SAINT-VENANT'S PRINCIPLE IN BLOW-UP FOR HIGHER-ORDER QUASILINEAR PARABOLIC EQUATIONS: HAMILTON-JACOBI ASYMPTOTICS

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ABSTRACT. We prove localization estimates for $2m$-th order quasilinear parabolic equations with boundary data blowing up in finite time, as $t \to T^-$. The analysis is based on energy estimates obtained from a system of functional inequalities expressing a version of Saint-Venant's principle from the theory of elasticity. We consider a special class of parabolic operators including those having fixed orders of algebraic homogeneity $p > 0$. This class includes the second-order heat equation and linear $2m$-th order parabolic equations, for which $p = 1$, as well as many other higher-order quasilinear ones with $p \neq 1$. Such homogeneous equations are invariant under a group of scaling transformations, but the corresponding critical (least) localized regional blow-up regimes are not group-invariant and exhibit typical exponential singularities $\sim e^{(T-t)^{-\gamma}} \to \infty$ as $t \to T^-$ with the optimal constant $\gamma = 1/[m(p+1)-1] > 0$. For some particular equations, we study the asymptotic blow-up behaviour described by perturbed first-order Hamilton-Jacobi equations, which shows that general estimates of exponential type are sharp.

1. Introduction: statement of the problem, blow-up localization, Saint-Venant's principle and energy estimates

Saint-Venant's principle formulated in the theory of linear elasticity in the middle of the XIX-th century, led to the concept of energy estimates, and in the second half of the XX-th century became a fundamental tool of the general theory of linear and nonlinear partial differential equations (PDEs). It is a method of introducing parameters playing a role of new independent variables which are "dual" relative to the original space-time ones. Typically, the corresponding dual dependent variables are then prescribed by integral non-local operators and are smoother than the original ones. It is of principle importance in nonlinear equations with singularities and measurable coefficients admitting essentially generalized (weak) solutions. In applications, choosing a suitable discrete partition in the independent dual variables, the PDE under consideration generates a system of functional inequalities, which can contain a number of differential, difference and other operators. The duality between PDEs and corresponding systems of functional inequalities, which are expected to inherit important properties of the original evolution models, plays a key role. For instance, the most complete description of A.N. Tikhonov - S. Täklind

Date: January, 2001.

1991 *Mathematics Subject Classification.* 35K55, 35K65.

Key words and phrases. $2m$-th order quasilinear parabolic equations, boundary blow-up, localization, S-regime, Saint-Venant's principle, energy estimates, asymptotic behaviour, Hamilton-Jacobi equation.

uniqueness classes for general higher-order linear parabolic equations and systems was obtained in the 1970's by a version of Saint-Venant's principle.

In this paper we present an application of Saint-Venant's principle to localization blow-up phenomena for a class of quasilinear $2m$-th order parabolic equations. The study of such singular blow-up processes needs delicate estimates near finite blow-up times. This leads to new systems of functional inequalities, where special choices of infinite partitions in dual independent variables are necessary.

We begin with the mathematical statement of the basic problem. A mechanical and physical basement of blow-up singularity formation problems and a survey of results on energy estimates representing Saint-Venant's principle will be presented next.

1.1. **Statement of the problem.** Let us formulate the basic example posed in a simple geometry. Let $\Omega = \{|x| > 1\}$ be the open complement of the unit ball $B = \{|x| < 1\}$ in $\mathbf{R}^N$. In the cylindrical domain $Q = \Omega \times (0, T)$ we consider the Cauchy-Dirichlet problem for a general quasilinear $2m$-th order parabolic equation, $m \geq 1$,

$$(|u|^{q-1}u)_t + \sum_{|\alpha|=m} (-1)^m D_x^\alpha a_\alpha(x, t, u, ..., D_x^m u) = 0, \quad q > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x) \in L_{q+1}(\Omega) \quad \text{in } \Omega, \tag{1.2}$$

$$D_x^\alpha (u - f) = 0 \quad \text{on } \partial\Omega \times (0, T) \text{ for any } |\alpha| \leq m - 1. \tag{1.3}$$

The functions $a_\alpha(x, t, \xi)$ are assumed to be continuous, and the elliptic operator on the left-hand side satisfies the following growth and coercivity conditions:

$$\sum_{|\alpha|=m} a_\alpha(x, t, \xi)\xi_\alpha \geq d_1 (\sum_{|\beta|=m} |\xi_\beta|)^{p+1}, \quad d_1 > 0, \ p \geq q, \tag{1.4}$$

$$|a_\alpha(x, t, \xi)| \leq d_2 (\sum_{|\beta|=m} |\xi_\beta|)^p \quad \text{for } (x, t, \xi) \in \bar{Q} \times \mathbf{R}^{n(m)}, \ d_2 > 0, \tag{1.5}$$

where $n(m)$ is the number of distinct multiindices of the length not exceeding m. We use the usual notations $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = \alpha_1 + ... + \alpha_N$,

$$D_x^\alpha = \partial^{\alpha_1 + ... + \alpha_N} / \partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}$$

and $D_x^m u = \{D_x^\alpha u, \ |\alpha| = m\}$. The function $f(x, t)$ determining the boundary values, is assumed to be extended from the lateral boundary $\partial\Omega \times (0, T)$ into the domain Q such that for any $T_0 < T$, there hold

$$f(\cdot, t) \in C([0, T_0]; \ L_{q+1}(\Omega)) \cap L_{p+1}(0, T_0; \ W_{p+1}^m(\Omega)), \tag{1.6}$$

$$f_t(\cdot, t) \in L_1(0, T_0; \ L_{q+1}(\Omega)) \cap L_r(0, T_0; L_r(\Omega)), \ r = (p+1)/(p+1-q). \tag{1.7}$$

The main feature of the problems under consideration is as follows. We assume that the boundary function $f(\cdot, t)$ *blows up* as $t \to T^-$. The rate of blow-up is characterized by the following function defined for $t \in (0, T)$:

$$F(t) = \sup_{0<\tau<t} \left(\int_\Omega |f(x,\tau)|^{q+1} dx \right) + \int_0^t \int_\Omega |D_x^m f(x,\tau)|^{p+1} dx d\tau$$

$$+ \left(\int_0^t \left(\int_\Omega |f_\tau(x,\tau)|^{q+1} dx \right)^{1/(q+1)} d\tau \right)^{q+1} \to \infty \text{ as } t \to T^-. \tag{1.8}$$

Without loss of generality, we always suppose that the support of $f(\cdot, t)$ is uniformly bounded for $t \in (0, T)$. Given $S \subset \partial\Omega$, by $W_{p+1}^m(\Omega, S)$ we denote, as usual, the closure in the norm of the Sobolev space $W_{p+1}^m(\Omega)$ of the subset of functions from $C^\infty(\Omega)$ vanishing in a neighbourhood of S. We set $W_{p+1}^m(\Omega, \emptyset) \equiv W_{p+1}^m(\Omega)$ and by $\langle \cdot, \cdot \rangle$ denote the duality product between $\left(W_{p+1}^m(\Omega, \partial\Omega)\right)^*$ and $W_{p+1}^m(\Omega, \partial\Omega)$.

Definition 1. A function $u(x,t)$ is said to be a *generalized energy solution* of the problem (1.1)-(1.3), if for any $T_0 < T$, there hold

(i) $u - f \in L_{p+1}(0, T_0; W_{p+1}^m(\Omega, \partial\Omega)) \cap C([0, T_0]; L_{q+1}(\Omega))$,

(ii) $(|u|^{q-1}u)_t \in L_{(p+1)/p}(0, T_0; (W_{p+1}^m(\Omega, \partial\Omega))^*)$,

(iii) u satisfies the initial condition (1.2) and the following integral identity:

$$\int_0^{T_0} \langle (|u|^{q-1}u)_t, \eta \rangle dt + \int_0^{T_0} \int_\Omega \sum_{|\alpha|=m} a_\alpha(x,t,u(x,t),...,D_x^m u(x,t)) D_x^\alpha \eta(x,t) dx dt = 0, \tag{1.9}$$

where $\eta(x,t) \in L_{p+1}(0, T_0; W_{p+1}^m(\Omega, \partial\Omega))$ is an arbitrary function.

An extensive literature is devoted to questions of local existence and uniqueness of generalized (weak) solutions of the mixed problem for equations (1.1). In the case $q = 1$, $f = 0$ problem (1.1)-(1.3) was studied in detail in [30]. In the case $m = 1$ and bounded domains Ω existence of energy solutions follows from [2] for arbitrary $T_0 < T$ by the second inclusions in (1.6)-(1.7). For $m > 1$ and $f = 0$ the solvability of problem under consideration was established in [9] in the case where equation (1.1) has a variational structure, in particular, if $a_\alpha(x,t,\xi) \equiv |\xi_\alpha|^{p-1}\xi_\alpha$. The results admit a natural extension to more general higher-order equations and boundary conditions.

1.2. **Localization of blow-up: definition, terminology and a survey.** The paper is devoted to the study of the localization of blow-up boundary regimes.

Definition 2. The set of points $y \in \overline{\Omega}$ satisfying for any small $\varepsilon > 0$

$$\sup_{0<t<T} \left(\int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,t)|^{q+1} dx + \int_0^t \int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,\tau)|^{p+1} dx d\tau \right) = \infty,$$

is said to be the *singularity* or the *blow-up subset*, denoted by $\Omega_S \equiv \Omega_S(u)$, of the energy solution $u(x,t)$ of problem (1.1)-(1.3). $\Omega_R \equiv \overline{\Omega} \setminus \Omega_S$ is the corresponding *non-singular subset*.

Actually, for sufficiently regular solutions, it is expected that $u(x_0, t)$ is uniformly bounded as $t \to T^-$ at any interior point $x_0 \in \Omega_R = \Omega \setminus \Omega_S$, and vise versa, $u(x_0, t)$

is not uniformly bounded as $t \to T^-$ for any $x_0 \in \Omega_S$. For such regular solutions, the blow-up set Ω_S is known as the *localization domain* of the solution (or of the boundary blow-up regime). In the one-dimensional case, $L = \operatorname{meas} \Omega_S$ is called the *localization length* of the blow-up regime, see Chapt. 3 in [43]. We also use the following terminology [43]. Given bounded initial data, we say that the blow-up regime, prescribed by the function $f(x,t)$ on $\partial\Omega$, is *localized* (respectively, *non-localized*) if $\operatorname{meas} \Omega_S < \infty$ (respectively, $\operatorname{meas} \Omega_S = \infty$). Localized blow-up regime is called S-regime if $\operatorname{meas} \Omega_S \in (0, \infty)$ (regional blow-up) and LS-regime if $\operatorname{meas} \Omega_S = 0$ (this includes the case of single point blow-up). A non-localized blow-up regime with $\operatorname{meas} \Omega_S = \infty$ is also called HS-regime (global blow-up).

A general treatment of blow-up processes naturally occurred in the 1930's - 1950's in the context of N.N. Semenov's chain reaction theory (1930's), adiabatic explosion and combustion theory (we mention D.A. Frank-Kamenetskii's equation (1938) admitting blow-up in the non-stationary version and the first blow-up analysis by O.M. Todes (1933)), see [18] and [53]. A strong influence was due to blow-up singularities in gas dynamics, the intense explosion (focusing) problem with second kind self-similar solutions considered by K. Bechert, K.G. Guderley and L.I. Sedov in the 1940's; see [3], p. 127, and [54]. Another classical area of blow-up processes occurring in the 1960's is nonlinear optics. Here the main model is the nonlinear (cubic) Schrödinger equation admitting blowing up self-focusing solutions. See references in the book [51] and in surveys in [29], [31].

A great interest to blow-up phenomena in the 1960's was also generated by highly nonstationary and nonlinear problems of the Controlled Thermonuclear Fusion (CTF) and in the Inertial Confinement Fusion (ICF). N.G. Basov and O.N. Krohin [4] (1964) proposed to use concentrated (focusing) laser beams for heating of a deuterium-tritium (DT) pellet to create plasma of super-high temperatures. A remarkable feature of boundary blow-up regimes in CTF-problems was revealed numerically by I. Nuckolls, L. Wood, A. Thiessen and G. Zimmerman in [32] (1972) showing a principle possibility of the laser blow-up-like compression of a DT drop to super-high densities *without shock waves* (these crucial results were first announced by E. Teller in Montreal in 1972). Such a compression was expected to initiate a thermonuclear burning (by the Lawson criterion the rate of produced thermonuclear energy at a fixed ion temperature is proportional to the product of plasma density and confinement time).

The electron and ion conductivity of fully ionized plasma are known to depend essentially on the temperature. A simplest mathematical model including the heat propagation only (gas dynamics phenomena are known to be inertial and under certain hypotheses can be neglected at the purely heat stage) consists of a quasilinear second-order heat equation

$$u_t - (k(u)u_x)_x = 0 \quad \text{in } \mathbf{R}_+ \times (0,T), \tag{1.10}$$

with given bounded initial data (temperature) $u(x,0) = u_0(x)$, and a blow-up boundary regime at $x = 0$:

$$u(0,t) = f(t) \to \infty \quad \text{as } t \to T^- < \infty. \tag{1.11}$$

Here $k(u) > 0$ is the given heat conductivity coefficient satisfying $k(u) \to \infty$ as $u \to \infty$ ($k(u) \sim u^{5/2}$ for the fully ionized plasma) and $f(t)$ is a prescribed temperature on the surface of the DT-target, which is generated by laser beams and increases in a blowing up (peaking) regime.

A first detailed analysis of the striking effect of space *localization* of blow-up boundary regime (*S-regime of blow-up* generated by a blow-up *Standing wave*) in the quasilinear diffusion equation (1.10) with the power law $k(u) = u^\sigma$, $\sigma > 0$, was performed by A.A. Samarskii and I.M. Sobol' (1963) in [44] (such explicit blow-up solutions were used before as parabolic barriers, see typical references in A.S. Kalashnikov's survey [20]). In the 1970's the effect of space localization in Ω_S, where temperature $u(x,t)$ tends to infinity as $t \to T$, became a popular subject and posed a number of typical problems for quasilinear parabolic PDEs. In particular, the problem of localization of blow-up solutions in reaction-diffusion equations was first proposed by S.P. Kurdyumov [25] in 1974, see Chapters 3 and 4 in [43] devoted to localization analysis. It was shown that, in a quasilinear model, the heat and burning localization arises in plasma with the electron heat conductivity and heat source due to amalgamation of DT nuclei [55]. Such a model, as well as different aspects of localization effects, are discussed in [27]. An extensive list of references on localization of blow-up dissipative structures with the historical review can be found in Kurdyumov's survey paper [26]. Main mathematical results on existence and nonexistence of localization of boundary blow-up and singularity formation for equation (1.10) with rather arbitrary monotone conductivity coefficients $k(u)$ are summarized in [43], Chapt. 3, see also surveys in [12], [13]. A necessary and sufficient condition of localization for the porous medium equation (1.10) with $k(u) = u^\sigma$, $\sigma > 0$, was established in [19]. For general N-dimensional quasilinear second-order parabolic equations ($m = 1$), localization conditions were obtained in [47], [48] by means of the method of energy estimates we are going to use below for arbitrary higher-order equations.

In the present analysis, the blow-up singularity is generated on the boundary by means of a blow-up function. On the other hand, interior blow-up can be generated by extra source-type lower-order operators, if, for instance, we consider the Cauchy problem for the quasilinear heat equation

$$u_t = \nabla \cdot (k(u) \nabla u) + Q(u), \tag{1.12}$$

where $Q(u) \geq 0$ has a superlinear growth as $u \to \infty$. Both types of blow-up (boundary and interior) phenomena are essentially related to each other, the localization terminology stays the same, though blow-up in (1.12) is more delicate, see the book by J. Bebernes and D. Eberly [5], H.A. Levine's survey [29] and Chapt. 4 in [43] for main results and related references. In general, localization problems of interior blow-up for such higher-order semilinear and quasilinear parabolic equations remain open.

1.3. **Main results.** We prove localization of boundary blow-up for general $2m$-th order quasilinear parabolic equations. Higher-order semilinear and quasilinear diffusion operators occur in several applications including thin film theory, nonlinear diffusion, lubrication theory, flame and wave propagation, phase transition at critical Lifschitz points and bi-stable systems (e.g., the Kuramoto-Sivashinskii equation and the extended Fisher-Kolmogorov equation). See a number of models and a list of references in the recent book by L.A. Peletier and W.C. Troy [38].

It is convenient to present the main result in two theorems. It turns out that in the special case $p = q$, when the parabolic operators exhibit some kind of "linear" properties (due to coinciding algebraic homogeneities), the asymptotic behaviour of blow-up solutions is the most nonlinear. One can see from the theorems below that estimates on the right-hand sides do not exhibit any scaling invariance, see also further comments below. The localized blow-up regimes admit the following general description for any $2m$-th order parabolic equations. We will present examples showing that such exponential estimates of localized blow-up regimes are sharp and cannot be improved.

Theorem 1.1. *Let $p = q > 0$. Given a blow-up boundary regime f, let the corresponding function* (1.8) *satisfy as $t \to T^-$,*

$$F(t) \le c \exp\{A(T-t)^{-1/[m(p+1)-1+\xi_0]}\}, \tag{1.13}$$

where c, A are arbitrary positive constants and the constant $\xi_0 > 0$ can be arbitrarily small. Then, for arbitrary constant $\delta > 0$, any energy solution $u(x,t)$ of the problem (1.1)-(1.3) *satisfies*

$$\int_{\{|x|>1+\delta\}} |u(x,t)|^{p+1} dx < C = C(\delta) < \infty, \quad t \in (0,T), \tag{1.14}$$

i.e., the singularity set Ω_S is concentrated on the boundary $\partial\Omega$ and f is a localized blow-up LS-regime.

The unit ball Ω can be replaced by any bounded domain with sufficiently smooth boundary and the result says again that for such boundary blow-up, the singularity set is concentrated on the boundary.

Theorem 1.2. *Let $p = q > 0$ and boundary blow-up regime satisfy as $t \to T^-$,*

$$F(t) \le c \exp\{w(T-t)^{-1/[m(p+1)-1]}\}, \tag{1.15}$$

with some positive constants c and w. Then there exists a constant $L = L(w) < \infty$ depending also on other parameters of the problem (1.1)-(1.3)*, such that for any energy solution,*

$$\int_{\{|x|>L\}} |u(x,t)|^{p+1} dx < C < \infty \quad \textit{for all } t \in (0,T). \tag{1.16}$$

The uniform estimate (1.16) means that the blow-up set of solution $u(x,t)$ is contained in the ball of radius L. Hence, any boundary regime satisfying (1.15) is, at most, a localized blow-up S-regime.

1.4. **Method: Saint-Venant's principle and energy estimates.** Saint-Venant's principle and problem [39] - [41] formulated in the 1850's play a fundamental role in the linear theory of elastic equilibrium. First rigorous energy estimates on exponential decay of a parameterized strain energy and a strain inequality for a cylinder with a loaded cross section on the end were established by R.A. Toupin (1965) [52] and by J.K. Knowles (1966) [21] (in these papers principle earlier references can be found). In particular, a detailed analysis in [21] established second-order estimates on smooth solutions of the biharmonic (fourth-order elliptic) equation

$$\Delta^2 \varphi = 0 \ \text{ in } \ R, \quad \varphi = \varphi_x = \varphi_y = 0 \ \text{ on } \ \partial R, \tag{1.17}$$

where $R \subset \mathbf{R}^2$ is a bounded simply connected domain (the cross section of the cylinder) with smooth boundary ∂R. It is proved that the strain energy

$$E(z) = \iint_{R_z} (\varphi_{xx}^2 + \varphi_{yy}^2 + 2\varphi_{xy}^2) dxdy$$

contained in the subdomain $R_z = R \cap \{x \geq z\}$ ($z \geq 0$ is a parameter) decays at least exponentially with z. Pointwise derivatives estimates were also proved by using mean value theorems for biharmonic functions. Such energy estimates in a weak form applied to more general second-order elliptic equations [22] and to the heat (second-order parabolic) equation [23].

Estimates of behaviour of the energy integrals over the families of inner subdomains of diameter $r \to 0$ (i.e., those having the structure of Saint-Venant's principle, we do not mention other important approaches) play important role in the study of regularity of generalized solutions of elliptic and parabolic equations. Such estimates for second-order equations were derived by S. Campanato [10]. V.A. Solonnikov [49], [50] proposed a method of obtaining such estimates for energy solutions of quasilinear divergent elliptic higher-order equations, which is based on the use of E. Hopf's cut-off functions. The behaviour of energy integrals for solutions of a general class of higher-order elliptic equations near the boundary was studied in [24], where references to earlier papers of the 1970's on polyharmonic and equations from elasticity theory can be found.

A systematic extension of ideas of different *a priori* estimates via introducing a parameter for classes of partial differential equations (PDEs) is due to O.A. Oleinik and collaborators, G.A. Iosif'yan, V.A. Kondratiev and E.V. Radkevich (1974-82), see [33] - [37], [24] and a full list of references in the survey in [37], where, in particular, the uniqueness in Tikhonov - Täcklind classes for linear second and higher-order parabolic equations and systems was proved. These were advanced applications of Saint-Venant's principle to linear higher-order PDEs. Extra references and further comments on energy methods for quasilinear higher-order PDEs can be found in Section 4 of this paper, where we begin our analysis of general higher-order parabolic equations.

1.5. **Plan of the paper.** Theorems 1.1 and 1.2 are proved in Sections 4 - 7 by a modification of the energy estimate method of the type [37]. We use a version of energy estimates for quasilinear $2m$-th order parabolic equations developed in [46] and [1]. An important feature of our asymptotic analysis is that, in order to exhibit the localization of blow-up, we derive an infinite functional system of inequalities for a series of suitable energy functions by means of a specifically organized spatial-time partition of the (x,t)-domain concentrated near blow-up singularity.

