STABILITY OF PERTURBED NONLINEAR PARABOLIC EQUATIONS WITH STURMIAN PROPERTY

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ABSTRACT. We study stability of an equilibrium f^* of autonomous dynamical systems under asymptotically small perturbations of the equation. We show that such stability takes place if the domain of attraction of the equilibrium f^* contains a one-parametric ordered family $\mathcal{F}^*$. In the stability analysis we need a special S-relation (a kind of "restricted partial ordering") to be preserved relative to the family $\mathcal{F}^*$. This S-relation is inherited from the Sturmian zero set properties for linear parabolic equations. As main applications, we prove stability of the self-similar blow-up behaviour for the porous medium equation, the p-Laplacian equation and the dual porous medium equation in $\mathbf{R}$ with nonlinear lower-order perturbations. For such one-dimensional parabolic equations the S-relation is Sturm's Theorem on the nonincrease of the number of intersections between the solutions and particular solutions with initial data in $\mathcal{F}^*$. This Sturmian property plays a key role and is true for the unperturbed PME, but is not true for perturbed equations.

1. Introduction: perturbed dynamical systems and applications to blow-up problems

We consider a dynamical system in a metric space X,

$$v_t = \mathbf{A}(v) \quad \text{for } t > 0, \tag{1.1}$$

and a non-autonomous equation which is an asymptotically small perturbation of (1.1) written in the form

$$u_t = \mathbf{B}(u, t) \quad \text{for } t > 0. \tag{1.2}$$

The main goal is to show that, under some hypotheses on the equilibrium f^* of the unperturbed equation (1.1) and an extra topological structure of the flow in its neighbourhood, compact orbits of (1.2) in an appropriate class converge to f^*. Hence, the ω-limit set of the perturbed equation (1.2) is given by f^* meaning the stability of the ω-limit set of (1.1) under arbitrary small perturbation of the dynamical system.

1. Application: blow-up in non-linear heat equations. As main applications induced the abstract scheme, we study the problem of stability of self-similar blow-up for

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a number of one-dimensional heat equations of the form

$$u_t = \mathbf{C}(u) + g(x,t,u,u_x) \quad \text{in } \mathbf{R}\times(0,T), \tag{1.3}$$

where $\mathbf{C}$ is a scaling invariant nonlinear operator of the form

$$\mathbf{C}(u) = (u^m)_{xx} + u^p \quad \text{(the porous medium operator)}, \tag{1.4}$$

$$\mathbf{C}(u) = (|u_x|^{m-1}u_x)_x + u^p \quad \text{(the p-Laplacian operator)}, \tag{1.5}$$

$$\mathbf{C}(u) = |u_{xx}|^{m-1}u_{xx} + u^p \quad \text{(the dual porous medium operator)}, \tag{1.6}$$

with exponents $m>1$ and $p>1$ (other more general nonlinearities can be considered). The term $g(\cdot)$ in the right-hand side of (1.3) stays for a perturbation of the scaling invariant flow given by the autonomous equation

$$u_t = \mathbf{C}(u) \quad \text{in } \mathbf{R}\times(0,T). \tag{1.7}$$

Such unperturbed nonlinear heat equations are directly related to well-established models from combustion and detonation theory [24] describing finite time blow-up processes; see references in [4] and [21].

In some parameter ranges, the stable generic blow-up in the unperturbed equation (1.7) is described by self-similar solutions. Obviously, the autonomous equations are invariant under a group of scaling transformations and admit self-similar solutions of the standard form

$$u_*(x,t) = (T-t)^{-\alpha}f(\xi), \quad \xi = x/(T-t)^{\beta}, \ \ \alpha = 1/(p-1), \tag{1.8}$$

where $T>0$ is the blow-up time of the solution. One can calculate the exponents β: $\beta=(p-m)/2(p-1)<1/2$ for operator (1.4), $\beta=(p-m)/(p-1)(m+1)$ for (1.6) and $\beta=(p-m)/2m(p-1)$ for (1.6). Then $f\geq 0$ satisfies the following nonlinear ODEs:

$$\mathbf{A}(f) \equiv \mathbf{C}(f) - \beta\xi f' - \alpha f = 0, \quad \xi\in\mathbf{R}; \ \ f(\xi)\to 0 \ \text{ as } \xi\to\infty. \tag{1.9}$$

Existence and uniqueness of a unique similarity profile $f^* = f^*(\xi)$ in the range $\beta<0$ is well-known; see [10] for (1.4) and [6] for general operators. Stability of such similarity profiles relative to perturbation of initial data has been proved in [12] for (1.4) and in [7] for (1.4) - (1.6) by different evolution methods.

Consider now a solution $u(x,t)$ of the perturbed equation (1.3) blowing up at a finite $T>0$. It is natural to introduce the rescaled variables inherited from similarity solutions

$$\theta(\xi,\tau) = (T-t)^{\alpha}u(x,t), \ \xi = x/(T-t)^{\beta}, \ \ \tau = -\ln(T-t). \tag{1.10}$$

The rescaled solution $\theta(\xi,\tau)$ satisfies the perturbed equation

$$\theta_\tau = \mathbf{B}(\theta,\tau) \equiv \mathbf{A}(\theta) - G(\xi,\tau,\theta,\theta_\xi) \quad \text{in } \mathbf{R}_+\times(\tau_0,\infty), \quad \tau_0 = -\ln T, \tag{1.11}$$

and the perturbation takes the form

$$G(\xi,\tau,\theta,\theta_\xi) = e^{-(\alpha+1)\tau}g(\xi e^{-\beta\tau}, T-e^{-\tau}, e^{\alpha\tau}\theta, e^{(\alpha+\beta)\tau}\theta_\xi) \tag{1.12}$$

We assume that

$$G(\cdot,\tau,\cdot,\cdot)\to 0 \quad \text{as } \tau\to\infty \tag{1.13}$$

on compact orbits. We thus arrive at equation (1.2) which is an asymptotically small perturbation of the autonomous one (1.1). We then need to prove that small perturbations do not change the asymptotic behaviour of compact orbits. We will discuss some other applications to finite-time extinction problems.

