.$$

Therefore, $\beta$ is a root of the algebraic system

$$-Q_{2m}(1, \beta) = \gamma\beta P_{2m}(1, \beta), \;\; (-1)^{m+1} P_{2m}(1, \beta) > 0, \quad \beta > 0. \tag{5.20}$$

Denote by $[z]$ the entire part of a number $z \in \mathbf{R}$.

**Proposition 5.1.** *For any $m > 1$, problem (5.20) has exactly $K = [m/2]$ different solutions $0 < \beta_1 < ... < \beta_K$.*

*Proof.* Setting $\beta = \tan s$ in (5.20), we arrive at the problem

$$\tan(2ms) = -\gamma \tan s, \;\; (-1)^{m+1}\cos(2ms) > 0, \quad s \in (0, \pi/2). \tag{5.21}$$

One can check that it admits $K$ solutions $s_1 = \tan^{-1}\beta_1 < ... < s_K = \tan^{-1}\beta_K$. $\square$

Observe that $K = 1$ for $m = 2$ and 3, so that such a solution is unique. In the case $m = 2$, where

$$P_4(1, \beta) = 1 - 4\beta^2 + \beta^4, \;\; Q_4(1, \beta) = 4\beta - 4\beta^3, \;\; \gamma = \frac{1}{3},$$

the first equation in (5.20) takes the form

$$\beta^4 - 16\beta^2 + 13 = 0.$$

Among the two positive roots $\beta_\pm^2 = 8 \pm \sqrt{51}$, the largest one $\beta_+$ does not satisfy the second condition in (5.20), $P_4(1, \beta_\pm) \equiv 85 \pm 12\sqrt{51} < 0$, which is true for the unique minimal root

$$\beta_1 = \sqrt{8 - \sqrt{51}}.$$

The existence of such local solutions with the prescribed asymptotic behaviour as $r \to 0$ is established by a standard reduction of the first-order system to a nonlinear integral equation to which Banach's Contraction Principle applies. In order to extend local solutions defined near the origin $r = 0$, we need an *a priori* bound, which guarantee that the real part $f(r)$ always vanishes at a finite point.

**Proposition 5.2.** *There exists a constant $a_m > 0$ depending on $m$ only, such that $f(r)$ is strictly decreasing in its bounded positivity interval $(0, l_m)$, $f'(r) < 0$ for $r \in (0, l_m)$, and $f(r) \le Y(r)$, where $Y(r) \ge 0$ is a unique compactly supported solution of the real ODE*

$$a_m (Y')^{2m} = \gamma Y - Y^{2m}, \quad r > 0; \quad Y'(0) = 0. \tag{5.22}$$

*Proof.* It follows from (5.18) that, setting $g' = \beta f'$,

$$\begin{aligned}(-1)^{m+1} P_{2m}(f', \beta f') &\equiv (f')^{2m}(-1)^{m+1} P_{2m}(1, \beta) \\ &\le (f')^{2m} \sup_{\beta \in \mathbf{R}} (-1)^{m+1} P_{2m}(1, \beta) \equiv a_m (f')^{2m},\end{aligned}$$

and we arrive at the ordinary differential inequality

$$a_m (f')^{2m} \ge (\gamma f - f^{2m}) > 0, \quad f \in (0, \gamma^\gamma).$$

Hence, $f' < 0$ on this interval, and the result follows by a standard comparison for first-order ODEs. An estimate of the support of $f$ is derived by integrating the ODE (5.22), as it has been done above. $\square$

It follows from this proposition that on $(0, l_m)$ the system (5.16) is not degenerate, and moreover $f'(r) < 0$ on $(0, l_m]$, so that both real and imaginary parts are strictly monotone decreasing on $[0, l_m]$. We observe that $f'(l_m) < 0$ unlike the real valued analysis above.

Finally we extend the real part $f$ by setting $f(r) = 0$ for $r > l_m$, and arrive at a continuous compactly supported solution $f = f(x)$. The imaginary part $g$ describing oscillatory properties of the solution $V_*(x, t)$ as $t \to T$ for any $x$ in the localization domain (see the behaviour presented below), does not admit a continuous extension.

We thus obtain a finite spectrum of complex-valued Hamilton-Jacobi profiles $\{\theta = \theta_k,\ k = 1, ..., K\}$, which can occur in the blow-up behaviour. Let us note again that for the fourth-order, $m = 2$, and the sixth-order, $m = 3$, equations (as well as in the second-order case $m = 1$, where the profile is real-valued with $\beta_1 = 0$), there exists a *unique* suitable radial complex profile $\theta_1(x)$.