Before proceeding with proofs of two theorems, in order to anticipate typical crucial features of boundary blow-up in the given class of equations, we study some special scaling invariant quasilinear and nonlinear second and higher-order parabolic equations, which will be proved to admit non-invariant blow-up asymptotics. In Sections 2 (second-order quasilinear equations) and 3 ($2m$-th order linear parabolic equations) we give more detailed characteristics of such blow-up singularities including the study of the geometric shape of localization domains and the asymptotic behaviour of blow-up solutions. In particular, we prove that in these two classes of problems, the exponential blow-up function on the boundary

$$f(t) = \exp\{(T-t)^{n_*}\}, \quad \text{with the critical exponent } n_* = -\frac{1}{m(p+1)-1}, \tag{1.18}$$

cf. (1.15), always leads to a localized S-regime with the localization domain of bounded positive measure, $\operatorname{meas}\Omega_S \in \mathbf{R}_+$. Therefore, Theorem 1.1 establishes the optimal characterization of LS-regimes. Moreover, we show that an arbitrary negative perturbation of the critical exponent n_* transforms this localized S-regime into the non-localized HS-regime with the boundary blow-up

$$f(t) = \exp\{(T-t)^{n}\}, \quad n < n_*, \tag{1.19}$$

where $\operatorname{meas}\Omega_S = \infty$ and $u(x,t) \to \infty$ as $t \to T^-$ uniformly on compact subsets.

The class of quasilinear equations with $p = q$ is a special one. It follows from (1.13) or (1.15) that the main blowing up regimes are not scaling invariant (though in examples the homogeneous parabolic operators are invariant under a group of scaling transformations). Formally, scaling invariant asymptotics are obtained by an extra nonlinear logarithmic transformation $v = \ln u$. In the next two sections we show that the asymptotic blow-up behaviour in terms of $v(x,t)$ is described by a blow-up similarity solution of a nonlinear equation of Hamilton-Jacobi type so that the higher-order parabolic terms form singular perturbations of first-order equations.

For the linear heat equation with $m = 1$ and for weakly quasilinear equations (1.10), $N = 1$, the above results on the blow-up singularity formation via Hamilton-Jacobi equations are well-known, see first results in [42] and a detailed analysis in [43], Chapt. 3.

In the case $p > q$, to be considered in a forthcoming paper [14], the main blow-up asymptotics are given by scaling-invariant self-similar solutions of model equations.

2. Second-order parabolic equations

If $m = 1$, then the Maximum Principle applies, and for some typical equations we can study the blow-up behaviour in more detail, including the asymptotic behaviour and the geometry of localization domains. Let $\Omega \subset \mathbf{R}^N$ be a bounded connected domain with the complement $E_\Omega = \mathbf{R}^N \setminus \overline{\Omega}$. In this section we consider the exterior problem with $m=1$, $p>0$, where for convenience we set $T=1$ (and $T=0$ later on):

$$(|u|^{p-1}u)_t = \Delta_p u \equiv \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-1}\frac{\partial u}{\partial x_i}\right) \quad \text{in } \; Q = E_\Omega \times (0,1), \tag{2.1}$$

$$u(x,t) = f(x,t) \quad \text{on } \; S = \partial\Omega \times (0,1), \quad u(x,0) = u_0(x) \quad \text{in } \; E_\Omega. \tag{2.2}$$

We assume that $f(\cdot,t)$ blows up at $t=1^-$. Without loss of generality, we suppose that $f \ge 1$ and $u_0(x) \ge 1$, hence $u \ge 1$ in Q by the Maximum Principle. Such quasilinear parabolic equations of the second order are well known in the literature, see the classical book [30] by J.L. Lions, Kalashnikov's survey [20] and E. DiBenedetto's book [11]. By an energy solution we mean a weak one. Since the operator of the parabolic equation is not monotone, the uniqueness of such weak solution is not straightforward. We then assume that $u(x,t)$ is a unique maximal solution, which is constructed by constructed by monotone smooth approximation. It satisfies the Maximum Principle and the usual comparison holds. Such a construction of extended limit semigroups of extremal (maximal and minimal) solutions is an essential and necessary feature for nonlinear parabolic equations admitting strong blow-up or extinction-like finite-time singularities, see Sect. 2 in [17] and a list of references therein.

In order to describe the asymptotic behaviour of main blow-up singularities for this equation, as in the linear case $p=1$ [42] (see details in [43], Chapt. 3), we perform the change of the dependent variable

$$u = e^v, \quad v \ge 0, \tag{2.3}$$

which leads to the quasilinear equation with a typical Hamilton-Jacobi operator and the diffusion p-Laplacian term

$$v_t = |\nabla v|^{p+1} + p^{-1}\Delta_p v. \tag{2.4}$$

The main idea is to prove that the asymptotic behaviour of this blow-up singularities is described by the corresponding *Hamilton-Jacobi equation*

$$V_t = |\nabla V|^{p+1}. \tag{2.5}$$

More precisely, we will show that the S-regime of boundary blow-up regional localization is given by its separate-variable solution

$$V_*(x,t) = (1-t)^{-1/p}\theta(x), \tag{2.6}$$

where $\theta \ge 0$ solves the stationary Hamilton-Jacobi equation

$$|\nabla\theta|^{p+1} - p^{-1}\theta = 0 \quad \text{in } \; E_\Omega. \tag{2.7}$$

2.1. **S-regime in one dimension.** If $N = 1$ and $E_\Omega = \mathbf{R}_+$, (2.7) becomes a first-order ODE of the form

$$|\theta'|^{p+1} - p^{-1}\theta = 0, \tag{2.8}$$

which gives the following sufficiently smooth compactly supported solution:

$$\theta = (1 - x/x_0)_+^{(p+1)/p}, \quad x \in \mathbf{R}_+, \tag{2.9}$$

where x_0 is the *effective localization length*,

$$x_0(p) = (p+1)p^{-p/(p+1)} \tag{2.10}$$

of the boundary blow-up S-regime prescribed at the origin $x = 0$,

$$v(0,t) = (1-t)^{-1/p}, \quad t \in (0,1). \tag{2.11}$$

On the other hand, it also satisfies the Neumann boundary condition

$$-v_x(0,t) = c_0(1-t)^{-1/p}, \quad c_0 = -\theta'(0) > 0.$$

In what follows, without loss of generality, we fix the blow-up Dirichlet boundary condition (2.11). The function θ is sufficiently smooth, $|\theta'|^{p+1} \in C^1$, and indeed it is Kruzhkov's entropy (viscosity) solution [28] of the Hamilton-Jacobi equation (2.5).

We prove that in the parabolic equation (2.4) the elliptic p-Laplacian operator is negligible on the asymptotic blow-up stage $t \to 1^-$ and forms a singular perturbation of the Hamilton-Jacobi equation. We introduce the rescaled function

$$g(x,\tau) = (1-t)^{1/p} v(x,t), \tag{2.12}$$

where $\tau = -\ln(1-t) \to \infty$ as $t \to 1^-$ is the new time variable, and arrive at the perturbed Hamilton-Jacobi equation

$$g_\tau = |\nabla g|^{p+1} - p^{-1} g + e^{-\tau/p}\Delta_p g \quad \text{for } \tau > 0, \quad g(0,\tau) \equiv 1. \tag{2.13}$$

The initial function $g_0(x) \geq 0$ is assumed to be bounded.

Theorem 2.1. *(i) There holds*

$$g(x,\tau) = \theta(x) + o(1) \quad as \ \tau \to \infty \ \ uniformly. \tag{2.14}$$

(ii) The effective localization length of $u(x,t)$ is bounded and positive:

$$L(u) = \text{meas}\{x > 0 : \limsup_{t \to 1} u(x,t) = \infty\} = x_0(p). \tag{2.15}$$

In the proof we derive an exact rate of convergence in (2.14) which is different in the parameter ranges $p > 1$, $p = 1$ and $p < 1$.

Proof. (i) The main difficulty is that the p-Laplacian is not bounded and is not well defined on the compactly supported function $\theta(x)$ at the interface $x = x_0$. Therefore, we first compare two solutions $\bar v$ and $\bar V_*$ of the problem posed in the domain $Q_1 = (0, 2x_0) \times (0,1)$ with the same boundary condition at $x = 0$ and an extra symmetric condition on the right-hand lateral boundary:

$$\bar v(2x_0, t) \equiv v(0,t) = (1-t)^{-1/p}.$$

The corresponding solution of the Hamilton-Jacobi equation

$$\bar{V}_*(x,t) = (1-t)^{-1/p}\bar{\theta}(x), \quad \text{where } \bar{\theta}(x) = |1 - x/x_0|^{(p+1)/p},$$

is indeed V_* reflected relative to $x = x_0$. As we show, this reflected solution is sufficiently regular for comparison by the Maximum Principle since $\Delta_p \bar{V}_*(x_0, t)$ makes sense.

If $p \le 1$, then $\theta(x) \in C^2$ and we can use simpler straightforward computations. In any case, for any $p > 0$ and especially for $p > 1$, we assume that the solution $\bar{v}$ is sufficiently smooth using a standard approximation via a suitable regularization of the p-Laplacian

$$\Delta_p v \mapsto \Delta_{p,\varepsilon} v = \nabla \cdot [(|\nabla v|^2 + \varepsilon^2)^{(p-1)/2} \nabla v], \;\; \varepsilon > 0.$$

Consider the difference $w = \bar{v} - \bar{V}_*$ satisfying the parabolic equation

$$w_t = |\bar{v}_x|^{p+1} - |(\bar{V}_*)_x|^{p+1} + p^{-1}\Delta_{p,\varepsilon}(\bar{V}_* + w).$$

The right-hand side can be written down in the form

$$w_t = \mathbf{B}w + p^{-1}\Delta_{p,\varepsilon}(\bar{V}_*),$$

where on regularized solutions the operator $\mathbf{B}w = a w_{xx} + b w_x$ is a differential second-order one with sufficiently smooth coefficients. Passing to the limit $\varepsilon \to +0$ relative to the regularizing parameter, we construct a weak plane supersolution $\bar{w}(t)$ depending on the variable t only and satisfying the differential inequality

$$\bar{w}' = p^{-1} \sup_x \Delta_p \bar{V}_*(x,t) \equiv c_p (1-t)^{-1/p}, \quad c_p = p^{-1} \sup_{x>0} \Delta_p \bar{\theta}(x) = [p(p+1)]^{-1},$$

where, as we have seen, the finiteness of the constant c_p plays a key role in the regularization argument. Choosing $\bar{w}(0) = C_0 \gg 1$, by the Maximum Principle we conclude that if $p \ne 1$

$$|\bar{v}(x,t) - \bar{V}_*(x,t)| \le \bar{w}(t) = C_0 + \overline{c}_p (1-t)^{(p-1)/p}$$

for any bounded initial data $\bar{v}(x,0)$. For $p = 1$ we get the logarithmic term $c_p |\ln(1-t)|$ in the right-hand side (cf. a general approach on p. 379 in [43]). Assume now that $p \ne 1$. In particular, this implies that at the end point of the localization domain $x = x_0$ there holds

$$\bar{v}(x_0,t) \le C_0 + \overline{c}_p (1-t)^{(p-1)/p}.$$

Consider now the solution $v(x,t)$ of the original problem in the half-line $\{x > 0\}$. One can see that $v \le \bar{v}$ in Q_1, so that

$$v \le C_0 + \overline{c}_p (1-t)^{(p-1)/p} \quad \text{in } (0, x_0] \times (0,1).$$

On the other hand, using standard monotonicity (in x) properties of large solutions of parabolic equations, we have that by the Maximum Principle for any small $\varepsilon > 0$ as $t \to 1^-$,

$$v(x,t) \le v(x_0 - \varepsilon, t) \quad \text{for any } x \ge x_0.$$

Passing to the limit $\varepsilon \to 0$, we then obtain that for $t \approx 1^-$

$$v(x,t) \le C_1 + \overline{c}_p (1-t)^{(p-1)/p} \quad \text{for } x \ge x_0.$$

One can see that $\Delta_p \theta \geq 0$ so that the compactly supported function $V_*(x,t)$ is a subsolution of the parabolic equation and hence the following estimate from below holds:

$$v(x,t) \geq -C_2 + V_*(x,t).$$

Summing up all the above estimates, we arrive at the rate of convergence in (2.14).

(ii) The calculations leading to the asymptotic expansion (2.14) mean that as $t \to 1^-$,

$$u(x,t) = e^{(1-t)^{-1/p}\theta(x)} e^{O((1-t)^{(p-1)/p})} \tag{2.16}$$

and

$$C_2 e^{(1-t)^{-1/p}\theta(x)} \leq u(x,t) \leq C_3 e^{O((1-t)^{(p-1)/p})} e^{(1-t)^{-1/p}\theta(x)}.$$

If $p = 1$, then estimates take the form

$$C_2 e^{(1-t)^{-1/p}\theta(x)} \leq u(x,t) \leq C_3 (1-t)^{-c_p} e^{(1-t)^{-1/p}\theta(x)}.$$

Obviously, this implies that $L \geq x_0(p)$. On the other hand, setting here $x = x_0$, where $\theta(x_0) = 0$, we get

$$u(x_0,t) \leq C_3 (1-t)^{-c_p} \quad \text{as } t \to 1^-. \tag{2.17}$$

Any power-like blow-up on the boundary

$$u(0,t) = (1-t)^n, \quad n < 0, \tag{2.18}$$

is known to correspond to a blow-up LS-regime with the zero effective localization length. This follows from the general Theorem 1.1 stated in the introduction to be proved later on. On the other hand, it can be proved by the standard comparison with the corresponding localized self-similar solution of the form

$$u_*(x,t) = (1-t)^n f(\eta), \quad \eta = x/(1-t)^{1/(p+1)},$$

where $f > 0$ solves the ODE

$$(|f'|^{p-1} f')' - (p+1)^{-1} (f^p)' \eta + np f^p = 0, \quad \eta > 0; \quad f(0) = 1.$$

The self-similar strictly monotone decreasing C^∞-profile $f(\eta) > 0$ for all $\eta > 0$ satisfies the following asymptotic behaviour:

$$f(\eta) = C_* \eta^{n(p+1)} (1 + o(1)) \to 0 \quad \text{as } \eta \to \infty \quad (C_* > 0).$$

Then the solution u_* is effectively localized at $x = 0$ in the sense that for any $x > 0$, as $t \to 1^-$,

$$u_*(x,t) \to u_*(x, 1^-) = C_* x^{n(p+1)} < \infty.$$

By the usual comparison of two solutions $u(x,t)$ and $u_*(x - x_0, t)$, which are assumed to be smooth enough by monotone regularization, this implies that $u(x,t)$ is also uniformly bounded for any $x > x_0$, so that $L = x_0$. $\square$

2.2. **Non-localized HS-regime in one dimension.** Let us prove that there exist blow-up functions from the same exponential class staying above the S-regime leading to non-localized HS-regimes (i.e., blow-up regimes which are "Higher than S"). We thus consider the blow-up boundary condition

$$u(0,t) = e^{(1-t)^n} \quad \text{with } n < -1/p. \tag{2.19}$$

Theorem 2.2. *Let* (2.19) *hold. Then* $u(x,t)$ *blows up as* $t \to 1^-$ *uniformly on the indefinitely expanding subset*

$$\{0 \le x \le c_0(1-t)^\beta\}, \quad \beta = [np+1]/(p+1) < 0, \tag{2.20}$$

with a constant $c_0 = c_0(p,n) > 0$.

Proof. The new functions $v = \ln u$ satisfies the boundary condition

$$v(0,t) = (1-t)^n.$$

The proof is based on comparison from below of the given solution $v(x,t)$ with the self-similar solution of the Hamilton-Jacobi equation (2.5)

$$V_*(x,t) = (1-t)^n h(\eta), \quad \eta = x/(1-t)^\beta, \tag{2.21}$$

where $h \ge 0$ solves the first-order ODE

$$|g'|^{p+1} - \beta g' \eta + ng = 0, \ \ \eta > 0; \quad g(0) = 1.$$

This ODE admits a scaling invariance and reduces to an autonomous ODE, which is integrated in quadratures. See a similar analysis of the case $p = 1$ in [43], p. 171. As the result, we obtain that for $n < -1/p$, such a monotone profile g exists and vanishes at a finite point

$$\eta_0 = \text{meas supp}\, g < \infty. \tag{2.22}$$

We then set $g(\eta) \equiv 0$ for all $\eta \ge \eta_0$ so that $g(\eta)$ becomes a continuous (but non-smooth) compactly supported rescaled viscosity solution of the Hamilton-Jacobi equation. The comparison of v and V_* in the domain $\{x < \eta_0(1-t)^n\} \times (0,1)$, where both solutions are sufficiently smooth, is performed as in the case of the S-regime above (see also [43], p. 382) and gives uniform blow-up on any unbounded subset (2.20) with arbitrary $c_0 < \eta_0$. □

A general stability approach to the asymptotic degeneracy of parabolic flows into the Hamilton-Jacobi ones was developed in [15] and [16]. By using there results, it is not difficult to show that the self-similar solution (2.21) of the Hamilton-Jacobi equation (2.5) actually describes the asymptotic behaviour of the HS-regime for the parabolic equation (2.4). This is similar to the behaviour in the linear case $p = 1$, [43], p. 381.

2.3. **S-regime in the N-dimensional geometry.** By $\Omega \subset \mathbf{R}^N$ we again denote a bounded smooth domain with complement E_Ω. Consider the following blow-up condition:

$$u(x,t) = e^{(1-t)^{-1/p}} \Longrightarrow v(x,t) = (1-t)^{-1/p} \quad \text{on } \partial\Omega, \ t \in (0,1).$$

The rescaled function $g(x,\tau)$ satisfies (2.13) with the boundary condition

$$g(x,\tau) = 1 \quad \text{on } \partial\Omega, \ \tau > 0.$$

The corresponding Hamilton-Jacobi stationary profile solves the stationary exterior problem

$$|\nabla\theta|^{p+1} - p^{-1}\theta = 0 \quad \text{in } E_\Omega, \quad \theta(x) = 1 \text{ on } \partial\Omega.$$

One can expect that this solution describes the asymptotic behaviour of the parabolic equation. Then the support of θ,

$$\operatorname{supp}\theta = \{x \in E_\Omega : \ \theta(x) > 0\},$$

is the effective localization domain of the solution $V_*(x,t)$. We now compare this localization domain for the Hamilton-Jacobi equation (2.5) and the localization domain

$$\Omega_S(v) = \{x \in E_\Omega : \ \limsup_{t\to 1} v(x,t) = \infty\}$$

of solutions $v(x,t)$ of (2.4) with bounded initial data v_0.

Theorem 2.3. *There hold: (i) $\Omega_S(v)$ is bounded, and (ii) the x_0-neighbourhood of Ω,*

$$N_{x_0}(\Omega) = \{y \in \mathbf{R}^N \setminus \Omega : \ \operatorname{dist}(y,\partial\Omega) \le x_0(p)\},$$

is contained in $\Omega_S(v)$.

Proof. (i) Let Ω_* be any convex smooth domain such that $\Omega \subseteq \Omega_*$. We use comparison with one-dimensional solutions. Fix a point $y_0 \in \partial\Omega_*$ and let $l(y_0)$ be the hyperplane tangent to $\partial\Omega_*$ at $y = y_0$. Denote by z the distance to $l(y_0)$. Consider the one-dimensional solution $v_1(z,t)$ of equation (2.4) with the blow-up function $v_1(0,t) = (1-t)^{-1/p}$ prescribed on the hyperplane, with sufficiently large initial data $v_1(z,0)$. By Theorem 2.1, $v_1(z,t)$ is effectively localized in the infinite strip of the width $x_0(p)$ with the outer plane lateral boundary parallel to $l(y_0)$. Since $v \le v_1$ in the outer half-space $\{z > 0\}$, the solution $v(x,t)$ is uniformly bounded at any x beyond the localization strip. Thus, $\Omega_S(v)$ is contained in the intersection of all such tangent strips constructed for all points $y_0 \in \partial\Omega_*$ which implies boundedness of the localization domain and some extra estimates to be improved below.

(ii) In order to prove the second result, we will compare the solutions with the radially symmetric self-similar solutions

$$V_*(r,t) = (1-t)^{-1/p}\theta(r), \quad r = |x|,$$

of the Hamilton-Jacobi equation, where θ solves the same stationary equation (2.7) which in the radial geometry becomes the same ODE (2.8) with $' = d/dr$. Then we obtain the same solution (2.9). Fix a point $y_0 \in \Omega$ and a small $\varepsilon > 0$ such that the ball $B_\varepsilon(y_0) \subset \Omega$. Consider the radial solution $v_\varepsilon(r,t)$, $r = |x - y_0|$, of the parabolic equation in the cylinder

$B_\varepsilon(y_0) \times (0,1)$, $v_\varepsilon = (1-t)^{-1/p}$ on ∂B_ε with bounded radial initial data. Let $V_*(r,t)$ be the corresponding approximate self-similar solution described above. The comparison from below establishes that the domain of localization $\{\varepsilon < r < \varepsilon + x_0\}$ of V_* is contained in the domain of localization of v_ε.

As the last step, we compare v_ε with the solution v: $v \geq v_\varepsilon$ in Q by the Maximum Principle. Since $y_0 \in \Omega$ and $\varepsilon > 0$ are arbitrary we obtain the result. □

In the case of the *convex* domain Ω, this simple comparison from above with plane (one-dimensional) self-similar solutions V_* and the comparison from below with the radial ones establish the equality

$$\Omega_S(v) = N_{x_0}(\Omega). \tag{2.23}$$

It is not difficult to see that the equality holds for more general domains Ω. It is curious that though $\partial\Omega$ can be arbitrarily smooth, the boundary of the blow-up set $\Omega_S(v)$ is not necessarily C^1-smooth, see p. 167 in [43].

