COMPOSITE STRUCTURE OF GLOBAL UNBOUNDED SOLUTIONS OF NONLINEAR HEAT EQUATIONS WITH CRITICAL SOBOLEV EXPONENTS

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ABSTRACT. We describe special asymptotic structures of solutions of the semilinear heat equation

$$u_t = \Delta u + u^p \quad \text{in } \Omega \times (0,T), \quad u = 0 \quad \text{on } \partial\Omega \times (0,T),$$

in the unit ball $\Omega = \{r = |x| < 1\} \subset \mathbf{R}^N$ in dimensions $N \geq 3$ with positive symmetric initial data $u_0(r)$. It is known [41] that in the critical Sobolev case $p = p_s = (N+2)/(N-2)$ the Cauchy problem with specially chosen initial data u_0 admits global unbounded solutions (GUSs) $u(\cdot,t)$, which are uniformly bounded in $L^1(\Omega)$ but are not bounded in $L^\infty(\Omega)$ for all $t > 0$. In the radial geometry, we establish the following asymptotic behaviour of such solutions as $t \to \infty$:

$$\ln \|u(\cdot,t)\|_\infty \sim \pi^2 t/4 \quad \text{for } N = 3, \quad \ln \|u(\cdot,t)\|_\infty \sim 2\sqrt{t} \quad \text{for } N = 4,$$

and $\|u(\cdot,t)\|_\infty \sim \gamma_0 t^{(N-2)/2(N-4)}$ for $N \geq 5$, where $\gamma_0 = \gamma_0(N) > 0$ is a constant independent of initial data. The large-time behaviour is not self-similar and is obtained by matching of inner and outer asymptotic expansions. The phenomenon of GUSs is shown to be a common feature of a number of quasilinear and fully nonlinear parabolic equations with scaling invariant operators in the critical Sobolev case.

1. Quasilinear heat equations, critical Sobolev exponents and mysterious global unbounded solutions

In this paper we describe the asymptotic structure of special global unbounded solutions of quasilinear reaction-diffusion equations related to typical models from combustion and nonstationary filtration theory. The main feature of such asymptotic patterns is that they do not exhibit purely self-similar behaviour and their construction is based on matching of asymptotic expansions in two different, inner and outer, regions.

1.1. Discovery of mysterious solutions of the semilinear heat equation. Existence of such solutions was proved in 1984 by W.-M. Ni, P.E. Sacks and J. Tavantzis [41], who studied the Dirichlet problem for the classical semilinear heat equation in the superlinear blow-up range $p > 1$

$$u_t = \Delta u + u^p \quad \text{in } \Omega \times \mathbf{R}_+, \quad u = 0 \quad \text{on } \partial\Omega \times \mathbf{R}_+, \tag{1.1}$$

Date: September, 2001.

1991 *Mathematics Subject Classification.* 35K55, 35K65.

Key words and phrases. Semilinear heat equation, Sobolev exponent, blow-up, global unbounded solutions, asymptotic behaviour, matched expansion.

where $\Omega \subset \mathbf{R}^N$, $N \geq 3$, is a bounded star-shaped domain with smooth boundary $\partial\Omega$. Let us present some details. Take a smooth function $\varphi(x) > 0$ in Ω, $\varphi = 0$ on $\partial\Omega$, fix an arbitrary $\mu > 0$ and let $u(x,t;\mu)$ be the solution of (1.1) with initial data

$$u(x,0) = \mu\varphi(x) \quad \text{in } \Omega. \tag{1.2}$$

Kaplan's eigenfunction method [30] shows that $u(r,t;\mu)$ blows up in finite time for all $\mu \gg 1$, while for any $0 < \mu \ll 1$ there is a global solution which converges exponentially fast to the trivial stationary solution $S \equiv 0$ of equation (1.1). It follows from parabolic theory that the subsets

$$\Lambda^+ = \{\mu > 0 : \ u(r,t;\mu) \text{ blows up}\} \quad \text{and} \quad \Lambda^- = \{\mu > 0 : \ u(r,t;\mu) \to 0 \ \text{ as } \ t \to \infty\}$$

are open, and there exists [41]

$$\mu_* = \inf \Lambda^+ > 0 \tag{1.3}$$

such that $u_*(r,t) \equiv u(r,t;\mu_*)$ does not belong to either of the classes (of blowing up or exponentially decaying solutions) and exhibits another asymptotic behaviour. What can we say about typical evolution properties of such solutions $u_*(r,t)$? Are these global or blowing-up solutions? These are questions to start with.

In what follows, in order to describe the structure of solutions in details, we assume that $\Omega = \{r = |x| < 1\}$ is the unit ball in $\mathbf{R}^N$ ($N \geq 3$) and impose the condition of radial symmetry $\varphi = \varphi(r)$. Then, $u(r,t;\mu)$ is also radially symmetric for all $t > 0$ by uniqueness. We now describe further properties of solutions to (1.1), (1.2) under this radial symmetry assumption (some of these are also true for more general solutions).

Subcritical Sobolev range. The questions can be addressed most easily for $p \in (1, p_s)$, where

$$p_s = (N+2)/(N-2) \tag{1.4}$$

is the critical Sobolev exponent for the stationary elliptic operator $\Delta u + u^p$. Since (1.1) is a smooth gradient dynamical system admitting the Lyapunov function

$$L(u)(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}, \tag{1.5}$$

which is strictly monotone on evolution orbits,

$$\frac{d}{dt}L(u)(t) = -\|u_t(t)\|_2^2 \leq 0,$$

it follows that $u_*(r,t)$ stabilizes to a bounded stationary solution $S(r) > 0$ in Ω satisfying

$$\Delta S + S^p = 0 \quad \text{in } \Omega, \quad S = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

The existence of such a solution S follows from the classical variational theory; see [45]. The compact embedding of the functional spaces $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, corresponding to two operators in (1.6), is true in the subcritical range $p \in (1, p_s)$ by the classical embedding theorems; see the first chapters in [32].

Supercritical Sobolev range. The asymptotic structure of $u_*(r,t)$ is more delicate in the range $p \geq p_s$ where, by Pohozaev's identity [45] for star-shaped domains Ω, the problem (1.6) admits no nontrivial bounded stationary solution $S \geq 0$. By using these nonexistence results and additional estimates, the authors of [41] established that $u_*(x,t)$ is a global weak solution, which is uniformly bounded in $L^1(\Omega)$ but is not bounded in $L^\infty(\Omega)$ for all $t > 0$. This poses a natural question about the actual asymptotic structure of such special space-time patterns $\{u_*\}$ admitted by the classical semilinear heat equation (1.1).

We shall refer to u_* as a *global unbounded solution* (GUS, for short). These are indeed *mysterious* solutions and, for a decade, almost nothing was known about their specific unusual evolution properties.

It turns out that in the supercritical Sobolev range $p > p_s$, the answer can be stated as follows: $u_*(r,t)$ can blow up at a finite instant $t = T < \infty$ and becomes bounded for small $t - T > 0$. Moreover, it was established in [22] that, in the Cauchy problem for (1.1) in $\mathbf{R}^N \times \mathbf{R}_+$, there exist GUSs exhibiting a standard type of self-similar behaviour

$$u_*(r,t) = s^{-1/(p-1)}\theta(\eta), \quad \eta = x/\sqrt{s}, \tag{1.7}$$

in both singular limits $s = T - t \to +0$, i.e. blow up as $t \to T^-$, and $s = t - T \to +0$ describing the collapse of the blow-up singularity as $t \to T^+$. Such a self-similar construction as $t \to T^-$ and the unique minimal self-similar extension for $t > T$ were performed [22] in the range $p \in (p_s, p_p)$, where

$$p_p = 1 + 6/(N-10) \quad \text{for } N \geq 11 \quad (p_p = \infty \ \text{for } N \leq 10)$$

(the critical exponent p_p, occurring in the ODE analysis of rescaled similarity profiles, was first introduced in [33]). Such self-similar GUSs are known to exist [31] also for the non-stationary Frank-Kamenetskii equation in $\mathbf{R}^3 \times \mathbf{R}_+$ with the exponential nonlinear source

$$u_t = \Delta u + e^u. \tag{1.8}$$

There are several open questions concerning the asymptotics as $t \to T^\pm$ of such "peaking" blow-up solutions satisfying $u_*(\cdot,t) \in L^\infty(\Omega)$ for all $t \in \mathbf{R}_+ \setminus \{t = T\}$, especially when the assumption of radial symmetry is dropped. Nevertheless, it can be said that for $p \in (p_s, p_p)$, the asymptotic pattern $u_*(r,t)$ is well understood, at least in its peaking form which is expected to be generic. Of course, it is not guaranteed that such patterns always describe the singularity formation for arbitrary solutions. It is widely recognized any kind of complete classifications of all possible singularity patterns is a hard problem. It is open for most semilinear and quasilinear heat equations (especially for those in multi-dimensional geometries and with critical and supercritical nonlinearities), which can admit both countable spectra (see [3], [7], [13], [26], [39], [50], [51], for example) and uncountable ones (see [17] and [7]) in different types of singularity formation; see also references in Chapter 4 of [47] and in [22].

