LYAPUNOV FUNCTIONALS IN SINGULAR LIMITS FOR PERTURBED QUASILINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract. As a key example, we study the asymptotic behaviour near finite focusing time $t = T$, of radial solutions of the porous medium equation with absorption

$$u_t = \Delta u^m - u^p \quad \text{in } \mathbf{R}^N \times (0, T),$$

with bounded compactly supported initial data $u(x, 0) = u_0(|x|)$, and exponents $m > 1$ and $p > -m$ larger than a certain critical value $p_c = p_c(m, N)$. We show that under certain assumptions, the behaviour of the solution as $t \to T^-$ near the origin is described by self-similar Graveleau solutions of the porous medium equation $u_t = \Delta u^m$. In the rescaled variables we deal with an exponential non-autonomous perturbation of a quasilinear parabolic equation, which is shown to admit an approximate Lyapunov functional. The result is optimal, and in the critical case $p = p_c > -m$ an extra $\ln(T - t)$ factor is shown to occur in the asymptotic focusing behaviour. Other types of self-similar and non self-similar focusing patterns are discussed.

1. Introduction

This paper is devoted to applications of the Lyapunov functional method to non-autonomous quasilinear degenerate parabolic equations arising in finite-time blow-up singularity formation phenomena. Let us begin with the basic example exhibiting the main peculiarities and difficulties of such an asymptotic analysis.

1.1. **Focusing for perturbed porous medium equation.** We study the asymptotic behaviour of focusing blow-up type in reaction-absorption media by applying the classical Lyapunov method to quasilinear degenerate parabolic equations with non-autonomous perturbations. The basic example, which we use in order to explain the main ingredients of the Lyapunov analysis of perturbed dynamical systems, is the phenomenon of focusing for the porous medium equation (PME) with absorption

$$u_t = \Delta u^m - u^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+ \tag{1.1}$$

in the space dimension $N \geq 2$ with the diffusion exponent $m > 1$ and the absorption one satisfying the first necessary restriction $p > -m$. It is known that for $p \leq -m$ no compactly supported solutions $u \not\equiv 0$ exist [22]. These restrictions guarantee finite

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propagation with free boundaries. Actually, our goal is to study the PME with a general lower-order perturbation term

$$u_t = \Delta u^m + g(|x|, u, |\nabla u|), \tag{1.2}$$

provided that $g(\cdot)$ forms near the focusing point an asymptotically small perturbation relative to typical focusing similarity behaviour for the unperturbed PME. Equation (1.1), where both nonlinear operators on the right-hand side are scaling invariant, is a convenient example to show that blow-up asymptotics crucially depend on two parameters m and p of the diffusion and absorption operators respectively.

In the focusing problem for the radial restriction of (1.1), we consider the Cauchy problem in which the initial distribution u_0 is supported outside a ball with the centre at the origin. Then the solution $u = u(r,t)$ with $r = |x| \ge 0$ has the inner interface $r_0(t)$. As time increases, we observe a flow into the ball and we assume that there exists a certain finite focusing time $T > 0$ such that the flow reaches the origin, $r_0(T) = 0$, $u(x,T) > 0$ for all $x \ne 0$ near the origin and $u(x,t) = 0$ for all $x \approx 0$ if $t < T$. We also assume that $u(0,t) > 0$ for arbitrarily small $t - T > 0$ that chooses a strong focusing behaviour; see comments below. As usual in the PME studies, it is convenient to introduce the pressure variable

$$v(r,t) = (c\, u)^{m-1}(r,t), \quad c = m/(m-1),$$

satisfying the degenerate parabolic equation

$$v_t = (m-1)v\left(v_{rr} + \frac{N-1}{r}v_r\right) + (v_r)^2 - k\, v^q, \tag{1.3}$$

where

$$k = (m-1)c^{-(p-1)/(m-1)} \quad \text{and} \quad q = (m+p-2)/(m-1).$$

Without loss of generality, we assume that the initial pressure $v_0(r) = c\, u_0^{m-1}(r)$ is continuous, monotone increasing and sufficiently smooth in the positivity domain, for instance,

$$|v_0| \le C \quad \text{and} \quad |v_0'| \le C. \tag{1.4}$$

Then the solution $v(r,t)$ is monotone increasing in r by the Maximum Principle. Once the asymptotic behaviour of such monotone solutions is known, the extension of the results to bounded compactly supported initial functions can be done by intersection comparison with such a family of monotone solutions.

1.2. **Focusing Graveleau similarity solutions and their stability.** The study of focusing problems for the radial PME

$$v_t = (m-1)v\left(v_{rr} + \frac{N-1}{r}v_r\right) + (v_r)^2 \tag{1.5}$$

began with the work of J. Graveleau in the 1970's. A rigorous construction of such similarity solutions was done in [4], where references to first original papers are available. The PME (1.5) was shown to admit a one-parameter family $\{G_\mu(r,t), \mu > 0\}$ of self-similar solutions with focusing time $T = 0$

$$G_\mu(r,t) = r^2\phi(\mu\xi)/(-t), \quad \xi = (-t)r^{-\alpha}, \qquad t < 0, \tag{1.6}$$

where $\mu \in \mathbf{R}_+$ is a parameter. The function $\phi = \phi(\xi)$ is a solution of a nonlinear degenerate ordinary differential equation (ODE) with initial values $\phi(0) = 0$ and $\phi'(0) = -1$. The anomalous exponents α is the unknown parameter of such self-similarity of the *second kind*, a term introduced by Ya.B. Zel'dovich in 1946; see [6]. It was shown [4] that there exists a unique value $\alpha = \alpha(N, m)$ such that the corresponding profile ϕ satisfies $\phi > 0$ on $(\gamma, 0)$, $\phi(\gamma) = 0$, $\phi'(\gamma)$ is finite, where $\gamma = \gamma(\mu) < 0$. We set $\phi = 0$ on $(-\infty, \gamma)$. Then for every $\mu \in \mathbf{R}^+$, G_μ is a generalized solution to (1.5) on $[0, \infty) \times (-\infty, 0]$. The function $G_\mu(r, t)$ is increasing in r in the positivity set for all $t < 0$, and its interface is given by $\rho_\mu(t) = (\mu t/\gamma)^{1/\alpha}$.

Rewriting the similarity solution in the form $G_\mu(r, t) = r^{2-\alpha}\phi(\mu\xi)/(-\xi)$, we have that

$$G_\mu(r, 0) = \mu\, r^{2-\alpha}, \tag{1.7}$$

with exponent $\alpha \in (1, 2)$ if $N \geq 2$ [4]. The behaviour of G_μ at the focusing time has important implications for the optimal regularity of solutions to the PME. It follows from (1.7), that the velocity $\nabla G_\mu(r, 0)$ is singular at the origin if the space dimension is greater than 1. Indeed, $G_\mu(r, 0)$ is Hölder continuous in the class $C^{2-\alpha}$ with the exponent $2 - \alpha \in (0, 1)$. The focusing phenomenon describes a special type of finite time singularity formation for the PME. The interface regularity also drops at the focusing instance. It was proved in [2] that the rescaled pressure of every radially symmetric compactly support solution to the PME, which focuses at $t = 0$, converges near the focusing point to a unique profile from the family $\{G_\mu, \mu \in \mathbf{R}_+\}$, which gives local intermediate asymptotics of the focusing phenomenon. The self-similar focusing singularity formation is a general feature of the focusing phenomena and is valid for a wide class of radial focusing solution.

The stability analysis of focusing in [2] is essentially based on the intersection comparison with the family of Graveleau similarity solutions. Such geometric ideas use Sturm Theorems on zeros of solutions of linear parabolic equations and play a key role in a number of singularity problems for quasilinear reaction-diffusion equations in one spatial dimension, where finite-time blow-up singularities occur; see references in books [7] and [37].

1.3. **Main result by the classical Lyapunov method. Rescaled variables and equation.** Let $T > 0$ be the finite focusing time of the solution $u(r, t)$. It is convenient to consider similarity solutions G_μ with the given focusing time T in the form

$$G_\mu(r, t) = (T - t)^{(2-\alpha)/\alpha} g_\mu(\xi), \tag{1.8}$$

where $\xi = r(T - t)^{-1/\alpha}$ is the rescaled spatial variable and g_μ is a certain function related to ϕ. By the scaling invariance of the PME there holds

$$g_\mu(\xi) \equiv \mu^2 g_1(\xi/\mu) \quad \text{for all } \mu > 0. \tag{1.9}$$

Given the pressure $v(r, t)$, we introduce the rescaled solution

$$\theta(\xi, \tau) = (T - t)^{-(2-\alpha)/\alpha} v(r, t), \quad \tau = -\ln(T - t) \to \infty \quad \text{as} \quad t \to T^-. \tag{1.10}$$

Then $\theta \geq 0$ solves a degenerate quasilinear parabolic equation of the form

$$\theta_\tau = \mathbf{A}(\theta) - k e^{-\gamma\tau}\theta^q, \tag{1.11}$$

where $\mathbf{A}$ is the operator of the rescaled PME

$$\mathbf{A}(\theta) = (m-1)\theta\theta_{\xi\xi} + (m-1)\frac{(N-1)}{\xi}\theta\theta_\xi + (\theta_\xi)^2 - \frac{1}{\alpha}\xi\theta_\xi + \frac{(2-\alpha)}{\alpha}\theta, \tag{1.12}$$

and the exponent γ in the absorption term in the right-hand side of (1.11) takes the form

$$\gamma = 1 + (2-\alpha)(p-1)/\alpha(m-1).$$

The rescaled monotone increasing initial function prescribed at $\tau_0 = -\ln T$ is given by

$$\theta(\xi, \tau_0) = \theta_0(\xi) \equiv T^{-(2-\alpha)/\alpha} v_0(\xi T^{1/\alpha}). \tag{1.13}$$

By $a(\tau) > 0$ for all $\tau \geq \tau_0$ we denote the finite inner interface of the rescaled solution $\theta(\xi, \tau)$.

Main result: classical and geometric Lyapunov methods. As the main hypothesis, we assume that the non-autonomous perturbation in (1.11) is exponentially small as $\tau \to \infty$ on locally bounded rescaled orbits. This means that $\gamma > 0$ and we fall into the scope of the *supercritical* absorption range

$$p > \max\{-m, p_c\}, \quad \text{where } p_c = (2-\alpha m)/(2-\alpha). \tag{1.14}$$

In terms of the exponent q of the pressure equation (1.3) the supercritical range reads

$$q > \max\{-2/(m-1), q_c\}, \quad q_c = 2(1-\alpha)/(2-\alpha) < 0.$$

We show that the asymptotic behaviour as $t \to T^-$ of solution $v(r,t)$ is described by focusing self-similar solutions (1.8) with the same focusing time T. Denote by

$$\omega(\theta_0) = \{h \geq 0, h \in C(\mathbf{R}_+) : \exists \ \{\tau_j\} \to \infty, \ \theta(\cdot, \tau_j) \to h(\cdot) \text{ as } j \to \infty\}, \tag{1.15}$$

the ω-limit set of the rescaled orbit satisfying the Cauchy problem (1.11), (1.13), with uniform convergence on compact subsets in ξ. The problem of the asymptotic behaviour of $v(r,t)$ near the focusing time T reduces to the stabilization as $\tau \to \infty$ to stationary solutions of the perturbed equation (1.11). The corresponding unperturbed equation

$$\theta_\tau = \mathbf{A}(\theta) \tag{1.16}$$

is autonomous and indeed is the rescaled PME with g_μ being its weak stationary solutions, $\mathbf{A}(g_\mu) = 0$ for any $\mu > 0$. On the other hand, there exists another stationary profile

$$g_*(\xi) = c_*\xi^2, \quad c_* = [2(N(m-1)+2)]^{-1}. \tag{1.17}$$

Profile (1.17) corresponds to the separate variable solution of the pressure PME (1.5)

$$v_*(r,t) = (T-t)^{-1} c_* r^2 \quad \text{in } \mathbf{R}^N \times (0,T). \tag{1.18}$$

Our main goal is to prove that, for equation (1.1) in the supercritical absorption range $p > \max\{-m, p_c\}$, with the critical exponent (1.14), the asymptotic focusing behaviour of radially symmetric solutions for (1.1) is described by self-similar family $\{G_\mu(r,t)\}$. Hence, near the focusing time and focusing point, the absorption term becomes negligible and the behaviour of the solution is determined by the purely diffusive equation (1.5). In the asymptotic analysis, we then arrive at an exponentially small non-autonomous nonlinear perturbation of the rescaled PME.

