Non-Autonomous Systems: Asymptotic Behaviour and Weak Invariance Principles*

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Results pertaining to asymptotic behaviour of solutions of non-autonomous ordinary differential equations with locally integrably bounded right-hand sides are presented. Ramifications for weakly asymptotically autonomous systems and adaptively controlled systems are highlighted.

Key Words: adaptive control, asymptotic behaviour, differential inclusions, invariance, non-autonomous differential equations, nonlinear systems, stability.

1. INTRODUCTION

The initial-value problem for a non-autonomous ordinary differential equations of the form $\dot{x} = f(t,x)$, $x(0) = \xi$ is considered, where f is of Carathéodory class (and so the initial-value problem has a solution and every solution can be maximally extended). Motivated by [18], one of the basic questions addressed in the paper is the following: if x is a global (forward-time) solution of the initial-value problem and $g \circ x \in L^1$ for some function g, then what can one deduce about the asymptotic behaviour of x? In the autonomous case $\dot{x} = f(x)$ with locally Lipschitz right-hand side and denoting, by φ, the generated flow, the following observation is a consequence of results in [9]: let $\alpha : [0,\infty) \to [0,\infty)$ be a continuous function with $\alpha(s) = 0 \Leftrightarrow s = 0$ and $\liminf_{s\to\infty} \alpha(s) > 0$; if $\alpha(\|\varphi(\cdot,\xi)\|) \in L^1$, then $x(t) = \varphi(t,\xi) \to 0$ as $t \to \infty$ (and so, in particular, for $1 \le p < \infty$, L^p trajectories of flows converge to zero as $t \to \infty$). By contrast, if x is a global solution of the non-autonomous initial-value problem with Carathéodory

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right-hand side, then $\alpha(\|x(\cdot)\|) \in L^1$ (for some α with the above properties) is not a sufficient condition for convergence to zero of x: a simple counterexample is provided by the scalar problem $\dot{x}(t) = m(t)$, $x(0) = 0$, where m is a locally integrable function: clearly, there are many choices of m such that its primitive $\int_0^{\cdot} m =: M \ (= x)$ is L^1, but $M(t) = x(t) \not\to 0$ as $t \to \infty$; however, note that, if m has primitive $M \in L^1$ which is also uniformly continuous, then (by an observation [5, Lemma p.269], frequently referred to as Barbălat's lemma) $M(t) = x(t) \to 0$ as $t \to \infty$. This simple example serves to illustrate that the class of locally integrable functions with uniformly continuous primitives may play a rôle in identifying a subclass of Carathéodory right-hand sides for which one can infer, from convergence of certain integrals, asymptotic behaviour of solutions of the non-autonomous initial-value problem. In turn, the issue of convergence of integrals is intimately connected with the concept of meagreness of functions (introduced in [9]), a concept which underpins the approach of the paper.

In the context of autonomous systems (and with the ubiquitous LaSalle principle as exemplar), the invariance property of ω-limit sets of bounded solutions has been widely exploited in analyses of asymptotic behaviour (see, for example, [1, 6, 7, 9, 12, 13, 14, 17] and references therein). A second focus of the current paper is an investigation of extensions of such results to systems that are weakly asymptotically autonomous (with associated "limit" systems in the form of autonomous differential inclusions). This notion of weak asymptotic autonomy will be made precise in due course: loosely speaking, the concept emcompasses cases of non-autonomous right-hand sides f with the property that, for some suitably regular set-valued map F and for all x, $f(t,x)$ approaches the set $F(x)$ as $t \to \infty$. Some control theoretic consequences for systems with inputs and outputs are highlighted.

2. NOTATION AND TERMINOLOGY

Throughout, $\mathbb{R}_+ := [0,\infty)$, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^N$ and μ denotes Lebesgue measure on $\mathbb{R}_+$. Let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be a Lebesgue measurable function: if $1 \le p < \infty$ and the function $t \mapsto \|x(t)\|^p$ is Lebesgue integrable (respectively, locally Lebesgue integrable), then we write $x \in L^p$ (respectively, $x \in L^p_{\mathrm{loc}}$); if the function $t \mapsto \|x(t)\|$ is essentially bounded (respectively, locally essentially bounded), then we write $x \in L^\infty$ (respectively, $x \in L^\infty_{\mathrm{loc}}$).

For a function $g : X \subset \mathbb{R}^N \to \mathbb{R}^P$ and $U \subset \mathbb{R}^P$, $g^{-1}(U)$ denotes the preimage of U under g, *i.e.* $g^{-1}(U) := \{z \in X \mid g(z) \in U\}$; g is termed a Borel function if, for every open set $U \subset \mathbb{R}^P$, $g^{-1}(U)$ is a Borel set; we record that, if g is a Borel function and $x : \mathbb{R}_+ \to X$ is Lebesgue measurable, then

$g \circ x$ is Lebesgue measurable. For notational simplicity, we write $g^{-1}(0)$ in place of the more cumbersome $g^{-1}(\{0\})$. $\mathcal{J}$ denotes the class of Borel functions $\alpha : \mathbb{R}_+ \to \mathbb{R}$ such that $\alpha^{-1}(0) = \{0\}$ and $\inf_{s\in[\lambda,\infty)} \alpha(s) > 0$ for all $\lambda > 0$. $\mathcal{K} \subset \mathcal{J}$ denotes the subclass of those $\mathcal{J}$-functions that are continuous and strictly increasing. $\mathcal{KL}$ denotes the class of functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $t \in \mathbb{R}_+$, $\beta(\cdot, t)$ is a $\mathcal{K}$-function and, for each $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing with $\beta(s, t) \downarrow 0$ as $t \to \infty$.

For non-empty $C \subset \mathbb{R}^N$, $d_C : \mathbb{R}^N \to \mathbb{R}_+$ denotes its Euclidean distance function given by $d_C(v) := \inf\{\|v - c\| \mid c \in C\}$. The function d_C is globally Lipschitz of rank 1: $|d_C(v) - d_C(w)| \le \|v - w\|$ for all $v, w \in \mathbb{R}^N$ (see, for example, [8], Proposition 2.4.1). A function $x : \mathbb{R}_+ \to \mathbb{R}^N$ is said to approach $C \subset \mathbb{R}^N$ if $d_C(x(t)) \to 0$ as $t \to \infty$. For non-empty $C \subset \mathbb{R}^N$ and $\varepsilon > 0$, $\mathbb{B}_\varepsilon(C) := \{z \mid d_C(z) < \varepsilon\}$ (the ε-neighbourhood of C): for $c \in \mathbb{R}^N$, we write $\mathbb{B}_\varepsilon(c)$ in place of the more cumbersome $\mathbb{B}_\varepsilon(\{c\})$. The closure of a set $C \subset \mathbb{R}^N$ is denoted by $\overline{C}$.

2.1. Meagre functions

The following concept was introduced in [9].

Definition 2.1. A Lebesgue measurable function $x : \mathbb{R}_+ \to \mathbb{R}^N$ is said to be *meagre* if $\mu(\{t \in \mathbb{R}_+ \mid \|x(t)\| \ge \lambda\}) < \infty$ for all $\lambda > 0$.

The first proposition provides two characterizations of meagreness.

Proposition 2.1. *Let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be Lebesgue measurable. The following statements are equivalent.*

(i) x is meagre.
(ii) There exists $\alpha \in \mathcal{J}$ such that $\alpha(\|x(\cdot)\|) \in L^1$.
(iii) There exists $\alpha \in \mathcal{K}$ such that $\alpha(\|x(\cdot)\|) \in L^1$.