The self-similar behaviour (1.7) as $t \to T^\pm$ leads to more or less standard stabilization problems for second-order quasilinear parabolic equations in the corresponding rescaled form. In radial geometry such rescaled dynamical systems are gradient and admit Lyapunov functions; see [13] for the semilinear case and [15], [19] for the quasilinear one. In the non-radial case, symmetrization arguments of moving planes and spheres based on Aleksandrov's Reflection Principle [1] (see [23], [8], [35], [52] and references therein) can play an important role in the asymptotic study of such localized singularity formation phenomena.

In summary, only the critical case $p = p_s$ was left entirely open and we now can describe our main result concerning non-similarity structure of GUSs in this case.

1.2. **The main result: asymptotic structure of GUSs in the critical Sobolev case.** Let $p = p_s$. The first result [22] towards the description of the asymptotic behaviour of GUSs was a by-product of the analysis of complete blow-up phenomena. It was proved that, unlike the supercritical range $p > p_s$, the GUS $u_*(r,t)$ does not blow-up in finite time and is a global classical solution, which is unbounded as $t \to \infty$. Indeed, finite-time blow-up of any radial solution for $p = p_s$ is always complete [22]. Hence, if $u_*(r,t)$ blows up at a finite $t = T^-$, then the proper minimal continuation of the solution satisfies $u_{\min}(r,t) \equiv \infty$ for $t > T$, in contrast to the result in [41] which states that $u_*(r,t) \ge u_{\min}(r,t)$ is uniformly bounded in $L^1(\Omega)$ for all $t > T$.

Thus, $u_*(r,t)$ is a classical bounded solution in $\Omega \times [0,T]$ for any $T > 0$ but $\|u(\cdot,t)\|_\infty$ is unbounded as $t \to \infty$. The main question remaining open is then as follows:

$$\text{What is the behaviour of } \alpha(t) = \|u(\cdot,t)\|_\infty \text{ as } t \to \infty?$$

We answer this question in the following theorem.

Theorem 1.1. *For $p = p_s$, nonnegative radially symmetric GUSs $u_*(r,t)$ of (1.1) exhibit the following asymptotic behaviour as $t \to \infty$:*

$$N = 3: \quad \ln \alpha(t) = \frac{\pi^2}{4}\, t(1 + o(1)), \tag{1.9}$$

$$N = 4: \quad \ln \alpha(t) = 2\sqrt{t}(1 + o(1)), \quad \textit{and} \tag{1.10}$$

$$N \ge 5: \quad \alpha(t) = \gamma_0 t^{(N-2)/2(N-4)}(1 + o(1)), \tag{1.11}$$

where the constant $\gamma_0 > 0$ depends on the dimension N only.

We compute that for $N = 3$, the multiplier $\pi^2/4$ is the square of the first positive root of Bessel's function $J_{-1/2}(\zeta) = \sqrt{2/\pi\zeta}\cos\zeta$ occurring in the asymptotic behaviour. It follows that the function (1.9) is the first one in an infinite sequence $\{\alpha_k(t)\}$ corresponding to a countable spectrum of GUSs $\{u_k(r,t)\}$ of the equation with the reaction term $|u|^{p-1}u$ changing sign precisely $k = 1, 2, \ldots$ times. The constant γ_0 in (1.11) will be calculated formally below as follows:

$$\gamma_0^{2(N-4)/(N-2)} = [N(N-2)]^{(N-2)/2} \frac{(N-4)^2\Gamma(N)}{N(N+2)\Gamma^2(N/2)}. \tag{1.12}$$

The theorem is proved in Sections 2 and 3, where we also describe the asymptotic behaviour of $u_*(r,t)$ as $t \to \infty$. We use a matching approach to treat such non-uniform asymptotic structures of GUSs which applies for all dimensions $N \ge 3$. The scheme of such an asymptotic analysis consists of two steps:

(i) We first derive the behaviour in the inner region near the origin $r = 0$ on small shrinking (r, t)-subsets determined via rescaled variables incorporating an unknown function $\alpha(t)$, which is proved to satisfy certain conditions of slow growth as $t \to \infty$.

(ii) As the second step, we introduce the second rescaled variables in order to extend the inner behaviour into the outer region (the boundary layer) near the boundary point $r = 1$. In turns out that we can match the extended inner expansion directly with the boundary condition $u = 0$ on $\partial\Omega$, which leads to a first-order ODE for the matching function $\alpha = \alpha(t)$. We can thus avoid any independent asymptotic expansion in the outer region.

Matched asymptotic expansion techniques are well known in asymptotic theory and have been recently used in the literature devoted to blow-up for the semilinear equation (1.1), see [3], [7], [13], [26], [39], [50], [51] and references therein, as well as in other related singularity formation phenomena. It was established that the generic inner blow-up behaviour as $t \to T^-$ in the subcritical range $p \in (1, p_s)$ can be obtained by a centre or stable manifold analysis, based on strongly dominant spectral properties of the linearized Hermite-like operator. Then the inner analysis (i) can be performed independently of the outer one, the spectrum of possible functions $\{\alpha_k(t)\}$ being determined by the inner analysis alone. The structure of GUSs u_* for $p = p_s$ belongs to a different type of expansion. For any $N \geq 4$, regardless of the direction of matching and extension (inner to outer or *vice versa*) we always need to introduce a matching quantity, which is $\alpha(t)$ if we start an analysis with the inner region. In the three-dimensional case $N = 3$ the outer-inner matching is shown to be governed by the first eigenfunction mode generated by a discrete spectrum of the linearized operator Δ in a special functional space. We find certain similarities of the composite structures of GUSs with those already known for the fast diffusion equation in a ball in $\mathbf{R}^N$, $N \geq 3$ [16], and the Frank-Kamenetskii equation (1.8) in a ball in dimension $N > 10$ [10].

In Section 4 we present a more detailed formal matching establishing precise values of exponents and constants in the asymptotic expansions, including $\gamma_0(N)$ in (1.12). These calculations require the careful consideration of a few terms in the asymptotic expansion and making these results entirely rigorous would need refined techniques of analysis beyond the scope of the current paper.

In Section 5 we show that composite structures for u_* occur for a wide range of different scaling invariant reaction-diffusion equations and these represent not an exceptional type of asymptotic behaviour but rather a widespread, but delicate, one for classical nonlinear heat equations. We consider equations with the quasilinear "porous medium" operator and the p-Laplacian one, as well as a fully nonlinear heat equation with the dual porous medium operator. We also discuss the formal possibility of extending the main ideas to higher-order parabolic equations.