In order to justify the passage to the limit in such perturbed dynamical systems (DSs), we use a classical Lyapunov type analysis based on an implicit construction of a Lyapunov functional (LF) for the unperturbed DS (1.16), which is proved to be an approximate LF for the perturbed one. Here we apply the idea of T.I. Zelenyak [40], who showed that since quasilinear second-order ordinary differential operators like (1.12) are potential, any smooth uniformly parabolic equation with such operators admits LFs, and, moreover, the omega-limit set of any compact orbit consists of a single element only. For degenerate equations like (1.16), (1.12), which are not uniformly parabolic and the LF does not poses necessary good properties, some extensions are needed [14].

As we have pointed out, the asymptotic analysis of the autonomous unperturbed rescaled PDE (1.16) was performed in [2] by the intersection comparison techniques via the Sturm Theorem [38]. This approach can be treated as construction of a discrete (or geometric) Lyapunov function, which has been used in a number of blow-up problems last twenty years. In particular, let us mention that the stability of the minimal travelling wave (TW) in the classical KPP paper [28] was proved by the intersection comparison with a family of TWs in the basic geometry admitting a single intersection only. H. Matano in [32] presented another proof of Zelenyak's result for semilinear parabolic equations by a geometric Lyapunov construction via counting the intersections. A crucial advantage of the geometric approach is that it applies to any smooth autonomous fully nonlinear parabolic equation on a bounded interval $\theta_\tau = F(\xi, \theta, \theta_\xi, \theta_{\xi\xi})$ satisfying the uniform parabolicity condition $F_r(\xi, p, q, r) \geq a_0 > 0$ (unlike the classical LF methods, where a quasilinear structure of the right-hand side is necessary).

Nevertheless, in the study of the exponentially perturbed rescaled PME (1.11) we found quite difficult to use intersection comparison ideas and typical techniques, which turned out to be quite effective for autonomous parabolic equations. Though the perturbation is asymptotically small, typical techniques of intersections comparison cannot be applied, since the self-similar solutions *are not* exact solutions of such perturbed equations, and the approach based on the Sturm Theorem on non-increase of the number of intersections between the solution $u(r,t)$ and similarity ones from $\{G_\mu(r,t)\}$ fails. On the other hand, the present focusing problem is posed in a special geometry (the unbounded half-space, limit unbounded rescaled profiles, etc.), and this makes hard using other geometric techniques like constructing a Lyapunov function by means of time-dependent deformations of stationary profiles from a given smooth family, cf. [17].

In the statement of the main result we describe the asymptotic behaviour of rescaled solutions in the following functional class:

$$H_* = \{f \geq 0 : \quad f(\xi) \leq c_* \xi^2 \ \text{ for } \ \xi \geq \xi_*\}, \tag{1.19}$$

where $\xi_* \gg 1$ is a constant.

Theorem 1.1. *Let $m > 1$ and $p > \max\{-m, p_c\}$. Assume that $\theta(\cdot, \tau) \in H_*$ for $\tau \gg 1$. Then*

$$\omega(\theta_0) \subseteq S_0 = \{g_\mu, \ \mu \geq 0\}. \tag{1.20}$$

The hypothesis on the behaviour in the functional class (1.19) is necessary to avoid an unbounded slow evolution of the rescaled orbit $\{\theta(\cdot,\tau)\}$ on S_0, when, roughly speaking, $\theta(\cdot,\tau) \approx g_{\mu(\tau)}(\cdot)$ with a slow growing function $\mu = \mu(\tau) \to \infty$ as $\tau \to \infty$. Such a possibility actually exists for some asymptotically small perturbations of non-exponential types. Unlike the unperturbed case, where the Sturmian intersection comparison applies, such a behaviour needs a more delicate asymptotic expansion analysis near the corresponding centre subspace of the linearized operator; see a comment at the end of Section 8. Note that for such non-autonomous perturbed DSs, no standard stable-centre manifold results are available. This assumption has a crucial importance and chooses a strong (generic) type of focusing behaviour. As typical in blow-up problems for nonlinear evolution equations including two or more nonlinear operators, we expect that in some parameter ranges there exist other finite or even countable spectra of different weaker focusing patterns. A possibility of other types of focusing behaviour in the present problem is discussed in Section 7.

Let us summarize the main steps of the analysis. In Section 2 we describe some auxiliary properties of the stationary profiles to be used in the Lyapunov construction. In Section 3 we present some estimates on the rescaled orbit.

Sections 3 and 4 are devoted to the stability theory based on the construction of an LF. We follow the general approach in [40] for the uniformly parabolic case and [14] for degenerate equations of the PME type. We construct an LF for the limit unperturbed equation (1.16), which is degenerate and is not of divergence form. Explicit LFs with sufficiently good properties are not available. We present a sequence of regularized problems and their corresponding LFs which provide us with an approximate LF for the original perturbed problem. We use the constructed sequence of LFs to obtain an estimate on the time derivative $\theta_\tau(\cdot,\tau)$ for $\tau \gg 1$, which is known to play a key role in the passage to the limit $\tau \to \infty$ establishing the stationary structure of the ω-limit set. However, in the case of degenerate (singular) dynamical systems, this estimate does not complete the stability analysis. In fact, when the weight function ρ associated with the LF, vanishes, $\rho \equiv 0$ pointwise on a "bad" subset $B^* \subset \mathbf{R}^3_+$, the key estimate does not guarantee any significant information on the asymptotic behaviour. Moreover, it is proved that this actually happens on any stationary profile g_μ of the limit equation (1.16), and therefore, no standard passage to the limit is available here unlike uniformly parabolic non-singular equations as in [40]. Our approach is based on a careful study of two sets, where $\rho = 0$ (including the stationary solutions of (1.16)) and $\rho \neq 0$ respectively).

It follows from the rescaled spatial variable in (1.10) that the focusing behaviour of the radial PME with absorption forms a scaling invariant interface singularity

$$r_0(t) = \mu(T-t)^{1/\alpha}(1+o(1)) \quad \text{as } t \to T^-, \tag{1.21}$$

where μ depends on initial data. It is important that this can be true in the supercritical range (1.14) only. In Section 8 we show that in the critical case $p = p_c > -m$, an extra logarithmic factor occurs in the asymptotic interface behaviour

$$r_0(t) = \mu_1(T-t)^{1/\alpha}|\ln(T-t)|^{(2-\alpha)/2}(1+o(1)) \quad \text{as } t \to T^-, \tag{1.22}$$

where $\mu_1 = \mu_1(m, N) > 0$ is independent of initial data. This is a typical phenomenon of non-scaling invariant asymptotics which can occur in the critical cases of nonlinear scaling invariant operators; see Chapt. 2 in [37], [20] and references therein.

Note that Theorem 1.1 does not establish that the ω-limit set of suitable compact orbits actually consists of a unique single profile g_μ. The approximate LF is not sufficiently regular to provide such a uniqueness unlike in the smooth case of uniformly parabolic equations, where convergence is always exponentially fast [40]. This is not the case for quasilinear degenerate equations. We conclude Section 8 with a comment showing that for some asymptotically small perturbations, we cannot expect the uniqueness.

2. **On stationary profiles**

We begin with a description of nonnegative stationary similarity profiles studied in [4] by the phase-plane analysis of the first-order ODE derived from the stationary one

$$\mathbf{A}(g) \equiv (m-1)|g|\Delta g + \operatorname{sign} g (g')^2 - \frac{1}{\alpha} g' \xi + \frac{2-\alpha}{\alpha} g = 0, \quad \xi > 0. \tag{2.1}$$

Here we extend the operator to negative values $g < 0$, so that this ODE describes focusing profiles for the sign PME $u_t = \Delta(|u|^{m-1}u)$. One can use the following scaling symmetry of the ODE (2.1): for any $\mu > 0$,

$$g(\xi) \text{ is a solution } \Longrightarrow \mu^2 g(\xi/\mu) \text{ is a solution.} \tag{2.2}$$

The one-parametric family of Graveleau profiles $S_0 = \{g_\mu \geq 0, \mu \geq 0\}$ given by (1.9) (we set $g_0 \equiv 0$ by continuity) is continuous relative to the parameter μ with the following total order of oriented intersections: for any $\mu < \nu$, $g_\nu(\xi)$ intersects $g_\mu(\xi)$ exactly ones, transversely and g_ν has a greater slope than g_μ at the intersection point. This implies a total ordering in S_0 denoted by $g_\mu \prec g_\nu$, which can be extended to more general solutions forming a restricted ordering, i.e., without the transitivity property.

We mention again that the explicit stationary profile (1.17) corresponds to the unbounded separate variable solution (1.18) and, moreover, represents the case of global blow-up:

$$v_*(r,t) \to \infty \quad \text{as } t \to T^- \text{ uniformly in } \mathbf{R}^N. \tag{2.3}$$

Relative to the growth as $r \to \infty$, it belongs to a boundary of the class of global solutions of the PME; see [9] and references in [27]. This implies that such a stationary profile g_* cannot occur in the ω-limit set of any suitable rescaled orbit corresponding to a bounded solution.

At this stage, we do not need a complete description of all the stationary solutions of the ODE (2.1). Since it admits the scaling symmetry (2.2), by the transformation

$$g(\xi) = \xi^2 \phi(\eta), \quad \eta = \ln \xi, \quad \psi = \phi', \tag{2.4}$$

it reduces to a first-order ODE studied on the $\{\phi, \psi\}$ phase-plane; see [4] and similar examples in [5] and [37], p. 136. Such equations with the negative coefficient $-1/\alpha$ of the linear term $g'\xi$ (corresponding to localized, single point singularities) are known to exhibit oscillatory behaviour about $g \equiv 0$, which is a typical property for ODEs arising in blow-up analysis of reaction-diffusion equations; see Chapt. 4 in [37] and references therein. In particular, ODEs like that are known to admit limit cycle solutions (2.4) oscillating infinitely many times as $\xi \to 0$, which can be interpreted as infinite order zeros for the sign PME; see [15], [39] and [12]. The phase-plane shows that there exists a two-parametric family of positive strictly increasing stationary solutions $g(\xi) > 0$ for $\xi > 0$, which oscillate around $g_*(\xi)$ as $\xi \to 0$ and $\xi \to \infty$. The change of variables (2.4) implies that any stationary profile g with $g(0) = 0$ satisfies

$$|g(\xi)| \le C\xi^2 \quad \text{in } \mathbf{R}_+$$

for some constant $C \ge 0$.

3. **First estimates on rescaled solution**

In what follows, in the case $p \ge 0$ we deal with unique weak solutions $u(x,t) \ge 0$ of the PME with absorption with well known regularity properties; see detailed Kalashnikov's survey [27] and DiBenedetto's book. In the singular absorption range $p \in (-m, 0)$, $u(x,t) \ge 0$ is a unique continuous maximal solution constructed by a regular monotone decreasing approximation; see details and references in [22] and [24], where the most singular cases of blow-up and extinction are considered.

We begin with first estimates of the rescaled solution $\theta(\xi,\tau)$.

Proposition 3.1. *On every compact subset $K = [0, \xi_0]$, $\theta(\xi,\tau) \le C$ for all $\tau > \tau_0$.*

Proof. If for some $\xi_0 > 0$ there exists a sequence $\{\tau_j\} \to \infty$ as $j \to \infty$ such that $\theta(\xi_0, \tau_j) \to \infty$, then a standard rescaling argument typical for the PME theory shows, after passing to the limit $\tau_j + s \to \infty$, that the rescaled PME (1.16) admits a global solution with initial datum being unbounded in $\{\xi > \xi_0\}$ contradicting the existence theory; see [9], [27]. □

It then follows from Bernstein estimates for quasilinear degenerate equations [27] that for $\tau \gg 1$, $|\theta_\xi| \le C$ on any compact subset $K \cap \{\theta \ge \delta\}$ for any fixed constant $\delta > 0$. This coincides with a typical optimal rescaled pressure estimate for the PME in the domain of the strict positivity, where the coefficient of the rescaled equation (1.11) are sufficiently smooth. For the strong absorption term θ^q with $q < 1$, this Bernstein estimate is not optimal for $\theta \approx 0$ (though is true) [26], [19], so that we impose the extra restriction $\theta \ge \delta$.

Regularity results for quasilinear parabolic equations of the PME type [27], [13] imply compactness of the orbit $\{\theta(\cdot,\tau), \tau > \tau_0\}$ satisfying (1.11). Given a sequence $\{\tau_j\} \to \infty$, we have that along a subsequence $\theta(\cdot, \tau_j) \to h(\cdot)$. Then $\theta(\cdot, \tau_j + s) \to f(\cdot, s)$ in $L^\infty_{\rm loc}(\mathbf{R}_+ : C_{\rm loc}(\mathbf{R}_+))$, where f solves the unperturbed equation (1.16) with initial data h. Let us establish an additional improved estimate on the orbit to be used in the stability theory.