Using a class of perturbed blow-up problems, we show that the intersection comparison principles based on Sturm's Theorem can be applied to the asymptotic analysis of non-autonomous perturbed equations in order to study omega-limit sets of compact orbits. This is done by a modification of the notion of stability of f^* which now includes an oriented intersection S-relation inherited by the limit autonomous equation. In Sections 2 we present an abstract framework of the approach. In Sections 3 and 4 we consider applications to stability in blow-up and extinction problems.

2. **Hypotheses and the S-Theorem**

We begin with the main notions, properties and hypotheses related to the corresponding counterparts occurring in the above applications to blow-up problems.

Let X be a complete metric space with the distance function $d(\cdot,\cdot)$. In the application to parabolic blow-up problems, where the Sturm Theorem on zero sets plays a key role, the space $X = C^1(I)$, where $I \subset \mathbf{R}$ is a bounded closed interval, is a natural metric for using Sturmian intersection properties. Then transversal intersections with stationary profiles in the autonomous evolution are stable by the standard parabolic regularity results and the Maximum Principle. In main applications, due to a special geometric structure of solutions and the standard regularity theory, we also can take $X = C(I)$.

We deal with a bounded class $\mathcal{L}$ of solutions $u \in C([0,\infty) : X)$ of (1.2) defined for every $t > 0$ with values in X. By $U = \{u_0\}$ we denote the corresponding bounded subset of admissible initial data. Actually, the analysis is based on metric-topology arguments applied to families of curves $\{u(t)\}$ and $\{v(t)\}$, which are formally treated as solutions of the abstract equations (1.2) and (1.1) respectively. Therefore, we do not need to specify in which sense solutions satisfy the equations and deal just with continuous solutions.

Denote by $\omega(u_0)$ the ω-limit set of an orbit $\{u(t), t > 0\} \subset \mathcal{L}$ of equation (1.2) with initial data $u_0 \in U$

$$\omega(u_0) = \{f \in X : \text{ there exists a sequence } \{t_j\} \to \infty \text{ such that } u(t_j) \to f\},$$

which is assumed to be compact subset of X.

By $\{\varphi_t^*\}$ we denote a continuous semigroup induced by the autonomous equation (1.1), globally defined on a bounded subset V^* of admissible initial data. The corresponding bounded class of solutions $v(t) = \varphi_t^*(v_0) \in C([0,\infty) : X)$ with $v_0 \in V^*$ is denoted by $\bar{\mathcal{L}}^*$. However, we only deal with a "restricted" class $\mathcal{L}^* \subseteq \bar{\mathcal{L}}^*$ and its corresponding subset U^* of initial data. Both subsets are characterized later on. By $\omega^*(v_0)$ with $v_0 \in U^*$, we denote the corresponding ω-limit set. Let f^* be an equilibrium

$$\varphi_t^*(f^*) \equiv f^*.$$

Let us present the main hypotheses.

(H1) COMPACTNESS OF THE ORBITS OF (1.2). We assume that, for any data $u_0 \in U$, orbit $\{u(t),\, t > 0\}$ is relatively compact in X, and if

$$u^s(t) \equiv u(t+s), \quad t, s > 0,$$

then the sets $\{u^s\}$ are relatively compact in $L^\infty_{\rm loc}([0,\infty) : X)$.

(H2) CONVERGENCE OF EQUATIONS. This means that $\mathbf{B}(\cdot, t)$ is a small perturbation of $\mathbf{A}(\cdot)$ in the sense that given a solution $u(t) \in \mathcal{L}$ of (1.2), if for a sequence $\{t_j\} \to \infty$ the sequence $\{u(t_j + t)\}$ converges in $L^\infty_{\rm loc}([0,\infty) : X)$ as $j \to \infty$ to a function $v(t)$, then $v(t) \in \mathcal{L}^*$ is a solution of (1.1).

Next, we introduce key hypotheses including a topological (oriented intersection, in applications) S-relation of partial ordering induced by the non-perturbed evolution driven by equation (1.1). We present first the hypotheses related to the autonomous equation.

(H3) ORDERED INVARIANT ONE-PARAMETRIC FAMILY FROM DOMAIN OF STABILITY. Let $W^s(f^*)$ be the domain of attraction (asymptotic stability) of the equilibrium f^*

$$W^s(f^*) = \{u_0 \in U : \ \varphi_t(u_0) \to f^* \ \text{ as } \ t \to \infty\}.$$

We assume that there exists a one-parametric continuous set $\mathcal{F}^* = \{f_\mu,\, \mu \in (\mu_1, \mu_2)\} \subset W^s(f^*)$ such that $f_{\mu^*} = f^*$ for some $\mu^* \in (\mu_1, \mu_2)$. Each closed subinterval $\{f_\mu,\, \mu_1 < a \le \mu \le b < \mu_2\}$ is relatively compact in X.

The family $\mathcal{F}^*$ is one-parametric and we assume that it admits a total ordering denoted by $\preceq$ in the sense that $f_\mu \preceq f_\nu$ (or $f_\nu \succeq f_\mu$) for all $\mu \le \nu$. Moreover, $f_\mu \prec f_\nu$ for all $\mu < \nu$, i.e., $f_\mu \ne f_\nu$.

For any $\mu \in (\mu_1, \mu_2)$, denote

$$F_\mu(t) \equiv \varphi_t(f_\mu) \to f^* \quad \text{as} \quad t \to \infty. \tag{2.1}$$

The *invariance* of the family means that for every $\mu \in (\mu_1, \mu_2)$ and $t > 0$, $F_\mu(t) \in \mathcal{F}^*$. i.e., $F_\mu(t) \equiv f_{\rho(t)}$ for some continuous function $\rho(t)$ with $\rho(0) = \mu$. This implies that for the autonomous equation (1.1) we need to specify two orbits $\{F_\pm(t),\, t \in \mathbf{R}\}$, $F_\pm(t) \to f^*$ as $t \to \infty$ satisfying $F_-(t_1) \prec F_-(t_2)$ and $F_+(t_1) \succ F_+(t_2)$ for any $t_1 < t_2$ and $F_-(t) \prec f^* \prec F_+(t)$ for any $t \in \mathbf{R}$.