2. Asymptotic behaviour in the inner region

In what follows, we omit the star subscript in the GUS, writing $u(r,t)$ instead of $u_*(r,t)$. Using the strong Maximum Principle and the established phenomenon of eventual monotonicity for autonomous radial quasilinear heat equations in bounded domains (see [40], [20] and references therein), we may assume that $u(r,t)$ is strictly monotone decreasing in r, i.e.

$$\partial u/\partial r < 0 \quad \text{in } (0,1)\times \mathbf{R}_+ \implies \|u(\cdot,t)\|_\infty = u(0,t).$$

2.1. **Auxiliary properties of the family of stationary solutions in $\mathbf{R}^N$.** As in [16], we show that in the inner region close to the origin, the GUS $u(r,t)$ evolves as $t\to\infty$ via a *slow motion* over a smooth one-parameter family of positive stationary solutions $\{S_\lambda(|x|),\ \lambda>0\}$ satisfying the radial restriction of the semilinear elliptic equation

$$\mathbf{B}(S) \equiv \Delta S + S^p = 0 \quad \text{in } \mathbf{R}^N. \tag{2.1}$$

These are famous solutions invariant under conformal and projective transformations [28], [37], namely

$$S_\lambda(r) = \lambda\left[\frac{N(N-2)}{N(N-2)+\lambda^{4/(N-2)}r^2}\right]^{(N-2)/2} \equiv \lambda\, S_1(r\lambda^{2/(N-2)}), \tag{2.2}$$

where S_1 satisfies $S_1(0)=1$ and the parameterization of the family implies $S_\lambda(0)=\lambda$ for $\lambda>0$. The ODE (2.1) is the Emden-Fowler equation introduced about a century ago [12], [14] and subsequently studied in detail in a number of monographs and textbooks on ODEs. Let us note some straightforward asymptotic properties of functions $\{S_\lambda\}$. Firstly,

$$S_1(r) = c_1 r^{2-N} - c_2 r^{-N} + \dots \quad \text{as } r\to\infty, \tag{2.3}$$

where $c_1 = [N(N-2)]^{(N-2)/2}$ and $c_2 = N(N-2)^2 c_1/2$, and for a fixed $r>0$,

$$S_\lambda(r) = \frac{c_1}{\lambda}\, r^{2-N} + \dots \quad \text{as } \lambda\to\infty. \tag{2.4}$$

Secondly, we describe related properties of S_λ for $\lambda \gg 1$ and $\lambda \ll 1$.

Proposition 2.1. *As $\lambda\to\infty$,*

$$S_\lambda(r)\to 0 \quad \textit{uniformly on any subset } [\delta,\infty) \textit{ with } \delta>0, \tag{2.5}$$

$$S_\lambda'(r) \equiv \frac{d}{dr}S_\lambda(r) \to -\infty \quad \textit{on any level set } \{S_\lambda=\varepsilon\} \textit{ with } \varepsilon>0, \tag{2.6}$$

$$S_\lambda(r)\to\infty \quad \textit{uniformly on any subset } \{r = o(\lambda^{-1/(N-2)})\}. \tag{2.7}$$

In the class of radially symmetric functions, we identify $L^2(\Omega)$ with the weighted space $L^2_\rho((0,1))$, $\rho(r)=r^{N-1}$, in which the inner product has the form

$$(u,v)_\rho = \int_0^1 r^{N-1}u(r)v(r)\,dr.$$

Proposition 2.2. *(i) As $\lambda \to \infty$, in the sense of distributions*

$$(S_\lambda)^\beta \to 0 \quad if \;\; 1 < \beta < 2N/(N-2), \quad and \tag{2.8}$$

$$(S_\lambda)^{2N/(N-2)} \to d_1\delta(x), \quad d_1 = \int_0^\infty \xi^{N-1} S_1^{2N/(N-2)}(\xi)d\xi < \infty. \tag{2.9}$$

(ii) As $\lambda \to 0$,

$$S_\lambda, \;\; S_\lambda' \to 0 \quad uniformly \; in \; \Omega. \tag{2.10}$$

2.2. **Intersection analysis and preliminary properties of $\alpha(t)$.** We begin our inner asymptotic analysis with some elementary intersection comparison arguments. For a fixed $t \geq 0$, denote by $\mathrm{Int}(t, \lambda)$ the number of intersections in $r \in (0, 1)$ of the spatial profiles $u(r, t)$ and $S_\lambda(r)$, i.e. the number of sign changes of the difference $w(r, t) = u(r, t) - S_\lambda(r)$ for $r \in (0, 1)$.

Proposition 2.3. *There holds*

$$\mathrm{Int}(t, \lambda) > 0 \quad for\; any \; t \geq 0 \; and \; \lambda > 0. \tag{2.11}$$

Proof. One can see that if $\mathrm{Int}(t_0, \lambda_0) = 0$ for some $t_0 \geq 0$ and $\lambda_0 > 0$, then $u(r, t_0) \leq S_{\lambda_0}(r)$ for all $r \in (0, 1)$ and hence $u \leq S_{\lambda_0}$ for $t > t_0$, contradicting the fact that $u(\cdot, t)$ is not bounded as $t \to \infty$. □

Under the monotonicity assumption on $u(r, \cdot)$, we define

$$\alpha(t) \equiv u(0, t) = \|u(\cdot, t)\|_\infty \quad \text{for all} \;\; t \geq 0, \tag{2.12}$$

where $\alpha(t)$ is sufficiently smooth and positive.

Proposition 2.4. *$\alpha(t)$ is bounded away from zero: there exists a constant $\lambda_0 > 0$ such that*

$$\alpha(t) \geq \lambda_0 > 0 \quad for \; any \;\; t \geq 0. \tag{2.13}$$

Proof. By property (2.10) and the regularity of $u_0(r)$, there exists a small $\lambda_0 > 0$ such that

$$\mathrm{Int}(0, \lambda_0) = 1. \tag{2.14}$$

By the first Sturm Theorem (see e.g. references in [19], [22] and Chapter 4 of [47]) the number of intersections $\mathrm{Int}(t, \lambda)$ does not increase with time, so that (2.14) implies that

$$\mathrm{Int}(t, \lambda_0) = 1 \quad \text{for all} \;\; t \in (0, T). \tag{2.15}$$

This gives (2.13). □

Proposition 2.5. *$\alpha(t)$ is unbounded as $t \to \infty$.*

Proof. Indeed, if $\alpha(t)$ were uniformly bounded, the classical Lyapunov argument for gradient systems (equation (1.1) admits the Lyapunov function (1.5) for any exponent $p > 1$), would imply that regular orbits $\{u(\cdot, t),\ t > 0\}$ would stabilize as $t \to \infty$ to a classical stationary solution $S(r)$ of the elliptic problem (2.1). Then S is a nontrivial solution by Proposition 2.4. This contradicts Pohozaev's non-existence result in the critical case [45]. □

Proposition 2.6. *There holds*

$$\alpha'(t) > 0 \quad for \;\; t \gg 1. \tag{2.16}$$

Proof. It follows from the properties (2.5)–(2.7) that for $t = 0$

$$\mathrm{Int}(0, \lambda) = 2 \quad \text{for all} \;\; \lambda \gg 1.$$

Indeed, for any sufficiently smooth monotone $u_0(r)$ with $u_0'(1) < 0$ (the strict inequality for arbitrarily small $t > \delta$ is guaranteed by the Oleinik-Hopf Boundary Lemma for uniformly elliptic and parabolic equations [43], [27]), we have $\mathrm{Int}(0, \lambda) = 2$ for $\lambda \gg 1$. One can see that both intersections cannot disappear at once.