3. **Blow-up localization in linear $2m$-th order equations**

The linear $2m$-th order parabolic equation in one dimension

$$u_t = (-1)^{m+1} D_x^{2m} u \quad \text{in } \; Q = \mathbf{R}_+ \times (0,1), \tag{3.1}$$

belongs to the case $p = q = 1$ of equations with Hamilton-Jacobi blow-up singularities. These problems can be studied using the convolution representation of the general solutions. Without loss of generality, we consider the following blow-up boundary condition:

$$u(0,t) = f(t) \to \infty, \;\; t \to 1^-; \;\; u_x = ... = D_x^{m-1} u = 0, \;\; x = 0, \;\; t \in (0,1), \tag{3.2}$$

and take zero initial data. Let $b(x,t)$ be the fundamental solution of (3.1) of the self-similar form

$$b(x,t) = t^{-1/2m} F(\xi), \quad \xi = x/t^{1/2m}, \tag{3.3}$$

where $F(\xi)$ is a unique symmetric solution of the linear ODE

$$(-1)^{m+1} F^{(2m)} + \frac{1}{2m}(F\xi)' = 0 \quad \text{in } \mathbf{R}, \qquad \int_{-\infty}^{\infty} F(\xi)\, d\xi = 1. \tag{3.4}$$

By the cosine Fourier transform, we have that

$$F(\xi) = \pi^{-1} \int_0^\infty e^{-s^{2m}} \cos(s\xi)\, ds.$$

The solution of (3.1), (3.2) is given by the convolution potential

$$\begin{aligned} u(x,t) &= 2(-1)^m \int_0^t f(\tau) D_x^{2m-1} b(x, t-\tau)\, d\tau \\ &\equiv 2(-1)^m \int_0^t f(\tau) F^{(2m-1)}[x/(t-\tau)^{1/2m}](t-\tau)^{-1}\, d\tau. \end{aligned} \tag{3.5}$$

For the heat equation, $m = 1$, classification of boundary blow-up via the potential is easy, see Chapters 3 and 6 in [43].

3.1. **S-regime of localization, HS-regimes.** It follows from (3.5) that as $t \to 1^-$, the asymptotic behaviour of the blow-up boundary regime $f(t)$ essentially depends on the asymptotic behaviour of the rescaled kernel $F(\xi)$ as $\xi \to \infty$. It is well-known (and can be easily seen from the ODE (3.4)) that the rescaled kernel $F(\xi)$ is oscillatory as $\xi \to \infty$ for $m > 1$. A standard asymptotic analysis shows that (3.4) admits the solutions with the asymptotic behaviour

$$F(\xi) \sim e^{-a|\xi|^\alpha} \quad \text{as } \xi \to \infty, \tag{3.6}$$

where $\alpha = 2m/(2m-1) \in (1,2)$, and $a \in \mathbb{C}$ satisfies the algebraic equation

$$(-1)^m(\alpha a)^{2m-1} + \frac{1}{2m} = 0.$$

It has $2m-1$ different roots $\{a_k\}$, and $F(\xi)$ for $\xi \gg 1$ exhibits the behaviour corresponding to (3.6) with the maximal $\mathrm{Re}\, a_k < 0$. The leading asymptotic terms of F are given by two linearly independent expressions of the type

$$\sim e^{-a_m|\xi|^\alpha}\cos(b_m|\xi|^\alpha) \text{ (or } \sin(b_m|\xi|^\alpha)) \quad \text{as } \xi \to \infty, \tag{3.7}$$

$$a_m = (2m-1)(2m)^{-\alpha}\cos\left(\frac{\pi(m-1)}{2m-1}\right), \quad b_m = (2m-1)(2m)^{-\alpha}\sin\left(\frac{\pi(m-1)}{2m-1}\right).$$

These asymptotic estimates are enough to establish the following S and HS-regimes of blow-up for the linear parabolic equations.

Theorem 3.1. *(i) For any $m > 1$, the blow-up S-regime occurs for the boundary function*

$$f(t) = e^{(1-t)^{-1/(2m-1)}} \quad \text{as } t \to 1^-, \tag{3.8}$$

and the effective localization length is given by

$$L(u) = a_m^{-1/\alpha} > l_m = 2m(2m-1)^{-1/\alpha}. \tag{3.9}$$

(ii) The HS-regime with any exponent $n < -1/(2m-1)$,

$$f(t) = e^{(1-t)^n} \quad \text{as } t \to 1^-, \tag{3.10}$$

is not localized and $\Omega_S(u) = \mathbf{R}_+$.

For $m = 1$, we have the equality $L(u) = l_1 = 2$, [43], p. 132.

Proof. (i) The localization result follows from potential (3.5), where we use the above asymptotics of the rescaled kernel. We present an analysis proving (3.9) and also showing the asymptotic behaviour of the solutions as $t \to 1^-$ to be compared with the Hamilton-Jacobi limits given below. By shifting $t \mapsto t+1$, the blow-up moment becomes $T = -0$ and now $t < 0$. Using (3.6) in potential (3.5), we obtain that for $t \approx -0$, the main asymptotic expansion terms of solution $u(x,t)$ are composed from oscillatory integrals of the form

$$\int^t e^{(-\tau)^{-\gamma} - a_m x^\alpha (t-\tau)^{-\gamma}} \cos[b_m x^\alpha (t-\tau)^{-\gamma}]d\tau, \quad \gamma = 1/(2m-1). \tag{3.11}$$

Here we omit lower-order terms, which do not affect convergence or divergence of the integral as $t \to -0$ provided that the main prescribed exponential part stays dominant. Setting $\tau = \mu t$, $\mu > 1$, we arrive at integrals

$$\int^{(-t)^{-1}} e^{(-t)^{-\gamma} G(\mu)} \cos[b_m x^\alpha (-t)^{-\gamma} (\mu - 1)^{-\gamma}] d\mu, \tag{3.12}$$

where the function

$$G(\mu) = \mu^{-\gamma} - a_m x^\alpha (\mu - 1)^{-\gamma}, \quad \mu > 1, \tag{3.13}$$

has a typical bell-shaped form with a single maximum at

$$\mu_* = (1 - a_m^{1/\alpha} x)^{-1} \quad \text{if } \; x < L \equiv a_m^{-1/\alpha},$$

and

$$G(\mu_*) = g(x) \equiv (1 - x/L)^\alpha.$$

Thus, if $x < L$, by the Cauchy criterion, integrals of the type (3.11) diverge as $t \to -0$ in view of the strong dominance of the singular multiplier with fast exponential growth. Using Taylor's expansion of $G(\mu)$ in a neighbourhood of $\mu = \mu_*$, we conclude that, in the first approximation, the corresponding non-oscillatory part (the envelope of the oscillating functional family) can be asymptotically estimated by the exponential function

$$e^{(-t)^{-\gamma} G(\mu_*)} \equiv e^{(-t)^{-\gamma} g(x)} \to \infty \tag{3.14}$$

as $t \to -0$ for any $x < L$.

One can see from (3.13) that for any fixed $x > L$, as $t \to -0$, integrals like (3.11) converge absolutely. The behaviour in the localization domain $x \in (0, L)$ is oscillatory so that $\limsup u(x,t) = \infty$ and $\liminf u(x,t) = -\infty$ as $t \to -0$, cf. Hamilton-Jacobi limits explained below.

(ii) The proof is similar. The divergence of the integral for any $x > 0$ is straightforward for any $n < -1/(2m-1)$. A more precise estimate can also be obtained and it corresponds to the similarity solutions of the Hamilton-Jacobi equation. $\square$

3.2. **The Hamilton-Jacobi structure of blow-up singularities.** Let us show that the above exponential blow-up regimes have a Hamilton-Jacobi structure. This can be seen directly from the potential. On the other hand, we can use the same nonlinear transformation $u = e^v$, which leads to the perturbed complex Hamilton-Jacobi equation for the function $v : Q \to \mathbb{C}$,

$$v_t = (-1)^{m+1} (v_x)^{2m} + \mathbf{P}_m(v), \tag{3.15}$$

where $\mathbf{P}_m$ is a quasilinear polynomial $2m$-th order operator which can be computed. For instance,

$$\mathbf{P}_2(v) = -v_{xxxx} - 4 v_x v_{xxx} - -6(v_x)^2 v_{xx} - 3(v_{xx})^2.$$

The corresponding Hamilton-Jacobi equation composed from the leading operator of the maximal algebraic homogeneity order,

$$v_t = (-1)^{m+1} (v_x)^{2m}, \tag{3.16}$$

admits the solution in separate variables

$$V_*(x,t) = (1-t)^{-1/(2m-1)}\theta(x), \tag{3.17}$$

where $\theta : \mathbf{R} \to \mathbb{C}$ solves the complex ODE

$$(-1)^{m+1}(\theta')^{2m} - \frac{\theta}{2m-1} = 0 \quad \text{for } x > 0, \;\; \theta(0) = 1. \tag{3.18}$$

For odd $m = 1, 3, ...$, this ODE admits a nonnegative compactly supported solution

$$\tilde{\theta}(x) = (1 - x/l_m)_+^{\alpha}, \quad l_m = 2m(2m-1)^{-1/\alpha}. \tag{3.19}$$

The constant l_m is the same as that calculated via the potential. For $m = 1$, where $l_1 = 2$, function (3.19) actually occurs in the asymptotic behaviour as $t \to -0$, [43], Chapt. 3. Though for $m > 1$, as we have seen above, this non-oscillatory profile does not describe the asymptotic behaviour, $\tilde{\theta}$ correctly explains the behaviour of the envelope to a family of oscillatory rescaled blow-up profiles. In general, the ODE (3.18) admits complex valued solutions with compactly supported real part describing the oscillatory blow-up behaviour. For any $m > 1$, problem (3.18) has the complex solutions

$$\theta(x) = (1 - bx)^{\alpha},$$

where $b \in \mathbb{C}$ solves

$$(-1)^{m+1}(\alpha b)^{2m} = \frac{1}{2m-1},$$

so that $b = e^{i\varphi_k}/l_m$, $\varphi_k = \pi(2k+1)/2m$ for even m's, and $a = e^{i\varphi_k}/l_m$, $\varphi_k = \pi k/m$ for odd ones, $k = 0, 1, ..., 2m-1$. Then $\theta(x) = \rho^{\alpha} e^{i\alpha\psi}$, where

$$\rho^2(x) = 1 + \frac{x^2}{l_m^2} - 2\frac{x}{l_m}\cos\varphi_k, \quad \tan\psi(x) = -\frac{l_m x \sin\varphi_k}{l_m - x\cos\varphi_k}.$$

This gives a typical oscillating behaviour of the blowing up solution

$$u(x,t) \sim \operatorname{Re} e^{V_*(x,t)} \equiv e^{(1-t)^{-\gamma}\rho^{\alpha}(x)\cos(\alpha\psi(x))}\cos[(1-t)^{-\gamma}\rho^{\alpha}(x)\sin(\alpha\psi(x))]. \tag{3.20}$$

We again observe the exponential envelope to the oscillating family (3.20)

$$e^{(1-t)^{-\gamma}\rho^{\alpha}(x)\cos(\alpha\psi(x))} \quad \text{as } t \to 1^-.$$

In a similar way, we can construct blow-up HS-regimes with

$$v(0,t) = (1-t)^n, \quad n < -1/(2m-1).$$

Approximate self-similar solutions satisfying the Hamilton-Jacobi equation take the similarity form

$$V_*(x,t) = (1-t)^n h(\eta), \quad \eta = x/(1-t)^{\beta}, \quad \beta = [n(2m-1)+1]/2m < 0,$$

where $h : \mathbf{R} \to \mathbb{C}$ solves the complex ODE

$$(-1)^{m+1}(h')^{2m} - \beta h'\eta + nh = 0.$$

Complex-valued profiles h describe the oscillating character of global blow-up.

4. **Energy estimates and functional systems**

We return to the general problem (1.1)–(1.3).

4.1. **Energy estimates.** Let us introduce families of the subdomains of Q. For any fixed $s > 1$ and $\delta > 0$, we define

$$\Omega(s) = \{|x| > s\}, \quad \Omega(s,\delta) = \Omega(s) \setminus \Omega(s+\delta),$$
$$Q_a^b(s,\delta) = \Omega(s,\delta) \times (a,b), \quad 0 \le a \le b \le T; \quad Q_a^b(s) \equiv Q_a^b(s,\infty).$$

We will use the interpolation Gagliardo-Nirenberg inequalities for the domains $\Omega(s,\delta)$. For any $0 \le j < m$, there holds

$$\|D_x^j v\|_{r,\Omega(s,\delta)} \le k_1 \delta^{-j-n(r-g)/rg} \|v\|_{g,\Omega(s,\delta)} + k_2 \|D_x^m v\|^{\theta}_{p+1,\Omega(s,\delta)} \; \|v\|^{1-\theta}_{g,\Omega(s,\delta)}, \tag{4.1}$$

where $v(x)$ is an arbitrary function from the Sobolev space

$$W^m_{p+1}(\Omega(s,\delta)) \cap L_g(\Omega(s,\delta)), \quad \|v\|^h_{h,\Omega} = \int_\Omega |v|^h dx,$$

and $g > 0,\ r > 1$ fixed exponents. The constant $\theta \in [j/m, 1]$ is determined by

$$\frac{1}{r} - \frac{j}{n} = \theta \left(\frac{1}{p+1} - \frac{m}{n} \right) + \frac{1-\theta}{g}. \tag{4.2}$$

Positive constants k_1 and k_2 do not depend on $v(x)$, s and δ.

We next introduce cut-off functions. Let $\eta_0(h) \ge 0$ be a C^m-smooth function such that

$$\eta_0(h) = 0 \text{ for } h \le 0, \ \ \eta_0(h) = 1 \text{ for } h \ge 1, \ \ 0 \le \eta_0(h) \le 1 \text{ for any } h \in \mathbf{R}.$$

Given arbitrary $s > 1$ and $\delta > 0$, we define the cut-off function

$$\eta(x) = \eta_0((|x| - s)/\delta). \tag{4.3}$$

Lemma 4.1. *Let $u(x,t)$ be an arbitrary energy solution of the problem* (1.1) – (1.3). *Then for any $0 \le a < b < T$ and any $\varepsilon > 0$, there holds*

$$\int_{\Omega(s+\delta)} |u(x,b)|^{q+1} dx + \int_a^b \int_{\Omega(s+\delta)} |D_x^m u(x,t)|^{p+1} dx dt \le (1+\varepsilon) \int_{\Omega(s)} |u(x,a)|^{q+1} dx$$
$$+ \, \delta^{-m(p+1)} c(\varepsilon) \int_a^b \int_{\Omega(s,\delta)} |u(x,t)|^{p+1} dx dt. \tag{4.4}$$

The positive constant $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$ depends on parameters of the problem and is independent of $u(x,t)$, s and δ.

Proof. Let us fix an $s_1 \in (1,s)$, where s is as in (4.3), and introduce the cut-off function

$$\eta_1(x) = \eta_0(\frac{|x| - s_1}{\delta_1}), \quad \text{where } \delta_1 = s - s_1 > 0.$$

It is obvious that $\eta_1^r(x)\eta(x) \equiv \eta(x)$ for any exponent $r > 0$. Therefore, for any $t \in (0,T)$,

$$\langle b(u)_t, \eta u \rangle = \langle b(u)_t, \eta_1^q \eta u \rangle = \langle b(\eta_1 u)_t, \eta u \rangle = \langle b(\eta_1 u)_t, \eta \eta_1 u \rangle = \langle b(v)_t, \eta v \rangle,$$

where $b(u) = |u|^{q-1}u$ and $v = \eta_1 u$. As a consequence, similar to Proposition 3.2 in [8], we obtain the formula of integration by parts

$$(q+1)q^{-1}\int_a^b \langle (|u|^{q-1}u)_t, u\eta\rangle dt = \int_\Omega |u(x,b)|^{q+1}\eta(x)dx - \int_\Omega |u(x,a)|^{q+1}\eta(x)dx, \tag{4.5}$$

where η is given in (4.3). We substitute the test function $u(x,t)\eta(x)$ into the integral identity (1.9), where u is the given energy solution and $\eta(x)$ is the cut-off function (4.3). Using (4.5) and the structural conditions (1.4), (1.5), standard transformations yields

$$\int_{\Omega(s)} |u(x,b)|^{q+1}\eta dx + \int_a^b\int_{\Omega(s)} |D_x^m u|^{p+1}\eta dxdt \le \int_{\Omega(s)} |u(x,a)|^{q+1}\eta dx$$

$$+c\left(\int_{Q_a^b(s,\delta)} |D_x^m u|^{p+1}dxdt\right)^{\frac{p}{p+1}}\left(\sum_{i=1}^{m-1}\int_{Q_a^b(s,\delta)} |D_x^i u|^{p+1}|D_x^{m-i}\eta|^{p+1}dxdt\right)^{\frac{1}{p+1}}. \tag{4.6}$$

Let us estimate the second term on the right-hand side by using the interpolation inequality (4.1) with $j = 0,1,2,...,m-1$, $r = p+1$ and $g = q+1$. We also use the following obvious property of the cut-off function $\eta(x)$:

$$|D_x^i\eta| \le c\delta^{-i}, \quad i = 1,2,...,m. \tag{4.7}$$

As the result, for $i = 1,2,...,m-1$, we have

$$\int_{\Omega(s,\delta)} |D_x^i u(x,t)|^{p+1}|D_x^{m-i}\eta|^{p+1}dx \le c\delta^{-(m-i)(p+1)}\int_{\Omega(s,\delta)} |D_x^i u(x,t)|^{p+1}dx$$

$$\le c\delta^{-(m-i)(p+1)}\left[k_1^{p+1}\delta^{-i(p+1)}\int_{\Omega(s,\delta)} |u|^{p+1}dx + k_2^{p+1}\left(\int_{\Omega(s,\delta)} |D_x^m u|^{p+1}dx\right)^{\frac{i}{m}}\right.$$

$$\left.\times\left(\int_{\Omega(s,\delta)} |u|^{p+1}dx\right)^{\frac{m-i}{m}}\right] \le c\delta^{-m(p+1)}k_1^{p+1}\int_{\Omega(s,\delta)} |u|^{p+1}dx + ck_2^{p+1}\left(\int_{\Omega(s,\delta)} |D_x^m u|^{p+1}dx\right)^{\frac{i}{m}}$$

$$\times\left(\delta^{-m(p+1)}\int_{\Omega(s,\delta)} |u|^{p+1}dx\right)^{\frac{m-i}{m}} \le \varepsilon\int_{\Omega(s,\delta)} |D_x^m u|^{p+1}dx + c(\varepsilon)\delta^{-m(p+1)}\int_{\Omega(s,\delta)} |u|^{p+1}dx,$$

where at the last step we apply Young's inequality with a parameter $\varepsilon > 0$. Substituting these estimates into (4.6) and using Young's inequality where necessary, we obtain

$$\int_{\Omega(s)} |u(x,b)|^{q+1}\eta dx + \int_a^b\int_{\Omega(s)} |D_x^m u(x,t)|^{p+1}\eta dxdt \le \int_{\Omega(s)} |u(x,a)|^{q+1}\eta dx$$

$$+\varepsilon\int_{Q_a^b(s,\delta)} |D_x^m u(x,t)|^{p+1}dxdt + c(\varepsilon)\delta^{-m(p+1)}\int_{Q_a^b(s,\delta)} |u(x,t)|^{p+1}dxdt, \tag{4.8}$$

where $\varepsilon > 0$ can be arbitrarily small and $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Consider (4.8) as the relation between the following functions of argument s:

$$A_b(s) = \int_{\Omega(s)} |u(x,b)|^{q+1}dx, \quad A_a(s) = \int_{\Omega(s)} |u(x,a)|^{q+1}dx,$$

$$B(s)=\int_{Q_a^b(s)}|u(x,t)|^{p+1}dxdt,\quad H(s)=\int_{Q_a^b(s)}|D_x^m u(x,t)|^{p+1}dxdt.$$

It then follows from (4.8) that

$$A_b(s+\delta)+H(s+\delta)\le \varepsilon H(s)+c(\varepsilon)\delta^{-m(p+1)}[B(s)-B(s-\delta)]+A_a(s). \tag{4.9}$$

Since $s>1$ and $\delta>0$ are arbitrary continuous arguments in (4.9), the following inequalities hold for any $j=1,2,...,$

$$A_b(s+\frac{\delta}{2^{j-1}})+H(s+\frac{\delta}{2^{j-1}})$$

$$\le \varepsilon H(s+\frac{\delta}{2^j})+c(\varepsilon)(\frac{2^j}{\delta})^{m(p+1)}[B(s+\frac{\delta}{2^j})-B(s+\frac{\delta}{2^{j-1}})]+A_a(s+\frac{\delta}{2^j}), \tag{4.10}$$

which are derived from (4.9) replacing $s\mapsto s+\delta/2^j$ and $\delta\mapsto\delta/2^j$.