Proposition 3.2. *Under the above assumptions, there exists a positive constant $C > 0$ such that for all $\tau \ge \tau_0$,*

$$\theta(\xi,\tau) \le C\xi^2 \quad \textit{for } \xi \gg 1. \tag{3.1}$$

Proof. Assume for contradiction that there exist sequences $\{\xi_j\}$, $\{\tau_j\} \to \infty$ such that

$$\theta(\xi_j, \tau_j)/\xi_j^2 = C_j \to \infty \quad \text{as } j \to \infty. \tag{3.2}$$

We apply a scaling argument and begin with the case $q > 1$ (we omit the special case $q = 1$ when (1.1) reduces to the PME). Denote $r_j = \xi_j(T - t_j)^{1/\alpha}$ and consider a sequence of functions

$$v_j(r,t) = \frac{1}{v(r_j,t_j)} v(r_j r, t_j + \lambda_j t), \quad \lambda_j = \frac{r_j^2}{v(r_j,t_j)}, \quad j \geq 1.$$

Then, $v_j(1,0) = 1$ by construction, and by (3.2) the corresponding focusing time satisfies

$$T_j = (T - t_j)v(r_j,t_j)/r_j^2 \to \infty \quad \text{as } j \to \infty. \tag{3.3}$$

On the other hand, $v_j(r,t)$ satisfies the equation

$$v_t = (m-1)v\left(v_{rr} + \frac{N-1}{r} v_r\right) + (v_r)^2 - P_j v^q, \tag{3.4}$$

with $P_j = k r_j^2 v^{q-2}(r_j,t_j)$. From (3.2), we have that

$$P_j \to 0 \quad \text{as } j \to \infty \text{ if } q > 1.$$

Consider $w_0(r)$, a bounded function satisfying

$$w_0(r) \leq v_j(r,0) \quad \text{for } j \gg 1,$$

and let $w(r,t)$ be the solution of the PME with a linear absorption term

$$w_t = (m-1)w\left(w_{rr} + \frac{N-1}{r} w_r\right) + (w_r)^2 - w \tag{3.5}$$

and initial data $w(r,0) = w_0(r)$. One can see that $w(r,t)$ is a subsolution of (3.4) if j is large enough. Then, from the condition on initial data, by the Maximum Principle it follows that

$$v_j(r,t) \geq w(r,t) \quad \text{for } j \gg 1.$$

However, by the classical properties of the PME and hence of (3.5), there exists a finite time $t = T^*$ such that $w(0,T^*) > 0$. Therefore, $T_j \leq T^*$ for $j \gg 1$ contradicting (3.3).

In the case $q < 1$, we perform another rescaling

$$v_j(r,t) = (T - t_j)^{-(2-\alpha)/\alpha} \xi_j^{-2} v(r\xi_j(T - t_j)^{1/\alpha}, t_j + (T - t_j)t).$$

¿From hypothesis (3.2) we have that $v_j(1,0) = C_j \to \infty$ as $j \to \infty$. On the other hand, $v_j(r,t)$ solves equation (3.4) with

$$P_j = k\xi_j^{-2(1-q)}(T - t_j)^{(2-\alpha)/\alpha(q-1)+1} \to 0 \quad \text{as } j \to \infty.$$

If $q < 0$, then the perturbation in (3.4) is not small in the pointwise sense for $v \approx 0$, but as we have mentioned above it is small as $j \to \infty$ in the weak sense due to known differential properties of maximal solutions. We fix $C \gg 1$ and compare $v_j(r,t)$ with a solution $\bar{v}_j(r,t)$ with initial data $\bar{v}_j(r,0) = C$ for $r \geq 1$. By comparison, $v_j \geq \bar{v}_j$ for all $j \gg 1$. It suffices to suppose that the limit $\bar{v}_j \to \bar{v}$ as $j \to \infty$ exists and bounded, where $\bar{v}(r,t)$ solves the rescaled PME with initial data $\bar{v}(r,0) = C \gg 1$, so that the corresponding focusing time of $\bar{v}$ satisfies $T_C \ll 1$. Obviously, this contradicts the assumption that $T_j \equiv 1$ for any $j \geq 1$ by construction. $\square$

The key estimate (3.1) makes it possible to control the first-order derivatives $\theta_\xi(\xi,\tau)$ and $\theta_\tau(\xi,\tau)$ for $\xi \gg 1$. In addition we need the following crucial estimate establishing an optimal class of rescaled solutions.

Proposition 3.3. *If $\theta(\tau) \in H_*$ for all $\tau \gg 1$, then there exists positive constants C_0 and C_1 such that*

$$\theta(\tau) \in H^\alpha = \{f \geq 0 : \ f(\xi) \leq C_0(1+\xi)^{2-\alpha} \ \ for \ \ \xi \geq 0\}, \tag{3.6}$$

and

$$a(\tau) \leq C_1 \quad for \ \ \tau \gg 1. \tag{3.7}$$

Proof. First of all, the hypothesis on the growth of the solution $\theta(\xi,\tau)$ in (1.19) implies that there exists ξ_0 such that

$$\theta(\xi,\tau) \leq c_* \xi^2 \ \text{ for any } \ \xi \geq \xi_0 \ \text{ and } \ \tau \geq \tau_0.$$

¿From the hypothesis on initial data, we have that there exists μ_0 large enough such that for every $\mu \geq \mu_0$,

$$\theta(\xi,\tau_0) \leq g_\mu(\xi) \ \text{ for any } \ \xi \geq \xi_0(\mu), \tag{3.8}$$

where $\xi_0(\mu)$ denotes the first intersection between the profiles g_* and g_μ. It follows from previous arguments that the profile g_μ, which is a supersolution of equation (1.11), satisfies (3.8) and

$$\theta(\xi_0(\mu),\tau) \leq c_*(\xi_0(\mu))^2 = g_\mu(\xi_0(\mu)) \ \text{ for any } \ \tau \geq \tau_0.$$

By the Maximum Principle, the same inequality holds for every $\xi \geq \xi_0(\mu)$ and $\tau \geq \tau_0$, and (3.6) follows.

Finally, in order to obtain (3.7), consider a fixed large value $\mu \geq \mu_0$ and assume for contradiction that there exists a sequence $\{\tau_j\} \to \infty$ such that $a(\tau_j) \to \infty$ as $j \to \infty$. Then, if j_0 is large enough, by the uniform estimate (3.6) on $\theta(\xi,\tau)$,

$$a(\tau_{j_0}) > \xi_0(\mu) + 1 \ \text{ and } \ \theta(\xi,\tau_{j_0}) < g_\mu(\xi) \ \text{ for } \ \xi \geq \xi_0(\mu).$$

By continuity, the same inequality holds for the shifted profile $g_\mu(\xi - \varepsilon)$ if $\varepsilon \ll 1$. Then, we have that the function given by $\bar{\theta}(\xi,\tau) = g_\mu(\xi - \varepsilon e^{(\tau-\tau_{j_0})})$, is a supersolution of equation (1.11) which satisfies $\bar{\theta}(\xi,\tau_{j_0}) \geq \theta(\xi,\tau_{j_0})$. By the standard comparison, the same inequality holds for every $\tau \geq \tau_{j_0}$. In particular, $a(\tau) \geq \bar{a}(\tau)$ for every $\tau \geq \tau_{j_0}$. This implies that the solution in the original variables $v(r,t)$ does not focus at time T at the origin $x = 0$, whence a contradiction. $\square$

4. **Stationary dynamical system**

Consider the unperturbed rescaled PME (1.16). Let

$$\theta = \frac{m}{m-1} v^{(m-1)/m}, \tag{4.1}$$

where θ stays for the pressure. Then $v(\xi, \tau)$ solves the problem

$$v_\tau = \mathbf{D}(v) \equiv a(v) v_{\xi\xi} + b(\xi, v, v_\xi) \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+, \tag{4.2}$$

$$v(\xi, 0) = v_0(\xi) \equiv \theta_0^m(\xi) \quad \text{in } \mathbf{R}_+, \tag{4.3}$$

where the coefficients are given by

$$a(v) = m|v|^{(m-1)/m}, \tag{4.4}$$

$$b(\xi, v, w) = -\frac{1}{\alpha}\xi w + \frac{m(N-1)}{\xi}|v|^{(m-1)/m} w + \frac{(2-\alpha)m}{\alpha(m-1)} v. \tag{4.5}$$

The construction of a LF uses properties of the stationary solutions to the equation (4.2). For given three parameters $\xi_0 > 0$, v_0, w_0, consider $\varphi(\xi) \equiv \varphi(\xi_0, \xi, v_0, w_0)$, the solution on $\xi \in (0, \xi_0]$ of the corresponding stationary DS

$$\mathbf{D}(\varphi) \equiv a(\varphi)\varphi'' + b(\xi, \varphi, \varphi') = 0, \tag{4.6}$$

where primes denote differentiation with respect to ξ, with boundary conditions

$$\varphi|_{\xi=\xi_0} = v_0, \quad \varphi'|_{\xi=\xi_0} = w_0. \tag{4.7}$$

We need some auxiliary properties of these functions, such as existence, uniqueness and continuous dependence with respect to the parameters.

4.1. **Properties of functions φ.**

Proposition 4.1. *For any fixed $\xi_0 > 0$, v_0, w_0, there exists a weak solution $\varphi(\xi_0, \xi, v_0, w_0)$ of* (4.6) *and* (4.7)*, a C^1-function which is bounded with its derivative on any compact in ξ, and $\varphi \in C^\infty$ at any point where $\varphi \neq 0$.*

Proof. These follow from the standard local regularity results for ordinary differential equations which apply at any point of nondegeneracy. On the other hand, it is not difficult to see that the weak solution φ is extended to the half-space $(0, \infty)$. Consider an arbitrary compact set $K = [\xi_1, \xi_2]$ and assume that $\varphi \gg 1$ on K. It follows from (4.6) that possible local strict maxima of φ are attained if $\varphi(\xi) > 0$, while $\varphi(\xi) < 0$ at minima points. Therefore, we can assume, reducing the interval K if necessary, that $\varphi > 1$ and $\varphi' > 0$ on K. Under this assumptions, it follows from (4.6) that

$$\varphi'' \leq \alpha^{-1}\xi(\varphi^{1/m})'.$$

Integrating by parts over K, yields the estimate

$$\varphi(\xi) \leq \left[\frac{m-1}{2\alpha m}(\xi^2 + C_1\xi)\right]^{m/(m-1)} + C_2, \tag{4.8}$$

where constants C_i do not depend on ξ_1. A similar argument applies if $\varphi \ll -1$. This gives the global existence and typical estimates on functions φ. $\square$

The following result is a direct consequence of the structure of the degenerate equation (4.6).

Proposition 4.2. *Fix some $\xi_0 > 0$, v_0, w_0 and an arbitrary compact interval K such that $\xi_0 \in K$. Assume that the solution $\varphi(\xi_0, \xi, v_0, w_0)$ of (4.6) and (4.7) satisfies*

$$\varphi^2 + (\varphi')^2 \neq 0 \quad for\ any\ \ \xi \in K. \tag{4.9}$$

Then the functions $\varphi(\xi_0, \xi, v_0, w_0)$ and $\varphi'(\xi_0, \xi, v_0, w_0)$ depend continuously on K relative to small perturbations of the parameters ξ_0,v_0, w_0.

The uniqueness of solutions and the continuous dependence on parameters may not be true if (4.9) is not valid. However, we obtain a certain uniqueness property to the left and to the right if we restrict ourselves to positive solutions.

Proposition 4.3. *Let $\varphi(\xi)$ be an arbitrary weak solution to (4.6) on $[0, \xi_0]$ such that $\varphi(\xi_0) = 0$, $\varphi'(\xi_0) = 0$. Then $\varphi \equiv 0$ on $[0, \xi_0]$ and there exists a unique solution $\varphi^+(\xi_0, \xi, 0, 0)$ in a small right-hand neighbourhood of the point $\xi = \xi_0$.*

Proof. Both results are easily obtained from a local analysis in a small right-hand (left-hand) neighbourhood of ξ_0 of the equivalent integral equation via the Banach Contraction Principle. Moreover, the analysis to the right-hand side yields the following behaviour of the solution φ_0 for $\xi > \xi_0$:

$$\varphi^*(\xi_0, \xi, 0, 0) = [c(\xi - \xi_0)]^{m/(m-1)}(1 + o(1)) \quad \text{as } \ \xi \to \xi_0^+, \tag{4.10}$$

where $c = (m-1)\xi_0/\alpha m$, if $\xi_0 > 0$. For $\xi_0 = 0$ we obtain the following estimate on all possible solutions:

$$\varphi^+(0, \xi, 0, 0) \leq (C\xi^2)^{m/(m-1)} \quad \text{as } \ \xi \to 0^+. \qquad \square \tag{4.11}$$

Before dealing with the continuous dependence on parameters, we summarize "bad" properties of solutions to (4.6), (4.7) to be used later on.