Fix a $\lambda \gg 1$. By continuity, there exists a $t_\lambda > 0$ such that $\alpha(t_\lambda) = \lambda$. Therefore, by the strong Maximum Principle and, using a standard geometric argument, at $t = t_\lambda$ the minimal intersection disappears, so that

$$\mathrm{Int}(t_\lambda, \lambda) = 1 \quad \text{and} \;\; u(r, t_\lambda) > S_\lambda(r) \quad \text{for small} \;\; r > 0, \tag{2.17}$$

whence $\alpha'(t_\lambda) \geq 0$ by standard local comparison with the stationary solution S_λ. Since $\lambda \gg 1$ is arbitrary, we conclude that $\alpha'(t) \geq 0$ for all $t \gg 1$. In order to prove the strict inequality (2.16) we observe that the derivative $z = u_t$ satisfies a linear non-degenerate parabolic equation of the form

$$z_t = \Delta z + p u^{p-1} z. \tag{2.18}$$

In the positivity domain equations (1.1) and (2.18) have analytic coefficients and their solutions are analytic in r. Then the number of zeros of $z(r, t)$ is finite for any $t > 0$. By the strong Maximum Principle this implies that $z > 0$ for small $r \geq 0$. Note that the analyticity assumption is not necessary; see references in Chapter 4 of [47]. □

Propositions 2.5 and 2.6 imply that

$$\alpha(t) \to \infty \quad \text{as} \;\; t \to \infty. \tag{2.19}$$

2.3. **Inner expansion: first approximation.** In view of (2.19), using the scaling symmetry of equation (1.1), we introduce new independent rescaled variables $\{\xi, \tau\}$ and the dependent one $f = f(\xi, \tau)$ defined by

$$u(r, t) = \alpha(t) f(\xi, \tau), \quad \xi = r\, \alpha^{2/(N-2)}(t), \quad \tau = \int_0^t \alpha^{4/(N-2)}(s) ds : \mathbf{R}_+ \to \mathbf{R}_+. \tag{2.20}$$

The rescaled solution $f(\xi, \tau)$ satisfies the non-autonomous quasilinear equation

$$f_\tau = \mathbf{B}(f) + g(\tau)\, \mathbf{C} f, \quad \text{with} \;\; g(\tau) = \alpha^{-p}(t) \alpha'(t), \tag{2.21}$$

where $\mathbf{B}$ is the nonlinear stationary operator (2.1) and $\mathbf{C}$ is the linear first-order one

$$\mathbf{C}f = -\frac{2}{N-2} f_\xi \xi - f. \tag{2.22}$$

The boundary condition is transformed into

$$f = 0 \quad \text{at} \quad \xi = \xi_*(\tau) \equiv \alpha^{2/(N-2)}(t) \ \text{ for } \ \tau > 0. \tag{2.23}$$

By the definition (2.12) of $\alpha(t)$ there holds

$$f(0,\tau) \equiv 1 \quad \text{for all} \ \ \tau > 0. \tag{2.24}$$

As in the proof of Proposition 2.6, (2.17) implies the following condition.

Proposition 2.7. *For any fixed $\tau \gg 1$, the profile $f(\xi,\tau)$ intersects $S_1(\xi)$ exactly once, $f(\xi,\tau) > S_1(\xi)$ for small $\xi > 0$ and $f(\xi,\tau) < S_1(\xi)$ for $\xi \gg 1$.*

Equation (2.21) is a perturbation of order $O(g(\tau))$ of the autonomous equation

$$f_s = \mathbf{B}(f) \quad \text{in} \ \ \mathbf{R}_+ \times \mathbf{R}_+, \quad \text{with the condition} \ \ f(0,s) \equiv 1. \tag{2.25}$$

Proposition 2.8. *There hold: (i) the perturbation in* (2.21) *is uniformly bounded on smooth functions,*

$$0 < g(\tau) \equiv \alpha^{-p}(t)\alpha'(t) \le 1 \quad for \ \ \tau > 0, \tag{2.26}$$

and (ii) it is not integrable,

$$g \notin L^1(\mathbf{R}_+). \tag{2.27}$$

Proof. (i) Setting $r = 0$ in (1.1) and using the fact that $\Delta u \le 0$ at the maximum point $r = 0$, we get $u_t(0,t) \equiv \alpha'(t) \le \alpha^p(t)$ and (2.26) follows. (ii) We have

$$\int^\infty g(\tau)d\tau = \int^\infty \alpha^{-1}(t)\alpha'(t)dt = \infty. \tag{2.28}$$

$\square$

Differentiating equation (1.1) and using standard Bernstein estimates on u and on the time-derivative u_t, [4], [44], we have for $t \gg 1$ that

$$|\alpha''(t)| \le \text{const}\, \alpha'(1 + \alpha^{p-1}) \le \text{const}(\alpha^p + \alpha^{2p-1}).$$

This means that for $\tau \gg 1$

$$|g'(\tau)| = \alpha^{(N+6)/(N-2)}|\alpha'' - p\alpha^{-1}(\alpha')^2| \quad \text{is uniformly bounded.} \tag{2.29}$$

Proposition 2.9. *The perturbation term in* (2.21) *vanishes asymptotically,*

$$g(\tau) \to 0 \quad as \ \ \tau \to \infty. \tag{2.30}$$

Proof. Assume that (2.30) is not true and that there exists a monotone sequence $\{\tau_k\} \to \infty$ such that $g(\tau_k) \to \varepsilon > 0$. Using standard interior regularity results for uniformly parabolic equations [32], [44], and the estimate (2.29), we have that $f(\xi, \tau_k + s) \to h(\xi, s)$ as $k \to \infty$ in $L^\infty_{\rm loc}(\mathbf{R}_+; C^2(\mathbf{R}_+))$, where $h \ge 0$ satisfies

$$h_s = \mathbf{B}(h) + \varepsilon \mathbf{C} h \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+, \tag{2.31}$$

$$h(0, s) \equiv 1 \quad \text{for } s \ge 0. \tag{2.32}$$

By the strong Maximum Principle for autonomous uniformly parabolic equations, (2.32) implies that h is a nonnegative stationary solution, i.e.

$$\mathbf{B}(h) + \varepsilon \mathbf{C} h = 0 \quad \text{for } \xi > 0 \quad \text{and } h(0) = 1; \tag{2.33}$$

see Section 3.2 in [10]. We rewrite (2.33) in the form

$$\Delta h - 2\varepsilon h' \xi/(N-2) - \varepsilon h + h^p = 0 \quad \text{for } \xi > 0, \quad h(0) = 1, \; h'(0) = 0. \tag{2.34}$$

We thus have a contradiction with the non-existence of such a profile $h \ge 0$ in $\mathbf{R}_+$, which we state below as a separate proposition. □

Proposition 2.10. *For any $\varepsilon > 0$, the solution $h(\xi)$ of* (2.34) *vanishes at a finite point and cannot be extended as a positive solution in all of $\mathbf{R}_+$.*

Proof. Nonexistence follows from Pohozaev's identity [45] for the elliptic operator $\Delta h + h^{p_s}$ in $\mathbf{R}^N$, since we can check that the extra first-order linear operator in (2.34) does not interfere with the sign of either identities obtained via multiplying the equation by $\xi^{N-1} h$ and $\xi^N h'$ respectively and integrating over $\mathbf{R}_+$; see e.g. p. 224 of [47]. □

We thus obtain the first asymptotic result in the inner region.

Theorem 2.1. *There holds*

$$f(\xi, \tau) \to S_1(\xi) \quad \textit{as } \tau \to \infty \ \textit{uniformly}. \tag{2.35}$$

Next, we conclude that

$$g'(\tau) \to 0 \quad \text{as } \tau \to \infty. \tag{2.36}$$

Indeed, by regularity results [32], convergence (2.35) implies that the time derivatives $f_\tau,\ f_{\xi\tau},\ f_{\tau\tau}, ... \to 0$ as $\tau \to \infty$. Since $z = f_\tau$ solves the parabolic equation

$$z_\tau = m\Delta z + p f^{p-1} z + g(\tau)\mathbf{C} z + g'(\tau)\mathbf{C} f$$

and $f(\cdot, \tau)$ converges to $S_1(\cdot)$, as do the corresponding spatial derivatives, passing to the limit $\tau \to \infty$ in the above equation one deduces that $g'(\tau)\mathbf{C} S_1 = o(1)$, and (2.36) follows since $\mathbf{C} S_1 \not\equiv 0$.

2.4. **Refined inner expansion.** We now refine the inner expansion by calculating the first correction term and deriving the rate of convergence in (2.35). Setting

$$f(\xi,\tau) = S_1(\xi) + Y(\xi,\tau), \quad \text{where } Y(0,\tau) \equiv 0, \tag{2.37}$$

in (2.21), we obtain the equation

$$Y_\tau = \mathbf{A}Y + g(\tau)Z + g(\tau)\mathbf{C}Y + \mathbf{D}(Y), \tag{2.38}$$

where $\mathbf{A}$ is the linear operator

$$\mathbf{A}Y = \Delta Y + pS_1^{p-1}Y, \tag{2.39}$$

Z is the function

$$Z(\xi) = \mathbf{C}S_1(\xi) = c_1\xi^{2-N} + \ldots \quad \text{as } \xi \to \infty,$$

and $\mathbf{D}$ is the nonlinear operator

$$\mathbf{D}(Y) = (S_1 + Y)^p - S_1^p - pS_1^{p-1}Y. \tag{2.40}$$

This is a quadratic perturbation locally in ξ on bounded small functions $Y \in C^2$, i.e.

$$\mathbf{D}(Y) = O(Y^2) \quad \text{as } Y \to 0. \tag{2.41}$$

By the intersection comparison analysis (see the proof of Proposition 2.6), we have that, for $\tau \gg 1$, the function $Y(\xi,\tau)$ has a unique zero for $\xi > 0$ and $Y(\xi,\tau) > 0$ for sufficiently small $\xi > 0$ and $Y(\xi,\tau) < 0$ near the boundary point $\xi = \xi_*(\tau) \equiv \alpha^{2/(N-2)}(t)$.