Let us construct a process of successive estimates using (4.10). It is similar to the procedure used in [50]. To this end, consider it for $j=1$ and next estimate $H(s+\delta/2)$ on the right-hand side by means of the same inequality for $j=2$. Continue this process cyclically for $j=3,4,\ldots$. Combining all these estimates, we deduce that

$$A_b(s+\delta)+H(s+\delta)\le\varepsilon H(s+\frac{\delta}{2})+c(\varepsilon)(\frac{2}{\delta})^{m(p+1)}[B(s+\frac{\delta}{2})-B(s+\delta)]$$

$$+A_a(s+\frac{\delta}{2})\le\varepsilon^2H(s+\frac{\delta}{4})+c(\varepsilon)[(\frac{2}{\delta})^{m(p+1)}[B(s+\frac{\delta}{2})-B(s+\delta)]$$

$$+\varepsilon\,(\frac{2^2}{\delta})^{m(p+1)}[B(s+\frac{\delta}{2^2})-B(s+\frac{\delta}{2})]]+A_a(s+\frac{\delta}{2})+\varepsilon A_a(s+\frac{\delta}{2^2})\le\ \ldots$$

$$\le\varepsilon^jH(s+\frac{\delta}{2^j})+c(\varepsilon)(\frac{2}{\delta})^{m(p+1)}\sum_{k=0}^{j-1}(2^{m(p+1)}\varepsilon)^k[B(s+\frac{\delta}{2^{k+1}})-B(s+\frac{\delta}{2^k})]$$

$$+\sum_{k=0}^{j-1}\varepsilon^kA_a(s+\frac{\delta}{2^{k+1}})\le\varepsilon^jH(s+\frac{\delta}{2^j})+A_a(s+\frac{\delta}{2^j})\sum_{k=0}^{j-1}\varepsilon^k$$

$$+c(\varepsilon)(\frac{2}{\delta})^{m(p+1)}[B(s+\frac{\delta}{2^j})-B(s+\delta)]\sum_{k=0}^{j-1}(2^{m(p+1)}\varepsilon)^k.$$

Setting in this inequality

$$\varepsilon\le\varepsilon_0=2^{-m(p+1)-1} \tag{4.11}$$

and next passing to the limit $j\to\infty$, we obtain

$$A_b(s+\delta)+H(s+\delta)\le c(\varepsilon)\frac{2^{m(p+1)+1}}{\delta^{m(p+1)}}\,[B(s)-B(s+\delta)]+\frac{1}{1-\varepsilon}\,A_a(s). \tag{4.12}$$

Here we have used the continuity property of the functions $A_a(s)$ and $B(s)$. Inequality (4.12) is equivalent to (4.4). This completes the proof of the lemma. $\square$

4.2. **System of functional inequalities.** We introduce an arbitrary strictly monotone sequence

$$\{t_j\} \to T^-, \quad \Delta_j = t_j - t_{j-1} > 0 \quad \text{for } j = 1, 2, ..., \ t_0 > 0, \tag{4.13}$$

and define two families of energy functions of the energy solution $u(x,t)$ of problem (1.1)-(1.3) for $s > 1$

$$h_j(s) = \sup_{t \in (t_{j-1}, t_j)} \int_{\Omega(s)} |u(x,t)|^{q+1} dx,$$

$$I_j(s) = \int_{t_{j-1}}^{t_j} \int_{\Omega(s)} |u(x,t)|^{p+1} dx dt, \quad j = 1, 2, \dots, \tag{4.14}$$

$$I_0(s) = \int_0^{t_0} \int_{\Omega(s)} |u(x,t)|^{p+1} dx dt, \quad h_0(s) = \int_{\Omega(s)} |u(x,t_0)|^{q+1} dx.$$

Lemma 4.2. *Given an arbitrary energy solution of* (1.1)-(1.3)*, for any $s > 1$ and $\delta > 0$, functions* (4.14) *satisfy the following system of functional inequalities:*

$$h_j(s+\delta) \le (1+\varepsilon_j) h_{j-1}(s) + c(\varepsilon_j) \frac{\Delta I_j(s)}{\delta^{m(p+1)}}, \quad \Delta I_j(s) \equiv I_j(s) - I_j(s+\delta), \tag{4.15}$$

$$I_j(s+\delta) \le c_1 \Delta_j^{1-\theta_1} \left[(1+\varepsilon_j) h_{j-1}(s) + c(\varepsilon_j) \frac{\Delta I_j(s)}{\delta^{m(p+1)}} \right]^{1+\nu}, \quad j = 1, 2, \dots, \tag{4.16}$$

where $c_1 > 0$ is a constant,

$$\theta_1 = \frac{n(p-q)}{n(p-q) + m(p+1)(q+1)}, \quad \nu = \frac{(p-q)(1-\theta_1)}{q+1}, \tag{4.17}$$

$\{\varepsilon_j > 0\}$ is an arbitrary sequence and positive constants satisfy $c(\varepsilon_j) \to \infty$ as $\varepsilon_j \to 0$.

Proof. Setting in (4.4) $a = t_{j-1}$ and $b = \tilde{t}$, where $\tilde{t} \in (t_{j-1}, t_j]$ is arbitrary, we obtain (4.15). In order to prove (4.16), we write down the interpolation inequality (4.1) with $\delta = \infty$, $j = 0$, $r = p+1$, $g = q+1$ and replacing $s \mapsto s + \delta$:

$$\int_{\Omega(s+\delta)} |u|^{p+1} dx \le k_2^{p+1} \left(\int_{\Omega(s+\delta)} |D_x^m u|^{p+1} dx \right)^{\theta_1} \left(\int_{\Omega(s+\delta)} |u|^{q+1} dx \right)^{\frac{(1-\theta_1)(p+1)}{q+1}},$$

where θ_1 is taken from (4.17). Integrating this inequality in t and using estimates (4.15) and (4.4), we get

$$I_j(s+\delta) \le c_2 \left(\int_{t_{j-1}}^{t_j} \int_{\Omega(s+\delta)} |D_x^m u|^{p+1} dx dt \right)^{\theta_1} \left[\int_{t_{j-1}}^{t_j} \left(\int_{\Omega(s+\delta)} |u|^{q+1} dx \right)^{\frac{p+1}{q+1}} dt \right]^{1-\theta_1}$$

$$\le c_1 [R_j(s,\delta)]^{\theta_1} \Delta_j^{1-\theta_1} [R_j(s,\delta)]^{(p+1)(1-\theta_1)/(q+1)} \equiv c_1 \Delta_j^{1-\theta_1} [R_j(s,\delta)]^{1+(p-q)(1-\theta_1)/(q+1)}, \tag{4.18}$$

where

$$R_j(s,\delta) \equiv (1+\varepsilon_j) h_{j-1}(s) + c(\varepsilon_j) \frac{\Delta I_j(s)}{\delta^{m(p+1)}}.$$

Inequality (4.18) gives (4.16). □

Other types of infinite functional systems of inequalities for energy functions were introduced in [34] - [37]. The last paper, where a survey on such methods of introducing a parameter can be found, contains an abstract scheme of the method with applications to uniqueness and asymptotic properties of solutions to the second and higher-order elliptic and parabolic equations in unbounded domains. As we have mentioned in the introduction, papers [49], [50] present a version of energy methods in the study of differential properties of solutions of quasilinear higher-order elliptic equations. Later on, in applications to quasilinear higher-order parabolic equations, the energy methods were extended in [6] - [9] and in [45], [46].

Let us comment on types and "measures" of singularities covered by energy methods in earlier results. In [37] an infinite partition was arranged near the *initial moment* $t = 0$ in order to prove uniqueness in optimal Tikhonov-Täcklind classes of growing functions for higher-order linear parabolic equations and systems. In [1] optimal uniqueness classes were obtained by special energy estimates for quasilinear parabolic equations. Formally, this can be classified as an asymptotic (short-time) description of "initial singularities" posed in terms initial data $u_0(x)$ growing as $x \to \infty$.

Localization phenomena present another type of *eventual singularities*, which occur at finite blow-up time $t = T^-$ and essentially depend on both nonlinear properties of the equations and the blow-up boundary regime. We thus face new types of evolution singularities occurring in nonlinear dynamical systems. In the analysis of the evolution singularities, the type of partitions is not known *a priori* and, in fact, should be chosen according to unknown asymptotic properties of the unbounded blow-up solutions. Such an asymptotic analysis of infinite functional systems is delicate, and the total amount of computations we are dealing with increases dramatically due to an essential difference between the initially posed (fixed) singularity and the eventual evolution one with unknown properties.

4.3. **Preliminary estimates.**

Corollary 4.1. *For equation* (1.1) *with* $p = q$*, the main functional system* (4.15), (4.16) *reads as follows: for any* $s > 1$ *and* $\delta > 0$*,*

$$h_j(s+\delta) \le (1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\frac{\Delta I_j(s)}{\delta^{m(p+1)}}, \tag{4.19}$$

$$I_j(s+\delta) \le c_1\Delta_j(1+\varepsilon_j)h_{j-1}(s) + c_1 c(\varepsilon_j)\Delta_j\,\frac{\Delta I_j(s)}{\delta^{m(p+1)}}, \quad j = 1,2,\dots. \tag{4.20}$$

Fix an arbitrary strictly monotone decreasing sequence $\{\alpha_i > 0\}$. We define the normalized energy functions

$$H_j(s) = \alpha_j h_j(s) \quad \text{and} \quad J_j(s) = \alpha_j I_j(s). \tag{4.21}$$

In terms of them, system (4.19), (4.20) takes the form

$$H_j(s+\delta) \le \beta_j H_{j-1}(s) + c(\varepsilon_j)\,\frac{\Delta J_j(s)}{\delta^{m(p+1)}}, \quad \beta_j \equiv \frac{(1+\varepsilon_j)\alpha_j}{\alpha_{j-1}}, \tag{4.22}$$

$$J_j(s+\delta) \le c_1\beta_j\Delta_j H_{j-1}(s) + c_1 c(\varepsilon_j)\Delta_j\,\frac{\Delta J_j(s)}{\delta^{m(p+1)}}, \quad j=1,2,\dots. \tag{4.23}$$

We now iterate this system. Namely, we estimate $H_{j-1}(s)$ on the right-hand side of (4.23) by means of (4.22) replacing $j \mapsto j-1$. Assuming now that $\delta = \delta_j$ in (4.22) and (4.23) depend on j and denoting for convenience

$$d_j = \delta_j^{-m(p+1)},$$

we obtain

$$\begin{aligned} J_j(s+\delta_j+\delta_{j-1}) &\le c_1\beta_j\Delta_j H_{j-1}(s+\delta_{j-1}) + c_1 c(\varepsilon_j)\Delta_j \Delta J_j(s+\delta_{j-1}) d_j \\ &\le c_1\beta_j\beta_{j-1}\Delta_j H_{j-2}(s) + c_1[c(\varepsilon_j)\Delta_j d_j \Delta J_j(s+\delta_{j-1}) + c(\varepsilon_{j-1})\Delta_j\beta_j d_{j-1}\Delta J_{j-1}(s)]. \end{aligned}$$

We again estimate $H_{j-2}(s)$ on the right-hand side by means of (4.22) with $j \mapsto j-2$:

$$\begin{aligned} J_j(s+\delta_j+\delta_{j-1}+\delta_{j-2}) \le{}& c_1\beta_j\beta_{j-1}\beta_{j-2}\Delta_j H_{j-3}(s) + c_1[c(\varepsilon_j)\Delta_j d_j \Delta J_j(s+\delta_{j-1}+\delta_{j-2}) \\ &+ c(\varepsilon_{j-1})\Delta_j\beta_j d_{j-1}\Delta J_{j-1}(s+\delta_{j-2}) + c(\varepsilon_{j-2})\Delta_j\beta_j\beta_{j-1}d_{j-2}\Delta J_{j-2}(s)]. \end{aligned}$$

Repeating the same procedure, we arrive at

$$J_j(s+\sum_{i=0}^{j}\delta_i) \le c_1\beta_j\dots\beta_1\Delta_j H_0(s+\delta_0) + c_1\sum_{k=1}^{j} c(\varepsilon_k)\Delta_j\beta_j\dots\beta_{k+1}d_k\,\Delta J_k(s+\sum_{i=0}^{k-1}\delta_i),$$

or, replacing $s+\sum_{i=0}^{j}\delta_i$ by s,

$$\begin{aligned} J_j(s) \le{}& c_1\beta_j\dots\beta_1\Delta_j H_0(s-\sum_{i=1}^{j}\delta_i) + c_1 c(\varepsilon_j)\Delta_j d_j[J_j(s-\delta_j) - J_j(s)] \\ &+ c_1\sum_{k=1}^{j} c(\varepsilon_k)\Delta_j\beta_j\dots\beta_{k+1}d_k\,J_k(s-\sum_{i=k}^{j}\delta_i). \end{aligned}$$

Collecting like terms together, we derive the following result.

Lemma 4.3. *For $p=q$, normalized energy functions $J_j(s)$, $j=1,2,\dots$, given by* (4.14), (4.21), *satisfy the following functional system:*

$$\begin{aligned} J_j(s) \le{}& \frac{\gamma_j}{1+\gamma_j}J_j(s-\delta_j) + \frac{c_1}{1+\gamma_j}\sum_{k=1}^{j-1} c(\varepsilon_k)\Delta_j\beta_j\dots\beta_{k+1}\,J_k(s-\sum_{i=k}^{j}\delta_i)\,\delta_k^{-m(p+1)} \\ &+ \frac{c_1}{1+\gamma_j}\beta_j\dots\beta_1\Delta_j H_0(s-\sum_{i=1}^{j}\delta_i), \quad s \ge 1+\sum_{i=1}^{j}\delta_i, \quad \gamma_j = c_1 c(\varepsilon_j)\Delta_j\delta_j^{-m(p+1)}. \end{aligned} \tag{4.24}$$

We now establish a global *a priori* bound on the solution $u(x,t)$ via the boundary function F.

Lemma 4.4. *For any energy solution $u(x,t)$ of the problem* (1.1)-(1.3) *with $p = q$, the following global a priori estimate holds:*

$$\int_\Omega |u|^{p+1}dx + \int_0^t \int_\Omega |D_x^m u|^{p+1}dxd\tau \leq C[\|u_0\|^{p+1}_{L_{p+1}(\Omega)} + F(t)], \quad t \in (0,T). \tag{4.25}$$

Proof. Using conditions (1.6),(1.7) and the definition of solution $u(x,t)$, it is easy to extend the proof of the integration by parts formula [2] to arbitrary $m > 1$. We then obtain

$$\begin{aligned}\int_0^t \langle (|u|^{p-1}u)_\tau, u - f\rangle d\tau &= \frac{p}{p+1}\int_\Omega (|u(x,t)|^{p+1} - |u(x,0)|^{p+1})dx \\ &+ \int_0^t \int_\Omega (|u(x,\tau)|^{p-1}u(x,\tau) - |u(x,0)|^{p-1}u(x,0))f_\tau(x,\tau)dxd\tau \\ &- \int_\Omega (|u(x,t)|^{p-1}u(x,t) - |u(x,0)|^{p-1}u(x,0))f(x,t)dx.\end{aligned} \tag{4.26}$$

Substituting the test function $\eta(x,t) = u(x,t) - f(x,t)$ into the integral identity (1.9) and using (4.26) and the structural conditions (1.4), (1.5), we conclude that

$$\begin{aligned}\frac{p}{p+1}\int_\Omega |u(x,t)|^{p+1}dx + d_1\int_0^t\int_\Omega |D_x^m u|^{p+1}dxd\tau &\leq d_2\int_0^t\int_\Omega |D_x^m u(x,\tau)|^p|D_x^m f(x,\tau)|dxd\tau \\ + \int_\Omega (|u(x,t)|^{p-1}u(x,t) - |u(x,0)|^{p-1}u(x,0))f(x,t)dx &- \int_0^t\int_\Omega [|u(x,\tau)|^{p-1}u(x,\tau) \\ - |u(x,0)|^{p-1}u(x,0)]f_\tau(x,\tau)dxd\tau &+ \frac{p}{p+1}\int_\Omega |u(x,0)|^{p+1}dx,\end{aligned} \tag{4.27}$$

where d_1 is taken from (1.4). By Young's inequality we derive the following straightforward estimates of the terms given on the right-hand side of (4.27):

$$\begin{aligned}\int_0^t\int_\Omega |D_x^m u(x,\tau)|^p|D_x^m f(x,\tau)|dxd\tau &\leq \varepsilon_1\int_0^t\int_\Omega |D_x^m u(x,\tau)|^{p+1}dxd\tau \\ &+ c(\varepsilon_1)\int_0^t\int_\Omega |D_x^m f(x,\tau)|^{p+1}dxd\tau,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\int_\Omega \big||u(x,t)|^{p-1}u(x,t) - |u(x,0)|^{p-1}u(x,0)\big|\,|f(x,t)|dx &\leq \varepsilon_1[\|u(\cdot,t)\|^{p+1}_{L_{p+1}(\Omega)} \\ + \|u(\cdot,0)\|^{p+1}_{L_{p+1}(\Omega)}] &+ c_2(\varepsilon_1)\|f(\cdot,t)\|^{p+1}_{L_{p+1}(\Omega)},\end{aligned} \tag{4.29}$$

$$\int_0^t\int_\Omega |u(x,0)|^p|f_\tau(x,\tau)|dxd\tau \leq \varepsilon_1\|u_0\|^{p+1}_{L_{p+1}(\Omega)} + c_3(\varepsilon_1)\|f_\tau(\cdot,\tau)\|^{p+1}_{L_1(0,t;L_{p+1}(\Omega))}, \tag{4.30}$$

$$\int_0^t\int_\Omega |u(x,\tau)|^p|f_\tau(x,\tau)|dxd\tau \leq \sup_{\tau\in(0,t)}\|u(\cdot,\tau)\|^p_{L_{p+1}(\Omega)}\int_0^t\left(\int_\Omega |f_\tau(x,\tau)|^{p+1}dx\right)^{\frac{1}{p+1}}d\tau$$

$$\leq \varepsilon_1 \sup_{\tau\in(0,t)} \|u(\cdot,\tau)\|^{p+1}_{L_{p+1}(\Omega)} + c_4(\varepsilon_1)\|f_\tau(\cdot,\tau)\|^{p+1}_{L_1(0,t;L_{p+1}(\Omega))}. \tag{4.31}$$

Combining (4.28)-(4.31) and substituting them into (4.27), we arrive at (4.25). □

5. First step: proof of localization of "flat" LS-regime

The proof of Theorem 1.1 consists of two steps. In this section we perform the first step and consider the case of "flat" (lower than the optimal one in (1.13)) boundary blow-up regime meaning that function F in (1.8) satisfies

$$F(t) \leq c\exp\{A(T-t)^{-1/[m(p+1)+\xi_0]}\}, \quad t\in(0,T), \tag{5.1}$$

where c and A are arbitrary positive constants. The constant $\xi_0 > 0$ can be arbitrarily small. Localization estimates are derived from the functional system (4.24) with the following choice of its free parameters.

(i) Fix a small $\delta > 0$ ($\delta \to 0$ later on), and define a sequence of displacements (a *spatial partition*)

$$\delta_i = 2^{-i}\delta, \quad i\geq 1, \ \ \delta_0 = 0. \tag{5.2}$$

(ii) We choose unit normalized multipliers

$$\alpha_i = 1, \quad i \geq 1. \tag{5.3}$$

(iii) All the parameters ε_j in the system (4.19), (4.20) and in (4.24) are constant:

$$\varepsilon_i = \varepsilon = \text{const} > 0, \quad \text{hence } \beta_j = 1+\varepsilon, \quad j\geq 1, \tag{5.4}$$

with a small fixed $\varepsilon > 0$ to be determined later.