Proof. (i) $\Rightarrow$ (iii): Assume that (i) holds. Let $(\lambda_n)_{n\in\mathbb{Z}} \subset (0, \infty)$ be a bi-sequence with $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}$ and such that

$$\lim_{n\to-\infty} \lambda_n = 0\,, \quad \lim_{n\to\infty} \lambda_n = \infty\,.$$

Define the bi-sequence $(\Lambda_n) \subset [0, \infty)$ by

$$\Lambda_n := \mu(\{t \in \mathbb{R}_+ \mid \|x(t)\| \ge \lambda_n\}) \quad (\ < \infty \ \text{ by meagreness}).$$

Let $(\eta_n)_{n\in\mathbb{Z}} \subset (0, \infty)$ be any bi-sequence such that $\sum_{n\in\mathbb{Z}} \Lambda_n \eta_n < \infty$: for example, the bi-sequence given by

$$\eta_n := \begin{cases} \min\{1, \Lambda_n^{-1}\}/n^2, & \text{if } n\Lambda_n \ne 0 \\ 1, & \text{otherwise} \end{cases}$$

suffices. Define $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha(0) = 0\,, \qquad \alpha(\lambda_{n+1}) = \sum_{m=-\infty}^{n} \eta_m \quad \forall\, n \in \mathbb{Z}\,,$$

$$\alpha(\lambda) = \frac{1}{\lambda_{n+1} - \lambda_n}[(\alpha(\lambda_{n+1}) - \alpha(\lambda_n))\lambda + \lambda_{n+1}\alpha(\lambda_n) - \lambda_n\alpha(\lambda_{n+1})] \qquad \forall\, \lambda \in [\lambda_n, \lambda_{n+1}]$$

and so α interpolates linearly on $[\lambda_n, \lambda_{n+1}]$. Evidently, α is a $\mathcal{K}$-function and so, *a fortiori*, is a $\mathcal{J}$-function.
For each $m \in \mathbb{N}$, define $T_m := \{t \mid \|x(t)\| \in [\lambda_m, \lambda_{m+1})\}$. Then,

$$\begin{aligned}\infty > \sum_{n\in\mathbb{Z}} \Lambda_n \eta_n &= \sum_{n\in\mathbb{Z}} \eta_n \mu(\{t \mid \|x(t)\| \geq \lambda_n\}) \\ &= \sum_{n\in\mathbb{Z}} \eta_n \sum_{m=n}^{\infty} \mu(T_m) = \sum_{m\in\mathbb{Z}} \mu(T_m) \sum_{n=-\infty}^{m} \eta_n = \sum_{m\in\mathbb{Z}} \mu(T_m)\alpha(\lambda_{m+1}) \\ &\geq \sum_{m\in\mathbb{Z}} \int_{T_m} \alpha(\|x(t)\|)dt = \int_0^\infty \alpha(\|x(t)\|)dt\,.\end{aligned}$$

(iii) $\Rightarrow$ (ii): Since $\mathcal{K} \subset \mathcal{J}$, the claim is immediate.

(ii) $\Rightarrow$ (i): Assume (ii) holds. Let $\lambda > 0$ be arbitrary. Since $\alpha \in \mathcal{J}$, $\varepsilon := \inf\{\alpha(s) \mid s \geq \lambda\} > 0$. Therefore,

$$\begin{aligned}\mu(\{t \in \mathbb{R}_+ \mid \|x(t)\| \geq \lambda\}) &\leq \mu(\{t \in \mathbb{R}_+ \mid \alpha(\|x(t)\|) \geq \varepsilon\}) \\ &\leq \frac{1}{\varepsilon}\int_0^\infty \alpha(\|x(t)\|)dt < \infty\end{aligned}$$

and so x is meagre. ∎

Remark 2.1. If $x \in L^p$ and $1 \leq p < \infty$, then x is meagre. The converse statement evidently fails to hold. Consider, for example, the unbounded function $x : \mathbb{R}_+ \to \mathbb{R}$ defined by the properties: (i) for all $n \in \mathbb{N}$, $x(t) = n$ for all $t \in I_n := [n, n+(1/n^2))$; (ii) $x(t) = 0$ for all $t \notin \cup_{n\in\mathbb{N}} I_n$. It is readily seen that x is meagre, but x is not of class L^p for all $p \in [1, \infty)$.

2.2. Limit sets

A point $l \in \mathbb{R}^N$ is an ω-limit point of $x : \mathbb{R}_+ \to \mathbb{R}^N$ if there exists a sequence (t_n) with $t_n \to \infty$ and $x(t_n) \to l$ as $n \to \infty$; the set $\Omega(x)$ of all such ω-limit points is the ω-limit set of x. The ω-limit set $\Omega(x)$ is always closed; if x is bounded, then $\Omega(x)$ is non-empty, compact, is approached by x and is the smallest closed set so approached (moreover, if x is continuous, then $\Omega(x)$ is a connected set).

An easy consequence of the above is the following observation.

Remark 2.2. If $x : \mathbb{R}_+ \to \mathbb{R}^N$ is continuous and has non-empty bounded ω-limit set $\Omega(x)$, then x is bounded and approaches $\Omega(x)$.

3. NON-AUTONOMOUS SYSTEMS WITH UNIFORMLY LOCALLY INTEGRABLY BOUNDED RIGHT-HAND SIDES

Consider the initial-value problem for a non-autonomous ordinary differential equation

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = \xi\,. \tag{1}$$

Throughout, $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. Precisely: for all $z \in \mathbb{R}^N$, $f(\cdot, z)$ is measurable; for almost all $t \in \mathbb{R}_+$, $f(t, \cdot)$ is continuous; f is locally integrably bounded on compact sets, specifically, for each compact set $C \subset \mathbb{R}^N$, there exists $m \in L^1_{\mathrm{loc}}$ such that $\|f(t,z)\| \le m(t)$ for almost all $t \in \mathbb{R}_+$ and all $z \in C$. For each $\xi \in \mathbb{R}^N$, (1) has a solution (a locally absolutely continuous function $x : [0, \omega) \to X$ with $x(0) = \xi$ and satisfying the differential equation in (1) for almost all $t \in [0, \omega)$) and every solution can be maximally extended to a maximal interval $[0, \omega_x)$; moreover, if x is maximal and $\omega_x < \infty$, then x is unbounded. A maximal solution x is called global if $\omega_x = \infty$. As before, we denote, by $\Omega(x)$, the ω-limit set (possibly empty) of a global solution x and record that, if x is a bounded maximal solution, then $\omega_x = \infty$ and $\Omega(x)$ is a non-empty, compact, connected set and is approached by x.

A function $m : \mathbb{R}_+ \to \mathbb{R}$ is said to be *uniformly locally integrable* if m is locally integrable and, for all $\varepsilon > 0$, there exists $\tau > 0$ such that

$$\int_t^{t+\tau} |m(s)| ds < \varepsilon \quad \forall\, t \in \mathbb{R}_+\,.$$

Clearly, a locally integrable function $m : \mathbb{R}_+ \to \mathbb{R}$ is uniformly locally integrable if, and only if, the function $t \mapsto \int_0^t |m(s)|ds$ is uniformly continuous. It is a routine exercise to show that, if $m \in L^p$ for some p ($1 \le p \le \infty$), then m is uniformly locally integrable.

Definition 3.1. For non-empty $C \subset \mathbb{R}^N$, let $\mathcal{F}(C)$ be the class of Carathéodory functions $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ with the property that there exists a uniformly locally integrable function m such that $\|f(t,z)\| \le m(t)$ for almost all $t \in \mathbb{R}_+$ and all $z \in C$.

We state and prove a convenient artifact used in the proof of Theorem 3.1 below.

PROPOSITION 3.1. *Let $A, B \subset \mathbb{R}^N$ be non-empty sets with $\mathbb{B}_\delta(B) \subset A$ for some $\delta > 0$. Let $f \in \mathcal{F}(A)$. Then there exists $\tau > 0$ such that, for every global solution x of (1) and for all $t \in \mathbb{R}_+$,*

$$x(t) \in B \quad \Longrightarrow \quad x(s) \in \mathbb{B}_\delta(B) \quad \forall\ s \in [t, t+\tau]\,.$$

Proof. Since $f \in \mathcal{F}(A)$, there exists a uniformly locally integrable function m such that $\|f(t,z)\| \le m(t)$ for almost all $t \in \mathbb{R}_+$ and all $z \in A$. Choose $\tau > 0$ such that $\int_t^{t+\tau} m(s)ds < \delta/2$ for all $t \in \mathbb{R}_+$. Assume that x is a global solution of (1). Let $t \in \mathbb{R}_+$ and assume that $x(t) \in B$. Seeking a contradiction, suppose that $x(s) \notin \mathbb{B}_\delta(B)$ for some $s \in [t, t+\tau]$. Define

$$s^* := \inf\{s \in [t, t+\tau] \,|\, x(s) \notin \mathbb{B}_\delta(B)\} > 0\,.$$

Then $x(s) \in \mathbb{B}_\delta(B) \subset A$ for all $s \in [t, s^*]$ and $d_B(x(s^*)) = \delta$, whence the contradiction

$$\begin{aligned} 0 < \delta = d_B(x(s^*)) &= d_B(x(s^*)) - d_B(x(t)) \\ &\le \|x(s^*) - x(t)\| \le \int_t^{s^*} \|f(x(s), s)\| ds \le \int_t^{t+\tau} m(s) ds \le \delta/2\,. \end{aligned}$$