(H4) ASYMPTOTIC STRUCTURAL PROPERTIES AND INTERSECTION S-RELATION FOR (1.1). (i) *Asymptotic transversality of* $\mathcal{F}^*$: there exist $\mu_1 < \nu < \mu < \mu_2$ such that for any $v_0 \in U^*$,

$$f_\nu \prec v_0 \prec f_\mu. \tag{2.2}$$

(ii) *S-relation and S-semigroup.* We assume that the total ordering in $\mathcal{F}^*$ can be extended as a binary relation for solutions in $\mathcal{L}^*$. As we have mentioned, in the applications this S-relation is induced by the Sturmian intersection property. It can be classified as a "restricted partial ordering" of solutions $v(t) \in \mathcal{L}^*$ with data from U^* and elements of $\mathcal{F}^* = \{f_\mu\}$.

The S-relation $\preceq$ satisfies two properties of partial ordering for any $v, v_1, v_2 \in U^*$:
(i) (reflexivity) $v \preceq v$, and

(ii) (antisymmetry) $v_1 \preceq v_2$ and $v_2 \preceq v_1$ imply $v_1 = v_2$.

The constraint of S-relation induced by the subset $\mathcal{F}^*$ of particular elements does not satisfy the transitivity property, i.e., $v_1 \preceq f_\mu \preceq v_2$ does not imply that $v_1 \preceq v_2$ for any $v_1,\, v_2 \in U^*$. Actually, such relation is defined *relative* to the elements of $\mathcal{F}^*$ only, and we do not define any partial ordering in U^* or $\mathcal{L}^*$.

Later on we will use $\preceq$ as a standard ordering relation, so that $v \prec f_\mu$ means that $v \preceq f_\mu$ and $v \neq f_\mu$. The S-relation is assumed to be *closed* meaning that for any convergent sequence $\{v_n\} \subset U^*$ there holds

$$v_n \preceq f_\mu \quad \text{and} \;\; v_n \to \bar f \implies \bar f \preceq f_\mu. \tag{2.3}$$

Let us present the main hypothesis on the autonomous evolution: the semigroup φ_t^* on U^* induced by equation (1.1) preserves the S-relation relative to the set $\mathcal{F}^*$ (and is called an S-semigroup) in the following sense: given a $\mu > 0$,

$$v_0 \prec f_\mu \;(v_0 \succ f_\mu) \implies v(t) \prec F_\mu(t) \;(v(t) \succ F_\mu(t)) \;\; \text{for all} \;\; t > 0. \tag{2.4}$$

Moreover, although is not always necessary for the asymptotic analysis, in main applications the semigroup φ_t^* is *strong* S-semigroup, i.e., for any given $\mu > 0$, $t_0 \geq 0$ and arbitrarily small $\tau > 0$, there exists a $\delta > 0$ such that

$$v(t_0) \prec F_\mu(t_0) \;(v(t_0) \succ F_\mu(t_0)) \implies v(t_0+\tau) \prec F_{\mu-\delta}(t_0+\tau) \;(v(t_0+\tau) \succ F_{\mu+\delta}(t_0+\tau)).$$

Dynamical systems generating order-preserving semigroups satisfy a number of fundamental properties, and their asymptotic behaviour is well understood, see books [16], [22] and recent papers [18], [19].

Our applications to blow-up singularities in reaction-diffusion or reaction-absorption equations deal with dynamical systems admitting no partial ordering between solutions having the same blow-up time. One can see that the S-relation of restricted partial ordering in (H4) which mimics the Sturm Theorem on zero sets for linear parabolic equations, expresses purely geometric intersection properties of curves and cannot be extended to any partial ordering of solutions.

In the applications to reaction-diffusion parabolic problems, such a geometric topological constraint means that the functions $f_\mu(\xi) \in \mathcal{F}^*$ must intersect each other with a fixed number of intersections. In typical applications with radial symmetry, such intersection is always unique and transversal. The same is supposed to be true for more general solutions. Otherwise, if a sufficient number of intersections is not available, such solutions do not belong to a suitable functional class $\mathcal{L}$. The Maximum Principle for one-dimensional parabolic equations stated as the first Sturm Theorem on non-increase of the number of intersections between solutions, can be treated as a dissipativity-like property of parabolic evolution but is not an order-preserving property of parabolic flows. Under special conditions, the number of intersections $\mathrm{Int}(t)$ between two given solutions can be fixed for special geometric configurations, so that $\mathrm{Int}(t)$ *does not increase* by Sturm's Theorem and, on the other hand, *does not decrease* if it is the minimal admissible one generated by the topology structure of the set $\mathcal{F}^*$ (and the corresponding orbits on a stable manifold of f^*). Under these special circumstances, existence of a fixed number of intersections can be treated as a *oriented* intersection constraint. Given two solutions of the rescaled autonomous parabolic equation, $\theta(\xi,t)$ and $F_\mu(\xi,t)$, the total number of intersection $\mathrm{Int}(t)$ should be taken into account with a local character of intersections. We then count the intersections with signs. For instance, consider the case of a single intersection so that $|\mathrm{Int}(t)|=1$. We say that $\mathrm{Int}(t)=+1$ if the difference $\theta(\xi,t)-F_\mu(\xi,t)$ changes sign from $-$ to $+$ when increasing ξ crosses the intersection, and $\mathrm{Int}(t)=-1$ otherwise. The mutual geometric locations with $\mathrm{Int}(t)=+1$, denoted by $\theta(\cdot,t)\prec F_\mu(\cdot,t)$, and $\mathrm{Int}(t)=-1$, denoted by $\theta(\cdot,t)\succ F_\mu(\cdot,t)$, differ from each other. Moreover, these cannot be transformed into each other via evolution since by continuity and standard intersection comparison arguments we would conclude that $\mathrm{Int}(t_0)=0$ for some t_0, and hence $\mathrm{Int}(t)\equiv 0$ for any $t>t_0$ by the usual comparison (the MP) for parabolic equations. Moreover, this would mean that $\theta(\xi,t)\notin\mathcal{L}$.

We now state the main stability theorem (the S-Theorem, for short).

Theorem 2.1. *Under hypotheses* (H1)-(H4)*, for any $u_0\in U$ there hold*

$$\omega(u_0)=f^*.$$

We begin with a key proposition providing us with a consistency between metric and topology for the unperturbed equation restricted to the ω-limit set of the perturbed equation.