(iv) The space displacements for $j = 1, 2, ...$ (a *time partition*) are

$$\Delta_j = \frac{\delta_j^{m(p+1)}}{c_1\cdot c(\varepsilon)} = r_0^{j-1}\Delta_1, \quad \Delta_1 = \frac{1}{c_1 c(\varepsilon)}\left(\frac{\delta}{2}\right)^{m(p+1)}, \ \ r_0 = 2^{-m(p+1)} < 1, \tag{5.5}$$

where c_1 and $c(\varepsilon)$ are the constants from (4.24). Then

$$\gamma_j = 1, \quad j\geq 1. \tag{5.6}$$

The functional system (4.24) takes the form

$$I_j(s) \leq \frac{1}{2}I_j(s-\delta_j) + \frac{1}{2}\sum_{k=1}^{j-1}(1+\varepsilon)^{j-k}r_0^{j-k}I_k(s-\sum_{i=k}^{j}\delta_i)$$

$$+\frac{1}{2}c_1[(1+\varepsilon)r_0]^{j-1}\Delta_1 h_0(s-\sum_{i=1}^{j}\delta_i), \quad j\geq 1, \ \ s > 1+\sum_{i=1}^{j}\delta_i. \tag{5.7}$$

Let us calculate the value t_0, which corresponds to displacements $\{\Delta_i, i\geq 1\}$ defined in (5.5):

$$T - t_0 = \sum_{i=1}^{\infty}\Delta_i = \Delta_1\sum_{i=0}^{\infty}r_0^i = \frac{\Delta_1}{1-r_0} = \frac{\delta^{m(p+1)}}{c_1 c(\varepsilon)2^{m(p+1)}(1-r_0)},$$

so that

$$t_0 = T - \frac{\delta^{m(p+1)}}{c_1 c(\varepsilon) 2^{m(p+1)}(1-r_0)} \to T^- \quad \text{as } \delta \to 0 \text{ or } \varepsilon \to 0. \tag{5.8}$$

We now fix $\varepsilon > 0$ in (5.7) so small that

$$(1+\varepsilon) r_0 = r_1 < 1. \tag{5.9}$$

Then system (5.7) takes the final form

$$I_j(s) \le \frac{1}{2} I_j(s-\delta_j) + 2^{-1} \sum_{k=1}^{j-1} r_1^{j-k} I_k(s - \sum_{i=k}^{j} \delta_j)$$

$$+ \frac{1}{2} c_1 r_1^{j-1} \Delta_1 h_0(s - \sum_{i=1}^{j} \delta_i), \quad j \ge 1, \;\; s > 1 + \sum_{i=1}^{j} \delta_i. \tag{5.10}$$

Assume that s satisfies the following additional restriction:

$$s - \delta_j - \sum_{i=1}^{j} \delta_i \ge 1. \tag{5.11}$$

Then the first summand on the right-hand side of (5.10) can be estimated from above by means of the same inequality with the shifted argument $s \mapsto s - \delta_j$. This yields

$$I_j(s) \le 2^{-2} I_j(s - 2\delta_j) + 2^{-2} \sum_{k=1}^{j-1} r_1^{j-k} I_k(s - \delta_j - \sum_{i=k}^{j} \delta_j) + 2^{-1} \sum_{k=1}^{j-1} r_1^{j-k} I_k(s - \sum_{i=k}^{j} \delta_j)$$

$$+ c_1 r_1^{j-1} \Delta_1 \left[2^{-1} h_0(s - \sum_{i=1}^{j} \delta_i) + 2^{-2} h_0(s - \delta_j - \sum_{i=1}^{j} \delta_i) \right].$$

If we now assume that s satisfies

$$s - 2\delta_j - \sum_{i=1}^{j} \delta_i \ge 1,$$

then the first summand on the right-hand side is estimated by (5.10) with $s \mapsto s - 2\delta_j$. After similar l steps, assuming that

$$s - l\delta_j - \sum_{i=1}^{j} \delta_i \ge 1, \tag{5.12}$$

we obtain

$$I_j(s) \le 2^{-l} I_j(s - l\delta_j) + \sum_{k=1}^{j-1} r_1^{j-k} \sum_{i=1}^{l} 2^{-i} I_k(s - i\delta_j - \delta_{j-1} - \ldots - \delta_k)$$

$$+ c_1 r_1^{j-1} \Delta_1 \sum_{i=1}^{l} 2^{-i} h_0(s - i\delta_j - \delta_{j-1} - \ldots - \delta_1), \quad j \ge 1. \tag{5.13}$$

Observe that (5.12) holds provided that

$$s = s_0 \equiv 1 + 2\delta, \quad l = l_1 \equiv 2^j. \tag{5.14}$$

Indeed, for any $j \ge 1$,

$$s_0 - l_1\delta_j - \sum_{i=1}^{j-1} \delta_i = 1 + 2\delta - 2^j\delta_j - \delta \sum_{i=1}^{j-1} 2^{-i} = 1 + 2^{-j+1}\delta > 1.$$

From (5.13) we then derive the first main inequality

$$I_j(s_0) \le 2^{-l_1} I_j(s_0 - l_1\delta_j) + \sum_{k_1=1}^{j-1} r_1^{j-k} \sum_{i_1=1}^{l_1} 2^{-i_1} I_{k_1}(s_0 - i_1\delta_j - \delta_{j-1} - \ldots - \delta_{k_1})$$

$$+ c_1 r_1^{j-1} \Delta_1 \sum_{i_1=1}^{l_1} 2^{-i_1} h_0(s_0 - i_1\delta_j - \delta_{j-1} - \ldots - \delta_1) \equiv A_1^{(1)} + A_1^{(2)} + A_1^{(3)}. \tag{5.15}$$

Let us transform $A_1^{(2)}$ into the form

$$A_1^{(2)} = r_1 \sum_{i_1=1}^{l_1} 2^{-i_1} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1})$$

$$+ \sum_{k_1=1}^{j-2} r_1^{j-k_1} \sum_{i_1=1}^{l_1} 2^{-i_1} I_{k_1}(s_0 - i_1\delta_j - \delta_{j-1} - \ldots - \delta_{k_1}) \equiv A_{1,1}^{(2)} + A_{1,2}^{(2)}.$$

In order to estimate the summands of $A_{1,1}^{(2)}$ we use (5.13) with $j \mapsto j-1$,

$$I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1}) \le 2^{-l_2} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1})$$

$$+ \sum_{k_2=1}^{j-2} r_1^{j-1-k_2} \sum_{i_2=1}^{l_2} 2^{-i_2} I_{k_2}(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \delta_{j-2} - \ldots - \delta_{k_2})$$

$$+ c_1 r_1^{j-2} \Delta_1 \sum_{i_2=1}^{l_2} 2^{-i_2} h_0(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \delta_{j-2} - \ldots - \delta_1). \tag{5.16}$$

The restriction from above (5.12) for l_2 now takes the form

$$s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1} - \sum_{i=1}^{j-2} \delta_i \ge 1,$$

and hence

$$l_2\delta_{j-1} \le s_0 - 1 - i_1\delta_j - \sum_{i=1}^{j-1} \delta_i \equiv \delta - i_1\delta_j + \delta_{j-1}.$$

Substituting estimate (5.16) into the expression for $A_{1,1}^{(2)}$, after some manipulations in the right-hand side of (5.15) we arrive at

$$I_j(s_0) \le A_2^{(1)} + A_2^{(2)} + A_2^{(3)},$$

$$A_2^{(1)} = A_1^{(1)} + r_1 \sum_{i_1=1}^{l_1} 2^{-(i_1+l_2)} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1}),$$

$$A_2^{(2)} = A_{1,2}^{(2)} + \sum_{i_1=1}^{l_1} 2^{-i_1} r_1 \sum_{k_2=1}^{j-2} r_1^{j-1-k_2} \sum_{i_2=1}^{l_2} 2^{-i_2} I_{k_2}(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \delta_{j-2} - ... - \delta_{k_2})$$

$$\equiv \sum_{k_2=1}^{j-2} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} r_1^{j-k_2} 2^{-(i_1+i_2)} I_{k_2}(s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2} - ... - \delta_{k_2}),$$

$$A_2^{(3)} = A_1^{(3)} + c_1 r_1^{j-1} \Delta_1 \sum_{i_1=1}^{l_1} \sum_{i_2=1}^{l_2} 2^{-(i_1+i_2)} h_0(s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2} - ... - \delta_1)$$

$$\equiv c_1 r_1^{j-1} \Delta_1 \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} 2^{-(i_1+i_2)} h_0(s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2} - ... - \delta_1).$$

Let us separate the terms in $A_2^{(2)}$ corresponding to $k_2 = j-2$ in the first summand and write down the representation

$$A_2^{(2)} = A_{2,1}^{(2)} + A_{2,2}^{(2)}.$$

We estimate summands from $A_{2,1}^{(2)}$ by means of the main inequality (5.13) with j replaced by $j-2$ and s replaced by $s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2}$. By manipulations similar to those at the previous step, we obtain a new form of estimate (5.15)

$$I_j(s_0) \le A_3^{(1)} + A_3^{(2)} + A_3^{(3)}.$$

Performing these cyclic estimates $j-1$ times, we arrive at

$$I_j(s_0) \le A_{j-1}^{(1)} + A_{j-1}^{(2)} + A_{j-1}^{(3)}, \tag{5.17}$$

where

$$A_{j-1}^{(1)} = 2^{-l_1} I_j(s_0 - l_1\delta_j) + r_1 \sum_{i_1=1}^{l_1} 2^{-(i_1+l_2)} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1}) + ...$$

$$+ r_1^h \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} ... \sum_{i_h=0}^{l_h} 2^{-(i_1+...+i_h+l_{h+1})} I_{j-h}(s_0 - i_1\delta_j - ... - i_h\delta_{j-h+1} - l_{h+1}\delta_{j-h} - \sum_{i=j-h}^{j-1} \delta_i) + ...$$

$$+ r_1^{j-2}\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_{j-2}=0}^{l_{j-2}} 2^{-(i_1+...+i_{j-2}+l_{j-1})} I_2(s_0 - i_1\delta_j - ... - i_{j-2}\delta_3 - l_{j-1}\delta_2 - \sum_{i=2}^{j-1}\delta_i),$$

$$A_{j-1}^{(2)} = r_1^{j-1}\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_{j-1}=0}^{l_{j-1}} 2^{-(i_1+...+i_{j-1})} I_1(s_0 - i_1\delta_j - ... - i_{j-1}\delta_2 - \sum_{i=1}^{j-1}\delta_i),$$

$$A_{j-1}^{(3)} = c_1 r_1^{j-1}\Delta_1\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_{j-1}=0}^{l_{j-1}} 2^{-(i_1+...+i_{j-1})} h_0(s_0 - i_1\delta_j - ... - i_{j-1}\delta_2 - \sum_{i=1}^{j-1}\delta_i).$$

The values l_k satisfy the following restrictions from above:

$$s_0 - i_1\delta_j - i_2\delta_{j-1} - ... - i_{k-1}\delta_{j-k+2} - l_k\delta_{j-k+1} - \sum_{i=1}^{j-1}\delta_i \geq 1$$

$$\Longrightarrow i_1\delta_j + i_2\delta_{j-1} + ... + i_{k-1}\delta_{j-k+2} + l_k\delta_{j-k+1} \leq 1 + 2\delta - 1 - \sum_{i=1}^{j-1}\delta_i$$

$$\equiv 2\delta - \delta(1 - 2^{-j+1}) = \delta + \delta_{j-1}, \quad k = 1, 2, ..., j-1. \tag{5.18}$$

Our next goal is to establish appropriate estimates from above on $A_{j-1}^{(1)}$, $A_{j-1}^{(2)}$ and $A_{j-1}^{(3)}$. We begin with $A_{j-1}^{(1)}$. Let us write it down in the form

$$A_{j-1}^{(1)} = \sum_{h=0}^{j-2} r_1^h P_h, \tag{5.19}$$

where for $h = 2, 3, ..., j-2$,

$$P_h = \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_h=0}^{l_h} 2^{-(i_1+...+i_h+l_{h+1})} I_{j-h}(s_0 - i_1\delta_j - ... - i_h\delta_{j-h+1} - l_{h+1}\delta_{j-h} - \sum_{i=j-h}^{j-1}\delta_i),$$

$$P_0 = 2^{-l_1} I_j(s - l_1\delta_j), \quad P_1 = \sum_{i_1=1}^{l_1} 2^{-(i_1+l_2)} I_{j-1}(s_0 - i_1\delta_j - l_2\delta_{j-2} - \delta_{j-1}).$$

By condition (5.18) on the number of iteration, the following estimate on P_h holds:

$$P_h \leq I_{j-h}(1)\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_h=0}^{l_h} 2^{-(i_1+...+i_h+l_{h+1})}, \quad h \geq 2. \tag{5.20}$$

According to (5.18), we also conclude that l_{h+1} satisfies

$$i_1\delta_j + i_2\delta_{j-1} + ... + i_h\delta_{j-h+1} + l_{h+1}\delta_{j-h} \leq \delta + \delta_{j-1}.$$

Hence, by the definition of sequence $\{\delta_j\}$ there holds

$$\delta_j(i_1 + 2i_2 + 2^2 i_3 + ... + 2^{h-1} i_h + 2^h l_{h+1}) \leq \delta_j(2^j + 2).$$

Therefore, $l_{h+1} \le 2^{j-h} - 2^{-h}(i_1 + 2i_2 + 2^2 i_3 + ... + 2^{h-1} i_h) + 2^{1-h}$, so that we can set

$$l_{h+1} = \left[2^{j-h} - 2^{-h}(i_1 + 2i_2 + ... + 2^{h-1} i_h) + 2^{1-h}\right],$$

where $[\cdot]$ denotes the entire part. Obviously, there holds

$$l_{h+1} \ge 2^{j-h} - 2^{-h}(i_1 + 2i_2 + ... + 2^{h-1} i_h) - 1. \tag{5.21}$$

Extending estimate (5.20), we obtain

$$P_h \le 2I_{j-h}(1) \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{\infty} ... \sum_{i_h=0}^{\infty} 2^{-R(i_1, i_2, ..., i_h)}, \tag{5.22}$$

where

$$R(i_1, i_2, ..., i_h) = 2^{j-h} + i_1(1 - 2^{-h}) + i_2(1 - 2^{-(h-1)}) + ... + i_h(1 - 2^{-1}).$$

It then follows from (5.22) that

$$P_h \le 2I_{j-h}(1) \sum_{i_1=1}^{\infty} 2^{-(1-2^{-h})i_1} \prod_{l=2}^{h} \left(\sum_{i_l=0}^{\infty} 2^{-(1-2^{-(h-l+1)})i_l} \right) 2^{-2^{j-h}}$$

$$\le 2^{2^{-h}} 2^{-2^{j-h}} I_{j-h}(1) \prod_{l=1}^{h} b_i, \tag{5.23}$$

where $b_i = (1 - 2^{-(1-2^{-i})})^{-1}$. In order to estimate constants b_i, we observe that for arbitrarily small $\mu > 0$, there exists a number $k_0 = k_0(\mu)$ such that

$$b_i \le 2 + \mu \quad \text{for any } \; i > k_0. \tag{5.24}$$

Hence, there holds

$$\prod_{i=1}^{h} b_i \le (2 + \mu)^{h-k_0} g_0^{k_0}, \quad h > 1; \quad g_0 = \sqrt{2}/(\sqrt{2} - 1). \tag{5.25}$$

Extending (5.23), we obtain

$$P_h \le 2^{2^{-h}} I_{j-h}(1) 2^{-2^{j-h}} (2 + \mu)^{h-k_0} g_0^{k_0}. \tag{5.26}$$

We now estimate $I_{j-h}(1)$ from above. By the *a priori* estimate (4.25) from Lemma 4.4 it follows by assumption (5.1) that

$$I_{j-h}(1) \le cF(t_{j-h}) = c_3 \exp\{A(T - t_{j-h})^{-\beta}\}, \quad \beta = 1/[m(p + 1) + \xi_0]. \tag{5.27}$$

By the definition (5.5) of $\{\Delta_j\}$ there holds

$$T - t_{j-h} = \sum_{i=j-h+1}^{\infty} \Delta_j = \sum_{i=j-h+1}^{\infty} r_0^{i-1} \Delta_1 = \frac{\Delta_1 r_0^{j-h}}{1 - r_0} \equiv \frac{\Delta_1 2^{-m(p+1)(j-h)}}{1 - r_0}.$$

Hence, inequality (5.27) provides us with the estimate

$$I_{j-h}(1) \le c_3 \exp\{A[(1 - r_0)/\Delta_1]^{\beta} 2^{(j-h)(1-\xi_0 \beta)}\}. \tag{5.28}$$

Substituting this estimate into (5.26), we get

$$P_h \le c_3 2^{2^{-h}}(2+\mu)^{h-k_0} g_0^{k_0} \exp\{-2^{j-h}K(j-h,\xi_0,\delta)\}, \tag{5.29}$$

where the function $K(u,\xi_0,\delta)$ has the form

$$K(u,\xi_0,\delta) = \ln 2 - A\left(\frac{1-r_0}{\Delta_1 2^{u\xi_0}}\right)^{\beta} \equiv \ln 2 - A\left[\frac{(1-r_0)c_1c(\varepsilon)2^{m(p+1)}}{\delta^{m(p+1)}2^{u\xi_0}}\right]^{\beta}. \tag{5.30}$$

Here the constants c_1 and $c(\varepsilon)$ are given in (4.24), (5.5) and $\varepsilon > 0$ satisfies (5.9), and β is from (5.27). We need some properties of the functions $K(u,\xi_0,\delta)$. For arbitrarily small values of $\delta > 0$ and $\xi_0 > 0$, there exists finite $u_0 = u_0(\xi_0,\delta) > 0$ such that

$$K(u,\xi_0,\delta) \ge \ln 2/2 \quad \text{for any } u \ge u_0. \tag{5.31}$$

Hence, estimate (5.29) can be extended as follows:

$$\begin{gathered} P_h \le c_3 2^{2^{-h}}(2+\mu)^{h-k_0} g_0^{k_0} \exp\{-2^{j-h}\ln 2/2\} \quad \text{if } h \le j-u_0, \\ P_h \le c_3 2^{2^{-h}}(2+\mu)^{h-k_0} g_0^{k_0} \exp\{2^{1+u_0}B\delta^{-m(p+1)\beta}\} \quad \text{if } h > j-u_0. \end{gathered} \tag{5.32}$$

The constant $B = A[(1-r_0)c_1c(\varepsilon)]^{\beta}$ is independent of h, j and other constants in (5.32). Moreover,

$$k_0 \equiv k_0(\mu) \to \infty \ \text{ as } \mu \to 0, \quad u_0(\xi_0,\delta) \to \infty \ \text{ as } \xi_0 \to 0,\ \delta \to 0. \tag{5.33}$$

Estimate (5.32) was obtained for any $h \ge 2$. For $h = 0$ and $h = 1$, the proof is quite similar.

We now estimate $A^{(1)}_{j-1}$. In view of (5.32) it follows from (5.19) that

$$A^{(1)}_{j-1} \le \sum_{h=0}^{j-u_0} c_3 2^{2^{-h}} g_0^{k_0}[r_1(2+\mu)]^h(2+\mu)^{-k_0}\exp\{-2^{j-h}\ln 2/2\}$$

$$+ \sum_{h=j-u_0+1}^{j-2} c_3 2^{2^{-h}} g_0^{k_0}[r_1(2+\mu)]^h(2+\mu)^{-k_0}\exp\{2^{1+u_0}B\delta^{-m(p+1)\beta}\} \equiv A^{(1)}_{j-1,1} + A^{(1)}_{j-1,2}.$$

We recall that by (5.9), $r_1 = (1+\varepsilon)r_0 = (1+\varepsilon)2^{-m(p+1)}$. Therefore, for any $m \ge 1$ and $p > 0$, we can choose small fixed $\varepsilon > 0$ and the $\mu > 0$ from (5.24) such that

$$(2+\mu)r_1 = (2+\mu)(1+\varepsilon)/2^{m(p+1)} = \lambda_1 < 1. \tag{5.34}$$

Then we deduce the estimates

$$A^{(1)}_{j-1,1} \le 2c_3 g_0^{k_0}(2+\mu)^{-k_0}\sum_{h=0}^{j-u_0}\lambda_1^h \exp\{-2^{j-h}\ln 2/2\} \tag{5.35}$$

$$\le c_4\lambda_1^j \sum_{h=0}^{j-u_0}\exp\{-(\ln 2/2)2^{j-h} + (j-h)\ln\lambda_1^{-1}\} < c_4\lambda_1^j B_1,$$

$$B_1 = \sum_{i=1}^{\infty}\exp\{-(\ln 2/2)2^i + i\ln\lambda_1^{-1}\} = B_1(\lambda_1) = \text{const} < \infty, \quad c_4 = 2c_3 g_0^{k_0}(2+\mu)^{-k_0};$$

$$A^{(1)}_{j-1,2} \le c_4 \exp\{2^{1+u_0} B \delta^{-m(p+1)\beta}\} \sum_{h=j-u_0+1}^{j-2} \lambda_1^h \le c_4 \exp\{2^{1+u_0} B\delta^{-m(p+1)\beta}\} \tag{5.36}$$

$$\times \lambda_1^{j-u_0+1} \sum_{i=0}^{u_0-1} \lambda_1^i \le c_4 \exp\{2^{1+u_0} B\delta^{-m(p+1)\beta}\} \frac{\lambda_1^j}{(1-\lambda_1)\lambda_1^{u_0-1}}.$$

Let us estimate the term $A^{(2)}_{j-1}$. We have

$$A^{(2)}_{j-1} \le r_1^{j-1} \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{\infty} ... \sum_{i_{j-1}=0}^{\infty} 2^{-(i_1+...+i_{j-1})} I_1(1) = r_1^{j-1} I_1(1) 2^{j-2} = (2r_1)^{j-1} I_1(1)/2.$$

It follows from (5.34) that $2r_1 < \lambda_1 < 1$, and hence

$$A^{(2)}_{j-1} \le 2^{-1} \lambda_1^{j-1} I_1(1). \tag{5.37}$$

Finally, consider $A^{(3)}_{j-1}$,

$$A^{(3)}_{j-1} \le c_1 r_1^{j-1} h_0(1) \Delta_1 \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{\infty} ... \sum_{i_{j-1}=0}^{\infty} 2^{-(i_1+i_2+...+i_{j-1})}$$

$$\le c_1 (2r_1)^{j-1} \Delta_1 h_0(1)/2 \le c_1 \lambda_1^{j-1} \Delta_1 h_0(1)/2. \tag{5.38}$$

Thus, we obtain from (5.17) due to (5.35)-(5.38) that

$$I_j(s_0) = I_j(1+2\delta) \le c\lambda_1^j \;\; \text{for any } j = 1, 2, ... , \tag{5.39}$$

where c is independent of j and $\delta > 0$ can be chosen arbitrarily small. Due to (5.39) we have

$$I^{(t_j)}(s_0) = \sum_{i=1}^{j} I_i(1+2\delta) \le c \sum_{i=0}^{j-1} \lambda_1^i \le \frac{c}{1-\lambda_1}, \;\; j = 1, 2, ...$$

This uniform estimate completes the proof of Theorem 1.1 under the additional restriction (5.1).

6. End of proof of Theorem 1.1: "steep" LS-regime

We now consider optimal blow-up LS-regimes. Namely, according to Theorem 1.1, the boundary blow-up function satisfies for $t \in (0, T)$

$$F(t) \le c \exp\{A(T-t)^{-\phi}\}, \quad \text{where } \phi = 1/[m(p+1) - 1 + \xi_0] \text{ and } \xi_0 > 0. \tag{6.1}$$

We begin with the functional system (4.24), but unlike the previous section, we use a different choice of free parameters in partitions.

(i) Fix $\varepsilon_i = \varepsilon_0 = \text{const}, \;\; i \ge 1$, where ε_0 is sufficiently small.