5.7. **Oscillatory blow-up patterns: conjecture on convergence of the envelope.** Let us describe a key blow-up and localization properties of the solution $V_*$. It follows from the nonlinear scaling, that the real part of the *approximate self-similar solution*

$$\tilde{U}_*(x,t) = \mathrm{Re}\, e^{V_*(x,t)} = \exp\{(T-t)^{-\gamma} f(r)\} \cos((T-t)^{-\gamma} g(r)), \tag{5.23}$$

blows up in a finite ball $\{|x| < l_m\}$, $l_m = \mathrm{meas\,supp}\, f$, with highly oscillatory behaviour at any of its internal point $r \in (0, l_m)$ (recall that $g \neq 0$ on $(0, l_m]$). By the construction, $g(0) = 0$, so that at the symmetry point $x = 0$, we observe the blow-up growth

$$\tilde{U}_*(0,t) \sim \exp\{[\gamma(T-t)]^{-\gamma}\},$$

which is expected to describe the actual $L^\infty$-estimate of blow-up solutions to (1.5). Blow-up estimates from the previous section confirm such a behaviour.

We obtain that for higher-order equations with $m > 1$, the asymptotic behaviour of blow-up solutions with regional blow-up is much more complicated than that proved in [14], [15] for the non-oscillatory second-order case $m = 1$. Actually, we have shown that after the nonlinear scaling, a *pointwise* convergence makes no sense in view of the time-oscillatory behaviour of the solution (5.23) in the blow-up set. We then expect that a stability analysis should be oriented to the convergence of such a characteristic of the oscillatory solutions as the *envelope* of the family as $t \to T^-$. Namely, we consider the family of rescaled oscillatory radial solutions $\{\ln(1 + |u(r,t)|),\ t \in (0,T)\}$, and state the following conjecture of the asymptotic behaviour of its envelope.

**Conjecture.** (Convergence of the envelope) *There exists a finite limit*

$$(T-t)^\gamma \sup_{s<t} \ln(1 + |u(r,s)|) \to f(r) \quad \text{as} \quad t \to T^-,$$

*where $f(r) > 0$ on $[0, l_m)$ is the real part of a solution of the complex radial stationary Hamilton-Jacobi equation* (5.16).

Such a stability theory and the envelope conjecture remain open problems for any $m > 1$.

## 6. Stability analysis of the generic blow-up pattern

In order to study the stability of the generic blow-up pattern constructed by means of the Hamilton-Jacobi equation, we introduce the rescaled variables

$$v(x,t) = (T-t)^{-1/(2m-1)} h(x,\tau), \quad \tau = -\ln(T-t), \tag{6.1}$$

where $h$ solves the perturbed Hamilton-Jacobi equation

$$h_\tau = \bar{\mathbf{H}}_{2m}(h) + e^{-\tau}\mathbf{A}h + ... \quad (\mathbf{A} = -(-\Delta)^m), \tag{6.2}$$

where we keep the first linear term only of the polynomial operator $\mathbf{P}_{2m}$.

The last term in (6.2) forms a higher-order *singular* perturbation of the first-order equation. Then as in the classical boundary layer theory, we restrict the evolution to such small spatial-time subsets where the $2m$-th order perturbation in (6.2) becomes dominant. One can see that this can be achieved if we perform further rescaling

$$\xi = x e^{\tau/2m}.$$

Using also the linearisation

$$h(x, \tau) = \theta_* + Y(\xi, \tau), \quad \theta_* = \gamma^\gamma,$$

we arrive at a perturbed linear $2m$-th order parabolic equation (at this moment, we are assuming that $h$ is a real-valued function)

$$Y_\tau = \mathbf{B}Y + e^\tau (-1)^{m+1} |\nabla Y|^{2m} + ... + \mathbf{D}(Y), \tag{6.3}$$

with the linear operator

$$\mathbf{B} = \mathbf{A} - \frac{1}{2m} \xi \cdot \nabla + I. \tag{6.4}$$

The nonlinear perturbation $\mathbf{D}$ is locally quadratic:

$$\mathbf{D}(Y) \equiv (\theta_* + Y)^{2m} - \theta_*^{2m} - 2m\theta_*^{2m-1} Y = O(Y^2) \text{ as } Y \to 0.$$

we begin with the spectral properties of the linear operator in equation (6.3).