Proposition 4.4. *There exists a subset $B_0^* \subset \mathbf{R}_+^3 = \{\xi_0 > 0, v_0 \in \mathbf{R}, w_0 \in \mathbf{R}\}$ such that for any point $(\xi_0, v_0, w_0) \in B_0^*$, the corresponding solution $\varphi(\xi_0, \xi, v_0, w_0)$ to (4.6) satisfies: there exists a value $\xi^* \in [0, \xi_0]$ such that (4.9) is not valid, and by Proposition 4.3*

$$\varphi(\xi_0, \xi, v_0, w_0) \equiv 0 \quad for \ \ \xi \in [0, \xi_*].$$

We next show that the continuous dependence of $\varphi(\xi_0, \xi, v_0, w_0)$ relative to parameters also holds in B_0^*.

Proposition 4.5. *For any fixed $(\xi_0, v_0, w_0) \in B_0^*$, uniformly on $[0, \xi_0]$, the functions φ and φ' are continuous at ξ_0, v_0, w_0, and in particular*

$$\varphi(\xi_0, \xi, v, w) \to 0 \quad as \ \ v \to v_0, \ w \to w_0$$

uniformly on $[0, \xi^*]$, where ξ^* is defined as in Proposition 4.4.

Proof. The continuous dependence on any compact subset $K_\varepsilon = [\xi^* + \varepsilon, \xi_0]$ follows from standard results for ordinary differential equations. Now assume for contradiction that there is no continuous dependence of φ on $[0, \xi^*]$. Then this would mean a local non-uniqueness of solutions of the ODE at a point of degeneracy. This contradicts its unique solvability proved by a standard application of the Banach Contraction Principle to the equivalent integral equation; see further details in [14]. $\square$

5. **Approximate Lyapunov functional**

Since equation (4.6) is degenerate we do not have sufficient regularity properties of the trajectory subset $\{\varphi\}$ of the stationary DS for constructing a good smooth LF existing for uniformly parabolic equations. We first consider a regularized problem for (4.2), (4.3) and study the regularized LF for such uniformly parabolic problem. Then, the main estimate, which play a key role in the proof of Theorem 1.1, will be proved by passing to the limit $\varepsilon \to 0$ in the regularizing parameter.

As the next step, we show that such a regularization applies to the non-autonomous perturbed equation (1.11) and prove a key estimate on the time-derivative $\theta_\tau(\cdot, \tau)$ for $\tau \gg 1$ establishing the stabilization to stationary focusing profiles.

5.1. **The regularized problem.** Weak or maximal solutions of the nonlinear degenerate equation (4.2) can be constructed as the limit of a sequence of classical solutions to a regularized problem; see the survey [27], books [13], [37] and references therein. For a fixed $\varepsilon > 0$ small enough, consider the Cauchy problem in $\mathbf{R}_+ \times (\tau_0, T_\varepsilon)$ (the existence time satisfies $T_\varepsilon \to \infty$ as $\varepsilon \to 0$) for the regularized version of equation (4.2), where the function $a(v)$ is replaced by

$$a_\varepsilon(v) = m(\varepsilon^2 + v^2)^{(m-1)/2m},$$

b is replaced by

$$b_\varepsilon(\xi, v, w) = -\frac{1}{\alpha}\xi w + \frac{m(N-1)}{\xi + \varepsilon}(\varepsilon^2 + v^2)^{(m-1)/2m} w + \frac{(2-\alpha)m}{\alpha(m-1)} v$$

(no singularity is available at $\xi = 0$, which is not a crucial but a convenient regularization for the future analysis). Monotone initial data v_0 stay the same. ¿From general regularity results, it follows that

$$v_\varepsilon(\xi, \tau) \to v(\xi, \tau), \quad (v_\varepsilon)_\xi \to v_\xi \quad \text{as } \varepsilon \to 0$$

uniformly on any compact subset of $\mathbf{R}_+ \times (\tau_0, \infty)$. In particular, as usual for the PME-type diffusion operators

$$(v_\varepsilon^\beta)_\tau \to (v^\beta)_\tau, \quad \text{where } \beta = (m+1)/2m,$$

in $L^2_{\rm loc}(\mathbf{R}_+ \times (\tau_0, \infty))$. Notices that at any point of strict positivity (ξ, τ), where $v(\xi, \tau) > 0$ is the classical solution, we have that as $\varepsilon \to 0$ the function v_ε converges to v with first derivatives. By Bernstein estimates we also have that on any given compact subset of $\mathbf{R}_+ \times (\tau_0, \infty)$, as $\varepsilon \to 0$,

$$|v_\varepsilon| \le C_1, \quad |(v_\varepsilon)_\xi| \le C_2,$$

where constants $C_i > 0$ are independent of $\varepsilon > 0$.

The same regularization is performed for the perturbed equation (1.11), where we need to replace the singular absorption term θ^q by its locally Lipschitz continuous regularization $F_\varepsilon(\theta) \to \theta^q$ as $\varepsilon \to 0$ uniformly on closed intervals $[\delta, 1/\delta]$ with any small $\delta > 0$. By the classical results [27] we can guarantee the same regularity estimates on compact subsets where $\{\theta \ge \delta > 0\}$, where the rescaled solution is uniformly bounded away from zero.

5.2. **Regularized Lyapunov functional.** The regularized equation is uniformly parabolic and we can construct the classical LF which will be shown to be non-increasing on solutions $v_\varepsilon(\cdot, \tau)$. The construction is performed via the corresponding regularized stationary DS. For given $\xi_0 > 0$, v_0 and w_0, denote by $\varphi_\varepsilon(\xi) \equiv \varphi_\varepsilon(\xi_0, \xi, v_0, w_0)$ the unique classical solution to the nonlinear ordinary differential equation

$$\mathbf{D}_\varepsilon(\varphi_\varepsilon) \equiv a_\varepsilon(\varphi_\varepsilon)\varphi_\varepsilon'' + b_\varepsilon(\xi, \varphi_\varepsilon, \varphi_\varepsilon') = 0, \quad \varphi_\varepsilon|_{\xi=\xi_0} = v_0, \quad \varphi_\varepsilon'|_{\xi=\xi_0} = w_0.$$

For arbitrary $(\xi, v, w) \in \mathbf{R}_+^3$, we introduce sufficiently regular functions

$$\rho_\varepsilon(\xi, v, w) = (\xi + \varepsilon)^{N-1} \exp\left\{ -\frac{1}{\alpha m} \int_0^\xi \zeta \left[\varepsilon^2 + \varphi_\varepsilon^2(0, \zeta, v_0, w_0)\right]^{-\frac{m-1}{2m}} d\zeta \right\}, \tag{5.1}$$

where $v_0 = \varphi_\varepsilon(\xi, 0, v, w)$, $w_0 = \varphi_\varepsilon'(\xi, 0, v, w)$, and

$$\Phi_\varepsilon(\xi, v, w) = \int_0^w (w - \eta)\rho_\varepsilon(\xi, v, \eta)d\eta - \frac{(2-\alpha)}{\alpha(m-1)} \int_0^v \rho_\varepsilon(\xi, \mu, 0)\mu d\mu. \tag{5.2}$$

It follows that the regularized weight function satisfies the following elementary exponential estimate from below:

$$\rho_\varepsilon(\xi, v, w) \ge \xi^{N-1} \exp\left\{ -\frac{1}{2\alpha m} \varepsilon^{-(m-1)/m} \xi^2 \right\}, \tag{5.3}$$

which shows that for arbitrarily small $\varepsilon > 0$, ρ_ε is strictly positive everywhere except $\xi = 0$ (obviously, this is not the case for $\varepsilon = 0$, where (5.3) gives no positivity bound).

The functions ρ_ε and φ_ε satisfy the system of PDEs [40]

$$\frac{b}{a_\varepsilon}\rho_\varepsilon = (-\Phi_\varepsilon)_v + (\Phi_\varepsilon)_{\xi w} + w(\Phi_\varepsilon)_{vw}, \quad \rho_\varepsilon = (\Phi_\varepsilon)_{ww}, \tag{5.4}$$

and $\rho_\varepsilon(\xi, v, w)$ is obtained by the characteristic method applied to the first-order PDE

$$\frac{b}{a_\varepsilon(v)}(\rho_\varepsilon)_w - w(\rho_\varepsilon)_v - (\rho_\varepsilon)_\xi = -\frac{b_w'}{a_\varepsilon(v)}\rho_\varepsilon. \tag{5.5}$$

We now introduce the regularized LF

$$L_\varepsilon[v_\varepsilon](\tau) = \int_{\mathbf{R}_+} \Phi_\varepsilon(\xi, v_\varepsilon, (v_\varepsilon)_\xi)d\xi, \tag{5.6}$$

and derive main upper estimates on the weight multiplication factor ρ_ε for small $\varepsilon \ge 0$.

Proposition 5.1. *For any small $\varepsilon > 0$, there holds*

$$\rho_\varepsilon(\xi, v, 0) \le e^{-c_0 \xi^\alpha} \quad for \;\; \xi \gg 1, \;\; 0 \le v \le C_0(1+\xi)^{(2-\alpha)m/(m-1)}, \tag{5.7}$$

where the constant $c_0 > 0$ depends on $C_0 > 0$.

Proof. It is a straightforward consequence of (5.1), properties of stationary solutions and the regularization argument establishing necessary approximation properties $\varphi_\varepsilon \to \varphi$ as $\varepsilon \to 0$. As in a number of blow-up ordinary differential problems, we use strong oscillatory properties of operator $\mathbf{A}$. More precisely, using a suitable multiplying factor and a standard potential well analysis as in [37], pp. 188–194, we prove that due to the linear terms in $\mathbf{A}$, oscillations of the stationary profiles are increasing for $\xi \gg 1$, and this gives the following estimate the orbits:

$$|\varphi_\varepsilon(\zeta)| \le C_1 \xi^{(2-\alpha)m/(m-1)} \quad \text{for } \; 1 \ll \zeta \le \xi.$$

Substituting this estimate into (5.1) and integrating, we obtain (5.7). The behaviour of $\varphi_\varepsilon(\zeta)$ for small $\zeta \ge 0$ does not affect the final estimate. □

We now extend the exponentially decaying estimate on the weight function ρ_ε to the necessary class H^α of rescaled orbits. We say that $(\xi, v) \in H^\alpha$ if, in view of transformation (4.1), $\theta(\xi) \equiv \frac{m}{m-1} v^{(m-1)/m}(\xi) \in H^\alpha$, where H^α is given by (3.6).

Proposition 5.2. *There exists a constant $c_0 = c_0(C_0) > 0$ such that for $\xi \gg 1$ and $0 \le w \le C_2(1+\xi)^{(2-\alpha)m/(m-1)}$, $C_2 = C_2(C_0) > 0$,*

$$(\xi, v) \in H^\alpha \implies \rho_\varepsilon(\xi, v, w) \le e^{-c_0 \xi^\alpha}. \tag{5.8}$$

For $w = 0$, this estimate is presented above. Actually, this establishes that in the domain of attraction of similarity focusing profiles $\{g_\mu\}$ the weight Lyapunov multiplier $\rho(\xi, \cdot)$ is fast exponentially decaying $\sim e^{-c_0 \xi^\alpha}$ as $\xi \to \infty$. Since $\alpha \in (1, 2)$, this is slower than in typical linear cases associated with the weight $\tilde\rho(\xi) = \xi^{N-1} e^{-\xi^2/4}$ for Hermite-type ordinary differential hyper-geometric operators. More precisely, (5.8) with $\alpha = 2$ formally corresponds to the classical case of linear operators like $\mathbf{A}Y = \Delta_\xi Y - \frac{1}{2} Y' \xi$, which is self-adjoint in the weighted space $L^2_{\tilde\rho}(\mathbf{R}_+)$.

Proof of Proposition 5.2. The result follows from a similar analysis of stationary orbits satisfying the ODE (4.6) and having representation (2.4). It is again based on oscillatory properties of operator $\mathbf{A}$ (see [37], p.188), where an extra small perturbation occurs after multiplying by the factor and integrating over a bounded interval of a single oscillation for $\xi \gg 1$. It is important that the exponential decay of the weight function ρ is violated if and only if the orbit $\varphi(\zeta)$ approaches an orbit with the distinct behaviour $\varphi(\zeta) \sim \zeta^{2m/(m-1)}$ for $\zeta \gg 1$. Then, (5.1) can guarantee not exponential but a power-type asymptotics of the weight function (in fact, this describes an actual "thin" basin of stability of such stationary profiles). Since then $\varphi'(\zeta) \sim \zeta^{(m+1)/(m-1)}$ becomes arbitrarily large for $\zeta \gg 1$, the assumptions that $(\xi, v) \in H^\alpha$ and that $w \ge 0$ (staying for the derivative v_ξ) satisfies the same growth rate as v rule out such a behaviour of the orbit $\varphi(\zeta)$ of the stationary DS. Actually, one can see using (2.4) that (5.8) is valid for all $w \ge 0$ satisfying a less restrictive estimate $w \le \delta_0 \xi$ for $\xi \gg 1$, where $\delta_0 > 0$ is a small constant. □

In what follows, by classical Bernstein estimates [34], [3] (see some special features for the PME with strong absorption in [21]) we have that the derivative satisfies

$$|v_\xi| \le C_1(1+\xi)^{(2-\alpha)m/(m-1)} \quad \text{provided that } \theta(\tau) \in H^\alpha, \tag{5.9}$$

so that we can use exponential estimate on the weight function ρ in the given functional class of solutions.