Consider the linear operator (2.39), which is a perturbation of the Laplacian Δ in $\mathbf{R}^N$. The coefficient

$$pS_1^{p-1}(\xi) = c_3\xi^{-4} + \ldots, \quad c_3 = pc_1^{p-1}, \tag{2.42}$$

has a strong decay at infinity, so that $\mathbf{A}$ is a bounded asymptotically vanishing perturbation of the Laplacian Δ. The operator $\mathbf{A}$ defined in the domain

$$D(\mathbf{A}) = \{f \in W^{2,q}_{\rm loc}(\mathbf{R}^N) \text{ for any } q \geq 1 : \ \ f, \Delta f \in L^\infty(\mathbf{R}^N)\}$$

is sectorial in $L^\infty(\mathbf{R}^N)$ and generates a strong continuous semigroup $e^{\mathbf{A}\tau}$; see [38], Chapter 7 and [25]. We study the asymptotic behaviour of the perturbed equation (2.38) using classical linear perturbation theory. First, for convenience, we extend our rescaled initial-boundary value problem to the Cauchy problem for equation (1.1) in $\mathbf{R}_+ \times \mathbf{R}_+$. Consider the equivalent parabolic equation extended to negative values $u < 0$ by replacing the source term by $|u|^{p-1}u$. We define the solution for $r \in [1, 3/2]$ by smooth odd reflection with respect to $r = 1$: $u(r,t) = -u(2-r,t)$. We next put $u(r,t) = -u(1/2,t) \to 0$ as $t \to \infty$ for $r \in (2,3)$, and perform a smooth approximation on $(3/2, 2)$. Finally, we put $u(r,t) = -u(1/2,t)v(r)$ for $r \geq 3$, where $v(r) > 0$, $v(3) = 1$, is a smooth monotone decreasing function exponential decaying at infinity. As a result, after rescaling, we obtain a global smooth solution u of the Cauchy problem in $\mathbf{R}_+ \times \mathbf{R}_+$ for a parabolic equation coinciding with the original one (2.21) in the domain $\{\xi < \xi_*(\tau) = \alpha^{2/(N-2)}(t),\ \tau > 0\}$. In the complement, we put a perturbation into the equation which is uniformly bounded and moreover uniformly vanishing on any domain with $\xi \gg 1$ as $\tau \to \infty$.

This extension does not affect the asymptotic analysis presented below and we omit this negligible perturbation term in such remote domains of $\mathbf{R}_+ \times \mathbf{R}_+$.

The expansion (2.42) is sufficient to show that for $N \geq 3$ the fundamental solution of the perturbed heat equation $W_\tau = \mathbf{A}W$ in $\mathbf{R}^N \times \mathbf{R}_+$ has the decay rate $\tau^{-N/2} \in L^1((1,\infty))$ as $\tau \to \infty$ in $L^\infty(\mathbf{R}^N)$ similar to that for the heat equation $W_\tau = \Delta W$. Indeed, after the heat kernel scaling $W(\xi,\tau) = \tau^{-N/2}V(\eta,s)$, $\eta = \xi/\sqrt{\tau}$, $s = \ln \tau$, we arrive at the perturbed equation

$$V_s = \mathbf{B}V + \nu(\eta,s)V, \tag{2.43}$$

where

$$0 \leq \nu(\eta,s) = e^s S_1^{p-1}(\eta e^{s/2}) \leq C\,\frac{e^s}{1+\eta^4 e^{2s}} \in L^1(\mathbf{R}^N \times \mathbf{R}_+).$$

We note (and this is important for an easier proof of the asymptotic estimates) that the linear operator

$$\mathbf{B}V = \Delta V + \frac{1}{2}V_\eta \eta + \frac{N}{2}V \equiv \frac{1}{\rho}\nabla \cdot (\rho \nabla V) + \frac{N}{2}V, \quad \text{where } \rho = e^{\eta^2/4},$$

is self-adjoint in the weighted Hilbert space of radial functions $L^2_\rho(\mathbf{R}^N)$ with discrete spectrum $\{\lambda_k = -k,\ k = 0,1,2,...\}$ and a complete subset of eigenfunctions $\{\psi_k(\eta)\}$ [6]. The boundedness of solutions of the rescaled equation (2.43) follows from the eigenfunction expansion by the classical linear theory; see [32], Chapter 4, and [9].

By Theorem 2.1, Proposition 2.7 and by consrtruction of the extended solution f, we have that

$$|Y| \leq C\,\xi^{2-N}, \quad |Y_\xi| \leq C\,\xi^{1-N} \quad \text{for } \xi \gg 1, \tag{2.44}$$

and hence

$$Y(\xi,\tau) \to 0 \quad \text{as } \tau \to \infty \text{ uniformly in } \mathbf{R}_+ \tag{2.45}$$

together with the derivatives Y_ξ, $Y_{\xi\xi}$. We now use the semigroup $e^{\mathbf{A}\tau}$ with known asymptotic properties. It follows from (2.27) that the main perturbation term of order $g(\tau)$ in (2.38) provides us with a decay rate as $\tau \to \infty$ slower than $\tau^{-N/2}$, corresponding to the convolution with the fundamental solution of the unperturbed equation. In order to justify such asymptotic behaviour we perform the change of the dependent variable

$$Y(\xi,\tau) = g(\tau)W(\xi,\tau),$$

where, for $\tau \gg 1$, equation for W takes the form

$$W_\tau = \mathbf{A}W + Z + g(\tau)[\mathbf{C}W + O(W^2)] - q(\tau)W, \tag{2.46}$$

with the perturbation factor

$$q(\tau) = g'(\tau)/g(\tau).$$

The non-integrability condition (2.27) implies that there exists a sequence $\{\tau_k\} \to \infty$ such that $q(\tau_k) \to c = 0$. Assume for a moment that $c = 0$ along any subsequence. In view of estimates and known asymptotic properties of the semigroup with infinitesimal generator $\mathbf{A}$ and the structure of the first-order operator (2.22), we have that $W(\cdot,\tau)$ is uniformly bounded; see heat potential estimates in [32]. Let $\omega(W_0)$ be the omega-limit set defined in terms of uniform convergence on compact subsets of the orbit $\{W(\cdot,\tau),\, \tau > \tau_0\}$ with initial datum W_0. Passing to the limit in (2.46) along a sequence $\{\tau_k + s\} \to \infty$ such that $W(\tau_k) \to f \in \omega(W_0)$, we have that $W(\cdot, \tau_k + s) \to h(\cdot, s)$ in $L^\infty_{\rm loc}(\mathbf{R}_+; C^2_{\rm loc}(\mathbf{R}_+))$, where $h(\cdot, s)$ is a smooth solution of the linear non-homogeneous parabolic equation

$$h_s = \mathbf{A}h + Z \quad \text{for } \xi > 0,\ s > 0,$$

with the initial datum $h_0 \in \omega(W_0)$. Since by construction $h = h_\xi = 0$ at $\xi = 0$ for any $s \ge 0$, by the Sturm theorem on zeros of solutions of uniformly parabolic equations (see a similar analysis on p. 673 in [10]), we conclude that $h(s)$ is a stationary solution,

$$\text{(2.47)} \qquad h(\xi, s) \equiv R(\xi): \quad \mathbf{A}R + Z = 0 \ \text{ for } \xi > 0, \quad R(0) = R'(0) = 0.$$

Indeed, if $h(\xi, s_1) \not\equiv R(\xi)$ for some $s_1 > 0$, then by the strong Maximum Principle $h(\xi, s_1) \ne R(\xi)$ a. e. near the origin $\xi = 0$ and hence $h(0, s) \ne 0 = R(0)$ a. e. for $s > s_1$, contradicting the definition of h. It is easy to see (and this has been proved in Proposition 2.7 by the intersection comparison) that

$$\text{(2.48)} \qquad R(\xi) \quad \text{has a unique zero for } \xi > 0,$$

$R(\xi)$ is positive for small $\xi > 0$ and is negative for $\xi \gg 1$.