Proposition 2.1. *Given a solution $\{u(t)\}\in\mathcal{L}$ of* (1.2) *with $u_0\in U$, for any $\varepsilon>0$, there exists $\lambda_0=\lambda_0(\varepsilon)>0$ satisfying $\lambda_0(\varepsilon)\to 0$ as $\varepsilon\to 0$, such that for a $f\in\omega(u_0)$, there holds*

$$f_{\mu^*-\lambda}\preceq f\preceq f_{\mu^*+\lambda}\ \ for\ a\ \lambda\in(0,\lambda_0(\varepsilon)]\ \Longrightarrow\ d(f,f^*)<\varepsilon. \tag{2.5}$$

Proof. Assume for contradiction that there exists a constant $\varepsilon_0>0$ and sequences $\{\lambda_n\}\to 0$ and $\{f_n\}\subset\omega(u_0)$ such that

$$f_{\mu^*-\lambda_n}\preceq f_n\preceq f_{\mu^*+\lambda_n}\quad\text{but}\quad d(f_n,f^*)\geq\varepsilon_0. \tag{2.6}$$

Since by the assumption, $\{f_n\}$ is a compact sequence, there exists a converging subsequence $\{f_{n_k}\}\to\bar f\in X$. Since the restricted partial ordering is closed we obtain $f^*\preceq\bar f\preceq f^*$. Hence $\bar f=f^*$ and a contradiction with (2.6) follows. $\square$

Proof of Theorem 2.1. Consider a bounded orbit $\{u(t),\,t>0\}$ with $u_0\in U$. Let us prove that

$$u(t)\to f^*\text{ as }\ t\to\infty. \tag{2.7}$$

We argue by contradiction. Assume that there exist a constant $\varepsilon>0$ and a sequence $\{t_j\}\to\infty$ such that

$$d(u(t_j),f^*)>\varepsilon,\quad j=1,2,\ldots. \tag{2.8}$$

Fix $\lambda_0=\lambda_0(\varepsilon/2)$ with λ_0 being the function defined in Proposition 2.1. Let f_ν and f_μ be the profiles from (2.2), and consider the corresponding solutions $F_\nu(t)$ and $F_\mu(t)$. By

(2.1), there exists $t_0 \gg 1$ such that

$$F_\nu(t_0) \equiv f_{\rho_1(t_0)} \;\text{ and }\; F_\mu(t_0) \equiv f_{\rho_2(t_0)}, \tag{2.9}$$

where $0 < \mu^* - \rho_1(t_0) < \lambda_0$ and $0 < \rho_2(t_0) - \mu^* < \lambda_0$. Consider the sequence $\{\hat{t}_j = t_j - t_0\} \to \infty$ with $u(\hat{t}_j) \to g \in \omega(u_0)$ as $j \to \infty$ choosing a subsequence if necessary. From (2.2) we have that

$$f_\nu \preceq g \preceq f_\mu. \tag{2.10}$$

On the other hand, by (H2) choosing a subsequence, we have that $u(\hat{t}_j + t) \to v(t)$ locally, where $v(t)$ is a solution of (1.1) with initial data $v(0) = g \in \omega(u_0)$. Hence,

$$d(v(t), u(\hat{t}_j + t)) \le \varepsilon/2 \quad \text{ for } j \gg 1, \tag{2.11}$$

uniformly on $t \in [0, t_0]$.

It follows from (2.10) and (2.11) and the S-relation property in (H4), that

$$F_\nu(t) \preceq v(t) \preceq F_\mu(t), \quad t \in [0, t_0]. \tag{2.12}$$

Taking into account (2.9), we have that $v(t_0) \in \omega(u_0)$ satisfies

$$f_{\mu^*-\lambda} \preceq v(t_0) \preceq f_{\mu^*+\lambda} \quad \text{with a } \lambda \in (0, \lambda_0(\varepsilon/2)].$$

Then Proposition 2.1 guarantees that

$$d(v(t_0), f^*) < \varepsilon/2. \tag{2.13}$$

Hence, for all $j \gg 1$,

$$d(u(t_j), f^*) = d(u(\hat{t}_j + t_0), f^*) \le d(u(\hat{t}_j + t_0), v(t_0)) + d(v(t_0), f^*) \le \varepsilon \tag{2.14}$$

contradicting (2.8). $\square$

3. Applications to stability in blow-up problems

We apply the above S-Theorem to equation (1.3) with operators (1.4)–(1.6). After rescaling (1.10) we arrive at the perturbed equation (1.11) to be compared with the corresponding unperturbed, autonomous one

$$\theta_\tau = \mathbf{A}(\theta), \quad \tau > 0. \tag{3.1}$$

ODEs for the similarity profiles take the form (1.9). In what follows, we denote by $f^*(\xi) \ge 0$, $f^* \not\equiv 0$, a non-trivial rescaled similarity profile (equilibrium)

$$\mathbf{A}(f^*) = 0 \quad \text{in } \mathbf{R}, \quad f^*(\xi) \to 0 \text{ as } \xi \to \infty.$$

In the case of quasilinear operators (1.4) and (1.5), f^* is a weak solution, while for the fully nonlinear equation with operator (1.6), it is a mild solution or a proper maximal solution constructed by monotone approximations via uniformly parabolic equations; see [5], [15] and references therein. Same definitions apply to the rescaled solution $\theta(\xi, \tau)$ (and $u(x, t)$) so that θ is a unique weak solution of the first two equations and mild, maximal one for the third equation.

In the case $\beta < 0$, any stationary profile $f^*(\xi)$ is compactly supported since the ODEs do not admit positive solutions $f(\xi)$ vanishing at infinity. The proof in Chapter 4 [21] for the ODE with operator (1.4), $\beta < 0$, is extended to other ones. Moreover, by similar arguments one can establish that there exists a non-trivial, symmetric, compactly supported equilibrium f^* (see details in [6]) which will be proved to be unique and asymptotically stable.