5.3. **Approximate Lyapunov functional for the perturbed equation.** We consider the regularized perturbed equation, where v_ε now denotes the corresponding local in time solution of the regularized equation with the same initial-boundary data

$$v_\tau = a_\varepsilon(v)v_{\xi\xi} + b_\varepsilon(\xi, v, v_\xi) - ke^{-\gamma\tau}\tilde{F}_\varepsilon(v), \tag{5.10}$$

where $\tilde{F}_\varepsilon$ is a suitable approximation of the absorption term with the following behaviour at $\varepsilon = 0$ for small $v > 0$,

$$\tilde{F}_0(v) = O(v^{\tilde{q}}), \quad \text{where } \tilde{q} = 1 + (q-1)(m-1)/m \equiv (m+p-1)/m.$$

This is a non-autonomous equation, and we can construct an approximate LF only. The key estimate is as follows.

Proposition 5.3. *For any fixed $S > \tau_1 \gg 1$ and sufficiently small $\varepsilon > 0$,*

$$\int_{\tau_1}^{S} d\tau \int_{\mathbf{R}_+} \rho_\varepsilon(\xi, v_\varepsilon, (v_\varepsilon)_\xi)[(v_\varepsilon^\beta)_\tau]^2 d\xi \le C, \tag{5.11}$$

where $\beta = (m+1)/2m$ and the constant C does not depend on S and ε.

Proof. By the construction of the LF (5.6) of the unperturbed equation we have the following approximate monotonicity identity (cf. [40] and [14])

$$\frac{d}{d\tau}L_\varepsilon[v_\varepsilon](\tau) = -\int \frac{1}{a_\varepsilon(v_\varepsilon)}\rho_\varepsilon(\xi, v_\varepsilon, (v_\varepsilon)_\xi)[(v_\varepsilon)_\tau]^2 d\xi - ke^{-\gamma\tau}J_\varepsilon(\tau) \quad \text{for } \tau > \tau_0, \tag{5.12}$$

where $J_\varepsilon(\tau)$ in the exponential perturbation has the form

$$J_\varepsilon(\tau) = \int \frac{1}{a_\varepsilon(v_\varepsilon)}\tilde{F}_\varepsilon(v_\varepsilon)\rho_\varepsilon(\cdot)(v_\varepsilon)_\tau d\xi.$$

In order to apply integration by parts in deriving (5.12) we note that equation (5.10) and Bernstein bounds of derivatives for monotone solutions in H^α imply that $v_\tau(\xi,\tau)$ has at most power growth as $\xi \to \infty$.

Since $J_\varepsilon(\tau)$ can change sign, (5.12) does not guarantee the strict monotonicity of $L_\varepsilon[v_\varepsilon](\tau)$, i.e., it is not an LF. We are going to use the fact that the last term is exponentially small for $\tau \gg 1$, so that $L_\varepsilon[v_\varepsilon](\tau)$ is "almost" monotone and is an approximate LF. Integration over (τ_1, S) yields

$$L_\varepsilon[v_\varepsilon](S) - L_\varepsilon[v_\varepsilon](\tau_1) = -\int_{\tau_1}^{S} d\tau \int \frac{1}{a_\varepsilon(v_\varepsilon)}\rho_\varepsilon(\cdot)[(v_\varepsilon)_\tau]^2 d\xi - k\int_{\tau_1}^{S} e^{-\gamma\tau}J_\varepsilon(\tau)d\tau. \tag{5.13}$$

We now use the Hölder inequality in the last term

$$J_\varepsilon \le \left(\int \frac{\rho_\varepsilon}{a_\varepsilon}[(v_\varepsilon)_\tau]^2\right)^{1/2}\left(\int \frac{\rho_\varepsilon}{a_\varepsilon}\tilde F_\varepsilon^2\right)^{1/2} \le \frac{1}{2}\left(1+\int \frac{\rho_\varepsilon}{a_\varepsilon}[(v_\varepsilon)_\tau]^2\right)\left(\int \frac{\rho_\varepsilon}{a_\varepsilon}\tilde F_\varepsilon^2\right)^{1/2},$$

where

$$\left(\int \frac{\rho_\varepsilon}{a_\varepsilon}\tilde F_\varepsilon^2\right)^{1/2} \le C_2$$

is uniformly bounded by known regularity properties of maximal (weak) solutions [26], [19] and by the exponential estimate in Proposition 5.2. It follows that the last term in the right-hand side of (5.13) is uniformly bounded. Therefore,

$$(1-\delta_1)\int_{\tau_1}^S d\tau \int \frac{1}{a_\varepsilon(v_\varepsilon)}\rho_\varepsilon(\cdot)[(v_\varepsilon)_\tau]^2 d\xi + L_\varepsilon[v_\varepsilon](S) \le L_\varepsilon[v_\varepsilon](\tau_1) + C_3 \le C_4, \tag{5.14}$$

where $\delta_1 = C_2 e^{-\tau_1}/2 \ll 1$. Hence,

$$\int_{\tau_1}^S d\tau \int \frac{1}{a_\varepsilon(v_\varepsilon)}\rho_\varepsilon(\cdot)[(v_\varepsilon)_\tau]^2 d\xi \le C_5 + \int R_\varepsilon(v_\varepsilon(\xi,S))d\xi, \tag{5.15}$$

with

$$R_\varepsilon(v) = \frac{(2-\alpha)}{\alpha(m-1)}\int_0^v \rho_\varepsilon(\xi,\mu,0)\mu d\mu.$$

By Proposition 5.1 there holds

$$\int R_\varepsilon(v_\varepsilon(\xi,S))d\xi \le C_6, \tag{5.16}$$

where C_6 is independent of S and ε. Finally, substituting this estimate into (5.15) and using that for small $\varepsilon \ge 0$,

$$\frac{1}{a_\varepsilon(v_\varepsilon)}[(v_\varepsilon)_\tau]^2 \ge \frac{2m}{(m+1)^2}[(v_\varepsilon^\beta)_\tau]^2,$$

we derive the uniform estimate (5.11). □

5.4. **Passage to the limit** $\varepsilon \to 0$. We now consider the functions $\rho(\xi,v,w)$ and $\Phi(\xi,v,w)$ corresponding to (5.1) and (5.2) with $\varepsilon = 0$. The main estimate for the perturbed equation (5.10) with $\varepsilon = 0$ in the v-variable is now stated in a similar way.

Proposition 5.4. *For any fixed $S > \tau_1 \gg 1$ and $L > 0$, there holds*

$$\int_{\tau_1}^S \int_0^L \rho(\xi,v,v_\xi)[(v^\beta)_\tau]^2 d\xi \le C, \tag{5.17}$$

where C does not depend on S.

In order to obtain it, we begin with some extra properties of $\varphi_\varepsilon(\xi_0,\xi,v_0,w_0)$.

Proposition 5.5. *For arbitrary fixed $\xi_0 > 0$, $v_0 \in \mathbf{R}$, $w_0 \in \mathbf{R}$,*

$$\varphi_\varepsilon \to \varphi, \quad \varphi_\varepsilon' \to \varphi' \quad \textit{as } \varepsilon \to 0.$$

Proof. It is a straightforward consequence of the continuous dependence of φ_ε and $(\varphi_\varepsilon)'$ on parameter ε if the limit solution φ with $\varepsilon = 0$ satisfies $\varphi^2 + (\varphi')^2 \neq 0$ on $[0, \xi_0]$. If this is not the case and there exists a $\xi^* \in [0, \xi_0]$, where $\varphi^2(\xi^*) + (\varphi')^2(\xi^*) = 0$, whence $\varphi \equiv 0$ in $[0, \xi^*]$, the uniform limit holds again by the uniqueness result given in Proposition 4.3. $\square$

We now introduce the bad set

$$B^* = \{(\xi, v, w) \in \mathbf{R}^3_+ : \quad \rho(\xi, v, w) = 0\}, \tag{5.18}$$

and the good one $G^* = \mathbf{R}^3_+ \setminus B^*$.

Proposition 5.6. *There hold:*

$$B_0^* \subseteq B^*, \tag{5.19}$$

$$\rho > 0 \quad \textit{in the open subset} \;\; G^*. \tag{5.20}$$

Proof. It follows from the definition of B_0^* that

$$\{(\xi_0, v_0, w_0) \in \mathbf{R}^3_+ : \varphi(\xi_0, \xi, v_0, w_0) \quad \text{satisfies (4.9) on } \; K = [0, \xi_0]\} \subseteq G^*. \tag{5.21}$$

On the other hand, by construction, ρ is a nonnegative function with $\rho(\xi_0, v_0, w_0) = 0$ if and only if

$$I(\xi_0, v_0, w_0) = \int_0^{\xi_0} \zeta |\varphi(\xi_0, \zeta, v_0, w_0)|^{-(m-1)/m} d\zeta \quad \text{diverges.} \tag{5.22}$$

Using easy local properties of the stationary DS, we have that (5.22) holds if and only if $\varphi(\xi_0, \xi, v_0, w_0)$ vanishes on $[0, \xi_0]$ including the case, when φ vanishes at $\xi = 0$ with its derivative. Therefore, $I(\xi_0, v_0, w_0) < \infty$ if (4.9) holds. Finally, if $(\xi_0, v_0, w_0) \in B^*$, it follows from (4.10) that either there exists $\xi^* \in [0, \xi_0]$ with $|\varphi(\xi, \zeta, v, w)| \sim (\zeta - \xi^*)^{m/(m-1)}$ as $\zeta \to \xi^*$ or

$$|\varphi(\zeta)| \le C \zeta^{2m/(m-1)} \quad \text{as } \; \zeta \to 0.$$

One can see that in both cases integral $I(\xi_0, v_0, w_0)$ diverges, and hence $\rho(\xi_0, v_0, w_0) = 0$, cf. [14]. By the definition in (5.21) and by the continuous dependence of φ and φ' on parameters $(\xi_0, v_0, w_0) \in G^*$, G^* is open. $\square$

It is important that the limit cycle solutions (2.4) or others satisfying in the pressure form $g(\xi) \to 0$ as $\xi \to 0$, which has a typical parabolic envelope $\sim \xi^2$ as $\xi \to 0$, always generate curves in B^*. See more detailed analysis of B^* given below.

By standard regularity results we have that

$$\rho \;\text{ and }\; \Phi \;\text{ are smooth functions on } G^*. \tag{5.23}$$

Therefore by construction, both functions satisfy a smooth DS in G_*

$$\rho \frac{b}{a} = -\Phi_v + \Phi_{\xi w} + w \Phi_{vw}, \quad \rho = \Phi_{ww}, \tag{5.24}$$

in the classical sense. The above analysis also establishes the following regularization property.

Proposition 5.7. *As $\varepsilon \to 0$,*

$$\rho_\varepsilon(\xi, v_\varepsilon, w_\varepsilon) \to \rho(\xi, v, w) \quad in \;\; G^*. \tag{5.25}$$

The actual regularity of ρ and Φ and convergence as $\varepsilon \to 0$ on the set B^* are unknown. The previous study shows only that both functions are continuous on $\mathbf{R}^3_+$. Therefore, since the functions ρ and Φ are expected to be non-smooth on sufficiently wide subset B^*, we cannot use directly the approximate LF

$$L[v](\tau) = \int_{\mathbf{R}_+} \Phi(\xi, v, v_\xi) d\xi. \tag{5.26}$$

However, by Proposition 5.7 and the definition of B_* we derive the following result.

Proposition 5.8. *As $\varepsilon \to 0$,*

$$\liminf_{\varepsilon \to 0} \rho_\varepsilon(\xi, v, w) \ge \rho(\xi, v, w). \tag{5.27}$$

Proof of Proposition 5.4. The proof of the main estimate in (5.17) is now a straightforward consequence of the approximation technique, cf. [14]. □

5.5. **Proof of Theorem 1.1.** We begin with the following property of the stationary solutions to equation (1.16), which follows from the definition of B^* and Proposition 4.3. Here and later on, dealing with stationary profiles g satisfying

$$\mathbf{A}(g) = 0, \tag{5.28}$$

we have to perform the transformation (4.1), i.e., we set

$$\bar{g} = \frac{m-1}{m} g^{m/(m-1)}. \tag{5.29}$$

For instance, $\bar{S}_0$ is a subset of stationary profiles $\{\bar{g}_\mu\}$ transformed according to (5.29).