Let us return to the assumption $c = 0$. Arguing by contradiction, assume that $q(\tau_k) \to -c < 0$ along a sequence $\{\tau_k\} \to \infty$. Then passing to the limit as $\tau_k + s \to \infty$ as above, we obtain that the limit stationary solution $h \equiv R(\xi)$ solves the ODE $\mathbf{A}R + Z + cR = 0$, implying oscillatory behaviour as $\xi \to \infty$ and contradicting property (2.48); see Proposition 2.7. Thus, there holds

$$\text{(2.49)} \qquad g'(\tau)/g(\tau) \to 0 \quad \text{as } \tau \to \infty.$$

We summarize these results as follows.

Proposition 2.11. *There holds*

$$\text{(2.50)} \qquad Y(\xi, \tau) = g(\tau)(R(\xi) + o(1)) \quad \textit{as } \tau \to \infty,$$

uniformly on compact subsets where, for $\xi \gg 1$,

$$\text{(2.51)} \qquad N = 3: \quad R(\xi) = -\frac{c_1}{2}\xi + ...,$$

$$\text{(2.52)} \qquad N = 4: \quad R(\xi) = -\frac{c_1}{2}\ln \xi + ...,$$

$$\text{(2.53)} \qquad N \ge 5: \quad R(\xi) = -c_* + ...\ (c_* > 0).$$

Proof. Expansions (2.51)–(2.53) follow from the radial equation (2.47) with appropriate expansions of the coefficients

$$R'' + \frac{N-1}{\xi}R' + c_3\xi^{-4}R + c_1\xi^{2-N} + ... = 0$$

for $\xi \gg 1$. $\square$

Note that since by (2.51)–(2.53) $R(\xi)$ changes sign once at a point $\xi_0 > 0$, we have that

$$Y(\xi, \tau) < 0 \quad \text{for } \xi \geq \xi_0 + 1, \ \tau \gg 1. \tag{2.54}$$

3. Proof of the theorem: extension of inner expansion and matching

In this section we complete the proof of Theorem 1.1. It follows from (2.50) that

$$f(\xi, \tau) = S_1(\xi) + g(\tau)(R(\xi) + o(1)) \quad \text{as } \tau \to \infty, \tag{3.1}$$

uniformly on compact subsets in ξ.

3.1. **Extension and matching for $N = 3$.** Substituting asymptotic expansions (2.3) and (2.51) into (3.1) and taking into account two main terms, we obtain that for $\xi \gg 1$

$$f(\xi, \tau) = \frac{c_1}{\xi}(1 - \frac{1}{2} g(\tau)\xi^2 + ...), \tag{3.2}$$

where $c_1 = \sqrt{3}$ and $c_2 = 3\sqrt{3}/2$. It follows from (2.51) that the linearization expansion is valid on subsets in $\xi \gg 1$ as $\tau \to \infty$, where $|Y(\xi, \tau)| \ll S_1(\xi)$, or, equivalently,

$$g(\tau)\xi^2 \ll 1. \tag{3.3}$$

This suggests the following change of the spatial variable in order to extend the expansion:

$$\zeta = \xi \sqrt{g(\tau)}. \tag{3.4}$$

Then the inner expansion (3.2) takes the form

$$f(\xi, \tau) = \sqrt{g(\tau)} \frac{c_1}{\zeta}(1 - \frac{1}{2}\zeta^2 + ...). \tag{3.5}$$

According to this structure, we introduce the new rescaled solution F given by

$$f(\xi, \tau) = \sqrt{g(\tau)} F(\zeta, \tau). \tag{3.6}$$

In general, this change of variables is intended to describe the asymptotic behaviour in the *intermediate region* determined in terms of compact subsets in ζ bounded away from zero. We will show that this intermediate region overlaps with the boundary one near $r = 1$. It follows from (2.51) and (2.54) that

$$F(\zeta, \tau) \equiv \frac{f(\xi, \tau)}{\sqrt{g(\tau)}} \leq \frac{S_1(\xi)}{\sqrt{g(\tau)}} \leq \frac{2c_1}{\xi\sqrt{g(\tau)}} \equiv \frac{2c_1}{\zeta} \tag{3.7}$$

for all $\zeta \geq (\xi_0 + 1)\sqrt{g(\tau)}$, $\tau \gg 1$, i.e. F is uniformly bounded on any sufficiently small compact subsets $\{0 < \delta_1 \leq \zeta \leq \delta_2\}$ bounded away from the origin.

One can derive a perturbed parabolic equation for the rescaled function $F = F(\zeta, s)$

$$F_s = \Delta F + \mathbf{C}F + g(\tau)F^5 - \frac{1}{2}(g^{-2}g')(\tau)(F_\zeta \zeta + F), \tag{3.8}$$

where the new time variable $s = s(\tau)$ is given by

$$g(\tau)d\tau = ds, \tag{3.9}$$

so that $s : \mathbf{R}_+ \to \mathbf{R}_+$ by the non-integrability condition (2.27).

In what follows, we need to consider the case of bounded perturbations in (3.8) and assume that along a subsequence $\{s_k\} \to \infty$, there exists a finite limit

$$(3.10) \qquad g^{-2}(\tau_k)g'(\tau_k) \to a_0 \le 0, \quad s_k = \int_0^{\tau_k} g(\tau)d\tau.$$

We will show later on that an infinite limit leads to contradiction. In view of (3.3), the inner expansion provides us with the following expansion near the origin: as $s \to \infty$,

$$(3.11) \qquad F(\zeta, s) = \frac{c_1}{\zeta}(1 - \frac{1}{2}\zeta^2 + ...) \quad \text{for sufficiently small } \zeta > 0.$$

Let us pass to the limit $s \to \infty$ in equation (3.8). The non-autonomous perturbations on the right-hand side are asymptotically small. By the standard parabolic regularity theory, it follows that

$$(3.12) \qquad F(\zeta, s_k + s) \to \bar{F}(\zeta, s) \quad \text{in } L^\infty_{\rm loc}(\mathbf{R}_+; C([\delta_1, \delta_2]))$$

as $k \to \infty$, where $\bar{F}$ is a weak and hence a classical solution of the unperturbed uniformly parabolic equation

$$(3.13) \qquad \bar{F}_s = \Delta_\zeta \bar{F} + \mathbf{C}\bar{F} - \frac{1}{2}a_0(F_\zeta \zeta + F)$$

with initial data $\bar{F}_0$ from the ω-limit set of the orbit $\{F(\cdot, s), s > 0\}$. Let us note that due to (2.49) $g(\tau)$ satisfies another typical condition of a slow (not exponentially fast) decaying function

$$(3.14) \qquad g(\tau_k + \tau)/g(\tau_k) \to 1 \quad \text{as } k \to \infty$$

uniformly on compact intervals in τ. This follows from (2.49) integrated over $(\tau_k, \tau_k + \tau)$ with uniformly bounded τ. This is valid for $g(\tau) \sim 1/\tau$ and other functions decaying more slowly as $\tau \to \infty$ than the exponential one $g(\tau) \sim e^{-\tau}$, for which (3.14) does not hold. Now passing to the limit in (3.11), we deduce that $\bar{F}(\zeta, s)$ satisfies the following singular stationary condition near the origin:

$$(3.15) \qquad \bar{F}(\zeta, s) = \frac{c_1}{\zeta}(1 - \frac{1}{2}\zeta^2 + ...) \quad \text{for small } \zeta > 0.$$

Actually, this means that there exists the limit

$$(3.16) \qquad (\zeta \bar{F})_{\zeta\zeta} = -c_1 \quad \text{at } \zeta = 0.$$

Since (3.13) is an autonomous uniformly parabolic equation, the strong Maximum Principle implies that $\bar{F}(\zeta, s) \equiv F_0(\zeta)$ (in view of (3.16) we rewrite the problem in terms of $\tilde{F} = \zeta F$) is a stationary solution satisfying the ODE problem

$$(3.17) \qquad \Delta F_0 - F_0'\zeta l_1 - l_2 F_0 = 0, \quad \zeta > 0; \quad (\zeta F_0)''(0) = -c_1,$$

where $l_1 = 2 + a_0/2$ and $l_2 = 1 + a_0/2$. Since F_0 will be shown to be unique, we have that $F(\zeta, s) \to F_0(\cdot)$ as $s \to \infty$ uniformly on compact intervals $[\delta_1, \delta_2]$.