3.1. **Application of the S-Theorem.** Let $X = C(I)$ be the space of bounded and continuous compactly supported functions defined on sufficiently large closed symmetric interval $I = [-k, k]$, $k \gg 1$, with the distance function given by the L^∞-norm $\| \cdot \|$. The solution class $\mathcal{L}$ is

$$\mathcal{L} = \{\theta(\tau) \geq 0 : \ \theta(\tau) \in X \text{ satisfies (1.11) and } c_1 \leq \|\theta(\xi, \tau)\| \leq c_2 \ \forall \ \tau \geq \tau_0\}, \tag{3.2}$$

where $c_1 = c_1(k) > 0$ is a sufficiently small constant and the constant $c_2 = c_2(k) > c_1$ is large enough. We denote by U the class of such smooth compactly supported initial data θ_0 such that the corresponding solution $\theta \in \mathcal{L}$. The class $\mathcal{L}^*$ of solutions $\theta^*(\tau)$ of the unperturbed equation (3.1) is defined in the same way, and by U^* we denote the bounded set of the corresponding initial data θ_0^*.

We next prove via the S-Theorem that for any positive k, there holds $\mathcal{L} \subset W^s(f^*)$. In order to apply the S-Theorem, we need to check hypotheses (H1)–(H4). To begin with, we note that interior regularity results for parabolic equations guarantee the compactness and convergence hypotheses (H1) and (H2). To see that $\omega(\theta_0) \in U^*$ for every $\theta_0 \in U$ we argue as follows. Assume for contradiction that there exists $f = \lim_{j \to \infty} \theta(\tau_j) \in \omega(\theta_0)$ which does not belong to U^*. This means that one of the bounds in the definition of $\mathcal{L}^*$ is not true for the corresponding solution $\theta^*(\xi, \tau)$ and some $\tau_1 > \tau_0$. By the convergence $\theta(\xi, \tau + \tau_j) \to \theta^*(\xi, \tau)$ uniformly on $[\tau_0, \tau_1] \times \mathbf{R}$ we deduce that the assumption holds for $\theta(\xi, \tau_0 + \tau_j)$ for any $\tau_j \gg 1$ contradicting the definition of $\mathcal{L}$.

Consider the crucial hypotheses (H3)–(H4) dealing with the class of solutions $\mathcal{L}^*$ for the unperturbed equation (3.1) (cf. [7]).
(H3) Let us introduce the family $\mathcal{F}^* \subset W^s(f^*)$. We define functions f_μ for every $\mu \in \mathbf{R}$ by translation

$$f_\mu(\xi) = f^*(\xi - \mu) \quad \text{in } \mathbf{R}.$$

The corresponding solution $F_\mu(\tau)$ of the unperturbed parabolic equation (3.1) is given explicitly

$$F_\mu(\tau) = \varphi_t(f_\mu) \equiv f^*(\xi - \mu e^{\beta\tau}) \quad \text{in } \mathbf{R} \times \mathbf{R}_+.$$

Since $\beta < 0$, we have that for any $\mu \in \mathbf{R}$, $F_\mu(\tau) \to f^*$ as $\tau \to \infty$ uniformly on compact subsets in $\mathbf{R}$, so that $\mathcal{F}^* \subset W^s(f^*)$. Thus,

$$\mathcal{F}^* = \{f_\mu, \, \mu \in \mathbf{R}\}$$

is a continuous one-parametric family of functions satisfying $f_{\mu^*} \equiv f^*$ for $\mu^* = 0$ and

$$f_\mu(\cdot) \to 0 = f_\infty \quad \text{as } \mu \to \pm\infty. \tag{3.3}$$

The total ordering $\preceq$ in $\mathcal{F}^*$ is straightforward. It characterizes the number and the character of intersection of profiles from $\mathcal{F}^*$. We say that $f_\lambda \prec f_\nu$ if these profiles intersect

each other exactly once and the difference $f_\nu(\xi) - f_\lambda(\xi)$ has a change of sign from $-$ to $+$ at the intersection. Then $f_\lambda \preceq f_\nu$ means that either $f_\lambda \prec f_\nu$ or $f_\lambda = f_\nu$.
(H4) (i) *Asymptotic transversality of $\mathcal{F}^*$.* The property (2.2) follows from obvious geometric structure of the family $\mathcal{F}^*$. In particular, it suffices to observe that the interfaces $a_\mu < b_\mu$ of the function $f_\mu(\xi)$ satisfy

$$a_\mu \to \pm\infty, \;\; b_\mu \to \pm\infty \;\; \text{as} \;\; \mu \to \pm\infty.$$

This implies that the asymptotic transversality assumption holds for any initial data with compact support.

(ii) *S-relation and S-semigroup.* We keep the same definition of $\prec$ and $\preceq$ as in (H3) for functions $\theta_0^* \in U^*$. Obviously, both the reflexivity and antisymmetry properties of the S-relation hold. In order to show that the S-relation is closed in the sense of (2.3) and that the S-semigroup property (2.4) holds, we need to prove special intersection properties characterizing solutions in $\mathcal{L}^*$. The proof relies on the existence of a two-parametric family of functions $\mathcal{G}^* = \{f_\mu^\lambda\}$ from the local domain of unstability $W^u(f^*)$ of the equilibrium f^*. These functions are defined as follows:

$$f_\mu^\lambda(\xi) = \lambda^{-\alpha} f^*((\xi - \mu)\lambda^{-\beta}), \quad \mu \in \mathbf{R}, \;\; \lambda \in \mathbf{R}.$$

The corresponding solutions of the PDE (3.1) are given explicitly

$$F_\mu^\lambda(\xi, \tau) = (1 - (1 - \lambda)e^\tau)^{-\alpha} f^*((\xi - \mu e^{\beta\tau})(1 - (1 - \lambda)e^\tau)^{-\beta}). \tag{3.4}$$

These rescaled solutions are obtained from the self-similar ones for the original PDE by translations in both independent variables x and t. We have that for any fixed μ, solutions (3.4) stabilize as $\tau \to \infty$ to the trivial one $f_\infty \equiv 0$ if $\lambda > 1$ and blow-up in finite time if $\lambda < 1$. For $\lambda = 1$ we are given the family $\mathcal{F}^*$ from the domain of stability of f^*.

We now state the main intersection property of parabolic evolution in $\mathcal{L}^*$. Given a solution $\theta \in \mathcal{L}^*$, by $I_\mu(\tau) \equiv \mathrm{Int}(\tau, f_\mu)$ we denote the number of intersections of $\theta(\xi, \tau)$ and the function $f_\mu(\xi)$.