Proposition 5.9. *Fix an arbitrary $\xi_0 > 0$ and let $g(\xi)$ be a nonnegative stationary solution of* (5.28) *on* $(0, \xi_0]$. *Fix $\bar{g} = (m-1)g^{m/(m-1)}/m$ and define the subset*

$$\mathcal{L}^*(\bar{g}, \xi_0) \equiv \{(\xi, \bar{g}(\xi), \bar{g}'(\xi)), \; \xi \in (0, \xi_0]\}. \tag{5.30}$$

Then, either

$$B^* \cap \mathcal{L}^*(\bar{g}, \xi_0) = \emptyset, \tag{5.31}$$

or

$$\mathcal{L}^*(\bar{g}, \xi_0) \subset B^*. \tag{5.32}$$

Since the good set G^* is open, we have the following property.

Lemma 5.1. *Let $\bar{g}(\xi) \ge 0$ satisfy* (5.31). *Then, for any point $S_0 = (\xi, \bar{g}(\xi), \bar{g}'(\xi))$ with $\xi \in (0, \xi_0]$, there exists its neighbourhood $Q(S_0) \subset \mathbf{R}^3_+$ such that*

$$B^* \cap Q(S_0) = \emptyset. \tag{5.33}$$

We now establish a version of Proposition 5.9 for any function from the ω-limit set $\omega(\theta_0)$.

Proposition 5.10. *Let $h \in \omega(\theta_0)$. Then, for any $\xi_0 > 0$, either*

$$B^* \cap \mathcal{L}^*(h, \xi_0) = \emptyset, \tag{5.34}$$

or

$$\mathcal{L}^*(h, \xi_0) \subset B^*. \tag{5.35}$$

Proof. Let $h \in \omega(\theta_0)$ and let $\{\tau_j\} \to \infty$ be a sequence such that $v(\cdot, \tau_j) \to h(\cdot)$ as $j \to \infty$ uniformly on $[0, 2\xi_0]$. By the above regularity results, we have $v(\cdot, \tau_j + s) \to f(\cdot, s)$ as $j \to \infty$ in $L^\infty_{\rm loc}(\mathbf{R}_+ : C_{\rm loc}(\mathbf{R}_+))$, where $f(\cdot, s)$ is a unique nonnegative solution of the Cauchy problem for the unperturbed equation $f_s = \mathbf{D}(f)$ and initial data $f(\xi, 0) = h(\xi)$ in $\mathbf{R}_+$.

Now assume that there exists a $\bar{\xi} \in (0, \xi_0]$ such that $S = (\bar{\xi}, h(\bar{\xi}), h'(\bar{\xi})) \in G^*$. Since G^* is open, by the continuity of $h(\xi)$, we have a small neighbourhood, $E_0(\bar{\xi}) = (\bar{\xi} - \delta, \bar{\xi} + \delta)$, and $s_0 > 0$ such that $f(\xi, s) \ge \delta_0 > 0$ and $\{(\xi, f(\bar{\xi}, s), f_\xi(\bar{\xi}, s)),\ \xi \in E_0, s \in [0, s_0]\} \subset G^*$. On the other hand, from Lemma 5.1 it follows that

$$(\xi, v(\xi, \tau_j + s), v_\xi(\xi, \tau_j + s)) \in G^*$$

for $j \gg 1$ and $\xi \in E_0$, $s \in [0, s_0]$. We can assume that $\rho \ge \rho_0 > 0$ and $v(\xi, \tau_j + s) \ge \delta_0/2 > 0$ on $\bar{E}_0 \times [0, s_0]$ for $j \gg 1$.

We now follow the idea of the approximate Lyapunov-type analysis; see [29] and applications to unperturbed and perturbed PME-like equations in [14] and [18]. The main estimate (5.17) implies that

$$\int_{\tau_j}^{\tau_j + s_0} \int_{E_0(\bar{\xi})} \rho(\xi, v, v_\xi)[(v^\beta)_\tau]^2 d\xi d\tau \le C. \tag{5.36}$$

It then follows that $f(\cdot, s)$ is a stationary solution to (1.16) on $E(\bar{\xi})$. Indeed, (5.36) means that for $s \in [0, s_0]$,

$$\|v^\beta(\cdot, \tau_j + s) - v^\beta(\cdot, \tau_j)\|^2_{L^2(E(\bar{\xi}) \times (0, s_0))} \equiv \int_{B(\bar{\xi})} \int_0^{s_0} |v^\beta(\xi, \tau_j + s) - v^\beta(\xi, \tau_j)|^2 d\xi ds$$

$$\le \int_{B(\bar{\xi})} \int_{\tau_j}^{\tau_j + s_0} [(v^\beta)_\tau]^2 d\xi d\tau \le \int_{B(\bar{\xi})} \int_{\tau_j}^{\infty} [(v^\beta)_\tau]^2 d\xi d\tau \to 0$$

as $j \to \infty$. Therefore, $v^\beta(\cdot, \tau_j + s) \to f^\beta(\cdot)$ as $j \to \infty$ in $L^2(E(\bar{\xi}) \times (0, s_0))$, so that $f(\xi, s)$ does not depend on s.

We now deal with a continuation argument to the left and right-hand side of the interval $E(\bar{\xi})$. Assume for contradiction that

$$\xi^* = \sup\{\xi_1 \in (0, \xi_0] : h \equiv g \quad \text{on } [\bar{\xi}, \xi_1]\} < \infty.$$

Using continuity of $h(\xi)$ and Proposition 4.3, we have that $h(\xi) = g(\xi)$ on $[\bar{\xi}, \xi^*]$, and $h > 0$ on $[\bar{\xi}, \xi^* + \delta]$ for a positive δ. Then, from the uniform convergence and the regularity including the first-order derivatives of h in the positivity set $\{h > 0\}$, it follows that $h'(\xi^*) = g'(\xi^*)$, whence $\rho(\xi^*) > 0$. Using again the good properties of the subset G^*, we conclude that $h \equiv g$ on the extended interval $[\bar{\xi}, \xi^* + \delta]$ and we arrive at a contradiction.

A similar argument applies to the continuation to the left-hand side leading to the equality

$$\inf\{\xi_1 \in (0,\xi_0] : h \equiv g \quad \text{on } [\xi_1,\bar\xi]\} = 0.$$

This completes the proof of Proposition 5.10. $\square$

We now establish the following intermediate property. By the interior parabolic regularity results, any $h(\xi) \in \omega(\theta_0)$ is at least C^2 smooth in its positivity domain.

Proposition 5.11. *If $h \in \omega(\theta_0)$, then $\bar h = (m-1)h^{m/(m-1)}/m$ satisfies*

$$\{(\xi, \bar h(\xi), \bar h'(\xi)),\ \xi > 0\} \subseteq B_* \equiv \{\rho(\cdot) = 0\}. \tag{5.37}$$

Proof. Indeed, if $(\xi_0, \bar h(\xi_0), \bar h'(\xi_0)) \in G_*$ for some $\xi_0 > 0$, then by the above analysis h is a stationary profile satisfying (5.28) which entirely lies in G_* and has interface at some $\bar\xi \ge 0$. Such stationary profiles are not available. $\square$

Recall that

$$\rho(\cdot) = 0 \quad \text{on profiles from } \bar S_0 \cup \{\bar g_*\}. \tag{5.38}$$

It is curious that the bad set B_* has rather complicates structure and is not necessarily generated by stationary profiles only. For instance, consider the smooth envelope $l_*(\xi)$ of the family S_0,

$$l_*(\xi) = \sup_{\mu>0} g_\mu(\xi).$$

Then by the definition (5.18), for any $\xi > 0$,

$$(\xi, \bar l_*(\xi), (\bar l_*)'(\xi)) \in B^*.$$

Since each $g_\mu(\xi)$ intersects $g_*(\xi)$ exactly at two points, by the scaling property (1.9) we have $l_*(\xi) = c^*\xi^2$, with a constant $c^* > c_*$. It follows from symmetry (2.2) that any stationary profile g from B^* (nonnegative or changing sign) generates its own family of stationary solutions $\{\mu^2 g(\xi/\mu)\}$ and hence the corresponding envelope, which has the form $c_g\xi^2$, where the constant c_g depends of the generating function g. Thus, B^* contain envelopes $l_g(\xi)$ of different families of stationary solutions of the quadratic form:

$$l_g(\xi) = c_g\xi^2,$$

and these are not stationary profiles if $c_g \ne c_*$. There exist infinitely many other curves $l(\xi)$ in B^* which are not stationary solutions. By uniqueness for the ODE in the positivity domain, such C^2-smooth non-stationary curves (called envelops) never locally coincide with stationary profiles and at any tangency point $\xi = \xi_0$, where $g(\xi_0) = l_g(\xi_0) > 0$ and $g_\xi(\xi_0) = (l_g)_\xi(\xi_0)$, we have $g_{\xi\xi}(\xi_0) \ne (l_g)_{\xi\xi}(\xi_0)$. Therefore, all non-stationary nonnegative curves $l(\xi)$ in B^* are strictly positive and have quadratic growth

$$l(\xi) > 0 \quad \text{for } \xi > 0, \quad l(\xi) \sim \xi^2 \quad \text{as } \xi \to \infty. \tag{5.39}$$

End of Proof of Theorem 1.1. By (5.39) Proposition 3.3 rules out all non-stationary curves from the ω-limit set as well as other strictly positive stationary profiles. We thus obtain that $\omega(\theta_0) \subseteq S_0 \equiv \{g_\mu,\ \mu \ge 0\}$, and this completes the proof of Theorem 1.1. $\square$

6. **Final comments on Lyapunov analysis**

1. Existence of a smooth LF $L(v)(\tau)$ for any one-dimensional quasilinear uniformly parabolic equation was established in [40]. Moreover, for equations on bounded spatial intervals, this classical Lyapunov approach proves not only that the ω-limit set of any compact orbit consists of a single stationary profile, but also establishes that the rate of convergence is exponentially fast (without studying the spectrum of the corresponding linearized problem).

For the degenerate (or singular) quasilinear equations like (4.2) the situation becomes more complicated since the LF is obtained from a degenerate nonlinear dynamical system admitted weak non-smooth solutions. It was shown in [14] that the multiplication factor $\rho(\cdot)v_\tau$, which leads to a typical formal identity (we consider unperturbed equation and assume necessary regularity of L)

$$dL(v)/d\tau = -\int \rho(\cdot)(v^\beta)^2_\tau \le 0 \quad \text{for } \tau > 0, \tag{6.1}$$

can be degenerated. Namely $\rho(\cdot) = 0$ on curves generating a non-empty bad subset $B^* \subset \mathbf{R}^3_+$, where the technique cannot control the time derivative v_τ for $\tau \gg 1$. This is a principle difference from the uniformly parabolic case, where $\rho(v) \ge \rho_0 > 0$ uniformly on any orbit [40]. Fortunately, in typical blow-up problems for the PME with source

$$u_t = (u^m)_{xx} + u^p \quad \text{in } \mathbf{R} \times (0,T), \quad m > 1, \;\; p > m,$$

the corresponding factor $\rho(\cdot)$ is strictly positive in a neighbourhood of the set of stationary profiles so that an extension argument applies [14].

In the present paper we consider a new case of the asymptotic behaviour, where the degeneracy, $\rho(\cdot) = 0$, holds not only on any stationary profile from a one-parametric family $S_0 = \{g_\mu\}$ but also on infinitely many envelope-like curves which are not stationary ones at all. For such autonomous equations admitting an LF satisfying (6.1), La-Salle's Invariance Principle saying that the ω-limit sets of a compact orbit is contained in a maximal invariant subset of $\{v : dL(v)/d\tau = 0\}$ does not rule out the possibility of stabilizing to a non-stationary envelope provided that those form an invariant subset (which is the case in the focusing problem).

In the perturbed non-autonomous equation under consideration, Propositions 5.9, 5.10 and Lemma 5.1 establish that the topology structure of the bad subset B^* is not sufficient to spoil the stabilization to stationary profiles. This kind of asymptotic analysis can be applied to a number of other perturbed reaction-diffusion equations admitting non-unique and non-disjoint subset of limit stationary profiles, on which the corresponding weight $\rho(\cdot)$ vanishes identically.