6.1. **Discrete point spectrum of the non self-adjoint operator B.** We study spectral properties of the linear operator (6.4). If $m = 1$, it admits a natural symmetric representation

$$\mathbf{B} = \frac{1}{\mu} \nabla \cdot (\mu \nabla) + I, \quad \mu(\xi) = e^{-|\xi|^2/4}. \tag{6.5}$$

Therefore, since $-\mathbf{B} + I \geq 0$ is semibounded from below, there exists a unique Friedrichs extension, which is a self-adjoint operator in the weighted Hilbert space $L^2_\mu(\mathbf{R}^N)$ with the domain $\mathcal{D}(\mathbf{B}) = H^2_\mu(\mathbf{R}^N)$ and a discrete point spectrum, see [1]. The eigenfunctions form an orthonormal basis in $L^2_\mu(\mathbf{R}^N)$ and the classical Hilbert-Schmidt theory applies.

For any $m > 1$, operator (6.4) is not self-adjoint. In the one-dimensional geometry, $N = 1$, where it becomes the even order ordinary differential operator

$$\mathbf{B}y = (-1)^{m+1} y^{(2m)} - \frac{1}{2m} \xi y' + y, \tag{6.6}$$

it cannot be written down as a self-adjoint differential expression of the form $(p_0 y^{(m)})^{(m)} + ... + p_m y$ and is not self-adjoint. See Chapt. 1 in [22]. Let us introduce the weight function

$$\rho(\xi) = e^{-d_* |\xi|^\alpha} > 0 \quad (\alpha = 2m/(2m-1)), \tag{6.7}$$

where $d_* = d_*(m, N) \in (0, d]$ is a small constant.

**Lemma 6.1.** *(i) Operator* $\mathbf{B}$ *in* $L^2_\rho(\mathbf{R}^N)$ *has the discrete real point spectrum*

$$\sigma_p(\mathbf{B}) = \{\lambda_\beta = 1 - |\beta|/2m, \ \ |\beta| = 0, 1, 2, ...\}, \tag{6.8}$$

*where* $\beta = (\beta_1, ..., \beta_N)$ *is a multiindex,* $|\beta| = \beta_1 + ... + \beta_N$*. Each eigenfunction has the separable decomposition*

$$\psi_\beta(\xi) = \psi_{\beta_1}(\xi_1)...\psi_{\beta_N}(\xi_N), \tag{6.9}$$

*where* $\psi_{\beta_k}(\xi_k)$ *is an eigenfunction (a Kummer-type polynomial) of the one-dimensional operator* (6.6) *with* $\xi = \xi_k$*.*

*(ii) The subset* $\{\psi_\beta\}$ *is complete in* $L^2_\rho(\mathbf{R}^N)$*.*

The proof in [10] is based on the analysis of the strong continuous (analytic) semigroup generated by $\mathbf{B}$, which is constructed by the backward rescaling of the parabolic one $b(t) * u_0$ with well-known contractivity and analytic properties. $\mathbf{B}^{-1}$ is an integral operator with an $L^2_\rho(\mathbf{R}^N)$-kernel and hence is a compact operator with a point spectrum only.

6.2. **A formal stability analysis.** It follows from Lemma 6.1 that $\mathbf{B}$ has unstable (if $|\beta| < 2m$) and centre (if $|\beta| = 2m$) finite-dimensional subspaces. As usual in rescaled blow-up problems, the unstable subspace corresponds to instabilities of the parabolic flow relative to space-time shifting and also reflects "degenerate" eigenfunctions which are eigenfunctions of the first-order operator in (6.4).

We next use the standard orthogonality properties of the complete subset of normalized eigenfunctions $\Phi = \{\psi_\beta\}$ of operator $\mathbf{B}$ and of $\Phi^* = \{\psi^*_\beta\}$ of the adjoint operator $\mathbf{B}^*$ with the same point spectrum:

$$(\psi_\beta, \psi^*_\gamma) = \delta_{\beta\gamma},$$

where $(\cdot, \cdot)$ is the inner (duality) product in $L^2(\mathbf{R}^N)$. Then one can use a standard discrete eigenfunctions expansion of the solution

$$Y(\tau) = \sum_{(\beta)} a_\beta(\tau)\psi_\beta,$$

and derive a dynamical system on the expansion coefficients $\{a_k(\tau)\}$ by the corresponding Schmidt procedure. Such a linearized stability approach is effective for the rescaled semilinear equation (1.2) and gives a classification of blow-up patterns in terms of the spectrum of Kummer's polynomials. Then the generic blow-up pattern corresponds to the behaviour on the centre manifold with $|\beta| = 2m$.