∎

THEOREM 3.1. *Let $S \subset \mathbb{R}^N$ be non-empty and closed. Let $g : \mathbb{R}^N \to \mathbb{R}^P$ be a Borel function with the properties: (i) $C := S \cap g^{-1}(0)$ is closed; (ii) $\inf_{z \in K} \|g(z)\| > 0$ for every closed set $K \subset S$ with $K \cap C = \emptyset$. Assume that x is a global solution of (1) with trajectory in S and $g \circ x$ is meagre. Then $C \neq \emptyset$. If $f \in \mathcal{F}(\mathbb{B}_r(C))$ for some $r > 0$, then x approaches C and $\Omega(x) \subset C$. If, in addition, C is compact, then x is bounded and $\Omega(x) \neq \emptyset$.*

Proof. We first prove that $C \neq \emptyset$. Seeking a contradiction, suppose that $C = \emptyset$. Then, by closedness of S, it follows from property (ii) of g that $\inf_{z \in S} \|g(z)\| > 0$ and hence $t \mapsto \|g(x(t))\|$ is bounded away from 0, contradicting meagreness of $g \circ x$.
To prove the remaining assertions, it is sufficient to show that x approaches C (since, by the closedness of C it then follows immediately that $\Omega(x) \subset C$;

moreover, if C is compact, then boundedness of x follows, which in turn implies non-emptiness of $\Omega(x)$). Since, by assumption, the trajectory of x is contained in S, it is immediate that x approaches C if $C = S$. Consider the remaining case wherein $C \subsetneq S$. By closedness of C, there exists $\eta \in (0, r/3)$ such that $S \setminus \mathbb{B}_\eta(C) \neq \emptyset$. For $\theta \in (0, \eta)$, define

$$\iota(\theta) := \inf\{\|g(z)\| \mid z \in S \setminus \mathbb{B}_\theta(C)\} > 0\,,$$

wherein positivity is a consequence of the closedness of S and of property (ii) of g. Seeking a contradiction, suppose that $\lim_{t\to\infty} d_C(x(t)) \neq 0$. Then there exist $\delta \in (0, \eta)$ and a sequence (t_n) with $t_n \to \infty$ as $n \to \infty$ and $d_C(x(t_n)) \geq 3\delta$ for all $n \in \mathbb{N}$. By meagreness of $g \circ x$, there exists a sequence (s_n) with $s_n \to \infty$ as $n \to \infty$ and $\|g(x(s_n))\| < \iota(\delta)$ for all $n \in \mathbb{N}$ and so $d_C(x(s_n)) \leq \delta$ for all $n \in \mathbb{N}$. Extracting subsequences of (t_n) and (s_n) (which we do not relabel), we may assume that $s_n \in (t_n, t_{n+1})$ for all $n \in \mathbb{N}$. We now have

$$d_C(x(t_n)) \geq 3\delta, \quad d_C(x(s_n)) \leq \delta, \quad s_n \in (t_n, t_{n+1}) \quad \forall\, n \in \mathbb{N}$$

and so, by continuity, for each $n \in \mathbb{N}$ there exists $\sigma_n \in (t_n, s_n)$ such that $x(\sigma_n) \in B := \{y \in S \mid d_C(y) = 2\delta\}$. Extracting a subsequence (which, again, we do not relabel), we may assume that $\sigma_{n+1} - \sigma_n \geq 1$ for all $n \in \mathbb{N}$. Noting that $\mathbb{B}_\delta(B) \subset \mathbb{B}_r(C)$ and invoking Proposition 3.1 (with $A = \mathbb{B}_r(C)$), we may conclude the existence of $\tau \in (0, 1)$ such that, for all $n \in \mathbb{N}$,

$$d_C(x(t)) \geq \delta \quad \forall\, t \in [\sigma_n, \sigma_n + \tau].$$

Therefore, $\{t \in \mathbb{R}_+ \mid \|g(x(t))\| \geq \iota(\delta)\} \supset \cup_{n\in\mathbb{N}}[\sigma_n, \sigma_n + \tau]$, which (on noting that the intervals $[\sigma_n, \sigma_n + \tau]$, $n \in \mathbb{N}$, are each of length $\tau > 0$ and form a mutually disjoint family) contradicts meagreness of $g \circ x$. Therefore, x approaches C. ▮

An immediate consequence of the above theorem is the following result in [18] (which posits *a priori* boundedness of a solution of (1)).

THEOREM 3.2. *Assume $f \in \mathcal{F}(C)$ for every compact set $C \subset \mathbb{R}^N$. Let $g : \mathbb{R}^N \to \mathbb{R}$ be continuous. If x is a bounded global solution of (1) and $g \circ x \in L^1$, then $g(x(t)) \to 0$ as $t \to \infty$.*

Proof. Let $S := \overline{x(\mathbb{R}_+)}$ (the closure of the trajectory of x). Then the hypotheses of Theorem 3.1 hold and the result follows. ▮

Theorem 3.2 is also a consequence of part (ii) of Theorem 18 in [9].

The next result infers asymptotic behaviour without positing *a priori* boundedness of solutions and, in particular, implies that, for (1) and under the relatively weak condition that f be of class $\mathcal{F}(C)$ for some (arbitrarily small) open neighbourhood C of $\{0\}$, global meagre solutions (and so, *a fortiori* global solutions of class L^p, $1 \leq p < \infty$) of (1) are necessarily bounded and, moreover, approach zero.

THEOREM 3.3. *Assume that, for some $\varepsilon > 0$, $f \in \mathcal{F}(\mathbb{B}_\varepsilon(0))$. If x is a global solution of (1) and x is meagre, then $x(t) \to 0$ as $t \to \infty$.*

Proof. Defining $g : \mathbb{R}^N \to \mathbb{R}$ by $g(z) := \|z\|$, the hypotheses of Theorem 3.1 hold with $S = \mathbb{R}^N$. Therefore, $\|x(t)\| \to 0$ as $t \to \infty$. ∎

4. WEAKLY ASYMPTOTICALLY AUTONOMOUS SYSTEMS

Let $\mathcal{U}$ denote the class of set-valued maps $x \mapsto F(x) \subset \mathbb{R}^N$, defined on $\mathbb{R}^N$, that are upper semicontinuous[1] at each $x \in \mathbb{R}^N$ and take non-empty convex compact values.

In this section, we study the asymptotic behaviour of non-autonomous differential equations of the form (1), where the Carathéodory function $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be *weakly asymptotically autonomous* in the sense that there exists $F \in \mathcal{U}$ such that the following property holds:

(AA) For all compact $C \subset \mathbb{R}^N$ and all $\varepsilon > 0$, there exists $T \geq 0$ such that

$$\text{ess-sup}_{t \geq T} d_{F(x)}(f(t,x)) < \varepsilon\,, \quad \forall\, x \in C\,.$$

If (AA) holds with F singleton-valued (that is, with $F : x \mapsto \{f^*(x)\}$ for some continuous function $f^* : \mathbb{R}^N \to \mathbb{R}^N$), then we say that f is *asymptotically autonomous.*

Remark 4.1. (i) Property (AA) says that, as $t \to \infty$, $f(t,x)$ essentially approaches $F(x)$ locally uniformly with respect to x.
(ii) Noting that, by properties of F, the set $\cup_{x \in C} F(x)$ is compact whenever C is compact (see, for example, [2: Proposition 1.1.3]), the following is readily deduced: if f is of Carathéodory class and (AA) holds for some $F \in \mathcal{U}$, then $f \in \mathcal{F}(C)$ for every compact C.

[1] F is upper (respectively, lower) semicontinuous at $x \in \mathbb{R}^N$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $F(y) \subset \mathbb{B}_\varepsilon(F(x))$ (respectively, $F(x) \subset \mathbb{B}_\varepsilon(F(y))$ for all $y \in \mathbb{B}_\delta(x)$; F is continuous at $x \in \mathbb{R}^N$ if it is both upper and lower semicontinuous at x.

4.1. The limit system

Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be of Carathéodory class and such that (AA) holds for some $F \in \mathcal{U}$. Intuitively, one would expect some relationship between the asymptotic behaviour of the solution of the non-autonomous initial-value problem (1) and behaviour of solutions of the differential inclusion (which we loosely refer to as the limit system)

$$\dot{x} \in F(x), \tag{2}$$

where, by a solution of (2), we mean a locally absolutely continuous function $x : [0, \omega) \to \mathbb{R}^N$ satisfying (2) almost everywhere.