Another essential part of the Lyapunov analysis deals with the estimate of the weight function $\rho(\xi, v, v_\xi)$, which is shown to be exponentially decaying,

$$\rho(\xi, \cdot) \le e^{-c_0\xi^\alpha} \quad \text{for } \xi \gg 1,$$

on solutions from the corresponding functional class H^α forming the domain of stability of Graveleau's similarity solutions. This is similar to the linear blow-up theory associated with the second-order symmetric operator

$$\mathbf{A} = \Delta - \frac{1}{2}\xi \cdot \nabla \equiv \frac{1}{\rho}\nabla \cdot (\rho \nabla) \quad \text{in } \mathbf{R}^N,$$

with the explicit exponential weight $\rho(\xi) = e^{-|\xi|^2/4}$ forming a wide stability domain in $L^2_\rho(\mathbf{R}^N)$. Spectral properties of this linear operator $\mathbf{A}$ are known to play a fundamental role in single point blow-up singularity formation phenomena for semilinear heat equations; see e.g., [7], [8], [12], [25], [33] and [37].

2. Consider the case of arbitrary perturbations

$$v_\tau = \mathbf{D}(v) + \nu(\tau) G(v, v_\xi, \tau),$$

where $\nu(\tau) \to 0$ as $\tau \to \infty$ and the operator G is bounded for $\tau \gg 1$ on suitable evolution orbits from the same functional class as above. Then the key estimate (5.17) is derived similarly by multiplying the equation by $\rho(\cdot)v_\tau$ and integrating provided that $\nu \in L^1(\mathbf{R}_+)$, i.e., $\omega(\theta_0) \subseteq S_0$ holds for any integrable perturbation (though we expect that the integrability hypothesis can be weakened; see the end of Section 8).

7. On self-similar and other focusing patterns

In this section we discuss other possible types of focusing behaviour for the PME with strong absorption in different parameter ranges. In particular, this shows the necessity of extra hypotheses imposed in previous sections in order to choose the generic type of focusing behaviour.

7.1. **Similarity focusing in the subcritical range.** Assume that the interval $p \in (-m, p_c)$ (note that $p_c < 0$) is non-empty. As we have mentioned, then in the case of generic strong focusing both operators in (1.1) are expected to be essentially and equally involved into the asymptotic behaviour so that this range is quite opposite to the above ones. In such cases, it is natural to look for a self-similar solution of the form

$$u_*(r,t) = (T-t)^{1/(1-p)} g(\xi), \;\; \xi = r/(T-t)^\beta, \;\; \beta = (m-p)/2(1-p) > 0, \tag{7.1}$$

where $g \geq 0$ solves a nonlinear ODE, which is a radial restriction of the elliptic equation

$$\mathbf{A}(g) \equiv \Delta g^m - \beta g_\xi \xi + \frac{1}{1-p}\, g - g^p = 0 \quad \text{in } \mathbf{R}^N. \tag{7.2}$$

A proper similarity profile $g = g(\xi)$ is supposed to have finite interface at some $\xi_0 > 0$, $g(\xi_0) = 0$, $g(\xi) > 0$ for $\xi > \xi_0$, and the following "minimal" behaviour at infinity: $g(\xi) = C_* \xi^{2/(m-p)}(1 + o(1))$ as $\xi \to \infty$, where $C_* > 0$ is a constant depending on m, p and N. It then follows that the self-similar solution (7.1) forms the following final-time profile:

$$u_*(r, T^-) = C_* r^{2/(m-p)} \quad \text{for all } r > 0. \tag{7.3}$$

Observe that it is above the Graveleau one, $r^{2/(m-p)} \gg r^{(2-\alpha)/(m-1)}$ as $r \to 0$ provided that $p < p_c$.

Existence and uniqueness of g satisfying (7.1) and having the simplest monotone spatial shape with finite interface remain open. This problem belongs to a wide class of similarity ODE problems occurring in blow-up analysis of reaction-diffusion equations studied in the literature from the middle of the 1970's; see references in Chapt. 4 in [37] and a survey in [12]. According to this similarity, we expect that the above ODE can admit a finite number of different solutions $\{g_k,\ k = 1, ..., M\}$ depending on the oscillatory character of solutions in a small neighbourhood of the exact stationary one $S(\xi) = c_0 \xi^{2/(m-p)}$ satisfying the equation $\Delta_\xi S^m - S^p = 0$, where with $|p| < m$. Then the spectral properties of the corresponding linearized self-adjoint operator and the distribution of the point spectrum play a key role.

7.2. **Non self-similar patterns.** Typically, nonlinear problems with blow-up singularities can admit an infinite countable or uncountable spectrum of different singularity patterns. In quasilinear reaction-diffusion equations, such spectra can be composed from similarity solutions (as mentioned above) or can be constructed by linearization techniques and matching procedures; see [8], [11], [12], [25], [33] and references therein. In most the cases, for essentially quasilinear equations, the general structure of such singularity formation spectra (possibly uncountable) is a hard problem remaining open. Below, we mainly concentrate on discussing of a stable generic blow-up behaviour occurring under certain assumptions on initial data and exponents of the equations. We expect that the focusing-like problem for the PME with strong absorption belongs to such a class of highly sensitive dynamical systems admitting infinitely many types of focusing. We still do not know which hypotheses on initial data rule out the spectrum of non-generic patterns and need to impose an extra condition on suitable functional classes of solutions (or on the interface evolution).

Let us present a brief formal derivation of a countable spectrum of distinct focusing patterns in both ranges $p > p_c$ and $p \le p_c$. Similar to (7.1), we introduce the rescaled solution

$$u(r,t) = (T-t)^{1/(1-p)} g(\xi,\tau), \;\; \xi = r/(T-t)^\beta, \tag{7.4}$$

satisfying the quasilinear equation with operator in (7.2)

$$g_\tau = \mathbf{A}(g) \quad \text{for } \tau > \tau_0. \tag{7.5}$$

We perform a standard linearization by setting $g(\xi,\tau) = S(\xi) + Y(\xi,\tau)$. Then Y solves the equation

$$Y_\tau = \mathbf{B}Y + \mathbf{D}(Y), \tag{7.6}$$

where $\mathbf{B}$ is the linear operator

$$\mathbf{B}Y = m\Delta(S^{m-1}Y) - \beta Y'\xi + \frac{1}{p-1}\,Y - pS^{p-1}Y$$

and $\mathbf{D}(Y) = O(Y^2)$ as $Y \to 0$ is a nonlinear operator (a quadratic perturbation). The second-order ordinary differential operator $\mathbf{B}$ admits a self-similar extension [30] and under extra conditions at singular points if necessary (to be justified by known general properties of the solution u), it has a real discrete point spectrum in a weighted space $L^2_\rho(\mathbf{R}_+)$ and a complete subset of eigenfunctions, $\sigma(\mathbf{B}) = \{\lambda_k,\ k = 1, 2, ...\}$ and $\Phi = \{\psi_k\}$, where $\lambda_k < 0$ for all sufficiently large k. Namely, $\lambda_1 > 0$ and we expect that $\lambda_2 < 0$ if $p < p_c$ and $\lambda_2 > 0$ if $p > p_c$. The last non-oscillating property implies that no similarity focusing exists in the supercritical range $p > p_c$ and the Graveleau one is supposed to be generic. We can fix a $k \gg 1$ and consider a linearized flow around the stationary solution tangent to the corresponding one-dimensional eigenspace,

$$f(\xi, \tau) = S(\xi) + Ce^{\lambda_k \tau}\psi_k(\xi) + ... \quad \text{as } \tau \to \infty, \tag{7.7}$$

uniformly on compact subsets in ξ. The outer matching for $\xi \gg 1$ is a well known problem similar to that in blow-up in reaction-diffusion equations; see papers referred above. We expect that such linearized patterns can be also matched for sufficiently small $\xi > 0$ creating finite focusing interfaces at $\xi = a_k(\tau) > 0$ satisfying $f(a_k(\tau), \tau) = 0$. A local analysis of the singular point $\xi = 0$ shows that eigenfunctions $\psi_k(\xi)$ are bounded at the origin, so that solving this equation using (7.7) we obtain that the corresponding interface exhibits the following behaviour:

$$a_k(\tau) = c_1 e^{\lambda_k (m-p)\tau/2}(1 + o(1)) \to 0 \quad \text{as } \tau \to \infty.$$

It follows that in the original variables we expect to have a countable spectrum of focusing patterns with the following interface behaviour

$$r_k(t) = c_2(T - t)^{\beta - \lambda_k(m-p)/2}(1 + o(1)) \quad \text{as } t \to T^-, \tag{7.8}$$

which tends to zero faster than the one of the Graveleau solution provided that $k \gg 1$, i.e., $\lambda_k < 0$ is sufficiently small negative. Observe that formally, such analysis applies to the whole parameter range $p \in (-m, m)$, where the stationary solution $S(r)$ exists. It does not matter that for $p \in (1, m)$ the self-similar solution (7.4) formally blows up as $t \to T^-$ globally in $\mathbf{R}^N$ since $\beta < 0$. Nevertheless, (7.8) shows that the interface of any such pattern focuses as $t \to T^-$ at the origin $x = 0$ provided that $\lambda_k < 2\beta/(m - p)$.

This construction shows that extra hypotheses are necessary to specify a suitable functional class of solutions with generic Graveleau focusing behaviour. It is clear that by the Sturmian argument, special focusing solutions $u_k(r, t)$ with the interface behaviour (7.8) must intersect the stationary profile $S(r)$ at least k times for all $t \in [0, T)$, and this is a necessary condition of their existence. We do not know if in the case $p > p_c$, a simple assumption that $u_0(r)$ intersect $S(r)$ at a single point near $r = 0$ rules out all such unstable types of focusing and guarantees convergence to Graveleau profiles.

8. **A centre subspace behaviour in the critical case**

In this section we consider the PME with absorption in the critical case

$$p = p_c = (2 - \alpha m)/(2 - \alpha).$$

This case makes sense, i.e., $p_c > -m$, provided that $\alpha < 1 + 1/m$. It is worth mentioning that $p_c < 2 - m$ for all $m > 1$, so that the interface propagation is governed by a second-order interface equation [19] (though this is not important for the asymptotic behaviour of solutions).

8.1. **Autonomous rescaled equation.** The critical case of a special interaction of scaling-invariant operators is characterized as follows: the exponent $p = p_c$ is the only one when both the PME and the PME with absorption,

$$u_t = \Delta u^m \quad \text{and} \quad u_t = \Delta u^m - u^{p_c},$$

are invariant under the same group of scaling transformations. Therefore, the rescaled solution

$$\theta(\xi, \tau) = (T-t)^{-(2-\alpha)/\alpha} v(r,t), \quad \xi = r/(T-t)^{1/\alpha},$$

satisfies the autonomous equation

$$\theta_\tau = \mathbf{A}(\theta) - k\theta^q, \quad \text{where } q = q_c = \frac{m + p_c - 2}{m-1} \equiv -\frac{2(\alpha-1)}{2-\alpha} < 0. \tag{8.1}$$

The main goal of our analysis is to show that a standard classification of global asymptotics as $t \to \infty$ for the quasilinear heat equations

$$u_t = \Delta u^m - u^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad m \ge 1, \;\; p > 1, \tag{8.2}$$

which is known for almost twenty years (see an extended list of references to Chapters 2 and 4 in [37]) applies to blow-up focusing singularities for equation (8.1). It is known that the most delicate behaviour for (8.2) occurs in the critical case $p_c = m + 2/N$, where an extra logarithmic factors $\ln t$ occur in the space-time asymptotic structure as $t \to \infty$. Such a behaviour was first established in [16] for $m = 1$; see [20] and references therein for similar critical behaviour in quasilinear equations.

8.2. **One-dimensional centre subspace.** We will show that in the critical case the asymptotic behaviour of the orbit $\{\theta(\tau), \tau > \tau_0\}$ corresponds to a slow motion in a neighbourhood of the stationary set $S^* = \{g_\mu, \, \mu \in \mathbf{R}_+\}$. This means that in the first approximation the solution can be represented as follows:

$$\theta(\xi, \tau) \approx g_{\mu(\tau)}(\xi) \;\text{ for }\; \tau \gg 1 \tag{8.3}$$

with the unknown sufficiently smooth function $\mu(\tau)$ having a slow (non-exponential) behaviour for $\tau \gg 1$. According to this structure, we introduce a new rescaled function using the scaling invariance of the PME

$$\theta(\xi, \tau) = \mu^2(\tau) g(\eta, \tau), \quad \eta = \xi/\mu(\tau), \tag{8.4}$$

where, for convenience, $\mu(\tau)$ stays for the interface of $\theta(\xi, \tau)$, so that $\eta = 1$ is the fixed interface of $g(\eta, \tau)$. It follows that g solves the following perturbed equation:

$$g_\tau = \mathbf{A}(g) - k\mu^{q-2}(\tau) g^q + \frac{\mu'}{\mu}(\tau)\mathbf{D}g, \tag{8.5}$$

where $\mathbf{D}g = g_\eta \eta - 2g$ is a first-order linear operator.