We will show that for perturbed equation (6.3) such an application based on the eigenfunctions expansion is not straightforward and leads to a *nonlinear stability analysis*. Indeed, equation (6.3) contains an exponentially large first-order perturbation. Therefore, the stability and the rate of convergence as $\tau \to \infty$ to the generic Hamilton-Jacobi blow-up pattern are described by its "resonance" approximate self-similar solution in separate variables

$$Y(\xi, \tau) = e^{-\tau/(2m-1)}[\psi(\xi) + o(1)]. \tag{6.10}$$

This corresponds the behaviour induced by the stable subspace

$$E_{2m-1}(0) = \mathrm{Span}\{\psi_\beta,\ |\beta| = 4m^2/(2m-1)\}.$$

Substituting it into (6.3) and passing to the limit $\tau \to \infty$, we obtain the following semilinear elliptic equation for the profile $\psi$:

$$\mathbf{C}_{2m}(\psi) \equiv \mathbf{A}\psi - \frac{1}{2m}\xi \cdot \nabla\psi + \frac{2m}{2m-1}\psi + (-1)^{m+1}|\nabla\psi|^{2m} = 0 \quad \text{in } \mathbf{R}^N. \tag{6.11}$$

For odd $m = 1, 3, ...$ this equation admits solutions with singularities at finite $\xi$, where $\psi \to -\infty$. Such singular behaviour is not suitable (not *minimal*) and does not provide us with a finite limit as $t \to T^-$. The minimal behaviour for (6.11) as $\xi \to \infty$ corresponds to a typical parabolic (algebraic) behaviour:

$$\psi(\xi) = \alpha^{-\alpha}|\xi|^\alpha + ..., \quad \xi \to \infty.$$

We thus conclude that on compact subsets in $\xi$, we observe an exponentially fast convergence of the order

$$e^{-\tau/(2m-1)} \equiv (T-t)^{1/(2m-1)} \to 0 \ \text{ as } \ t \to T^-,$$

to the self-similar profile $\theta$ satisfying $\theta(0) = \theta_*$.

Thus, as a conjecture, we conclude that the constant rescaled profile $\theta_*$ is stable with the rate of convergence $O(e^{-\tau/(2m-1)})$ and the corresponding nonlinear "eigenfunction" $\psi(\xi)$ is a solution of the semilinear elliptic equation (6.11). For this higher-order equation with $m > 1$, the existence and uniqueness of such a solution remain an open problem, unlike the case $m = 1$ presented below.

6.3. **Existence and uniqueness for the second-order equation.** In the second-order case $m = 1$, where we can consider rescaled function is real-valued, the resonance equation takes the form

$$\mathbf{C}_2(\psi) \equiv \Delta\psi - \frac{1}{2}\xi \cdot \nabla\psi + 2\psi + |\nabla\psi|^2 = 0. \tag{6.12}$$

Setting $\psi = \ln\phi$ yields the following semilinear equation for $\phi > 0$:

$$\Delta\phi - \frac{1}{2}\xi \cdot \nabla\phi + 2\phi\ln\phi = 0, \quad \text{in } \mathbf{R}^N, \tag{6.13}$$

where the nonlinear lower-order term $2\phi\ln\phi$, $\phi > 0$, has a typical single-well shape with a single minimum point at $\phi_m = 1/e$.

**Proposition 6.1.** *Equation* (6.13) *admits the explicit radial solution*

$$\phi(\xi) = e^{N/4}\, e^{-|\xi|^2/4}. \tag{6.14}$$

*In the radial setting,* (6.14) *is a unique positive solution.*

*Proof.* In terms of the original variable $\psi = \ln\phi$ we obtain the parabolic solution

$$\psi(\xi) = (N - |\xi|^2)/4, \tag{6.15}$$

which belongs to the linear subspace $W_2 = \mathrm{Span}\{1, |\xi|^2\}$ invariant under the quadratic operator of the equation (6.12) and the explicit solution follows. The uniqueness proof in [2] uses the exact solutions of the parabolic equation

$$v_t = \mathbf{C}_{2m}(v), \quad t > 0,$$

which belong to the invariant subspace: $v(\cdot, t) \in W_2$ for all $t > 0$. Similar solutions for semilinear equations with the Hamilton-Jacobi nonlinearity $|\nabla v|^2$ are well-known, see p. 94 in [24] and references therein. The uniqueness proof is based on the intersection comparison via the Sturmian argument. $\square$

Keeping the main term in the asymptotic expansion, we then conclude that uniformly on compact subsets in $\xi$

$$h(x, \tau) = \theta_* + e^{-\tau} \frac{1}{4}(N - |\xi|) + \dots .$$

This expansion with a fixed second term describes the transition to a unique Hamilton-Jacobi pattern existing for small $|x| > 0$, which has been described in the previous section. Actually, such a matched asymptotic analysis is expected to provide us with an alternative proof of the stability of the Hamilton-Jacobi pattern. For $N = 1$, another proof by means of a dynamical systems approach is presented in [14], [15] (no centre manifold or matched asymptotic analysis is involved).