Remark 4.2. Artstein [2, 3] has introduced a different concept of asymptotically autonomous systems with ordinary differential equations as limit systems. For purposes of comparison, if we restrict our approach to the case wherein F is singleton-valued, then Artstein's approach is based on a notion of convergence which is weaker than that used in (AA). However, it is difficult to compare the two approaches in a useful manner, given the greater generality of our class of limit systems (differential inclusions vis à vis differential equations).

We record some well known facts in Lemmas 4.1 and 4.2 which form a distillation of results in, for example, [4], [11] and [16].

LEMMA 4.1. *Let $F \in \mathcal{U}$. For each $\xi^* \in \mathbb{R}^N$, (2) has a solution with $x(0) = \xi^*$ and every such solution can be extended to a maximal interval $[0, \omega_x)$. If $\omega_x < \infty$, then x is unbounded.*

With respect to (2), a non-empty set $S \subset \mathbb{R}^N$ is said to be *weakly invariant* if, for each $\xi^* \in S$, (2) has at least one maximal solution $x : [0, \omega_x) \to \mathbb{R}^N$, with $x(0) = \xi^*$ and $x(t) \in S$ for all $t \in [0, \omega_x)$.

LEMMA 4.2. *Let $F \in \mathcal{U}$. If $x : [0, \omega_x) \to \mathbb{R}^N$ is a bounded maximal solution of (2), then $\omega_x = \infty$ and $\Omega(x)$ is non-empty, compact, connected, is approached by x (and is the smallest closed set so approached), and is weakly invariant with respect to (2).*

The essence of the ensuing proposition is the assertion that, if (AA) holds and x is a bounded global solution of (1), then the ω-limit set $\Omega(x)$ is weakly invariant with respect to the associated autonomous inclusion (2).

PROPOSITION 4.1. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be of Carathéodory class and such that (AA) holds for some $F \in \mathcal{U}$. Let $\xi \in \mathbb{R}^N$ and assume that x is a global solution of (1). If x is bounded, then the ω-limit set $\Omega(x)$ of x is non-empty, compact and connected, is approached by x and is weakly invariant with respect to (2).*

Proof. That $\Omega(x)$ is non-empty, compact, connected and approached by x is standard. We focus on establishing the weak invariance property. Let $l \in \Omega(x)$ and so there exists a sequence (t_n) such that $t_n \to \infty$ and $x(t_n) \to l$ as $n \to \infty$. For each $n \in \mathbb{N}$, define continuous $x_n : \mathbb{R}_+ \to \mathbb{R}^N$ and $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\begin{aligned} x_n(t) &:= x(t + t_n)\ \forall\ t \\ f_n(t) &:= d_{F(x_n(t))}(f(t + t_n, x_n(t)) + 1/n \quad \text{a.a. } t \geq 0. \end{aligned}$$

We claim that f_n is measurable. To this end note that, for each n, upper semicontinuity of the map $t \mapsto F(x_n(t))$, together with compactness of its values, ensures that, for each $z \in \mathbb{R}^N$, the map $t \mapsto d_{F(x_n(t))}(z)$ is measurable; the function $t \mapsto f(t + t_n, x_n(t))$ is measurable and so is the pointwise limit of a sequence of simple functions (i.e. measurable functions with finite images); it follows that f_n is the pointwise limit of a sequence of measurable functions and so is itself measurable.

By (AA), together with boundedness of x, it follows that, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\text{ess-sup}_{t \geq 0} f_n(t) < \varepsilon \quad \forall\ n > n_\varepsilon\,.$$

Observe that $x_n(0) \to l$ as $n \to \infty$ and

$$\dot{x}_n(t) \in \mathbb{B}_{f_n(t)}(F(x_n(t))) \quad \text{a.a. } t \geq 0.$$

For $m \in \mathbb{N}$, let I_m denote the interval $[m-1, m]$. By [8], Theorem 3.1.7, the sequence of restricted functions $(x_n|_{I_1})$ has a subsequence $(x_{\sigma_1(n)}|_{I_1})$ converging uniformly to an absolutely continuous function $x_1^* : I_1 \to \mathbb{R}^N$, satisfying (2) almost everywhere and with $x_1^*(0) = l$. By the same argument, the sequence $(x_{\sigma_1(n)}|_{[1,2]})$ contains a subsequence $(x_{\sigma_2(n)}|_{[1,2]})$ converging uniformly to an absolutely continuous function $x_2^* : I_2 \to \mathbb{R}^N$, satisfying (2) almost everywhere and with $x_2^*(1) = x_1^*(1)$. By induction, there exists a nested sequence $((x_{\sigma_m(n)}))_{m \in \mathbb{N}}$ of subsequences of (x_n) such that, for each m, $(x_{\sigma_m(n)})$ converges uniformly on $I_m = [m-1, m]$ to an absolutely continuous function $x_m^* : I_m \to \mathbb{R}^N$ that satisfies (2) almost everywhere and with $x_m^*(m-1) = x_{m-1}^*(m-1)$ $(x_1^*(0) = l)$. Define $x^* : \mathbb{R}_+ \to \mathbb{R}^N$ by the property that

$$t \in I_m \quad \Longrightarrow \quad x^*(t) = x_m^*(t).$$

Then x^* is a solution of (2) with initial data $x^*(0) = l$. Moreover, the "diagonal" sequence $(x_{\sigma_k(k)}) \subset (x_n)$ converges uniformly on each interval $[0,m]$ to x^*. That $x^*(t) \in \Omega(x)$ for all $t \in \mathbb{R}_+$ is now an immediate consequence of the observation that the sequence $(t_{\sigma_k(k)})$ is such that $t_{\sigma_k(k)} \to \infty$ and $x(t + t_{\sigma_k(k)}) = x_{\sigma_k(k)}(t) \to x^*(t)$ as $k \to \infty$. ∎

We record a corollary that specializes Proposition 4.1 to the case wherein the limit system (2) coincides with an autonomous ordinary differential equation with locally Lipschitz right-hand side (in which case, the limit system generates a flow).

COROLLARY 4.1. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is of Carathéodory class and such that (AA) holds with $F : z \mapsto \{f^*(z)\}$ for some locally Lipschitz function $f^* : \mathbb{R}^N \to \mathbb{R}^N$. Let $\xi \in \mathbb{R}^N$ and assume that x is a global solution of (1). If x is bounded, then the ω-limit set $\Omega(x)$ of x is non-empty, compact and connected, is approached by x and is invariant under the flow generated by f^*.*

We remark that Corollary 4.1 can also be obtained from Artstein's results [2, 3].

4.2. Weak invariance principles

The following theorem and corollary are now easy consequences of Theorem 3.1, in conjunction with Remark 4.1, Proposition 4.1 and Corollary 4.1.

THEOREM 4.1. *Let $S \subset \mathbb{R}^N$ be non-empty and closed. Let $g : \mathbb{R}^N \to \mathbb{R}^P$ be a Borel function with the properties: (i) $C := S \cap g^{-1}(0)$ is compact; (ii) $\inf_{z \in K} \|g(z)\| > 0$ for every closed set $K \subset S$ with $K \cap C = \emptyset$. Assume that $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is of Carathéodory class and such that (AA) holds for some $F \in \mathcal{U}$. If x is a global solution of (1) with trajectory in S and $g \circ x$ is meagre, then $C \neq \emptyset$, x is bounded and x approaches the largest subset of C that is weakly invariant with respect to (2).*

Proof. It follows from Theorem 3.1 that $C \neq \emptyset$. Moreover, in view of Remark 4.1 and compactness of C, $f \in \mathcal{F}(\mathbb{B}_r(C))$ for every $r > 0$. By Theorem 3.1, x is bounded with non-empty ω-limit set $\Omega(x) \subset C$. By Proposition 4.1, $\Omega(x)$ is weakly invariant with respect to (2) and is approached by x. ∎

COROLLARY 4.2. *Let $S \subset \mathbb{R}^N$, C and g be as in Theorem 4.1. Assume that $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is of Carathéodory class and such that (AA) holds with $F : z \mapsto \{f^*(z)\}$ for some locally Lipschitz function $f^* : \mathbb{R}^N \to \mathbb{R}^N$. If x is a global solution of (1) with trajectory in S and $g \circ x$ is meagre, then $C \neq \emptyset$, x is bounded and x approaches the largest subset of C that is invariant under the flow generated by f^*.*

Example 4.1. Consider the two-dimensional non-autonomous inhomogeneous linear system

$$\dot{x}(t) = [A(t) + B(t)]x(t) + h(t), \quad A(t) = \begin{bmatrix} f_1(t) & f_3(t) \\ -f_3(t) & f_2(t) \end{bmatrix} \tag{3}$$

where the functions f_i, B and h are of class L^1_{loc}. We assume: (i) f_1, f_2, f_3 approach compact intervals I_1, I_2, I_3 respectively, that is, for $i = 1, 2, 3$, $d_{I_i}(f_i(t)) \to 0$ as $t \to \infty$; (ii) $f_1 \in L^1$; (iii) $I_2 \subset (-\infty, 0)$; (iv) $0 \notin I_3$; (v) $h \in L^1$ with $h(t) \to 0$ as $t \to \infty$; (vi) $\|B(\cdot)\| \in L^1$ with $\|B(t)\| \to 0$ as $t \to \infty$. We will show that, for each $\xi \in \mathbb{R}^2$, the (unique) maximal solution x of (3), with initial value $x(0) = \xi$, is such that $x(t) \to 0$ as $t \to \infty$.