Lemma 3.1. *Let $\theta(\tau) \in \mathcal{L}^*$ and $\theta_0 \notin \mathcal{F}^*$. Then*
(i) for any $\mu \in \mathbf{R}$, $I_\mu(\tau) > 0$ for $\tau \geq 0$, and
(ii) $\mathrm{Int}(g, f^*) > 0$ for any $g \in \omega(\theta_0)$.

Proof. (i) Assume for contradiction that there exist $\tau_0 > 0$ and ν such that $I_\nu(\tau_0) = 0$ so that the profiles do not intersect each other and hence are ordered uniformly in $\mathbf{R}$. We may assume without loss of generality that $\nu > 0$.

We begin with the case $\theta(\xi, \tau_0) \leq f_\nu(\xi)$ for every $\xi \in \mathbf{R}$. Then, by the usual comparison we have that $I_\nu(\tau) = 0$ and $\theta(\tau) \leq F_\nu(\tau - \tau_0)$ for every $\tau \geq \tau_0$. Assume for a moment that at $\tau = \tau_0$ no contact points are available on the interfaces. Then, by applying the strong Maximum Principle to solutions in the domain of strict positivity (and hence sufficient regularity), we have that the solutions are strictly separated at $\tau = \tau_0 + \delta$ with $\delta > 0$ being arbitrarily small. By continuity the same is true for θ and f_ν^λ if $\lambda \approx 1^-$. By considering the associated non-stationary solution $F_\nu^\lambda(\xi, \tau - \tau_0)$ after shifting in time, we get that $\theta(\cdot, \tau) \to 0$ as $\tau \to \infty$ uniformly in $\mathbf{R}$. This contradicts the assumption $\theta \in \mathcal{L}$.

If $\theta(\xi, \tau_0) \geq f_\nu(\xi)$ in $\mathbf{R}$, the proof is similar by choosing $\nu \approx 1^+$. In this case we obtain that the solution blows up in finite time again contradicting the same assumption.

Finally, let us prove that there exists a $\tau_1 > \tau_0$ when interfaces are separated and hence the previous argument applies at the moment $\tau_1 + \delta$ for the profile f_γ with $\gamma = \nu$ or $\gamma \approx \nu$. If interfaces are not separated for every $\tau \geq \tau_0$, we fix $\tau^* \gg 1$ such that function $F_\nu^\lambda(\tau - \tau^*)$ has a unique intersection with θ_0 by choosing $\lambda \approx 1^-$. It follows from this construction and obvious properties of the family $\mathcal{G}^*$ that $I_\mu^\lambda(\tau) \equiv \mathrm{Int}(\tau^*, f_\mu^\lambda) \geq 2$ whence a contradiction.

(ii) Given a sequence $\{\tau_k\} \to \infty$, we pass to the limit $\tau_k + \tau \to \infty$ so that $\theta(\tau_k + \tau) \to \tilde{\theta}(\tau)$. Then the same argument applies to the solution $\tilde{\theta} \in \mathcal{L}^*$ of the equation (1.11) with initial data $g \in \omega(\theta_0)$. $\square$

Let us show that the S-relation is closed in the sense of (2.3). Indeed, if $\bar{f} \preceq f_\mu$ does not hold, then, by construction, the only possibility is that $\bar{f} \not\equiv f_\mu$ does not intersect f_μ. Since $\bar{f} \in U^*$, this contradicts Lemma 3.1, (i).

Finally, we prove that $\{\varphi_t^*\}$ is an S-preserving semigroup preserving the S-relation. This is a consequence of the Sturm Theorem and Lemma 3.1. Assume that $\theta_0^* \preceq F_\mu(0)$. This means that $I_\mu(0) = +1$, i.e., there exists a unique intersection of the profiles and the difference $F_\mu(\xi, 0) - \theta_0^*(\xi)$ changes sign from $-$ to $+$ at the intersection. For degenerate equations admitting weak solutions, intersections can be a point or an interval. By the Sturm Theorem and Lemma 3.1, it follows that $I_\mu(\tau) \equiv +1$ for every $\tau > 0$ and hence, the same local character of the intersection is preserved in time. Therefore $\theta^*(\tau) \prec F_\mu(\tau)$ for any $\tau > 0$. By repeating the arguments relative to the profile f_ν and the solution $F_\nu(\tau)$, we obtain the opposite estimate $\theta^*(\tau) \succ F_\nu(\tau)$ for all $\tau > 0$.

Hence, (H1)–(H4) hold and this provide us with the following conclusion.

Theorem 3.1. *Any bounded and compactly supported rescaled solution $\theta(\xi, \tau)$ of the parabolic equation* (1.11)*, with operators* (1.4)*,* (1.5) *and* (1.6) *satisfies one of the following properties: as $\tau \to \infty$,*

$$\textit{either} \ \ (i) \ \ \theta(\tau) \to 0 \quad \textit{or} \quad (ii) \ \ \theta(\tau) \to f^*$$

uniformly in $\mathbf{R}$.

As a straightforward consequence, we have that if $\theta(\tau) \in \mathcal{L}$, then the solution satisfies (ii). The estimates appearing in the definition of $\mathcal{L}$ may strongly depend on the perturbation term G of the equation and the same is true in constructive characterizations of the sets U and $\mathcal{L}$. In the next section, we choose a special case showing such constructive characterizations. The same technique is extended to more general perturbations G satisfying (1.13) though this can cause some specific technical difficulties depending on G. The application below shows how the invariance properties of the semigroup φ_t can be used in order to get estimates for a "good" set of initial data. This set provides us with a family $\hat{\mathcal{F}}^*$ satisfying (H3) and (H4) with respect to the perturbed equation (1.11). Hence, although this equation is not autonomous, the classical intersection comparison arguments apply to any solution of (1.11) and the solutions with initial data in the family

$\hat{\mathcal{F}}^*$. We note that now we are comparing solutions of the same equation (1.11). Then, the stability theory in [6] applies without changes, and the convergence result follows for a wide class of initial data. Such an approach allows to avoid rather delicate calculations and estimates for general orbits.