(ii) We define the sequence $\{\alpha_j\}$ of normalized multipliers in (4.21) as follows:

$$\alpha_j = \alpha_{j-1} r_2, \;\; r_2 = r_3/(1+\varepsilon_0), \;\; r_3 = \text{const} < 1, \tag{6.2}$$

so that it is a geometric series

$$\alpha_j = \alpha_1 r_2^{j-1} = r_2^j, \;\; j \ge 1 \; (\alpha_0 = 1).$$

(iii) Then

$$\beta_j = r_3 \quad \text{for any} \;\; j \ge 1. \tag{6.3}$$

(iv) Let us fix a large constant $K > 0$ (in what follows we pass to the limit $K \to \infty$), and define the partitions

$$\delta_i = (K+i)^{-1-\xi_1/m(p+1)}, \quad \Delta_i = (K+i)^{-m(p+1)-\xi_1}, \;\; i \ge 1 \tag{6.4}$$

(we set $\delta_0 = 0$), where the constant $\xi_1 = \xi_1(\xi_0) > 0$ will be determined below. Under the above definitions, the functional system (4.24) takes the form

$$\begin{aligned} J_j(s) \le \lambda_1 J_j(s-\delta_j) &+ \lambda_1 \sum_{k=1}^{j-1} \frac{\Delta_j}{\Delta_k} r_3^{j-k} J_k(s - \sum_{i=k}^{j} \delta_i) \\ &+ c_2 r_3^j \Delta_j H_0(s - \sum_{i=1}^{j} \delta_i), \quad j \ge 1, \end{aligned} \tag{6.5}$$

where

$$\lambda_1 = c_1 c(\varepsilon_0)/[1 + c_1 c(\varepsilon_0)] < 1, \quad c_2 = c_1/[1 + c_1 c(\varepsilon_0)]. \tag{6.6}$$

Recall that

$$J_j(s) = I_j(s) r_2^j = I_j(s) \exp\{-\ln r_2^{-1} j\}. \tag{6.7}$$

System (6.5) looks similar to (5.10) for the flat LS-regime and we apply the same iteration procedure. Assuming (5.11), the term $J_j(s-\delta_j)$ in (6.5) is estimated from above by the same inequality with $s \mapsto s - \delta_j$ so that

$$\begin{aligned} J_j(s) \le \lambda_1^2 J_j(s - 2\delta_j) &+ \sum_{k=1}^{j-1} \frac{\Delta_j}{\Delta_k} r_3^{j-k} \sum_{i=1}^{2} \lambda_1^i J_k(s - (i-1)\delta_j - \sum_{h=k}^{j} \delta_h) \\ &+ c_2 r_3^j \Delta_j \sum_{i=1}^{2} \lambda_1^{i-1} H_0(s - (i-1)\delta_j - \sum_{h=1}^{j} \delta_h). \end{aligned} \tag{6.8}$$

Performing this procedure l times and assuming that (5.12) holds, we obtain

$$\begin{aligned} J_j(s) \le \lambda_1^l J_j(s - l\delta_j) &+ \sum_{k=1}^{j-1} \frac{\Delta_j}{\Delta_k} r_3^{j-k} \sum_{i=1}^{l} \lambda_1^i J_k(s - i\delta_j - \sum_{h=k}^{j-1} \delta_h) \\ &+ c_2 r_3^j \Delta_j \sum_{i=1}^{l} \lambda_1^{i-1} H_0(s - i\delta_j - \sum_{h=1}^{j-1} \delta_h), \;\; j \ge 1. \end{aligned} \tag{6.9}$$

Let us comment on a possible number of iterations l in (6.9). Denote

$$\delta = \delta(K,\xi_1) = \sum_{i=1}^{\infty} \delta_j \equiv \sum_{i=1}^{\infty} (K+i)^{-1-\xi_1/m(p+1)}. \tag{6.10}$$

We have

$$\int_{K+1}^{\infty} x^{-1-\xi_1/m(p+1)} dx \le \delta \le \int_{K}^{\infty} x^{-1-\xi_1/m(p+1)} dx,$$

and

$$\xi_1^{-1} m(p+1)(K+1)^{-\xi_1/m(p+1)} \le \delta(K,\xi_1) \le \xi_1^{-1} m(p+1) K^{-\xi_1/m(p+1)}, \tag{6.11}$$

so that for any fixed $\xi_1 > 0$

$$\delta(K,\xi_1) \to 0 \quad \text{as } K \to \infty. \tag{6.12}$$

Denote

$$s_0 = 1 + 2\delta(K,\xi_1), \tag{6.13}$$

and determine the following number of iterations at the fist step:

$$l_1 = l_1(j) = [\delta/\delta_j], \quad j \ge 1. \tag{6.14}$$

Then

$$s_0 - l_1\delta_j - \sum_{i=1}^{j} \delta_i = 1 + 2\delta - l_1\delta_j - \sum_{i=1}^{j} \delta_i \ge 1 + 2\delta - \delta - \sum_{i=1}^{j} \delta_i = 1 + \sum_{i=j+1}^{\infty} \delta_i > 1.$$

Hence, (6.9) holds with $l = l_1$,

$$\begin{aligned} J_j(s_0) &\le \lambda_1^{l_1} J_j(s_0 - l_1\delta_j) + \sum_{k_1=1}^{j-1} \frac{\Delta_j}{\Delta_{k_1}} r_3^{j-k_1} \sum_{i_1=1}^{l_1} \lambda_1^{i_1} J_{k_1}(s_0 - i_1\delta_j - \sum_{h=k_1}^{j-1} \delta_h) \\ &+ c_2 r_3^j \Delta_j \sum_{i_1=1}^{l_1} \lambda_1^{i_1-1} H_0(s_0 - i_1\delta_j - \sum_{h=1}^{j-1} \delta_h) \equiv A_1^{(1)} + A_1^{(2)} + A_1^{(3)}. \end{aligned} \tag{6.15}$$

We again decompose $A_1^{(2)}$ in a manner similar to that in the previous section to get

$$\begin{aligned} A_1^{(2)} &= \frac{\Delta_j}{\Delta_{j-1}} r_3 \sum_{i_1=1}^{l_1} \lambda_1^{i_1} J_{j-1}(s_0 - i_1\delta_j - \delta_{j-1}) \\ &+ \sum_{k_1=1}^{j-2} \frac{\Delta_j}{\Delta_{k_1}} r_3^{j-k_1} \sum_{i_1=1}^{l_1} \lambda_1^{i_1} J_{k_1}(s_0 - i_1\delta_j - \sum_{h=k_1}^{j-1} \delta_h) \equiv A_{1,1}^{(2)} + A_{1,2}^{(2)}. \end{aligned} \tag{6.16}$$

We estimate the summands in $A_{1,1}^{(2)}$ by using (6.9),

$$
\begin{aligned}
J_{j-1}(s_0 - i_1\delta_j - \delta_{j-1}) &\le \lambda_1^{l_2} J_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1}) \\
&+ \sum_{k_2=1}^{j-2} \frac{\Delta_{j-1}}{\Delta_{k_2}} r_3^{j-1-k_2} \sum_{i_2=1}^{l_2} \lambda_1^{i_2} J_{k_2}(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \sum_{h=k_2}^{j-2} \delta_h) \\
&+ c_2 r_3^{j-1} \Delta_{j-1} \sum_{i_2=1}^{l_2} \lambda_1^{i_2-1} H_0(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \sum_{h=1}^{j-2} \delta_h),
\end{aligned}
\tag{6.17}
$$

Substituting estimate (6.17) into the expression for $A_{1,1}^{(2)}$, by some obvious manipulations with the summands on the right-hand side in (6.15) we obtain

$$
J_j(s_0) \le A_2^{(1)} + A_2^{(2)} + A_2^{(3)}, \quad j \ge 1, \tag{6.18}
$$

where

$$
\begin{aligned}
A_2^{(1)} &= \lambda_1^{l_1} J_j(s_0 - l_1\delta_j) + \frac{\Delta_j}{\Delta_{j-1}} r_3 \sum_{i_1=1}^{l_1} \lambda_1^{i_1+l_2} J_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1}), \\
A_2^{(2)} &= \sum_{k_2=1}^{j-2} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \frac{\Delta_j}{\Delta_{k_2}} r_3^{j-k_2} \lambda_1^{i_1+i_2} J_{k_2}(s_0 - i_1\delta_j - i_2\delta_{j-1} - \sum_{h=k_2}^{j-1} \delta_h),
\end{aligned}
$$

$$
A_2^{(3)} = c_2 r_3^{j} \Delta_j \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \lambda_1^{i_1+i_2-1} H_0(s_0 - i_1\delta_j - i_2\delta_{j-1} - \sum_{h=1}^{j-1} \delta_h).
$$

After $j-1$ steps of such calculations, similar to the previous section we deduce that

$$
J_j(s_0) \le A_{j-1}^{(1)} + A_{j-1}^{(2)} + A_{j-1}^{(3)}, \quad j \ge 1, \tag{6.19}
$$

$$
\begin{aligned}
A_{j-1}^{(1)} &= \lambda_1^{l_1} J_j(s_0 - l_1\delta_j) + \frac{\Delta_j}{\Delta_{j-1}} r_3 \sum_{i_1=1}^{l_1} \lambda_1^{i_1+l_2} J_{j-1}(s_0 - i_1\delta_j - l_2\delta_{j-1} - \delta_{j-1}) \\
&+ \frac{\Delta_j}{\Delta_{j-2}} r_3^2 \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \lambda_1^{i_1+i_2+l_3} J_{j-2}(s_0 - i_1\delta_j - i_2\delta_{j-1} - l_3\delta_{j-2} - \delta_{j-1} - \delta_{j-2}) + \dots + \frac{\Delta_j}{\Delta_2} r_3^{j-1}
\end{aligned}
$$

$$\times \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_{j-2}=0}^{l_{j-2}} \lambda_1^{i_1+i_2+\ldots+i_{j-2}+l_{j-1}} J_2(s_0 - i_1\delta_j - \ldots - i_{j-2}\delta_3 - l_{j-1}\delta_2 - \sum_{h=2}^{j-1}\delta_h),$$

$$A_{j-1}^{(2)} = \frac{\Delta_j}{\Delta_1} r_3^{j-1} \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_{j-1}=0}^{l_{j-1}} \lambda_1^{i_1+i_2+\ldots+i_{j-1}} J_1(s_0 - i_1\delta_j - \ldots - i_{j-1}\delta_2 - \sum_{h=1}^{j-1}\delta_h),$$

$$A_{j-1}^{(3)} = c_2 r_3^j \Delta_j \lambda_1^{-1} \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_{j-1}=0}^{l_{j-1}} \lambda_1^{i_1+i_2+\ldots+i_{j-1}} H_0(s_0 - i_1\delta_j - \ldots - i_{j-1}\delta_2 - \sum_{h=1}^{j-1}\delta_h).$$

Note that the values of $l_k = l_k(i_1, i_2, ..., i_{k-1})$, $k = 1, 2, ..., j-1$, are determined from (5.18), where now s_0 is given in (6.13) and $\{\delta_i\}$ are defined in (6.4). Thus, we conclude that

$$A_{j-1}^{(1)} = \sum_{h=0}^{j-2} \frac{\Delta_j}{\Delta_{j-h}} r_3^h P_h,$$

where

$$P_0 = \lambda_1^{l_1} J_j(s_0 - \lambda_1\delta_j), \quad P_1 = \sum_{i_1=1}^{l_1} \lambda_1^{i_1+l_2} J_{j-1}(s_0 - i_1\delta_j - l_2\delta_{j-1} - \delta_{j-1}),$$

$$P_h = \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_h=0}^{l_h} \lambda_1^{i_1+i_2+\ldots+i_h+l_{h+1}} J_{j-h}(s_0 - i_1\delta_j - \ldots - i_h\delta_{j-h+1} - l_{h+1}\delta_{j-h} - \sum_{k=j-h}^{j-1}\delta_k)$$

for $h \geq 2$. By (5.18) we have

$$P_h \leq J_{j-h}(1) \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_h=0}^{l_h} \lambda_1^{i_1+i_2+\ldots+i_h+l_{h+1}}, \quad h \geq 2. \tag{6.20}$$

Using (5.18) again, we derive the condition determining possible values of l_{h+1},

$$l_{h+1} \leq (\delta - i_1\delta_j - i_2\delta_{j-2} - \ldots - i_h\delta_{j-h+1})/\delta_{j-h} \equiv a. \tag{6.21}$$

Hence, we can set $l_{h+1} = [a]$. It then follows that

$$i_1 + i_2 + \ldots + i_h + l_{h+1} \geq \frac{\delta}{\delta_{j-h}} + i_1(1 - \frac{\delta_j}{\delta_{j-h}}) + i_2(1 - \frac{\delta_{j-1}}{\delta_{j-h}}) + \ldots + i_h(1 - \frac{\delta_{j-h+1}}{\delta_{j-h}}) - 1.$$

Therefore, the following estimates on P_h holds:

$$P_h \leq \lambda_1^{-1} J_{j-h}(1) \exp\left\{-\ln\lambda_1^{-1}\frac{\delta}{\delta_{j-h}}\right\} \sum_{i_1=1}^{\infty} \lambda_1^{(1-\frac{\delta_j}{\delta_{j-h}})i_1} \sum_{i_2=0}^{\infty} \lambda_1^{(1-\frac{\delta_{j-1}}{\delta_{j-h}})i_2} \cdots \sum_{i_h=0}^{\infty} \lambda_1^{(1-\frac{\delta_{j-h+1}}{\delta_{j-h}})i_h}$$

$$\leq \lambda_1^{-1} J_{j-h}(1) \exp\left\{-\ln\lambda_1^{-1}\frac{\delta}{\delta_{j-h}}\right\} \Pi_h, \tag{6.22}$$

where

$$\Pi_h = \lambda_1^{-(1-\frac{\delta_{j-1}}{\delta_{j-h}})} \cdots \lambda_1^{-(1-\frac{\delta_{j-h+1}}{\delta_{j-h}})} (\lambda_1^{-(1-\frac{\delta_j}{\delta_{j-h}})} - 1)^{-1} \cdots (\lambda_1^{-(1-\frac{\delta_{j-h+1}}{\delta_{j-h}})} - 1)^{-1}. \tag{6.23}$$

In order to estimate Π_h from above we need bounds on the last factors. By Lagrange's formula

$$\lambda_1^{-(1-\frac{\delta_{j-k}}{\delta_{j-h}})} - 1 = \exp\left\{\ln\lambda_1^{-1}(1-\frac{\delta_{j-k}}{\delta_{j-h}})\right\} - 1 \geq \ln\lambda_1^{-1}\left(1-\frac{\delta_{j-k}}{\delta_{j-h}}\right). \tag{6.24}$$

Therefore, (6.23) implies that for any $h \geq 1$,

$$\Pi_h \leq \lambda_1^{-(h-1-\frac{\delta_{j-1}+\ldots+\delta_{j-h+1}}{\delta_{j-h}})}\left[(\ln\lambda_1^{-1})^h\prod_{k=0}^{h-1}\left(1-\frac{\delta_{j-k}}{\delta_{j-h}}\right)\right]^{-1}. \tag{6.25}$$

It follows from (6.4) that

$$\begin{aligned} 1-\frac{\delta_{j-k}}{\delta_{j-h}} &= 1-\left(\frac{K+j-h}{K+j-k}\right)^{1+\xi_1/m(p+1)} \\ &= 1-\left(1-\frac{h-k}{K+j-k}\right)^{1+\xi_1/m(p+1)} \geq \frac{h-k}{K+j-k}. \end{aligned} \tag{6.26}$$

Hence,

$$\prod_{k=0}^{h-1}\left(1-\frac{\delta_{j-k}}{\delta_{j-h}}\right) \geq \prod_{k=0}^{h-1}\frac{h-k}{K+j-k} = \frac{h!(K+j-h)!}{(K+j)!} \equiv (C_{K+j}^h)^{-1},$$

where the binomial coefficient satisfies $(C_{K+j}^h)^{-1} \geq 2^{-(K+j)}$. Therefore, there holds

$$\Pi_h \leq \mu_1^h \lambda_1^{(\delta_{j-1}+\delta_{j-2}+\ldots+\delta_{j-h})/\delta_{j-h}} C_{K+j}^h, \quad \mu_1 = (\lambda_1 \ln\lambda_1^{-1})^{-1}.$$

Substituting this estimate into (6.22), we obtain that

$$P_h \leq \lambda_1^{(\delta_{j-1}+\delta_{j-2}+\ldots+\delta_{j-h+1})/\delta_{j-h}}\mu_1^h C_{K+j}^h J_{j-h}(1)\exp\{-\ln\lambda_1^{-1}\frac{\delta}{\delta_{j-h}}\}. \tag{6.27}$$

Let us estimate the last two factors on the right-hand side. By (6.11) and by the definition of δ_i in (6.4) we have

$$\frac{\delta}{\delta_{j-h}} \geq \frac{m(p+1)(K+j-h)^{1+\xi_1/m(p+1)}}{\xi_1(K+1)^{\xi_1/m(p+1)}},$$

so that

$$\exp\left\{-\ln\lambda_1^{-1}\frac{\delta}{\delta_{j-h}}\right\} \leq \exp\left\{-\frac{\ln\lambda_1^{-1}m(p+1)(K+j-h)^{1+\xi_1/m(p+1)}}{\xi_1(K+1)^{\xi_1/m(p+1)}}\right\}. \tag{6.28}$$

We now estimate $J_{j-h}(1)$. By the *a priori* estimate (4.25), it follows from Lemma 4.4, condition (6.1) on the boundary blow-up regime, definition (4.21) of $J_j(s)$ and from (6.2) that

$$J_{j-h}(1) \equiv r_2^{j-h}I_{j-h}(1) \leq r_2^{j-h}cF(t_{j-h}) \leq cr_2^{j-h}\exp\{A(T-t_{j-h})^{-\phi}\}, \tag{6.29}$$

where $r_2 = r_3/(1+\varepsilon_0)$. Let us derive an estimate on $T - t_i$ for any $i \geq 1$,

$$T - t_i = \sum_{k=i+1}^{\infty} \Delta_k = \sum_{k=K+i+1}^{\infty} k^{-m(p+1)-\xi_1}.$$

Hence,

$$\int_{K+i+1}^{\infty} x^{-[m(p+1)+\xi_1]} dx \leq T - t_i \leq \int_{K+i}^{\infty} x^{-[m(p+1)+\xi_1]} dx,$$

so that for $i \geq 1$,

(6.30) $$\phi_1 (K+i+1)^{-1/\phi_1} \leq (T - t_i) \leq \phi_1 (K+i)^{-1/\phi_1}, \ \ \phi_1 = 1/[m(p+1) - 1 + \xi_1].$$

Therefore, (6.29) implies that

(6.31) $$J_{j-h}(1) \leq c r_2^{j-h} \exp\{A_1 (K + j - h + 1)^{\phi/\phi_1}\},$$

where $A_1 = A\phi_1^{-\phi}$. We now choose $\xi_1 = \xi_1(\xi_0) > 0$ such that

(6.32) $$\frac{\phi}{\phi_1} \equiv \frac{m(p+1) - 1 + \xi_1}{m(p+1) - 1 + \xi_0} = 1 + \frac{\xi_1}{2m(p+1)},$$

i.e.,

(6.33) $$\xi_1 = \xi_0 \frac{2m(p+1)}{m(p+1) + 1 - \xi_0},$$

and, in particular, for any $\xi_0 > 0$,

(6.34) $$\xi_1 > \gamma_0 \xi_0, \quad \gamma_0 = 2m(p+1)/[m(p+1)+1] > 1.$$

Then estimate (6.31) for $h \geq 1$ takes the form

(6.35) $$J_{j-h}(1) \leq c r_2^{j-h} \exp\{A_1 (K + j - h + 1)^{\nu_1}\}, \ \ \nu_1 = 1 + \xi_1/2m(p+1) > 1.$$

Substituting estimates (6.28) and (6.35) into (6.27), we obtain

(6.36) $$P_h \leq c\mu_1^h 2^{K+j} r_2^{j-h} \exp\{-A_2 (K + j - h)^{2\nu_1 - 1} + A_1 (K + j - h + 1)^{\nu_1}\},$$

where $A_2 = m(p+1) \ln \lambda_1^{-1} / \xi_1 (K+1)^{2(\nu_1 - 1)}$. We need some extra simple manipulations. Let us write down the exponential term in the right-hand side of (6.36) in the form

$$\Gamma_1 \equiv -(K + j - h)^{\nu_1} \{A_2 (K + j - h)^{\nu_1 - 1} - A_1 (1 + (K + j - h)^{-1})^{\nu_1}\}.$$

Consider the auxiliary function

$$\Gamma_2(u) = A_2 (K + u)^{\nu_1 - 1} - A_1 [1 + (K + u)^{-1}]^{\nu_1}.$$

One can see that there exists $u_0 = u_0(A_2, A_1, K, \xi_1)$ such that

(6.37) $$\Gamma_2(u) > 1 \quad \text{for all } u > u_0.$$

Denote

$$\Gamma_2^{(0)} \equiv \Gamma_2^{(0)}(A_2, A_1, K, \xi_1) = \max_{2 \leq u \leq u_0} |\Gamma_2(u)|.$$

By the definition of $\Gamma_2(u)$ we have

$$\Gamma_1 \leq -(K + j - h)^{\nu_1} \quad \text{if } \ j - h \geq u_0, \quad \Gamma_1 \leq \Gamma_2^{(0)} (K + u_0)^{\nu_1} \quad \text{if } \ j - h < u_0.$$

Thus, from (6.36) we derive the following estimate:

$$P_h \le c\mu_1^h\, 2^{K+j} r_2^{j-h} \exp\{-(K+j-h)^{\nu_1}\}, \ \ h \le j - u_0,$$

$$P_h \le c\mu_1^h\, 2^{K+j} r_2^{j-h} \exp\{\Gamma_2^{(0)}(K+u_0)^{\nu_1}\}, \ \ h > j - u_0. \tag{6.38}$$

This estimate holds for any $h \ge 2$. For $h = 0$ and $h = 1$ it is elementary. Let us return back to the estimate on

$$A_{j-1}^{(1)} = \sum_{h=0}^{j-2} \frac{\Delta_j}{\Delta_{j-h}} r_3^h P_h \equiv \sum_{h=0}^{j-u_0} (\cdot) + \sum_{h=j-u_0+1}^{j-2} (\cdot).$$