Clearly the function $(t, z) \mapsto [A(t)+B(t)]z+h(t)$ is Carathéodory and such that (AA) holds with $F \in \mathcal{U}$ given by

$$\begin{aligned} F(z) = F(z_1, z_2) := \{u_1 z_1 + u_3 z_2 |\ u_1 \in I_1, u_3 \in I_3\} \\ \times \{-u_3 z_1 + u_2 z_2 |\ u_3 \in I_3,\ u_2 \in I_2\}. \end{aligned}$$

Let $\xi \in \mathbb{R}^2$ be arbitrary and let x denote the unique maximal solution of (3) with $x(0) = \xi$. We first prove boundedness of x. Define $t \mapsto V(t) := \|x(t)\|^2$ and note that

$$\begin{aligned} \dot{V}(t) &= 2\langle x(t), [A(t) + B(t)]x(t) + h(t)\rangle \\ &\le 2|f_1(t)|x_1^2(t) + 2f_2(t)x_2^2(t) + 2\|B(t)\|\|x(t)\|^2 \\ &\qquad + \|x(t)\|^2\|h(t\| + \|h(t)\| \quad \text{a.a. } t \ge 0 \end{aligned} \tag{4}$$

By compactness of I_2 and properties (i) and (iii) of f_2, there exist constants $c_0, T > 0$ such that $2f_2(t) \le -c_0 < 0$ for almost all $t \ge T$. By (4),

$$\dot{V}(t) \le f_4(t)V(t) + \|h(t)\| \quad \text{a.a. } t \ge 0 \tag{5}$$

where $f_4(t) := \|h(t)\| + 2(|f_1(t)| + |f_2(t)|\chi_{[0,T)}(t) + \|B(t)\|)$ for almost all $t \ge 0$ ($\chi_{[0,T)}$ being the characteristic function of $[0, T)$ taking value 1 on $[0, T)$ and value 0 elsewhere). Therefore,

$$V(t) \le \exp\left(\int_0^t f_4\right)\left(V(0) + \int_0^t \exp\left(-\int_0^s f_4\right)\|h(s)\|ds\right) \quad \forall\, t \ge 0.$$

Since f_1, h, $\|B(\cdot)\|$ are of class L^1 and $f_2 \in L^1_{\mathrm{loc}}$, it follows that $f_4 \in L^1$. Therefore V (and hence x) is bounded. By boundedness of x, (4) and recalling that $2f_2(t) \le -c_0 < 0$ for almost all $t \ge T$, there exists a positive constant c_1 such that $\dot{V}(t) \le -c_0 x_2^2(t) + c_1[|f_1(t) + \|B(t)\| + \|h(t)\|]$ for almost all $t \ge T$, whence

$$\int_T^t c_0 x_2^2(s)ds \le V(T) - V(t) + c_1 \int_T^t (|f_1(s)| + \|B(s)\| + \|h(s)\|]ds \qquad \forall\ t \ge T.$$

By boundedness of V, and recalling that f_1, h and $\|B(\cdot)\|$ are of class L^1, we may conclude that x_2 is in L^2 and so, *a fortiori*, is meagre. On setting $S = \overline{x(\mathbb{R}_+)}$ and $g : y = (y_1, y_2) \mapsto y_2^2$, Theorem 4.1 implies that x approaches the largest subset $\mathcal{I}$ of $S \cap g^{-1}(0)$ that is weakly invariant with respect to the limit differential inclusion $\dot{z} \in F(z)$. Let $\hat{\xi} \in \mathcal{I}$ and so $\hat{\xi} = (\hat{\xi}_1, 0)$. By weak invariance, there exists a solution $z = (z_1, z_2)$ of the differential inclusion with $z(0) = \hat{\xi}$ and $z_2(t) = 0$ for all $t \ge 0$. Therefore, $0 = \dot{z}_2(t) \in \{-uz_1(t)|\ u \in I_3\}$ for all $t \ge 0$. By assumption (v), $0 \notin I_3$ and so $z_1(t) = 0$ for all $t \ge 0$. Therefore, $\hat{\xi} = 0$, $\mathcal{I} = \{0\}$ and $x(t) \to 0$ as $t \to \infty$.

Next, we present two variants of the LaSalle principle. For a locally Lipschitz function $z \mapsto V(z) \in \mathbb{R}$, let $V^\circ(z; v)$ denote its Clarke directional derivative at $z \in \mathbb{R}^N$ in direction $v \in \mathbb{R}^N$:

$$V^\circ(z; v) := \limsup_{\substack{y \to z \\ h \downarrow 0}} \frac{V(y + hv) - V(y)}{h}\,.$$

We record that: (a) $(z, v) \mapsto V^\circ(z; v)$ is upper semicontinuous; (b) for all z, $V^\circ(z; \cdot)$ is subadditive; (c) $V^\circ(z; \cdot)$ is globally Lipschitz, uniformly with respect to z in compact sets (see Proposition 2.1.1 in [8]).

THEOREM 4.2. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be of Carathéodory class and such that (AA) holds for some $F \in \mathcal{U}$. Assume that x is a bounded global solution of (1) and write $S := \overline{x(\mathbb{R}_+)}$. If there exist locally Lipschitz $V : \mathbb{B}_\varepsilon(S) \to \mathbb{R}$ (for some $\varepsilon > 0$), upper semicontinuous $g : S \to \mathbb{R}$ and $\gamma \in L^1$ such that*

$$V^\circ(z; f(t, z)) - \gamma(t) \le g(z) \le 0 \quad \text{for a.a. } t \in \mathbb{R}_+ \text{ and all } z \in S \tag{6}$$

then $C = S \cap g^{-1}(0) \ne \emptyset$ and x approaches the largest subset of C that is weakly invariant with respect to (2).

Proof. As the composition of locally Lipschitz V and locally absolutely continuous x, $V \circ x$ is locally absolutely continuous. Let $\mathcal{N} \subset \mathbb{R}_+$ be a set of measure zero with the property that $\dot{x}(t)$ and $(V \circ x)'(t)$ both exist for all $t \in \mathbb{R}_+ \backslash \mathcal{N}$. Let $L > 0$ be a Lipschitz constant for V on $\mathbb{B}_{\varepsilon/2}(S)$. Note that, if $t \in \mathbb{R}_+ \backslash \mathcal{N}$, then $V(x(t+h)) - V(x(t)) \le V(x(t) + h\dot{x}(t)) - V(x(t)) + L|x(t+h) - x(t) - h\dot{x}(t)|$ for all sufficiently small $h > 0$. Invoking (6), it follows that

$$(V \circ x)'(t) - \gamma(t) \le V^o(x(t); f(t, x(t)) - \gamma(t) \le g(x(t)) \le 0 \quad \text{a.a. } t \in \mathbb{R}_+$$

whence $\int_0^t |g(x(s))|ds \le V(x(0)) - V(x(t)) + \int_0^t |\gamma(s)|ds$ for all $t \in \mathbb{R}_+$. Therefore, by boundedness of $V \circ x$ and since $\gamma \in L^1$, we may conclude that $g \circ x \in L^1$. Hence, $g \circ x$ is meagre. In view of Theorem 4.1, to complete the proof it now suffices to show that C is closed and $\inf_{z \in K} |g(z)| > 0$ for every closed set $K \subset S$ with $K \cap C = \emptyset$. The latter follows immediately from the fact that $|g|$, being a lower semicontinuous function, achieves its infimum on compact sets. To show closedness of C, let $(z_n) \subset C$ be a convergent sequence with limit z. Clearly, $z \in S$ and, by upper semicontinuity of g at z, $0 = \limsup_{n \to \infty} g(z_n) \le g(z) \le 0$. Therefore, $g(z) = 0$ and so $z \in C$, showing that C is closed. ∎

Corollary 4.3. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be of Carathéodory class and such that (AA) holds for some $F \in \mathcal{U}$. Assume that x is a bounded global solution of (1) and write $S := \overline{x(\mathbb{R}_+)}$. If there exist $d \in L^1$ and locally Lipschitz $V : \mathbb{B}_\varepsilon(S) \to \mathbb{R}$ (for some $\varepsilon > 0$) such that*

$$\left.\begin{array}{l} \max_{z \in S} d_{F(z)}(f(t,z)) \le d(t) \quad \text{a.a. } t \in \mathbb{R}_+ \\ g(z) := \max_{v \in F(z)} V^\circ(z; v) \le 0 \quad \forall\, z \in S\,, \end{array}\right\} \tag{7}$$

then x approaches the largest subset of $S \cap g^{-1}(0)$ that is weakly invariant with respect to (2).