4. Global blow-up in nonlinear heat equations with source and other applications

In the first application, we consider a general autonomous perturbation of the form $g = g(u)$ and such that the rescaled perturbation G in (1.12) satisfies (1.13), i.e.,

$$h(u) = u^{-p} g(u) \to 0 \quad \text{as } u \to \infty. \tag{4.1}$$

We also assume that the function $h(u)$ is globally bounded. It is convenient that with this choice of general autonomous perturbations, some of the invariant properties of the semigroup φ_t^* remain valid for φ_t induced by the perturbed equation. In particular, we will use the following properties of (1.3):

(i) given symmetric and decreasing for $x > 0$ initial data u_0, the same holds for the corresponding solution $u(x,t)$ for every $t > 0$ (this follows from the Maximum Principle), and

(ii) solutions are translationally invariant, $u_\lambda^\gamma(x,t) = u(x+\lambda, t+\gamma)$ is also a solution of the equation.

Denote by $\mathcal{L}_k$ the subset $\mathcal{L}$ depending on the length $2k$ of the interval $[-k,k]$. We have the following characterization of the subset $\mathcal{L}$ corresponding to the space $C_0(\mathbf{R})$ of continuous compactly supported functions.

Theorem 4.1. *Define*

$$U = \{u_0 \in C_0(\mathbf{R}) \;\; \textit{such that} \;\; u(x,t) \;\; \textit{blows up at} \;\; T = 1\},$$

and let $\mathcal{L}$ be the set given by

$$\mathcal{L} = \{\theta(\tau) \in C_0(\mathbf{R}) \; \textit{is rescaled solution of} \; (1.11), \; (4.1) \; \textit{with initial data} \; \theta_0 \in U\}.$$

Then

$$\mathcal{L} = \cup_{k \in \mathbf{R}_+} \mathcal{L}_k \, .$$

Theorems 3.1 and 4.1 guarantee that any solution $u \in \mathcal{L}$ of the perturbed equations with arbitrary perturbations from the class (4.1) has the asymptotic self-similar behaviour with the unique rescaled similarity profile f^*.

Proof. Step 1: symmetric and decreasing initial data. We first deal with this kind of initial data for which the estimates involved in the definition of $\mathcal{L}_k$ can be obtained much simpler thanks to space symmetry and monotonicity properties.

Let $u_0 = \theta_0 \in U$. Let us prove that the corresponding rescaled solution $\theta(\xi,\tau)$ belongs to $\mathcal{L}_k$ for some sufficiently large $k > 0$. This means, keeping in mind the definition of $\mathcal{L}_k$, that it is globally bounded from above with compact support in I_k for every $\tau \geq 0$ and does not tend to the trivial zero solution, see (3.2). We may assume that the function h is negative. The same proof with minor changes in the construction of sub and

supersolutions applies without this extra assumption. The result is a consequence of the following estimates.
1. A lower bound. If $u_0 = \theta_0 \in U$, the following estimate holds:

$$\|\theta(\tau)\| > c^* = \alpha^{-\alpha} \quad \text{for any } \tau \geq 0.$$

Indeed, if this is false and $\|\theta(\tau_0)\| \leq c^*$ for some $\tau_0 \geq 0$, in the original variables we obtain that

$$u(x, t_0) \leq H(t_0) \equiv c^*(1-t_0)^{-\alpha}, \quad \text{with } t_0 = 1 - e^{\beta\tau_0}.$$

Hence, taking into account that $H(t+c)$ is a supersolution of equation (1.1) for every $c \in \mathbf{R}$ and by the strong Maximum Principle, we obtain that for a fixed positive $\delta \ll 1$ and arbitrarily small $\varepsilon > 0$, $u(x, t_0+\delta) \leq H(t_0 - \varepsilon + \delta)$. By comparison, the same inequality holds for $t \geq t_0 + \delta$ and hence $u(x,t)$ does not blow-up at time $T = 1$ contradicting the assumption.
2. An upper bound. Let $\pm a$ be the interfaces of the symmetric equilibrium $f^*(\xi)$. On one hand, let us prove that there exists $\tau_0 \geq 0$ such that

$$\theta(\xi, \tau) \leq f^*(0) + 1 \quad \text{for all } \tau \geq \tau_0 \quad \text{and every } |\xi| \geq a.$$

Assume that this is false for some $\xi_0 \geq a$ and a sequence $\{\tau_j\} \to \infty$. Then, by the decreasing and symmetric hypotheses, the same holds for every $\xi \in [-\xi_0, \xi_0]$. Using that $h(u)$ is globally bounded, we construct a subsolution in the following way. First of all, consider small enough fixed values of ε and ρ such that for every j,

$$\theta(\xi, \tau_j) \geq (1-\rho)^{-\alpha} f_\varepsilon(\xi(1-\rho)^{-\beta}), \tag{4.2}$$

where $f_\varepsilon(\xi)$ denotes the unique stationary solution to the nonlinear ODE

$$\underline{\mathbf{A}}(f) \equiv \mathbf{A}(f) - \varepsilon f^p = 0, \quad \xi \in \mathbf{R}, \quad f(\xi) \to 0 \text{ as } \xi \to \infty, \tag{4.3}$$

$\mathbf{A}$ being the operator defined in (1.9). Note that f_ε can be obtained from the profile f^* by rescaling. For instance, for the PME operator (1.4), the transformation is given by

$$f_\varepsilon(\xi) = \lambda f(\lambda^{(m-p)/2}\xi), \quad \lambda = (1-\varepsilon)^{-(p-1)}.$$

Finally, consider $j^* \gg 1$ such that $G(\tau) \leq \varepsilon$ for $\tau \geq \tau_{j^*}$. With the above assumptions the function

$$\underline{\theta}(\xi, \tau) = (1 - \rho e^{\tau - \tau_{j^*}})^{-\alpha} f_\varepsilon(\xi(1-\rho e^{\tau-\tau_{j^*}})^{-\beta})$$

is a subsolution of equation (1.11), (4.1) for $\tau \geq \tau_{j^*}$. Hence, it follows from (4.2) by the Maximum Principle that

$$\underline{\theta}(\xi, \tau) \leq \theta(\xi, \tau) \quad \text{for } \tau \geq \tau_{j^*}.$$

Since the subsolution $\underline{\theta}$ blows-up at a finite τ_0, the same holds for the solution θ. In terms of the original variable $u(x,t)$ this means that u blows up at time $t_0 < 1$ and a contradiction follows. In addition, we note that the upper bound in $[-a, a]$ can be obtained as follows. One has that for every $M \gg f^*(0)$, the symmetric and decreasing stationary profile f_M satisfying $f_M(0) = M$ is a supersolution of (1.11) in $[-a,a] \times \mathbf{R}_+$ and $f_M \to \infty$ in $[-a,a]$ as $M \to \infty$. Then the result follows by applying the Maximum Principle.