Hence, by (6.38) we have

$$A_{j-1}^{(1)} \le c2^{K+j} \sum_{h=0}^{j-u_0} \frac{\Delta_j}{\Delta_{j-h}} r_3^h r_2^{j-h} \mu_1^h \exp\{-(K+j-h)^{\nu_1}\}$$

$$+ c2^{K+j} \sum_{h=j-u_0+1}^{j-2} \frac{\Delta_j}{\Delta_{j-h}} r_3^h r_2^{j-h} \mu_1^h \exp\{\Gamma_2^{(0)}(K+u_0)^{\nu_1}\}$$

$$= c2^{K+j} r_2^j \sum_{h=0}^{j-u_0} \frac{\Delta_j}{\Delta_{j-h}} (1+\varepsilon_0)^h \mu_1^h \exp\{-(K+j-h)^{\nu_1}\}$$

$$+ c2^{K+j} r_2^j \sum_{h=j-u_0+1}^{j-2} \frac{\Delta_j}{\Delta_{j-h}} [(1+\varepsilon_0)\mu_1]^h \exp\{\Gamma_2^{(0)}(K+u_0)^{\nu_1}\}$$

$$\le c2^{K+j} r_2^j \sum_{h=0}^{j-u_0} \Lambda_1^h + cK_1 2^{K+j} r_2^j \sum_{h=j-u_0+1}^{j-2} \Lambda_1^h, \tag{6.39}$$

where $K_1 = \exp\{\Gamma_2^{(0)}(K+u_0)^{\nu_1}\}$, $\Lambda_1 = (1+\varepsilon_0)\mu_1$ and $\mu_1 = (\lambda_1 \ln\, \lambda_1^{-1})^{-1}$. As a natural continuation of (6.39), we finally obtain

$$A_{j-1}^{(1)} \le c2^{K+j} r_2^j K_1 \Lambda_1^{j-1} (\Lambda_1 - 1)^{-1} \equiv C_1 (2r_2\Lambda_1)^j, \quad j \ge 1. \tag{6.40}$$

Here and later on constants C_1, C_2, ... depend on the known parameters and, in particular, are independent of j. Next, we have

$$A_{j-1}^{(2)} \le c\frac{\Delta_j}{\Delta_1} r_3^{j-1} J_1(1) \sum_{i_1=1}^{\infty} \lambda_1^{i_1} \sum_{i_2=0}^{\infty} \lambda_1^{i_2} \ldots \sum_{i_{j-1}=0}^{\infty} \lambda_1^{i_{j-1}} \le c_1 r_2^{j-1} (1+\varepsilon_0)^{j-1} K^{m(p+1)+\xi_1}$$

$$\times (K+j)^{-m(p+1)-\xi_1} J_1(1) \lambda_1 (1-\lambda_1)^{-(j-1)} \le C_2 [r_2(1+\varepsilon_0)(1-\lambda_1)^{-1}]^j. \tag{6.41}$$

Hence, there holds

$$A_{j-1}^{(3)} \le c_2 r_3^j \Delta_j H_0(1) \sum_{i_1=1}^{\infty} \lambda_1^{i_1} \sum_{i_2=0}^{\infty} \lambda_1^{i_2} \ldots \sum_{i_{j-1}=0}^{\infty} \lambda_1^{i_{j-1}} \le c_2 r_2^j (1+\varepsilon_0)^j$$

$$\times(K+j)^{-m(p+1)-\xi_1}H_0(1)\lambda_1(1-\lambda_1)^{-(j-1)} \le C_3[(1+\varepsilon_0)r_2(1-\lambda_1)^{-1}]^j. \tag{6.42}$$

By (6.19) and (6.40)-(6.42) we have

$$J_j(s_0) \le C_4 r_2^j[2\Lambda_1+2(1+\varepsilon_0)(1-\lambda_1)^{-1}]^j \equiv C_4 r_2^j \Lambda_2^j. \tag{6.43}$$

Recalling the definition of the normalized functions $J_i(s)$ in (4.21) and (6.2), we obtain

$$I_j(s_0) \le C_4\Lambda_2^j. \tag{6.44}$$

It follows from (6.44) that for any $j \ge 1$,

(6.45)

$$I^{(t_j)}(s_0) \equiv \int_{t_0}^{t_j}\int_{\Omega(s_0)}|u|^{p+1}dxdt = \sum_{i=1}^{j} I_i(s_0) \le C_4\sum_{i=1}^{j}\Lambda_2^i \le C_5\Lambda_2^j = C_5\exp\{j\ln\Lambda_2\}.$$

By estimate (6.30), $K+i \le \phi_1^{\phi_1}(T-t_i)^{-\phi_1}$ for $i \ge 1$. Substituting into (6.45), we get

$$I^{(t_j)}(s_0) \le C_5\exp\{\Lambda_3(T-t_j)^{-\phi_1}\}, \quad \Lambda_3 = \phi_1^{\phi_1}\ln\Lambda_2. \tag{6.46}$$

Since $(T-t_j)/(T-t_{j-1}) \to 1$ as $j \to \infty$ monotone, the "discrete" estimate (6.46) implies the "continuous" one for any $t > t_0$, i.e.,

$$I^{(t)}(s_0) \le C_6\exp\{\Lambda_4(T-t)^{-\phi_1}\}, \quad \Lambda_4 = \Lambda_3[(T-t_1)/(T-t_2)]^{\phi_1}. \tag{6.47}$$

Let us note that by the *a priori* estimate (4.25) and condition (6.1), we have the "initial estimate"

$$I^{(t)}(1) \le cF(t) \le c\exp\{A(T-t)^{-\phi}\}, \quad t > t_0;\ \ \phi = 1/[m(p+1)-1+\xi_0]. \tag{6.48}$$

Comparing estimates (6.47) and (6.48) shows that we essentially reduce the singularity order at $s = s_0$ relative to that at $s = 1$ since by the construction $\xi_1 > \gamma_0\xi_0$, where $\gamma_0 > 1$.

We now start the next computational cycle by taking estimate (6.47) as the initial one instead of (6.48). Namely, for $s > s_0$, we introduce a sequence of new energy functions $I_j(s)$ keeping for convenience all the previous notations, so that for $i \ge 1$,

$$\Delta_i = \Delta_i(\xi_2) \equiv (K+i)^{-m(p+1)-\xi_2}\omega_0, \ \ \delta_i = \delta_i(\xi_2) \equiv (K+i)^{-1-\frac{\xi_2}{m(p+1)}}, \tag{6.49}$$

where $\delta_0 = 0$ by implication and ξ_2 is determined from the equality similar to (6.32)

$$\frac{m(p+1)-1+\xi_2}{m(p+1)-1+\xi_1} = 1+\frac{\xi_2}{m(p+1)} \implies \xi_2 = \xi_1\frac{2m(p+1)}{m(p+1)+1-\xi_1} > \gamma_0\xi_1, \tag{6.50}$$

where $\gamma_0 > 1$ is as in (6.34). The new energy functions are

$$I_j(s) = \int_{t_{j-1}}^{t_j}\int_{\Omega(s)}|u(x,t)|^{p+1}dxdt, \quad s \ge s_0,\ j \ge 1.$$

Here $\{t_j\}$ is the new time partition of the interval $(0,T)$ corresponding to $\{\Delta_i\}$ from (6.49). Instead of $\delta = \delta(K,\xi_1)$ in (6.10), we set

$$\delta = \delta(K,\xi_2) = \sum_{i=1}^{\infty}\delta_i(\xi_2), \tag{6.51}$$

so that similar to (6.11) there holds

$$m(p+1)\xi_2^{-1}(K+1)^{-\frac{\xi_2}{m(p+1)}} \le \delta(K,\xi_2) \le m(p+1)\xi_2^{-1}K^{-\frac{\xi_2}{m(p+1)}}. \tag{6.52}$$

We also define the new value

$$s_1 = s_0 + 2\delta = 1 + 2\delta(K,\xi_1) + 2\delta(K,\xi_2). \tag{6.53}$$

We perform calculations similar to those which led us from the initial estimate (6.48) to (6.47), but now with (6.47) as the initial one. We then obtain for any $t < T$

$$I^{(t)}(s_1) \le C_7 \exp\{\Lambda_5(T-t)^{-\phi_2}\}, \quad \phi_2 = 1/[m(p+1)-1+\xi_2]. \tag{6.54}$$

Using (6.54) as the initial estimate for the next cycle of calculations, etc., after a number l of such cycles we arrive at the estimate

$$I^{(t)}(s_{l-1}) \le C_8 \exp\{\Lambda_6(T-t)^{-\phi_l}\}, \quad \phi_l = 1/[m(p+1)-1+\xi_l],\ \Lambda_6 = \Lambda_6(l), \tag{6.55}$$

where

$$\xi_l \ge \gamma_0^l \xi_0,\ \gamma_0 > 1; \quad s_{l-1} = 1 + 2\delta(K,\xi_1) + ... + 2\delta(K,\xi_l), \tag{6.56}$$

and $\delta(K,\xi)$ satisfies (6.52). Choosing $l = l_0$ such that $\xi_{l_0} = 1+\nu$, $\nu > 0$, then $l_0 = [\ln(1+\nu) + \ln \xi_0^{-1}]/\ln \gamma_0$, as a consequence, we deduce that

$$\begin{aligned} s_{l_0-1} &\le 1 + 2\sum_{i=1}^{l_0} \delta(K,\xi_i) \le 1 + 2m(p+1)\sum_{i=1}^{l_0} \xi_0^{-1}\gamma_0^{-i} K^{-\xi_0\gamma_0^i/m(p+1)} \\ &\le 1 + 2m(p+1)\xi_0^{-1}(\gamma_0-1)^{-1}K^{-\xi_0\gamma_0/m(p+1)}. \end{aligned} \tag{6.57}$$

Thus, performing l_0 cycles, we obtain the estimate

$$I^{(t)}(s_{l_0-1}) \le C_9 \exp\{\Lambda_7(T-t)^{-1/[m(p+1)+\nu]}\},\ t < T,\ \nu > 0. \tag{6.58}$$

We now choose the value of K. Given an arbitrarily small number $\mu > 0$, let $K = K(\mu)$ be such that

$$2m(p+1)\xi_0^{-1}(\gamma_0-1)^{-1}K^{-\xi_0\gamma_0/m(p+1)} = \mu. \tag{6.59}$$

Then, in view of (6.57) and (6.59), estimate (6.58) yields

$$I^{(t)}(1+\mu) \le C_9 \exp\{\Lambda_7(T-t)^{-1/[m(p+1)+\nu]}\} \quad \text{with finite } \Lambda_7.$$

The constants C_9 and Λ_7 do not depend on t. Therefore, we arrive at a flat blow-up LS-regime posed on the lateral boundary of the domain $\{|x| > 1+\mu\} \times (0,T)$. By the results from Section 5, the blow-up set is contained in $\{|x| \le 1+\mu\}$. Since $\mu > 0$ can be chosen arbitrarily small, Theorem 1.1 follows.

7. **Boundary S-regime: proof of Theorem 1.2**

As above, we begin with estimate (4.24). First, we define the sequence of displacements $\{\Delta_i\}$. Namely, we fix two numbers $K > 0$, $\gamma > 0$, and define for $i = 1, 2, ...,$

$$\Delta_i = [(1+\gamma)^{-m(p+1)}w]^{m(p+1)-1}[(i+K)^{-1}(m(p+1)-1)]^{m(p+1)}, \tag{7.1}$$

where $w > 0$ is from condition (1.15). It is easy to check that for $j \geq 0$,

$$\left[\frac{w(m(p+1)-1)}{(j+K+1)(1+\gamma)^{m(p+1)}}\right]^{m(p+1)-1} \leq T - t_j \equiv \sum_{i=j+1}^{\infty} \Delta_i \tag{7.2}$$

$$\leq \left[\frac{w(m(p+1)-1)}{(j+K)(1+\gamma)^{m(p+1)}}\right]^{m(p+1)-1}.$$

The constant $K > 0$ is fixed to guarantee the inequality

$$T - t_0 \leq \left[\frac{w(m(p+1)-1)}{K(1+\gamma)^{m(p+1)}}\right]^{m(p+1)-1} \leq T \Longrightarrow K \geq K_0 = K_0(T, \gamma). \tag{7.3}$$

One can see that for $j \geq 0$,

$$w^{\frac{m(p+1)-1}{m(p+1)}} \Delta_j^{-\frac{1}{m(p+1)}} \leq \frac{w}{1+\gamma}(T-t_j)^{-\frac{1}{m(p+1)-1}}, \tag{7.4}$$

$$\frac{K}{K+1}\frac{w}{1+\gamma}(T-t_j)^{-\frac{1}{m(p+1)-1}} \leq w^{\frac{m(p+1)-1}{m(p+1)}} \Delta_j^{-\frac{1}{m(p+1)}} \tag{7.5}$$

$$\equiv (1+\gamma)^{m(p+1)-1}\frac{j+K}{m(p+1)-1}.$$

Let us now choose the weights α_i, which define the weighted energy functions $J_j(s)$ and $H_j(s)$ for $j = 0, 1, ...,$

$$\alpha_j = \exp\{-w^{\frac{m(p+1)-1}{m(p+1)}} \Delta_j^{-\frac{1}{m(p+1)}}\} = \exp\left\{-(1+\gamma)^{m(p+1)-1}\frac{j+K}{m(p+1)-1}\right\}. \tag{7.6}$$

Then for any $j \geq 1$,

$$\frac{\alpha_j}{\alpha_{j-1}} = \exp\left\{-\frac{(1+\gamma)^{m(p+1)-1}}{m(p+1)-1}\right\} = \text{const} < 1. \tag{7.7}$$

We now set in (4.24)

$$\varepsilon_k = \varepsilon_0 = \text{const} > 0, \quad k \geq 1,$$

and for arbitrary $j \in \mathbb{N}$ and all $i \leq j$ define displacements $\delta_i^{(j)}$ as follows:

$$\delta_i^{(j)} = \delta_j \equiv [c_1 c(\varepsilon_0)\Delta_j]^{\frac{1}{m(p+1)}} \omega_0^{-\frac{1}{m(p+1)}}, \tag{7.8}$$

where $\omega_0 > 0$ is a small parameter to be determined. It is easy to see that

$$\delta_j = c_2(1+\gamma)^{-[m(p+1)-1]} w^{\frac{m(p+1)-1}{m(p+1)}} (j+K)^{-1} \omega_0^{-\frac{1}{m(p+1)}}, \tag{7.9}$$

where $c_2 = [c_1 c(\varepsilon_0)]^{1/m(p+1)}$ and constants $c_1, c(\varepsilon_0)$ are as in (4.24). Substituting (7.8) into (4.24), by simple transformations we deduce that

$$\begin{aligned} J_j(s) \le \omega_1 J_j(s-\delta_j) + \omega_1 \sum_{k=1}^{j-1} \lambda_0^{j-k} J_k(s-(j-k+1)\delta_j) \\ + \; c\lambda_0^j \Delta_j H_0(s-j\delta_j), \quad s > 1 + j\delta_j,\, j \in \mathbb{N}, \end{aligned} \tag{7.10}$$

where $\omega_1 = \omega_0/(1+\omega_0)$, $c = c_1/(1+\omega_0)$ and

$$\lambda_0 = \lambda_0(\gamma, \varepsilon_0) = (1+\varepsilon_0)\frac{\alpha_j}{\alpha_{j-1}} = (1+\varepsilon_0)\exp\left\{-\frac{(1+\gamma)^{m(p+1)-1}}{m(p+1)-1}\right\}, \tag{7.11}$$

where $\lambda_0 \to 0$ as $\gamma \to \infty$. Therefore, setting $\gamma > \gamma_0(\varepsilon_0) > 0$, we can assume that

$$\lambda_0 = \lambda_0(\gamma, \varepsilon_0) < 1. \tag{7.12}$$

We now begin the analysis of the functional system (7.10). We iterate inequalities (7.10) in the same way as above. After l_1 cycles of estimates we obtain that for any $s > 1 + (l_1 + j - 1)\delta_j$,

$$\begin{aligned} J_j(s) \le & \,\omega_1^{l_1} J_j(s - l_1\delta_j) + \sum_{k_1=1}^{j-1} \lambda_0^{j-k_1} \sum_{i_1=1}^{l_1} \omega_1^{i_1} J_{k_1}(s-(i_1+j-k_1)\delta_j) \\ & + c\lambda_0^j \Delta_j \sum_{i_1=1}^{l_1} \omega_1^{i_1-1} H_0(s-(i_1+j-1)\delta_j) \equiv A_1^{(1)} + A_1^{(2)} + A_1^{(3)}. \end{aligned} \tag{7.13}$$

Consider $A_1^{(2)}(s)$ in the following form:

$$\begin{aligned} A_1^{(2)} = \lambda_0 \sum_{i_1=1}^{l_1} \omega_1^{i_1} J_{j-1}(s-(i_1+1)\delta_j) \\ + \sum_{k_1=1}^{j-2} \lambda_0^{j-k_1} \sum_{i_1=1}^{l_1} \omega_1^{i_1} J_{k_1}(s-(i_1+j-k_1)\delta_j) \equiv A_{1,1}^{(2)} + A_{1,2}^{(2)}. \end{aligned}$$

We estimate the terms from $A_{1,1}^{(2)}$ using (7.13) with $j \mapsto j-1$ and $s \mapsto s-(i_1+1)\delta_j$. For any $s > 1 + (i_1+1)\delta_j + (l_2+j-2)\delta_{j-1}$, there holds

$$\begin{aligned} J_{j-1}(s-(i_1+1)\delta_j) \le \omega_1^{l_2} J_{j-1}(s-(i_1+1)\delta_j - l_2\delta_{j-1}) \\ + \sum_{k_2=1}^{j-2} \lambda_0^{j-1-k_2} \sum_{i_2=1}^{l_2} \omega_1^{i_2} J_{k_2}(s-(i_1+1)\delta_j - (i_2+j-1-k_2)\delta_{j-1}) \\ + c\,\lambda_0^{j-1}\Delta_{j-1} \sum_{i_2=1}^{l_2} \omega_1^{i_2-1} H_0(s-(i_1+1)\delta_j - (i_2+j-1-1)\delta_{j-1}). \end{aligned} \tag{7.14}$$

Substituting (7.14) into the expression for $A^{(2)}_{1,1}$, by a simple transformation we get

$$J_j(s) \le A^{(1)}_2(s) + A^{(2)}_2(s) + A^{(3)}_2(s), \quad j \ge 1,$$

where

$$A^{(1)}_2 = A^{(1)}_1 + \lambda_0 \sum_{i_1=1}^{l_1} \omega_1^{i_1+l_2} J_{j-1}(s - (i_1+1)\delta_j - l_2\delta_{j-1}),$$

$$A^{(2)}_2 = A^{(2)}_{1,2} + \sum_{k_2=1}^{j-2} \sum_{i_1=1}^{l_1} \sum_{i_2=1}^{l_2} \omega_1^{i_1+i_2} \lambda_0^{j-k_2} J_{k_2}(s - (i_1+1)\delta_j - (i_2+j-1-k_2)\delta_{j-1}),$$

$$A^{(3)}_2 = A^{(3)}_1 + c\lambda_0^j \Delta_{j-1} \sum_{i_1=1}^{l_1} \sum_{i_2=1}^{l_2} \omega_1^{i_1+i_2-1} H_0(s - (i_1+1)\delta_j - (i_2+j-2)\delta_{j-1}).$$

Since $\delta_j \le \delta_{j-1}$, in view of monotonicity of functions $J_k(s)$ it follows that

$$A^{(2)}_2 \le \sum_{k_2=1}^{j-2} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \omega_1^{i_1+i_2} \lambda_0^{j-k_2} J_{k_2}(s - (i_1+1)\delta_j - (i_2+j-1-k_2)\delta_{j-1}). \tag{7.15}$$

Repeating this cycle of calculations q times, we derive q inequalities for each $j \in \mathbb{N}$,

$$J_j(s) \le A^{(1)}_h(s) + A^{(2)}_h(s) + A^{(3)}_h(s), \;\; h = 1, 2, ..., q, \tag{7.16}$$

where

$$A^{(1)}_h \equiv A^{(1)}_h(s) = \omega_1^{l_1} J_j(s - l_1\delta_j) + \lambda_0 \sum_{i_1=1}^{l_1} \omega_1^{i_1+l_2} J_{j-1}(s - (i_1+1)\delta_j - l_2\delta_{j-1}) \tag{7.17}$$

$$+\lambda_0^2 \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \omega_1^{i_1+i_2+l_3} J_{j-2}(s - (i_1+1)\delta_j - (i_2+1)\delta_{j-1} - l_3\delta_{j-2}) + ... + \lambda_0^{h-1}$$

$$\times \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} ... \sum_{i_{h-1}=0}^{l_{h-1}} \omega_1^{i_1+i_2+...+i_{h-1}+l_h} J_{j-h+1}(s - (i_1+1)\delta_j - ... - (i_{h-1}+1)\delta_{j-h+2} - l_h\delta_{j-h+1}),$$

$$A^{(2)}_h = \sum_{k_h=1}^{j-h} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} ... \sum_{i_h=0}^{l_h} \lambda_0^{j-k_h} \omega_1^{i_1+i_2+...+i_h} J_{k_h}(s - (i_1+1)\delta_j - (i_2+1)\delta_{j-1} - ... \tag{7.18}$$

$$-(i_{h-1}+1)\delta_{j-h+2} - (i_h+j-h-k_h+1)\delta_{j-h+1}),$$

$$A^{(3)}_h = c\lambda_0^j [\Delta_j \sum_{i_1=1}^{l_1} \omega_1^{i_1-1} H_0(s - (i_1+j-1)\delta_j) \tag{7.19}$$

$$+ \Delta_{j-1} \sum_{i_1=1}^{l_1} \sum_{i_2=1}^{l_2} \omega_1^{i_1+i_2-1} H_0(s - (i_1+1)\delta_j - (i_2+j-2)\delta_{j-1}) + ... + \; \Delta_{j-h+1}$$

$$\times \sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2}...\sum_{i_h=1}^{l_h}\omega_1^{i_1+i_2+...+i_h-1}H_0(s-(i_1+1)\delta_j-(i_2+1)\delta_{j-1}-...-(i_h+j-h)\delta_{j-h+1})].$$

We choose the total number q of such cycles by setting

$$q = j\xi, \tag{7.20}$$

with a constant $\xi \in (0,1)$ to be determined later.