Proof. The following two facts are consequences of assumption (AA) and properties of $F \in \mathcal{U}$ and V° : (a) g is upper semicontinuous (see proof of Theorem 2.11 in [17]); (b) for almost all $t \in \mathbb{R}_+$ and all $z \in S$, there exists $v \in F(z)$ such that $f(t,z) = v + w(t)$, where $\|w(t)\| = d_{F(z)}(f(t,z)) \le d(t)$. Therefore, invoking (7) together with subadditivity and the global Lipschitz property of $V^\circ(z; \cdot)$ (the latter being uniform with respect to z in compact sets), we may infer that there exists $c > 0$ such that

$$V^\circ(z; f(t,z)) \le g(z) + cd(t) \quad \text{for a.a. } t \in \mathbb{R}_+ \text{ and all } z \in S\,.$$

Writing $\gamma = cd$, Theorem 4.2 now applies to conclude the result. ∎

4.3. Limit systems with asymptotically stable compacta

Here, we investigate the behaviour of weakly asymptotically autonomous systems in the case wherein the limit system has a uniformly globally asymptotically stable compactum in the following sense. A non-empty compact set $C \subset \mathbb{R}^N$ is said to be uniformly globally asymptotically stable with respect to (2) if there exists a class $\mathcal{KL}$ function β such that, for all $\xi \in \mathbb{R}^N$,

$$d_C(x(t)) \leq \beta(d_C(\xi), t) \quad \forall\, t \geq 0$$

for every maximal solution x of (2) with initial data $x(0) = \xi$ (here, the terminology "uniform" refers to uniformity with respect multiple solutions of the initial-value problem). The following theorem is a counterpart, in the present context of weakly asymptotically autonomous systems, of a result by Artstein (Theorem 8.3 in [2]).

THEOREM 4.3. *Assume that $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is of Carathéodory class and such that (AA) holds with F continuous. Assume further that $C \subset \mathbb{R}^N$ is compact and uniformly globally asymptotically stable with respect to the limit system (2). If x is a global solution of (1) with $\Omega(x) \neq \emptyset$, then x approaches C.*

Proof. Note initially that, by continuity of F together with convexity and compactness of its values, there exists a continuous map $\hat{f} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ such that $F(z) = \hat{f}(z, B)$, where $B := \overline{\mathbb{B}_1}(0)$ is the closed unit ball centred at $0 \in \mathbb{R}^N$ (see, for example, [4], Ch.1, §8, Theorem 2). By Filippov's lemma [10], x is a solution of the differential inclusion $\dot{x} \in F(x)$ if, and only if, there exists measurable $t \mapsto d(t) \in B$ such that $\dot{x}(t) = \hat{f}(x(t), d(t))$. Since C is uniformly globally asymptotically stable with respect to the limit system (2), it follows that C is uniformly globally asymptotically stable (in the sense of [15]) with respect to the non-autonomous system $\dot{x}(t) = \hat{f}(x(t), d(t))$. Therefore, by Theorem 2.8 in [15], there exist smooth $V : \mathbb{R}^N \to \mathbb{R}$, unbounded functions $\alpha_1, \alpha_2 \in \mathcal{K}$ and a continuous function $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$, with $\alpha_3^{-1}(0) = \{0\}$, such that

$$\left.\begin{aligned} \alpha_1(d_C(z)) \leq V(z) \leq \alpha_2(d_C(z)) \\ \langle \nabla V(z), v\rangle \leq -\alpha_3(d_C(z)) \end{aligned}\right\} \forall\ v \in F(z),\ \forall\ z \in \mathbb{R}^N .$$

In view of Remark 2.2, in order to prove the theorem it suffices to show that $\Omega(x) \subset C$. Let $l \in \Omega(x)$ and, seeking a contradiction, suppose that $l \notin C$. By compactness of C and properties of V, we have $V(l) > 0$. For $r > 1$, define the (compact) "annulus"

$$\mathcal{A}(r) := \{z \in \mathbb{R}^N \,|\, r^{-1}V(l) \leq V(z) \leq rV(l)\}\,.$$

Note that $\mathcal{A}(r) \cap C = \emptyset$. Since, for any $r > 1$, $l \in \Omega(x) \cap \operatorname{int} \mathcal{A}(r)$, there exists (t_n), such that

$$t_n \to \infty \quad \text{as } n \to \infty \qquad \text{and} \qquad x(t_n) \in \mathcal{A}(2) \quad \forall\, n \in \mathbb{N}. \tag{8}$$

Define $\beta := \min_{z \in \mathcal{A}(3)} \alpha_3(d_C(z)) > 0$ and $\gamma := \max_{z \in \mathcal{A}(3)} \|\nabla V(z)\| > 0$. By (AA), there exists $N \in \mathbb{N}$ such that

$$\text{ess-sup}_{t \geq 0} d_{F(z)}(f(t + t_N, z)) < \beta/(2\gamma) \quad \forall\, z \in \mathcal{A}(3).$$

Define $\mathcal{T} := \{t \in \mathbb{R}_+ \mid x(t + t_N) \in \mathcal{A}(3)\}$ and observe that, for almost all $t \in \mathcal{T}$, there exists $v \in F(x(t + t_n))$ such that

$$d_{F(x(t+t_n))}(f(t + t_N, x(t + t_N))) = \|f(t + t_N, x(t + t_N)) - v\| < \beta/(2\gamma).$$

Therefore, we may conclude that, for almost all $t \in \mathcal{T}$, there exists $v \in F(x(t + t_N)$ such that

$$\begin{aligned} (V \circ x)'(t + t_N) &= \langle \nabla V(x(t + t_N)), f(t + t_N, x(t + t_N)) \rangle \\ &\leq \langle \nabla V(x(t + t_N)), v) \rangle + \gamma\beta/(2\gamma) \end{aligned}$$

whence

$$(V \circ x)'(t + t_N) \leq -\alpha_3(d_C(x(t + t_N))) + \beta/2 \leq -\beta + \beta/2 = -\beta/2 \qquad \text{for a.a. } t \in \mathcal{T}.$$

It follows that, for some $\tau > 0$, $x(\tau + t_N) \in \mathcal{A}(3) \backslash \mathcal{A}(2)$ and so $x(t + t_N) \notin \mathcal{A}(2)$ for all $t \geq \tau$, contradicting (8). Therefore, $\Omega(x) \subset C$. ▮

4.4. Systems with input and output

Consider the system with input u of class L^p_{loc} for some $1 \leq p \leq \infty$ and output y:

$$\left. \begin{array}{l} \dot{x}(t) = k(x(t), u(t)), \quad x(0) = \xi \in \mathbb{R}^N, \quad u(t) \in \mathbb{R}^M, \\ y(t) = c(x(t)) \in \mathbb{R}^P. \end{array} \right\} \tag{9}$$

We assume that $c : \mathbb{R}^N \to \mathbb{R}^P$ is continuous with $c(0) = 0$, $k : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ is locally Lipschitz with $k(0, 0) = 0$ and

(AA1) $\forall$ compact $C \subset \mathbb{R}^N$ $\exists\, \rho > 0$ such that
$$\|k(z, v) - k(z, 0)\| \leq \rho \|v\| \quad \forall\, (z, v) \in C \times \mathbb{R}^M.$$

For each $\xi \in \mathbb{R}^N$ and $u \in L^p_{\text{loc}}$, the initial-value problem (9) has a unique maximally-defined solution which we denote by $t \mapsto x(t) = \varphi_u(t, \xi)$.

Remark 4.3. Consider system (9) with input $u \in L^p_{\mathrm{loc}}$ for some $1 \le p \le \infty$ and such that (AA1) holds. The function $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, defined by $f(t,z) := k(z,u(t))$, is of Carathéodory class: moreover, if $u(t) \to 0$ as $t \to \infty$, then f satisfies (AA) with $F : z \mapsto \{k(z,0)\}$.