3. Localization of the support. Consider $\theta(\xi,\tau) \in \mathcal{L}$ and denote by $a(\tau)$ and $b(\tau) \equiv -a(\tau)$ its left-hand and right-hand interfaces. Let us show that there exists a constant $c > 0$ such that

$$-c \le a(\tau) < b(\tau) \le c \quad \text{for every } \tau \ge 0.$$

Consider the function $f^*(\xi - A)$ with shifted argument, where $A > 0$ is large enough and such that

$$\theta(A,\tau) < f^*(0) \ \text{ for every } \tau \ge 0, \ \text{ and } \ \theta(\xi,0) \le f^*(\xi - A) \text{ for } \xi \ge A.$$

Since $f^*(\xi - A)$ is a supersolution of (1.11) in $[A,\infty)\times\mathbf{R}_+$, the result follows by comparison.

The previous estimates provide us with the convergence and stability result for any symmetric and decreasing initial data from $\mathcal{L}$. We next extend the proof to arbitrary compactly supported initial data.

Step 2: end of the proof of Theorem 4.1. Consider arbitrary initial data θ_0 satisfying the previous hypotheses. Define a one-parametric family $\hat{\mathcal{F}}^* = \{\hat{f}_\lambda(\xi) = \theta_0(\xi + \lambda),\ \lambda \in \mathbf{R}\}$ belonging to the domain of stability of f^*. Then, following the main lines of the stability analysis of unperturbed equations in [6], Section 4, we see that this family $\hat{\mathcal{F}}^*$ satisfies the crucial hypotheses (H3)–(H4) relative to the perturbed equation (1.11), (4.1) and the sets $\mathcal{L}$ and U.

Therefore, the stability result [6] applies and proves the convergence of any rescaled solution from $\mathcal{L}$ (with $T = 1$) to the unique profile f^*. Finally, we mention that in Theorem 3.1 we proved that for $\theta(\xi,\tau) \in \mathcal{L}_k$ there holds $\theta(\xi,\tau) \to f^*$ as $\tau \to \infty$. This implies that

$$(1-t)^\alpha u(\xi(1-t)^{-\beta},t) \to f^*(\xi) \ \text{ as } \ t \to 1^-$$

with uniform convergence on compact on ξ, whence $u(x,t)$ blows up as $t \to 1^-$ and $u_0 \in U$. The proof is completed. □

The same stability analysis applies to more general perturbations under the assumptions that the associated semigroup admits translations in both x and t, as well as spatial symmetry and monotonicity properties.

4.1. **On other applications.** The abstract framework of the stability analysis admits other applications to nonlinear reaction-diffusion-absorption equations. We begin with one of them where the application is quite similar to the one presented above with an explicit family $\mathcal{F}^*$ associated to the unique profile f^* is provided.

Extinction for quasilinear heat equations. A similar singular asymptotic phenomenon occurs in the study of a perturbed similarity extinction behaviour for a simple one-dimensional model

$$u_t = (u^m)_{xx} - u^p + g(u) \quad \text{in } \mathbf{R}\times(0,T),$$

in the parameter range $0 < m < p < 1$. The function $g(u)$ again plays a role of a small perturbation of the scaling invariant PME with strong absorption, and one can see, by applying the stability theory above, that the asymptotic behaviour of finite-time extinction is described by the unique nontrivial similarity solution corresponding to the autonomous equation in the class of continuous solutions satisfying (2.2). The existence and uniqueness of the profile f^* was study in [9], where the authors derived estimates which allow us to define the relation $\prec$ in the family $\mathcal{F}^* = \{f_\mu = f(\xi + \mu), \mu \in \mathbf{R}\}$ constructed in a similar way as in the previous example.

On the other hand, although some extra ingredients are required in the analysis, the theory also applies if the profile f^* is assumed to be unique but an explicit family $\mathcal{F}^*$ is not available. In these cases, if the asymptotic behaviour for the autonomous equations is known for a class of initial data with an appropriate topological structure, it is possible to construct an alternative family $\hat{\mathcal{F}}^*$ and establish the corresponding stability result. Among other examples, we mention the blow-up problem studied as first application, with operator (1.4) but with parameter $\beta > 0$ (single point blow-up). A stability analysis for the unperturbed equation was done in [11] by constructing a Lyapunov function for a wide class U^* of symmetric initial data. We remark that for this problem, a geometrical approach to stability is not known and the uniqueness of the profile f^* remains an open problem. However, assuming that f^* is unique, it is possible to construct an alternative family $\hat{\mathcal{F}}^* \subset U^*$ and the stability result in an appropriate class of solution can be obtained.

4.2. **On further extensions of stability theory.** A natural extension would be the development of an analogous stability analysis when the uniqueness of the profile f^* is not available and instead there occurs a continuous one-parametric family of stationary solutions S^*. In this framework, some new ingredients have to be used in order to establish the stability results. However, key "topology-metric" ideas of this paper remain the same. We mention two of the main applications arising in this context. The first one deals with the well-known focusing problem for the porous medium equation

$$u_t = \Delta u^m \quad \text{in } \mathbf{R}^N \times (0, T) \tag{4.4}$$

with $m > 1$ in dimensions $N \geq 2$, where in the radial setting the second kind self-similar Graveleau solution [2] is known to be asymptotically stable [1]. The second one deals with the asymptotic analysis near the extinction time for the fast diffusion equation (4.4) with $m \in (0, 1)$. The stability analysis of this unperturbed problem was performed in [13].

In both problems there exists a one-parametric family S^* of mutually intersecting rescaled stationary profile inducing a suitable S-relation of restricted partial ordering. The study of the analogous singularity formations for the same equations with perturbation can be placed in the context of this stability theory.

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