Since $q = q_c < 2$, equation (8.5) contains two non-autonomous perturbations. Assume that they are asymptotically small and the orbit is compact in a suitable metric. We then pass to the limit in this perturbed dynamical system. The limit equation at $\tau = \infty$ is the rescaled PME $g_\tau = \mathbf{A}(g)$, and since the interface is fixed, $g(1,\tau) \equiv 0$, by the Strong Maximum Principle we conclude that $\omega(\theta_0) = \{g_1\}$. This also follows from analyticity results for the radial PME [1]. Equation (8.5) in not autonomous and we cannot apply known results on invariant manifolds for infinite-dimensional dynamical systems; see [31], Chapter 9, and [35]. Nevertheless, we recall that (8.5) is a rescaled version of the autonomous equation (8.1) though existence of the corresponding centre manifold for it is not straightforward.

Let us present an asymptotic analysis establishing a power rate of divergence of the scaling factor $\mu(\tau)$. We perform a standard linearized analysis of (8.5) by setting

$$g(\eta,\tau) = g_1(\eta) + Y(\eta,\tau),$$

where Y solves the equation

$$Y_\tau = \mathbf{A}'(g_1)Y - k\mu^{q-2}(t)g_1^q + \frac{\mu'}{\mu}(\tau)\mathbf{D}g_1 + ..., \tag{8.6}$$

where we omit higher-order terms. The linear operator takes the form

$$\mathbf{A}'(g_1)Y = (m-1)(g_1\Delta Y + Y\Delta g_1) + 2g_1'Y_\eta - \frac{1}{\alpha}Y_\eta\eta + \frac{2-\alpha}{\alpha}Y. \tag{8.7}$$

In the radial geometry, it is a second-order ordinary differential operator. Then it admits a symmetric representation of the form

$$\mathbf{A}'(g_1) = \frac{1}{\rho}\frac{d}{d\eta}\left(a\frac{d}{d\eta}\right) + (m-1)\Delta g_1 + \frac{2-\alpha}{\alpha}I, \tag{8.8}$$

where

$$a(\eta) = \eta^{N-1}g_1^{2/(m-1)}(\eta)\exp\left\{-\frac{1}{\alpha(m-2)}\int\frac{\eta d\eta}{g_1(\eta)}\right\}, \quad \rho(\eta) = \frac{a(\eta)}{(m-1)g_1(\eta)}. \tag{8.9}$$

As a semibounded operator, it admits a unique Friedrichs self-adjoint extension [10], so that $\mathbf{A}'(g_1)$ is self-adjoint in $L^2_\rho(I)$ with $I = (1,\infty)$. Let us describe some of the spectral properties of $\mathbf{A}'(g_1)$; see [30].

Proposition 8.1. *(i) If $m \in (1, 3/2]$, then operator (8.7) in $L^2_\rho(I)$ has a discrete spectrum $\sigma(\mathbf{A}) = \{\lambda_k,\ k = 1,2,...\}$ which is a strictly monotone decreasing sequence of eigenvalues. The corresponding eigenfunctions $\Phi = \{\psi_k\}$ form an orthonormal basis in $L^2_\rho(I)$.*

(ii) If $m > 3/2$, then the spectrum is discrete and Φ is complete in the class of functions from $L^2_\rho(I)$ satisfying an additional condition at the finite singular point

$$Y(\eta) \text{ and } Y'(\eta) \text{ are bounded as } \eta \to 1^+. \tag{8.10}$$

(iii) For any $m > 1$, the centre subspace E^c of $\mathbf{A}'(g_1)$ is one-dimensional and is spanned by the eigenfunction

$$\psi_2(\eta) = c_2 \frac{d}{d\mu} g_\mu(\eta)|_{\mu=1} \equiv -c_2 \mathbf{D} g_1(\eta) \quad (\lambda_2 = 0), \tag{8.11}$$

where c_2 is a normalization constant.

Proof. (i), (ii). Consider first the singular point $\eta = \infty$. It follows from (8.9) that there exists a strong exponential dominance of the weight function for $\eta \gg 1$:

$$\rho(\eta),\ a(\eta) \sim \exp\left\{-c_0 \eta^\alpha\right\}, \quad c_0 > 0,$$

so that $\eta = \infty$ is always in the limit-point case and does not create a continuous spectrum. Consider the finite singular point $\eta = 1$. Since

$$g_1(\eta) = \frac{1}{\alpha}(\eta - 1) + \dots \quad \text{as} \ \ \eta \to 1^+,$$

we have that the behaviour of the eigenfunctions near $\eta = 1$ in the first approximation is governed by the second-order operator

$$\mathbf{A}'(g_1) Y = (m-1)(\eta - 1) Y'' + Y' + (2-\alpha) Y + \dots .$$

Hence, there exist two different types of asymptotics as $\eta \to 1$: if $m \neq 2$, then

$$Y_1(\eta) \sim 1 \ \ (Y, Y' \text{ are bounded}) \quad \text{and} \ \ Y_2(\eta) \sim (\eta - 1)^\nu,$$

where $\nu = (m-2)/(m-1)$. If $m = 2$, then $Y_2(\eta) \sim \ln(\eta - 1)$. Estimating the weight function near the singular point,

$$\rho(\eta) \sim (\eta - 1)^{-\nu},$$

we obtain that $Y_1 \in L^2_\rho$ and

$$Y_2 \notin L^2_\rho \quad \text{provided that} \ \ m \leq 3/2.$$

In this case $\eta = 1$, is in the limit-point case and the spectrum is discrete. If $m > 3/2$ then both solutions Y_1 and Y_2 belong to L^2_ρ near $\eta = 1$ (the limit-circle case), and in order to cancel the behaviour corresponding to the essentially non-smooth function Y_2, we impose the boundary condition (8.10), which is a standard one for singular Sturm-Liouville problems. Completeness of eigenfunctions is well-known [30].

(iii) Differentiating in μ the equation for g_μ, $\mathbf{A}(g_\mu) = 0$, one can check that $(g_\mu)'_\mu|_{\mu=1}$ is an eigenfunction of $\mathbf{A}'(g_1)$ with the eigenvalue $\lambda = 0$. Since this eigenfunction has a unique zero on $(1, \infty)$, it is the second one, and we arrive at (8.11). $\square$

The first positive eigenvalue $\lambda_1 > 0$ with the eigenfunction ψ_1, which does not change sign in I, corresponds to the natural unstability of the rescaled flow with respect to perturbations of the scaling moment T. On the subset of global solutions obtained with a fixed focusing time T, such an unstability is not available, since the orbits under consideration have the same focusing time. Therefore, the asymptotic behaviour in the perturbed dynamical system is expected to occur near a one-dimensional centre subspace E^c of the linear operator $\mathbf{A}'(g_1)$. By the classical parabolic regularity, we may assume that $Y(\tau)$

is sufficiently smooth, and $Y(\tau) \in L^2_\rho(I)$. We are looking for a solution in terms of the uniformly converging eigenfunctions expansion

$$Y(\tau) = \sum_{k=2}^{\infty} a_k(\tau)\psi_k \to 0 \quad \text{as } \tau \to \infty,$$

where the expansion coefficients satisfy a perturbed infinite-dimensional dynamical system obtained by substituting into equation (8.6) and multiplying it by ψ_k in $L^2_\rho(\mathbf{R}_+)$, $k = 1, 2, \dots$. Under the assumption of the centre subspace dominance, the first equation for $k = 2$ takes the form

$$a_2'(\tau) = -k\mu^{q-2}(g_1^q, \psi_2)_\rho + \frac{\mu'}{\mu}(\mathbf{D}g_1, \psi_2)_\rho + \dots, \quad \tau \gg 1, \tag{8.12}$$

where $(\cdot,\cdot)_\rho$ denotes the inner product in the weighted space $L^2_\rho(I)$. The second perturbation on the right-hand side is not integrable on $(1, \infty)$ ($\mu(\tau)$ is not bounded as $\tau \to \infty$), and hence we obtain the following evolution equation describing the centre manifold behaviour

$$\mu' = \gamma_0\mu^{q-1} + \dots \quad \text{for } \tau \gg 1, \quad \gamma_0 = \frac{k(g_1^q, \psi_2)_\rho}{(\mathbf{D}g_1, \psi_2)_\rho} > 0. \tag{8.13}$$

It follows that

$$\mu(\tau) = \gamma_1\tau^{1/(2-q)} + \dots, \quad \gamma_1 = [\gamma_0(2-q)]^{1/(2-q)}. \tag{8.14}$$

We then arrive at the following result.

Proposition 8.2. *Let $\mu(\tau)$ be unbounded as $\tau \to \infty$ and let the rescaled orbit approaches the centre subspace in $L^2_\rho(I)$. Then* (8.14) *holds.*

The assumption on the evolution centre subspace dominance in the asymptotic behaviour is essential and cannot be dispensed with. Due to known spectral properties of $\mathbf{A}'(g_1)$, equation (8.5) is assumed to admit a countable spectrum of other exponentially decaying patterns (though more unstable) corresponding to the behaviour close to the infinite-dimensional stable subspaces.

Finally, going back to the original variable, we deduce that in the critical case $p = p_c > -m$, the solution still focuses as $t \to T^-$ having a spatial profile close to the Graveleau solution, but scaling contains extra unbounded factors of slow growth. In particular, the interface of focusing solution has the following behaviour:

$$r_0(t) = \gamma_1(T-t)^{1/\alpha}|\ln(T-t)|^{(2-\alpha)/2} + \dots, \ \text{as} t \to T^-,$$

where the constant $\gamma_1 = \gamma_1(m, N) >$ does not depend on the initial data.

On the final-time profile. Once the asymptotic behaviour of $v(r,t)$ on shrinking compact subsets governed by the rescaled variable $\eta = \xi/\nu(\tau) \equiv r/(T-t)^{1/\alpha}\mu(\tau)$ is obtained, a standard compactness argument applies to extend the behaviour to small subsets $\{0 < r << 1\}$ at the focusing instance $t = T$; see different examples in [11], [8], [25], and [23]. This gives the following final-time profile:

$$v(r, T^-) = \gamma_2(r|\ln r|^{\alpha/2})^{2-\alpha}(1 + o(1)) \quad \text{as } r \to 0,$$

where we again observe an extra logarithmic multiplier making it non-scaling invariant.

8.3. **On other types of perturbations.** Using the above analysis, we now discuss how the type of perturbation (inherited from the general equation (1.2)) can affect the structure of the ω-limit set of a given orbit. Instead of (8.1) we consider the equation

$$\theta_\tau = \mathbf{A}(\theta) - \nu(\tau)\theta^q,$$

where $\nu(\tau) \to 0$ as $\tau \to \infty$ characterizes the order of asymptotically small non-autonomous perturbation of the rescaled PME. Then assuming that the slow motion hypothesis still takes place, we perform the scaling (8.4) and will arrive at the equation (cf. (8.5))

$$g_\tau = \mathbf{A}(g) - \nu(\tau)\mu^{q-2}(\tau)g^q + \frac{\mu'}{\mu}(\tau)\mathbf{D}g.$$

Performing the same analysis on the centre subspace and assuming that $\mu(\tau)$ is unbounded, we obtain the equation

$$\mu' = \gamma_0\nu(\tau)\mu^{q-1} + ... \quad \text{for } \tau \gg 1.$$

It follows that such an unbounded growth of $\mu(\tau)$ is possible if $\gamma(\tau) > 0$ for $\tau \gg 1$ and

$$\int^\infty \gamma(\tau)d\tau = +\infty,$$

i.e., the perturbation in the equation is non-integrable. If $\gamma \in L^1(\mathbf{R}_+)$ (the integral converges) then this suggests that $\mu(\tau)$ has a finite limit $\mu_* \in \mathbf{R}_+$ as $\tau \to \infty$, and hence $\omega(\theta_0) = \{g_{\mu_*}\}$ consists of a unique stationary Graveleau profile. On the other hand, if the integral diverges absolutely and integrals over bounded intervals are uniformly bounded (e.g., $\nu(\tau) \sim \cos\sqrt{\tau}/\sqrt{\tau}$ for $\tau \gg 1$), then such an asymptotically small perturbation is expected to create infinitely many slowing oscillations in the stationary family S_0, so that $\omega(\theta_0) = \{g_\mu,\, \mu_1 \le \mu \le \mu_2\}$ is a bounded uncountable subset of S_0. This natural conclusion shows that for perturbed DSs, a more detailed description of the ω-limit set essentially depends on the decay order of perturbations.

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