Writing $k^*(\cdot) := k(\cdot,0)$, the system with zero input $u = 0$, that is, the autonomous system

$$\left.\begin{aligned} \dot{x}(t) &= k^*(x(t)), \quad x(0) = \xi \in \mathbb{R}^N \\ y(t) &= c(x(t)) \in \mathbb{R}^P \end{aligned}\right\} \tag{10}$$

will play a rôle in the sequel: we denote, by φ, the (local) flow generated by k^* and so the unique maximally-defined solution of the autonomous initial-value problem in (10) is given by $t \mapsto \varphi(t,\xi) \equiv \varphi_0(t,\xi)$. System (10) is said to be *zero-state observable* if, for each $\xi \in \mathbb{R}^N$, the following holds

$$y(\cdot) = c(\varphi(\cdot,\xi)) = 0 \quad \Longrightarrow \quad \xi = 0\,. \tag{11}$$

Zero-state observability of (10) is equivalent to the condition that $\{0\}$ is the only subset of $c^{-1}(0)$ that is invariant under the flow φ.

Theorem 4.4. *Let k be such that (AA1) holds, let $\xi \in \mathbb{R}^N$ and let u be of class L^p for some p, $1 \le p \le \infty$. Assume that the maximal solution $x(\cdot) = \varphi_u(\cdot,\xi)$ of (9) has interval of existence $\mathbb{R}_+$.*

(i) If x is meagre, then $x(t) \to 0$ as $t \to \infty$.

(ii) Assume that $\{0\}$ is a globally asymptotically stable equilibrium of the autonomous system $\dot{z} = k^*(z)$. If $u \in L^1$ and x is bounded, then $x(t) \to 0$ as $t \to \infty$.

(iii) Let $c : \mathbb{R}^N \to \mathbb{R}^P$ be continuous with $c(0) = 0$. Let $S \subset \mathbb{R}^N$ be non-empty and closed. Define $C := S \cap c^{-1}(0)$. Assume that $\inf_{z \in K} \|c(z)\| > 0$ for every closed set $K \subset S$ with $K \cap C = \emptyset$. If the trajectory of x is contained in S and $c \circ x$ is meagre, then $C \ne \emptyset$, $y(t) = c(x(t)) \to 0$ as $t \to \infty$ and x approaches C. If C is compact and, in addition, $u(t) \to 0$ as $t \to \infty$ and system (10) is zero-state observable, then $x(t) \to 0$ as $t \to \infty$.

Proof. Define $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ by $f(t,z) := k(z,u(t))$. Invoking (AA1), we may infer that, for each compact set $C \subset \mathbb{R}^N$, there exists a constant ρ such that

$$\|f(t,z)\| \le \rho[1 + \|u(t)\|] =: m(t) \quad \text{for all } z \in C \text{ and a.a. } t \in \mathbb{R}_+\,.$$

Since functions of class L^p, $1 \le p \le \infty$, are *a fortiori* uniformly locally integrable, we may conclude that m is uniformly locally integrable and so

$f \in \mathcal{F}(C)$ for every compact $C \subset \mathbb{R}^N$.

(i) Assertion (i) is an immediate consequence of Theorem 3.3.

(ii) By converse Lyapunov theory (for example, Theorem 2.8 in [15]), there exists a smooth function $V : \mathbb{R}^N \to \mathbb{R}_+$ and continuous $\hat{\alpha} : \mathbb{R}_+ \to \mathbb{R}_+$, with $\hat{\alpha}^{-1}(0) = \{0\}$, such that $\langle \nabla V(z), k^*(z)\rangle \leq -\hat{\alpha}(\|z\|)$ for all $z \in \mathbb{R}^N$. Let $s^* > 0$ be such that $\|x(t)\| \leq s^*$ for all $t \in \mathbb{R}_+$. Define $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by $\alpha(s) := \hat{\alpha}(s)$ for $s \in [0, s^*]$ and $\alpha(s) := \hat{\alpha}(s^*)$ for $s \in (s^*, \infty)$. Note that α is a $\mathcal{J}$-function and $\alpha(x(t)) = \hat{\alpha}(x(t))$ for all $t \in \mathbb{R}_+$. By (AA1) and boundedness of x, there exist constants $\rho_0, \rho_1 > 0$ such that

$$\begin{aligned}(V \circ x)'(t) &= \langle \nabla V(x(t)), k(x(t), u(t))\rangle \\ &\leq \langle \nabla V(x(t)), k^*(x(t))\rangle + \rho_0 \|\nabla V(x(t))\| \|u(t)\| \\ &\leq -\alpha(\|x(t)\|) + \rho_1 \|u(t)\| \quad \text{a.a. } t \in \mathbb{R}_+.\end{aligned}$$

Since $u \in L^1$, it now follows that $\int_0^\infty \alpha(\|x(t)\|)dt \leq \infty$ and so, by Proposition 2.1, x is meagre. Assertion (ii) now follows by Assertion (i).

(iii) By continuity of c and closedness of S, $C = S \cap c^{-1}(0)$ is closed. By Theorem 3.1, we conclude that $C \neq \emptyset$, $c(x(t)) \to 0$ as $t \to \infty$ and x approaches C.
Now assume that C is compact and impose the additional hypotheses that $u(t) \to 0$ as $t \to \infty$, and that system (10) is zero-state observable. Then, by Remark 4.3 and Corollary 4.2, x approaches the largest subset $\mathcal{I} \subset C$ that is invariant under the flow φ generated by k^*. By zero-state observability, $\{0\}$ is the only subset of $c^{-1}(0) \supset C$ that is invariant under the flow φ. Therefore, $\mathcal{I} = \{0\}$ and the proof is complete. ▮

Remark 4.4. If, in part (iii) of Theorem 4.4, the set S is compact, then "inf" assumption on c is redundant (insofar as the requisite property is a consequence of compactness of S). This situation is of interest if x is known to be bounded in which case S can be taken to be the closure of the trajectory of x.

5. NON-AUTONOMOUS SECOND-ORDER SYSTEMS

In this section, we study asymptotic behaviour of (non-autonomous) ordinary differential equations of second order of a type that occurs naturally in the study of mechanical systems with M degrees of freedom:

$$\ddot{y}(t) = k(t, y(t), \dot{y}(t)), \quad y(t) \in \mathbb{R}^M, \quad (y(0), \dot{y}(0)) = (y^0, v^0). \tag{12}$$

Writing $N := 2M$, the function $k : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^M$, $(t,d,v) \mapsto k(t,d,v)$, is assumed to be of Carathéodory class with the property that there exists an upper semicontinuous map $(d,v) \mapsto K(d,v) \subset \mathbb{R}^M$ from $\mathbb{R}^N$ to the non-empty convex compact subsets of $\mathbb{R}^M$ such that

$$d_{K(d,v)}(k(t,d,v) \to 0 \text{ as } t \to \infty \quad \text{uniformly with respect to } (d,v) \text{ in compact sets.} \tag{13}$$

By a solution of (12) we mean a solution x $(= (y,\dot{y}))$ of (1) with

$$f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N, \quad (t,d,v) \mapsto (v\,,\, k(t,d,v)) \quad \text{and} \quad \xi = (y^0, v^0). \tag{14}$$

The essence of the next result is an assertion that, if the "position" component y of a global solution $x = (y,\dot{y})$ of the non-autonomous initial-value problem (12) is bounded and the "velocity" component $\dot{y}$ is meagre, then the solution x is bounded and approaches the set of equilibria of the autonomous system $\dot{x} \in F(x)$, where

$$F : (d,v) \mapsto \{v\} \times K(d,v)\,. \tag{15}$$

Note that $\dot{x} \in F(x)$ is simply the canonical first order representation of the autonomous second order inclusion $\ddot{y} \in K(y,\dot{y})$.

Theorem 5.1. *Let $k : \mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$ be of Carathéodory class and such that (13) holds for some upper semicontinuous map $(d,v) \mapsto K(d,v)$ from $\mathbb{R}^M \times \mathbb{R}^M$ to the non-empty convex compact subsets of $\mathbb{R}^M$. Let $(y^0, v^0) \in \mathbb{R}^M \times \mathbb{R}^M$. Assume that $x = (y,\dot{y}) : \mathbb{R}_+ \to \mathbb{R}^M \times \mathbb{R}^M$ is a solution of the initial-value problem (12). If y is bounded and $\dot{y}$ is meagre, then the set*

$$\mathcal{E} := \{z \mid 0 \in K(z,0)\} \times \{0\}$$

is non-empty and is approached by $(y,\dot{y})$.