Let us write down the condition for the argument s, which guarantees (7.16),

$$\begin{aligned} s \geq 1+(l_1+j-1)\delta_j, \quad s \geq 1+(i_1+1)\delta_j+(i_2+1)\delta_{j-1}+\dots \\ +(i_{h-1}+1)\delta_{j-h+2}+(l_h+j-h)\delta_{j-h+1}), \quad h=2,\dots,q. \end{aligned} \tag{7.21}$$

Consider system (7.21). First, we show that for any $\xi \in (0,1)$, there exists a number $s_0 = s_0(\xi) < \infty$ independent of $j \in \mathbb{N}$, such that inequalities (7.21) are satisfied for all $j \in \mathbb{N}$, $s \geq s_0(\xi)$ and suitable values of $l_k = l_k(i_1,\dots,i_{k-1})$. Let us write the second inequality (7.21) in the form

$$s \geq 1+\sum_{i=j-h+2}^{j}\delta_i+(j-h)\delta_{j-h+1}+\sum_{k=1}^{h-1}i_k\delta_{j-k+1}+l_h\delta_{j-h+1}, \tag{7.22}$$

where $h=2,...,j\xi$ for any $j \in \mathbb{N}$. Due to definition (7.9) we have

$$\sum_{i=j-h+2}^{j}\delta_i = G_1\sum_{i=j-h+2}^{j}(i+K)^{-1} \leq G_1\ln(\frac{j}{j-h}), \tag{7.23}$$

where $G_1 = c_2(1+\gamma)^{-m(p+1)+1}w^{\frac{m(p+1)-1}{m(p+1)}}\omega_0^{-\frac{1}{m(p+1)}}$. In addition, we have by (7.9) that for any $h<j$,

$$(j-h)\delta_{j-h+1} = G_1(j-h)/(j-h+1+K)) \leq G_1.$$

Hence, (7.22) follows from the inequality

$$s \geq 1+G_1[1+\ln(1-\xi)^{-1}]+l_h\delta_{j-h+1}+\sum_{k=1}^{h-1}i_k\delta_{j-k+1} \tag{7.24}$$

$$=1+G_1\left[1+\ln(1-\xi)^{-1}+l_h(j-h+1+K)^{-1}+\sum_{k=1}^{h-1}i_k(j-k+1+K)^{-1}\right].$$

Fix now $\zeta>0$ (to be determined), set

$$s_1 = 1+G_1[1+\ln(1-\xi)^{-1}+\zeta], \tag{7.25}$$

and define

$$l_h = \left[\delta_{j-h+1}^{-1}(G_1\zeta-\sum_{k=1}^{h-1}i_k\delta_{j-k+1})\right], \quad h=2,3,...,j\xi, \quad l_1=[G_1\zeta/\delta_j]. \tag{7.26}$$

As usual, $[G_1\zeta/\delta_j]$ denotes the entire part. It follows from (7.26) that (7.24) holds for all $s > s_1$ with s_1 from (7.25). Our next goal is to establish suitable upper estimates of functions $A_h^{(1)}(s), A_h^{(2)}(s), A_h^{(3)}(s)$ with $h = j\xi$. By (7.24)-(7.26) we have

$$A_h^{(1)} \equiv A_h^{(1)}(s) \leq \omega_1^{l_1} J_j(1) + \sum_{i=1}^{h-1} \lambda_0^i \sigma_i J_{j-i}(1), \tag{7.27}$$

$$\sigma_1 = \sum_{i=1}^{l_1} \omega_1^{i+l_2}, \quad \sigma_k = \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} ... \sum_{i_k=0}^{l_k} \omega_1^{i_1+i_2+\cdot+i_k+l_{k+1}}.$$

Let us estimate σ_k from above. By (7.26) there holds

$$i_1 + i_2 + ... + i_k + l_{k+1} \geq i_1 + i_2 + ... + i_k + (G_1\zeta - \sum_{m=1}^{k} i_m \delta_{j-m+1})\delta_{j-k}^{-1} - 1$$

$$= G_1\zeta\delta_{j-k}^{-1} + i_1(1 - \frac{\delta_j}{\delta_{j-k}}) + ... + i_k(1 - \frac{\delta_{j-k+1}}{\delta_{j-k}}) - 1$$

$$= \zeta(j-k+K) + i_1\frac{k}{j+K} + i_2\frac{k-1}{j+K-1} + ... + i_k\frac{1}{j+K-k+1} - 1.$$

Therefore,

$$\sigma_k \leq \omega_1^{-1}\omega_1^{\zeta(j-k+K)} \sum_{i_1=1}^{\infty} \omega_1^{i_1\frac{k}{j+K}} \sum_{i_2=0}^{\infty} \omega_1^{i_2\frac{k-1}{j+K-1}} ... \sum_{i_k=0}^{\infty} \omega_1^{i_k\frac{1}{j+K-k+1}} \tag{7.28}$$

$$\leq \omega_1^{-\frac{j+K-k}{j+K}} \omega_1^{\zeta(j-k+K)} \prod_{h=1}^{k} \left(1 - \omega_1^{\frac{k-h+1}{j+K-h+1}}\right)^{-1}, \quad k = 1, 2, ..., h-1.$$

Let us estimate the last factor on the right-hand side. Since the function $f(x) = 1 - \omega_1^x$ is convex on $(0, 1)$, we have $f(x) \geq f(0) + x(f(1) - f(0)) = xf(1)$, so that

$$1 - \omega_1^{\frac{k-h+1}{j+K-h+1}} \geq \frac{k-h+1}{j+K-h+1}(1-\omega_1).$$

Then estimate (7.28) implies that

$$\sigma_k \leq \omega_1^{-1}\omega_1^{\zeta(j-k+K)}(1-\omega_1)^{-k} \prod_{h=1}^{k} \frac{j+K-h+1}{k-h+1} \tag{7.29}$$

$$= \omega_1^{-1}\omega_1^{\zeta(j-k+K)}(1-\omega_1)^{-k} \frac{(j+K)!}{k!(j+K-k)!} \leq 2^{j+K}\omega_1^{-1}\omega_1^{\zeta(j-k+K)}(1-\omega_1)^{-k}.$$

Substituting estimate (7.29) into (7.27), we conclude that for $h \leq j\xi$,

$$A_h^{(1)}(s) \leq \sum_{i=0}^{h-1} \lambda_0^i 2^{j+K} \omega_1^{-1} \omega_1^{\zeta(j-i+K)} (1-\omega_1)^{-i} J_{j-i}(1) \tag{7.30}$$

$$= 2^K \sum_{i=0}^{h-1} (\frac{2\lambda_0}{1-\omega_1})^i 2^{j-i} \omega_1^{\zeta(j-i+K)-1} J_{j-i}(1) = 2^K \omega_1^{\zeta K-1} \sum_{i=0}^{h-1} (\frac{2\lambda_0}{1-\omega_1})^i 2^{j-i} \omega_1^{\zeta(j-i)} J_{j-i}(1).$$

It remains to estimate $J_{j-i}(1)$ from above. By (4.25), conditions (2.14) and properties (7.4)-(7.7) we have

$$J_{j-i}(1) = I_{j-i}(1)\alpha_{j-i} \le CF(t_{j-i})\alpha_{j-i} \le C\exp\{w(T-t_{j-i})^{-\frac{1}{m(p+1)-1}}\} \tag{7.31}$$

$$\times \exp\{-w^{\frac{m(p+1)-1}{m(p+1)}} \Delta_{j-i}^{-\frac{1}{m(p+1)}}\} \le C\exp\{(1-\frac{K}{(K+1)(\gamma+1)})w(T-t_{j-i})^{-\frac{1}{m(p+1)-1}}\}$$

$$\le C\exp\{a_1(j-i+K)\},\ a_1 = \frac{K+1}{K}(\gamma+1)^{m(p+1)}[1-\frac{K}{(K+1)(\gamma+1)}](m(p+1)-1)^{-1},$$

where C is given in (4.25). Therefore, (7.30) yields

$$A_h^{(1)} \le C2^K\omega_1^{\zeta K-1}\exp\{a_1K\}\sum_{i=0}^{h-1}(\frac{2\lambda_0}{1-\omega_1})^i \exp\{-(j-i)(\zeta\ \ln\omega_1^{-1} - \ln 2 - a_1)\}. \tag{7.32}$$

Now we fix the value of the free parameter ζ from the equality

$$\zeta\ln\omega_1^{-1} - \ln 2 - a_1 = 1 \implies \zeta = \frac{1+a_1+\ln 2}{\ln\omega_1^{-1}}. \tag{7.33}$$

Then it follows from (7.32) that

$$A_h^{(1)}(s) \le C2^K\omega_1^{\zeta K-1}\exp\{a_1K\}\exp\{-(j-h+1)\}\sum_{i=0}^{h-1}\left(\frac{2\lambda_0}{1-\omega_1}\right)^i.$$

By our construction, $\lambda_0 = \lambda_0(\gamma,\varepsilon_0) \to 0$ as $\gamma \to \infty$. Let us fix $\gamma = \gamma_0$ such that $\lambda_0 = \lambda_0(\gamma_0,\varepsilon_0)$ satisfies the condition

$$\frac{2\lambda_0}{1-\omega_1} = \lambda_1 < 1. \tag{7.34}$$

In this case we obtain the estimate

$$A_h^{(1)}(s) \le C2^K\omega_1^{\zeta K-1}\exp\{a_1K\}\exp\{-(j-h+1)\}(1-\lambda_1)^{-1},$$

and, as a consequence, for any $j \in \mathbb{N}$,

$$A_{\xi j}^{(1)} \le CC_1, \quad C_1 = 2^K\omega_1^{\zeta K-1}\exp\{a_1K\}(1-\lambda_1)^{-1}. \tag{7.35}$$

Let us estimate $A_h^{(2)}$ for $h \le j\xi$. By (7.21) we have

$$A_h^{(2)}(s) \le \sum_{k_h=1}^{j-h}\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_h=0}^{l_h}\lambda_0^{j-k_h}\omega_1^{i_1+i_2+...+i_h}J_{k_h}(1) =$$

$$\sum_{k_h=1}^{j-h}\lambda_0^{j-k_h}J_{k_h}(1)\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_h=0}^{l_h}\omega_1^{i_1+i_2+...+i_h} \le \frac{\omega_1}{(1-\omega_1)^h}\sum_{k=1}^{j-h}\lambda_0^{j-k}J_k(1). \tag{7.36}$$

Therefore, it follows from (7.31) that

$$(7.37)\qquad A^{(2)}_{j\xi}(s) \le \frac{\omega_1}{(1-\omega_1)^{\xi j}} \sum_{k=1}^{j(1-\xi)} \lambda_0^{j-k} J_k(1) \le \frac{C\omega_1 \exp\{a_1 K\}}{(1-\omega_1)^{j\xi}} \sum_{k=1}^{j(1-\xi)} \lambda_0^{j-k} \exp\{a_1 k\}$$

$$= \frac{C\omega_1 \exp\{a_1 K\}\lambda_0^{j\xi}}{(1-\omega_1)^{j\xi}\exp\{-a_1(1-\xi)j\}} \sum_{k=1}^{j(1-\xi)} \left(\frac{\lambda_0}{e^{a_1}}\right)^{j(1-\xi)-k}$$

$$\le \frac{C\omega_1 \exp\{a_1 K\}\lambda_0^{j\xi}\exp\{a_1(1-\xi)j\}e^{a_1}}{(1-\omega_1)^{j\xi}(e^{a_1}-\lambda_0)}$$

$$= \frac{C\omega_1 \exp\{a_1(K+1)\}}{e^{a_1}-\lambda_0} \exp\left\{-j\left[\xi \ln\left(\frac{1-\omega_1}{\lambda_0}\right) - a_1(1-\xi)\right]\right\}.$$

We now fix the free parameter ξ. Namely, let ξ satisfy the inequality

$$(7.38)\qquad 1 > \xi > \frac{a_1}{\ln(\frac{1-\omega_1}{\lambda_0}) + a_1} > 0 \Longrightarrow \xi \ln\left(\frac{1-\omega_1}{\lambda_0}\right) - a_1(1-\xi) > 0.$$

Then it follows from (7.37) that

$$(7.39)\qquad A^{(2)}_{j\xi}(s) \le CC_2 \equiv \frac{C\omega_1 \exp\{a_1(K+1)\}}{(e^{a_1}-\lambda_0)}.$$

Finally, we estimate $A^{(3)}_h$ with $h = \xi j$,

$$A^{(3)}_h \le c\lambda_0^j H_0(1)[\Delta_j \sum_{i_1=1}^{l_1} \omega_1^{i_1-1} + \Delta_{j-1} \sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2} \omega_1^{i_1+i_2-1} + ...$$

$$+ \Delta_{j-h+1} \sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2}...\sum_{i_h=1}^{l_h} \omega_1^{i_1+i_2+...+i_h-1}] \le c\lambda_0^j H_0(1)\omega_1^{-1}[\Delta_j \frac{\omega_1}{1-\omega_1} + \Delta_{j-1}(\frac{\omega_1}{1-\omega_1})^2 + ...$$

$$(7.40)\qquad + \Delta_{j-h+1}(\frac{\omega_1}{1-\omega_1})^h] \le c\lambda_0^j H_0(1)\omega_1^{-1}\Delta_{j-h+1} \sum_{k=1}^{h} \left(\frac{\omega_1}{1-\omega_1}\right)^k.$$

Since

$$\frac{\omega_1}{1-\omega_1} = \frac{\frac{\omega_0}{1+\omega_0}}{1+\frac{\omega_0}{1+\omega_0}} = \omega_0 < 1,$$

we have

$$(7.41)\qquad A^{(3)}_h \le c\lambda_0^j H_0(1)\omega_1^{-1}\Delta_{j-h+1}\frac{\omega_0}{1-\omega_0} \le CC_3 < \infty.$$

Combining estimates (7.35), (7.39) and (7.41), due to (7.16) we deduce that for any $j \ge 1$ and $s \ge s_1$, $J_j(s) \le CC_4$, $C_4 = C_1 + C_2 + C_3$, where s_1 is prescribed in (7.25). In particular,

$$(7.42)\qquad J_j(s_1) \le CC_4 < \infty.$$

Due to definition (7.1) and properties (7.4)-(7.6), it follows from (7.42) that for $j \in \mathbb{N}$,

$$
\begin{aligned}
I_j(s_1) &\le CC_4 \alpha_j^{-1} = CC_4 \exp\{w^{\frac{m(p+1)-1}{m(p+1)}} \Delta_j^{-\frac{1}{m(p+1)}}\} \\
&= CC_4 \exp\left\{ \frac{(\gamma+1)^{m(p+1)-1}}{m(p+1)-1}(j+K) \right\}. \qquad (7.43)
\end{aligned}
$$

Therefore,

$$
I^{(t_j)}(s_1) = \int_0^{t_j} \int_{\Omega(s_1)} |u(x,t)|^{p+1} dx dt \le CC_4 \sum_{i=0}^{j} \exp\left\{ \frac{(\gamma+1)^{m(p+1)-1}}{m(p+1)-1}(i+K) \right\}
$$

$$
\le CC_5 \exp\left\{ \frac{(\gamma+1)^{m(p+1)-1}}{m(p+1)-1}(j+K) \right\}, \quad C_5 = C_4 \left[1 - \exp\left\{ \frac{-(\gamma+1)^{m(p+1)-1}}{m(p+1)-1} \right\} \right]^{-1}.
$$

Hence, for any $t \in [t_{j-1}, t_j)$,

$$
I^{(t)}(s_1) \le CC_5 \exp\left\{ \frac{(\gamma+1)^{m(p+1)-1}}{m(p+1)-1} \right\} \exp\left\{ \frac{(\gamma+1)^{m(p+1)-1}}{m(p+1)-1}(j+K-1) \right\}. \qquad (7.44)
$$

Using (7.44), it follows from (7.5) and (7.4) that for any $t < T$,

$$
I^{(t)}(s_1) \le CC_6 \exp\left\{ w_1 (T-t)^{-\frac{1}{m(p+1)-1}} \right\}, \; w_1 = w/(1+\gamma), \qquad (7.45)
$$

where $C_6 = C_5 \exp\{\frac{(\gamma+1)^{m(p+1)-1}}{m(p+1)-1}\}$. Let us note that the constant $C_6 < \infty$ depends on known parameters only and does not depend on the boundary function $F(t)$. Thus, estimate (7.45) is the final point of the first cycle of computations. This cycle was based on the starting *a priori* estimate

$$
I^{(t)}(s_0) \equiv I^{(t)}(1) \le C \exp\{w(T-t)^{-\frac{1}{m(p+1)-1}}\}. \qquad (7.46)
$$

As the second global cycle, we will use (7.45) as the starting estimate instead of (7.46). In this second cycle of computations we use the same restriction on new parameters as in the first one with w replaced by w_1. As a result, we derive the estimate

$$
\begin{aligned}
I^{(t)}(s) &\le CC_6 \cdot C_6 \exp\left\{ \frac{w_1}{1+\gamma}(T-t)^{-\frac{1}{m(p+1)-1}} \right\} \\
&= CC_6^2 \exp\left\{ \frac{w}{(1+\gamma)^2}(T-t)^{-\frac{1}{m(p+1)-1}} \right\}, \quad s > s_2 = s_1 + G_2[1 + \ln(1+\xi)^{-1} + \zeta], \qquad (7.47)
\end{aligned}
$$

where, in view of (7.23) and (7.46),

$$
G_2 \le c_2 (1+\gamma)^{-m(p+1)+1} w_1^{\frac{m(p+1)-1}{m(p+1)}} \omega_0^{-\frac{1}{m(p+1)}} = G_1 (1+\gamma)^{-\frac{m(p+1)-1}{m(p+1)}}.
$$

Then we use (7.47) as the starting estimate for the third cycle and so on. After j cycles we obtain

$$
I^{(t)}(s) \le CC_6^j \exp\left\{ \frac{w}{(1+\gamma)^j}(T-t)^{-\frac{1}{m(p+1)-1}} \right\} \qquad (7.48)
$$

$$\text{for } s > s_j = s_0 + [1 + \ln(1+\xi)^{-1} + \zeta] \sum_{i=1}^{j} G_i = s_0 + G_1 \sum_{i=1}^{j} \frac{[1 + \ln(1+\xi)^{-1} + \zeta]}{(1+\gamma)^{\frac{(m(p+1)-1)i}{m(p+1)}}},$$

where $s_0 = 1$. One can see that

$$s_j < s_\infty = s_0 + G_1 \frac{[1 + \ln(1+\xi)^{-1} + \zeta]}{(1+\gamma)^{\frac{(m(p+1)-1)}{m(p+1)}} - 1} < \infty, \quad j \in \mathbb{N}, \tag{7.49}$$

and $s_j \to s_\infty$ as $j \to \infty$. Then $I^{(t)}(s)$ is a continuous and monotone nonincreasing function with respect to $s > s_0 = 1$. Therefore, it follows from (7.48), (7.49) that for arbitrary $j \in \mathbb{N}$,

$$I^{(t)}(s_\infty) \le C C_6^j \exp\left\{ \frac{w}{(1+\gamma)^j} (T-t)^{-\frac{1}{m(p+1)-1}} \right\} \equiv D^{(j)}(t), \;\; t < T. \tag{7.50}$$

Let us fix arbitrary $t < T$ and find a number j so large that

$$(1+\gamma)^j = w(T-t)^{-\frac{1}{m(p+1)-1}} \Longrightarrow j = \frac{\ln[w(T-t)^{-\frac{1}{m(p+1)-1}}]}{\ln(1+\gamma)}.$$

Then

$$\begin{aligned} C_6^j \equiv \exp\{j \ln C_6\} &= \exp\left\{ \frac{\ln[w(T-t)^{-\frac{1}{m(p+1)-1}}] \ln C_6}{\ln(1+\gamma)} \right\} \\ &= w^{\frac{\ln C_6}{\ln(1+\gamma)}} (T-t)^{-\frac{\ln C_6}{(m(p+1)-1)\ln(1+\gamma)}}. \end{aligned} \tag{7.51}$$

Due to (7.51) it follows from (7.50) that

$$I^{(t)}(s_\infty) \le C\, e\, w^{\frac{\ln C_6}{\ln(1+\gamma)}} (T-t)^{-\frac{\ln C_6}{(m(p+1)-1)\ln(1+\gamma)}}, \quad t < T. \tag{7.52}$$

We use estimate (7.52) as the new starting estimate on the energy function. One can see that such a starting power (non-exponential) "boundary regime" satisfies the condition of Theorem 1.1. Hence,

$$I^{(t)}(s_\infty + \varepsilon) < C_7 = C_7(\varepsilon) < \infty, \quad t < T,$$

for arbitrary $\varepsilon > 0$. This completes the proof of Theorem 1.2.

Acknowledgement. The second author thanks Department of Mathematical Sciences of the University of Bath for its hospitality during his visits. This work has been partially supported by the INTAS project CEC-INTAS-RFBR96-1060.

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