## 7. **Three types of blow-up**

We now present a brief derivation of the asymptotic Hamilton-Jacobi theory for the semilinear equation

$$u_t = (-1)^{m+1} \Delta^m u + u |\ln |u||^\gamma, \quad \gamma > 1. \tag{7.1}$$

The same nonlinear scaling $u = e^v$ leads to the equation

$$v_t = \mathbf{H}_{2m}(v) + \mathbf{P}_{2m}(v), \tag{7.2}$$

where the Hamilton-Jacobi operator takes a similar form

$$\mathbf{H}_{2m}(v) = (-1)^{m+1} (\nabla v \cdot \nabla v)^m + |\mathrm{Re}\, v|^\gamma,$$

and the perturbation polynomial operator $\mathbf{P}_{2m}$ stays the same. Taking into account the algebraic dimensional-like analysis, we expect that the asymptotically dominant Hamilton-Jacobi equation takes the form

$$V_t = \mathbf{H}_{2m}(V) \equiv (-1)^{m+1} (\nabla V \cdot \nabla V)^m + |\mathrm{ReV}|^\gamma. \tag{7.3}$$

It admits the self-similar solution

$$V_*(x, t) = (T - t)^{-1/(\gamma - 1)} \theta(\eta), \quad \eta = x/(T - t)^\beta, \;\; \beta = \frac{\gamma - 2m}{2m(\gamma - 1)}. \tag{7.4}$$

The profile $\theta : \mathbf{R}^N \to \mathbb{C}$ then solves the stationary Hamilton-Jacobi equation

$$(-1)^{m+1} (\nabla \theta \cdot \nabla \theta)^m - \beta \eta \cdot \nabla \theta - \frac{1}{\gamma - 1}\, \theta + |\mathrm{Re}\, \theta|^\gamma = 0, \quad \eta \in \mathbf{R}^N, \tag{7.5}$$

and $\theta(0) = \theta_*$, $\nabla\theta(0) = 0$. The analysis via *a priori* bounds of the corresponding system of ODEs for the radial function $\theta(\eta) = f(\eta) + ig(\eta)$ is similar to that in the critical case $\gamma = 2m$. We then derive a majorizing real ODE which shows that $f$ is compactly supported if $\gamma < 2m$ (as for $\gamma = 2m$). For $\gamma > 2m$ we need to consider strictly positive profiles $f$ with a special algebraic decay at infinity described by the linear part of the equation $-\beta\eta f' - f/(\gamma - 1) = 0$. We observe that even in the case $m = 1$ of the real Hamilton-Jacobi approximation, such a positive profile $\theta(\eta) \equiv f(\eta)$ is not unique (see Chapt. 4 in [24]), but the one with analytic expansion at the origin is unique and stable [15].

It follows from the similarity structure of the scaling invariant in (7.4) that the blow-up behaviour essentially depends on the sign of the exponent $\beta$. Namely, as in the second-order order-preserving parabolic case $m = 1$, we have the following three cases:

(i) $\beta > 0$ ($\gamma > 2m$): the approximate self-similar solution describes single point blow-up with the following final-time profile of the solution:

$$\mathrm{Re}\, V_*(x, T^-) = C_* |x|^{-2m/(\gamma - 2m)} < \infty, \quad x \neq 0, \tag{7.6}$$

where $C_* \neq 0$ is a constant, so that $V_*$ blows up as $t \to T^-$ at the origin $x = 0$ only.

(ii) $\beta = 0$ ($\gamma = 2m$): regional blow-up in a bounded domain of non-zero measure, the case already studied.

(iii) $\beta < 0$ ($1 < \gamma < 2m$): global blow-up. The solution $V_*$ blows up as $t \to T^-$ at any point $x \in \mathbf{R}^N$ and blow-up is global (but not uniform with highly oscillating behaviour) on subsets of the form $\{|x| \leq \eta_0 (T-t)^\beta\}$ with sufficiently small $0 < \eta_0 < \mathrm{meas\,supp} f$.

**Final conclusions.** We summarize the main results of the paper as follows. It is established that some basic comparison techniques via majorizing order-preserving operators apply to semilinear higher-order parabolic equations and provide us with sharp blow-up $L^\infty$ estimates similar to those which are well established for the second-order semilinear heat equations. On the other hand, we present a formal theory of Hamilton-Jacobi blow-up asymptotics for the $2m$-th order semilinear equation which exhibits some striking similarities with the simple second-order case $m = 1$ but indeed is more complicated.

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