We now prove that $F_0(\zeta)$ given by series

$$F_0(\zeta) = c_1 \frac{1}{\zeta}(1 - \frac{1}{2}\zeta^2) + \sum_{k=2}^{\infty} C_k \zeta^k, \tag{3.18}$$

vanishes at a finite $\zeta = \zeta_0$ corresponding in the original independent variable to $1 - r = o(1)$. One can see that the even coefficients vanish, $C_{2l} \equiv 0$, and the odd ones with $k = 2l + 1$ satisfy

$$C_{k+2} = \frac{kl_1 + l_2}{(k+2)(k+3)} C_k, \quad k = 3, 5, ..., \quad \text{where } C_3 = -\frac{c_1}{24}(3 + a_0), \;\; c_1 = \sqrt{3}.$$

Since $|C_{k+2}/C_k| \le (1 + 2|l_1|)/k \to 0$ as $k \to \infty$, the series in (3.18) converges uniformly in $\mathbf{R}_+$. We are looking for the stationary profile $F_0(\zeta)$ which is oscillating as $\zeta \to \infty$ and then its first positive root $\zeta = \zeta_0 > 0$,

$$F_0(\zeta_0) = 0, \tag{3.19}$$

gives the necessary matching condition.

Let us show that the value $\zeta_0 = \zeta_0(a_0)$ does not affect the limit (3.10). We use the expansion in the intermediate region

$$u(r,t) \equiv \alpha(t) f(\xi, \tau) = \alpha(t)\sqrt{g(\tau)} F(\zeta, s) = \alpha(t)\sqrt{g(\tau)}[F_0(\zeta) + o(1)] \tag{3.20}$$

as $\tau \to \infty$ uniformly on compact subsets in ζ bounded away from the origin. Then (3.19) guarantees that the extended solution $u(r,t)$ changes sign and hence vanishes at a point $\zeta = \zeta_0 + o(1)$ which must coincide with $r = 1$. This is a natural representation of matching with the boundary Dirichlet condition only. Therefore, we arrive at the following matching condition at $r = 1$:

$$\zeta = \xi\sqrt{g(\tau)} \equiv r\alpha^2(t)\sqrt{g(\tau)}|_{r=1} = \zeta_0 + o(1). \tag{3.21}$$

Since $g(\tau) = (\alpha^{-5}\alpha')(t)$, we obtain the equation

$$\alpha^2(t)\sqrt{\alpha^{-5}(t)\alpha'(t)} = \zeta_0 + o(1) \implies \frac{\alpha'(t)}{\alpha(t)} = \zeta_0^2 + o(1).$$

This is a functional (actually, infinite-dimensional) dynamical system, which asymptotically can be studied as an ODE for the function $\alpha(t)$. Integrating it, we obtain $\ln \alpha(t) = \zeta_0^2 t(1 + o(1))$, cf. (1.9). It then follows that $g(\tau) = \alpha^{-5}(t)\alpha'(t) = \zeta_0^2\alpha^{-4}(t)(1 + o(1))$, and hence using that $dt/d\tau = \alpha^{-4}(t)$,

$$g^{-2}(\tau)g'_\tau(\tau) = g^{-2}g'_t\alpha^{-4} = -4 + o(1) \implies a_0 = -4.$$

Hence, $l_2 = 0$, $l_1 = -1$ and the ODE for $F_0(\zeta)$, $\Delta F_0 + F_0 = 0$, together with the singular boundary condition in (3.17) imply that

$$F_0(\zeta) = \sqrt{\frac{3\pi}{2\zeta}} J_{-1/2}(\zeta) \equiv \frac{\sqrt{3}}{\zeta}\cos\zeta,$$

where $J_{-1/2}(\zeta) = \sqrt{2/\pi\zeta}\cos\zeta$ is a Bessel function. Therefore,

$$\zeta_0 = \pi/2. \tag{3.22}$$

These results are true for any subsequence $\{s_k\} \to \infty$.

Finally, it is not difficult to show that the assumption that $a_0 = \infty$, after an additional natural scaling and passing to the limit $s_k + s \to \infty$, implies that the final profile F_0 satisfies the first-order ODE $F_0'\zeta + F_0 = 0$, which does not admit a solution satisfying the singular boundary condition (3.15) and (3.16).

Using the above change of time-variable, we estimate the perturbation term $g(\tau) \sim 1/4\tau$ for $\tau \gg 1$, which is a typical slowly decaying function. It is a non-integrable perturbation, $g \notin L^1(\mathbf{R}_+)$, as we have proved.

Remark 1: a countable spectrum of different GUSs changing sign. It follows from the construction of the oscillating limit profile $F_0(\zeta)$ that, besides $u_*(r,t) \ge 0$, there exists a countable spectrum $\{u_{*k}(r,t)\}$ of GUSs changing sign exactly k times with the following functions

$$\ln \alpha_k(t) = \zeta_k^2\, t(1 + o(1)), \quad k = 1, 2, \dots,$$

where $\{\zeta_k = \pi/2 + \pi k\}$ is the strictly increasing sequence of roots of the Bessel function $F_0(\zeta)$, i.e. of $\cos\zeta$.

Remark 2: on extension of boundary layer behaviour for $N = 3$. Bearing in mind that, roughly speaking, $\alpha(t) \sim e^{\gamma_0 t}$ for $t \gg 1$, we observe a typical exponential growth of the L^∞-norm of the GUS. Let us show that this exponential function can be explained by means of a stable subspace dominance in the outer region (as we have seen, no such dominance is available in the inner region). Indeed, since $u(\cdot,t) \to 0$ as $t \to \infty$ for all $r \in (0,1)$, the linearized flow

$$Y_t = \Delta Y$$

is governed by the radial Laplacian having a discrete negative spectrum $\{\lambda_k < 0,\ k = 0, 1, \dots\}$ in the space $L^2_{r0}(B_1)$ of radial functions satisfying an extra condition at the singular point $r = 0$

$$\lim_{r \to 0} (rY(r))' = 0.$$

It excludes bounded functions $Y(r)$ at $r = 0$ satisfying $Y(0) \ne 0$, and hence retains those with the purely fundamental asymptotics $Y(r) = Cr^{-1} + o(1)$ as $r \to 0$. This makes the spectrum discrete, [34]. Therefore, any positive GUS has the following behaviour for all $r \in (0,1)$:

$$u(r,t) = C_0 e^{\lambda_1 t}\psi_1(r) + \dots \quad \text{as } t \to \infty,$$

where $\psi_1(r) > 0$ on $(0,1)$ denotes the first eigenfunction of Δ in $L^2_{r0}(B_1)$ and the constant $C_0 > 0$ depends on initial data. By construction, $\psi_1(r) = a_1 r^{-1} + o(1)$ as $r \to 0$, where $a_1 > 0$ is a normalization constant. Hence, for any fixed small $r > 0$,

$$u(r,t) = a_1 C_0 r^{-1} e^{\lambda_1 t} + \dots \quad \text{as } t \to \infty.$$

Comparing with (2.4), by Theorem 2.1 we conclude that the first terms of these two expansions can be matched (after an extra slight extension of the inner expansion) if

$$\alpha(t) = (a_1 C_0)^{-1} e^{-\lambda_1 t}(1 + o(1)) \quad \text{as } t \to \infty. \tag{3.23}$$

This means that

$$\ln \alpha(t) = (-\lambda_1)t(1 + o(1)),$$

which coincides with (1.9). Thus, $\gamma_0(3) = -\lambda_1 > 0$, where $\lambda_1 < 0$ is the first eigenvalue of Δ in $L^2_{r0}(B_1)$. Observe that (3.23) exhibits the dependence of the L^∞-norm of the GUS on the initial data by means of the constant $C_0 = C_0(u_0)$. This is a typical stable manifold dominance in the outer region (the boundary layer), where the asymptotic structure of GUSs can be obtained most easier by matching into the inner region near $r = 0$. This is an example of a standard stable-centre manifold matching.