Proof. With f and ξ given by (14), (12) is equivalent to (1). Moreover, f is of Carathéodory class and satisfies (AA) with $F \in \mathcal{U}$ given by (15). Define $g : \mathbb{R}^N \to \mathbb{R}$, $(d,v) \mapsto \|v\|$. By hypothesis, $g \circ x$ is meagre. By boundedness of y, there exists a compact set $D \subset \mathbb{R}^M$ such that the trajectory of $x = (y,\dot{y})$ is contained in $S := D \times \mathbb{R}^M$. Therefore, $C := S \cap g^{-1}(0) = D \times \{0\}$ is compact and so property (i) of Theorem 4.1 holds: clearly, property (ii) also holds. It now follows from Theorem 4.1 that x approaches the largest subset $\mathcal{I} \subset C$ that is weakly invariant with respect to the autonomous system $\dot{z} \in F(z)$. Let $\hat{\xi} = (\theta, 0) \in \mathcal{I} \supset \Omega(x) \neq \emptyset$. By weak invariance of $\mathcal{I}$, there exists a solution $z = (d,v) : \mathbb{R}_+ \to \mathbb{R}^N$

of the inclusion $(\dot{d}, \dot{v}) = \dot{z} \in F(z) = \{v\} \times K(d, v)$ with $z(0) = \hat{\xi}$ and $v(t) = 0$ for all $t \in \mathbb{R}_+$. Therefore, $z(t) = (d(t), 0) = (\theta, 0) = \hat{\xi}$ for all $t \in \mathbb{R}_+$ and so $0 \in K(\theta, 0)$. Hence, $\hat{\xi} \in \mathcal{E}$ and the proof is complete. ▌

Remark 5.1. The hypothesis of boundedness of y in Theorem 5.1 may be removed at the expense of imposing a stronger hypothesis on $\dot{y}$, namely, $\dot{y} \in L^1$ (of which boundedness of y is a consequence).

5.1. Stabilization by adaptive "velocity" feedback

For continuous $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\pi^{-1}(0) = \{0\}$, let $\mathcal{S}(\pi)$ denote the class of systems, with control $u(t) \in \mathbb{R}^M$, of the form

$$\left.\begin{aligned} &\ddot{y}(t) + p(y(t), \dot{y}(t)) + q(y(t)) = b(u(t)), \\ &(y(0), \dot{y}(0)) = (y^0, v^0) \in \mathbb{R}^M \times \mathbb{R}^M , \end{aligned}\right\} \tag{16}$$

with the following properties:

(i) $b : \mathbb{R}^M \to \mathbb{R}^M$ is continuous, $b(0) = 0$ and there exists $\beta > 0$ such that

$$\langle b(u), u \rangle \geq \beta \|u\|^2 \quad \forall\ u \in \mathbb{R}^M ;$$

(ii) $p : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$ is continuous and such that, for some $\alpha > 0$,

$$\|p(d, v)\| \leq \alpha\pi(\|v\|) \quad \forall\ (d, v) \in \mathbb{R}^M \times \mathbb{R}^M ;$$

(iii) $q : \mathbb{R}^M \to \mathbb{R}^M$ is such that $q = \nabla Q$ for some C^1 function $Q : \mathbb{R}^M \to \mathbb{R}$ with

$$\|y\| \to \infty \quad \Longrightarrow \quad Q(y) \to \infty .$$

Note that (iii) above implies that $q^{-1}(0) \neq \emptyset$.

We assume that the "velocity" $\dot{y}(t)$ is available for feedback purposes and that the only other system data available to the control designer is knowledge of the function π. Define the continuous function $\hat{\pi} : \mathbb{R}^M \to \mathbb{R}^M$ by $\hat{\pi}(0) := 0$ and $\hat{\pi}(v) := \pi(\|v\|)v/\|v\|$ for all $v \neq 0$. Consider the adaptive feedback strategy

$$u(t) = -\theta(t)\hat{\pi}(\dot{y}(t)), \qquad \dot{\theta}(t) = \|\dot{y}(t)\|\pi(\|\dot{y}(t)\|), \quad \theta(0) = \theta^0 \in \mathbb{R} . \tag{17}$$

The essence of our final result is that, if $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $\pi^{-1}(0) = \{0\}$, then, for every system in $\mathcal{S}(\pi)$, the adaptive control (17) renders the set $q^{-1}(0) \times \{0\}$ globally attractive.

THEOREM 5.2. *Let $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous with $\pi^{-1}(0) = \{0\}$. Let (b, p, q) be such that (16) is in $\mathcal{S}(\pi)$. Let $(y^0, v^0, \theta^0) \in \mathbb{R}^{2M+1}$. Let $(y, \dot{y}, \theta) : [0, \omega) \to \mathbb{R}^{2M}$ be a maximal solution of the (feedback-controlled) initial-value problem (16)-(17). Then*

(i) $\omega = \infty$ and $(y, \dot{y}, \theta)$ is bounded;
(ii) $\lim_{t\to\infty} \theta(t)$ exists;
(iii) $(y, \dot{y})$ approaches the set $q^{-1}(0) \times \{0\}$.

Proof. Define $V : [0, \omega) \to \mathbb{R}$ by $V(t) := Q(y(t)) + \langle \dot{y}(t), \dot{y}(t)\rangle/2$ and note that V is bounded from below. Then,

$$\begin{aligned}\dot{V}(t) &= -\langle p(y(t), \dot{y}(t)), \dot{y}(t)\rangle + \langle b(-\theta(t)\hat{\pi}(\dot{y}(t)), \dot{y}(t)\rangle \\ &\le (\alpha - \beta\theta(t))\pi(\|\dot{y}\|)\|\dot{y}(t)\| \\ &= (\alpha - \beta\theta(t))\dot{\theta}(t) \quad \text{for a.a. } t \in [0, \omega),\end{aligned}$$

which, on integration, yields

$$\text{constant} \le V(t) \le V(0) + \alpha(\theta(t) - \theta^0) - \beta(\theta^2(t) - (\theta^0)^2)/2 \qquad \forall\, t \in [0, \omega)\,. \tag{18}$$

By (18), θ is bounded, implying the boundedness of V. It follows from the definition of V that $\dot{y}$ and $Q(y)$ are bounded. Invoking the radial unboundedness property of Q, we see that y is bounded. Boundedness of $(y, \dot{y}, \theta)$ implies that $\omega = \infty$ (assertion (i) of the Theorem). Assertion (ii) follows by boundedness and monotonicity of θ. It remains to establish assertion (iii).

Let s^* be such that $\|\dot{y}(t)\| \le s^*$ for all $t \in \mathbb{R}_+$. Define $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by $\alpha(s) := \pi(s)s$ for $s \in [0, s^*]$ and $\alpha(s) := \pi(s^*)s^*$ for $s \in (s^*, \infty)$. Then α is a $\mathcal{J}$-function and $\alpha(\|\dot{y}(t)\|) = \pi(\|\dot{y}(t)\|)\|\dot{y}(t)\|$ for all $t \in \mathbb{R}_+$. By boundedness of θ, $\int_0^\infty \pi(\|\dot{y}(t)\|)\|\dot{y}(t)\|\, dt = \int_0^\infty \alpha(\|\dot{y}(t)\|dt < \infty$ and so (by equivalence of (i) and (iii) in Proposition 2.1) $\dot{y}$ is meagre. Define $k : \mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$ by

$$k(t, d, v) := -p(d, v) - q(d) + b(-\theta(t)\hat{\pi}(v)),$$

for which (13) holds with $K : (d, v) \mapsto \{-p(d, v) - q(d) + b(-\theta^*\hat{\pi}(v))\}$, where $\theta^* := \lim_{t\to\infty} \theta(t)$. Moreover, $(y, \dot{y})$ is a solution of the equation $\ddot{y}(t) = k(t, y(t), \dot{y}(t))$ with y bounded and $\dot{y}$ meagre. By Theorem 5.1, it follows that $(y, \dot{y})$ approaches the set $\{z \mid 0 \in K(z, 0)\} \times \{0\} = q^{-1}(0) \times \{0\}$. This completes the proof. ∎

Corollary 5.1. *Let $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous with $\pi^{-1}(0) = \{0\}$. Let $\mathcal{S}_0(\pi)$ denote the subclass of $\mathcal{S}(\pi)$ systems with the additional condition that $q^{-1}(0) = \{0\}$. For every system in $\mathcal{S}_0(\pi)$, the feedback control (17) renders $\{0, 0\}$ globally attractive.*

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