This behaviour of GUSs in $\mathbf{R}^3$ exhibits some common features of stabilization to a singular stationary solution of semilinear and quasilinear parabolic equations studied in [10], where the Frank-Kamenetskii equation with the Dirichlet boundary condition

$$u_t = \Delta u + e^u \;\text{ in }\; B \times \mathbf{R}_+, \qquad u = 0 \;\text{ on }\; \partial B \times \mathbf{R}_+, \tag{3.24}$$

is considered in the ball $B = \{|x| < \sqrt{2(N-2)}\}$, with $N > 10$. The only stationary solution is the *singular* one

$$S_\infty(x) = \ln\left(2(N-2)/|x|^2\right) \quad (S_\infty(0) = \infty),$$

satisfying the zero Dirichlet boundary condition at $r = \sqrt{2(N-2)}$. There exists a family $\{S_\lambda(x)\}$, with the same parameterization $S_\lambda(0) = \lambda \in \mathbf{R}$, of bounded at the origin stationary profiles such that $S_\lambda(x) \to S_\infty(x) - 0$ as $\lambda \to \infty$ uniformly on compact subsets bounded away from the origin. In the problem (3.24), the rate of stabilization from below of a *bounded orbit* to the *singular stationary solution* is determined by matching of the outer (boundary) expansion and the inner one, where the solutions evolve on this smooth family of stationary profiles. For (3.24), the linearized operator in the outer (boundary) domain does not have a centre subspace, so that the outer behaviour corresponds to the stable invariant subspace. Then the stable manifold behaviour in the boundary domain can be obtained independently of the inner (centre manifold) one. The outer-inner matching consists in extending the known outer behaviour into the inner region, which is equivalent to a matching of stable and centre manifolds.

3.2. **Extension and matching for $N \geq 5$.** Though the new rescaled variables are different, the idea of matching is similar. We now substitute expansions (2.3) and (2.53) into (3.1) to obtain that for $\xi \gg 1$,

$$f(\xi, \tau) = \xi^{2-N}(c_1 - c_* g(\tau)\xi^{N-2} + \ldots). \tag{3.25}$$

Hence, the new rescaled variable of the extended expansion takes the form

$$\zeta = \xi g^{1/(N-2)}(\tau), \tag{3.26}$$

and the inner expansion (3.25) reads

$$f(\xi, \tau) = g(\tau)\zeta^{2-N}(c_1 - c_* \zeta^{N-2} + \ldots). \tag{3.27}$$

We next introduce the new rescaled solution

$$f(\xi, \tau) = g(\tau)F(\zeta, \tau), \tag{3.28}$$

where F satisfies the perturbed parabolic equation

$$F_s = \Delta_\zeta F + g^{2/(N-2)}(\tau) F^p + g^{(N-4)/(N-2)}(\tau) \mathbf{C} F - (g^{-N/(N-2)} g')(\tau) (\frac{1}{N-2} F_\zeta \zeta + F), \tag{3.29}$$

with the new time variable obtained from

$$g^{2/(N-2)}(\tau) = ds/d\tau. \tag{3.30}$$

As in the previous case $N = 3$, assuming that $g^{-N(N-2)}(\tau_k) g'(\tau_k) \to a_0 \le 0$, we conclude from final computations that $a_0 = 0$ (unbounded particular limits with $a_0 = \infty$ lead to a contradiction with singular boundary conditions). Hence, the orbit $\{F(\cdot, s), s > 0\}$ stabilizes as $s \to \infty$ to a unique stationary solution $F_0(\zeta)$ of the autonomous limit equation (3.29),

$$\Delta F_0 = 0 \ \text{ for } \ \zeta > 0, \quad F_0(\zeta) = c_1 \zeta^{2-N} - c_* + o(1) \ \text{ as } \ \zeta \to 0. \tag{3.31}$$

Obviously, this gives the unique stationary profile $F_0(\zeta) = c_1 \zeta^{2-N} - c_*$ vanishing at the point $\zeta_0 = (c_1/c_*)^{1/(N-2)}$. Hence, $r = 1$ corresponds to $\zeta = \zeta_0 + o(1)$, whence the following asymptotic ODE for $\alpha(t)$ holds as $t \to \infty$:

$$\zeta = \xi g^{1/(N-2)}(\tau) \equiv r \alpha^{2/(N-2)}(t) g^{1/(N-2)}(\tau)|_{r=1} = \zeta_0 + o(1),$$

or, equivalently, $\alpha^{(N-6)/(N-2)}(t) \alpha'(t) = \zeta_0^{N-2} + o(1)$. Integrating this equation, we obtain the asymptotic behaviour (1.11) (and hence $a_0 = 0$), where

$$\gamma_0 = \big[2(N-4) \zeta_0^{N-2}/(N-2)\big]^{(N-2)/2(N-4)}.$$

One can check that in this case $g(\tau) \sim 1/\tau$ for $\tau \gg 1$ satisfying all the necessary slow decay properties for a non-integrable perturbation.

3.3. **Extension and matching for $N = 4$.** We need a slight modification of the previous analysis, in which we put slow logarithmic factors $|\ln g(\tau)|$ in the computations, as required by the extra term $\ln \xi$ in the expansion (2.52). Substituting (2.3) and (2.52) into (3.1), we obtain that for $\xi \gg 1$

$$f(\xi, \tau) = \frac{8}{\xi^2} - 4 g(\tau) \ln \xi + ... \equiv \frac{8}{\xi^2} (1 - \frac{1}{2} g(\tau) \xi^2 \ln \xi + ...) \tag{3.32}$$

(here $c_1 = 8$). It follows that the new rescaled variable for $\tau \gg 1$ is

$$\zeta = \xi \sqrt{(g |\ln g|)(\tau)}, \tag{3.33}$$

and the inner expansion (3.32) on compact subsets in $\zeta > 0$ takes the form

$$f(\xi, \tau) = \sqrt{(g |\ln g|)(\tau)} \, \frac{8}{\zeta^2} (1 - \frac{1}{4} \zeta^2 + ...) \tag{3.34}$$

Hence, we define the rescaled solution by means of the rescaling

$$f(\xi, \tau) = \sqrt{(g |\ln g|)(\tau)} F(\zeta, \tau), \tag{3.35}$$

where F satisfies the following perturbed parabolic equation:

$$F_s = \Delta F + (g|\ln g|)(\tau)F^3 + \frac{1}{|\ln g(\tau)|}\mathbf{C}F - \frac{(g|\ln g|)'(\tau)}{(g^2|\ln g|^2)(\tau)}(\frac{1}{2}F_\zeta \zeta + F) \tag{3.36}$$

with the time variable given by

$$(g|\ln g|)(\tau) = ds/d\tau. \tag{3.37}$$

This is again a perturbation of the heat equation, and we can apply the arguments of the previous case. In particular, we conclude that the last perturbation on the right-hand side of (3.36) tends to zero as $\tau \to \infty$, corresponding to $a_0 = 0$. Finally, the asymptotic ODE for $\alpha(t)$ is obtained from matching with the boundary condition, implying that $r = 1 + o(1)$ corresponds to the unique positive root $\zeta_0 = 2$ of the limit stationary profile $F_0(\zeta) = 8\zeta^{-2} - 2$. Thus, for $\tau \gg 1$,

$$\zeta = \xi(g|\ln g|)^{1/2}(\tau) \equiv r\alpha(t)\sqrt{(g|\ln g|)(\tau)}|_{r=1} = 2 + o(1).$$

By definition of $g(\tau)$, $\alpha\sqrt{\alpha^{-3}\alpha'|\ln(\alpha^{-3}\alpha')|} = 2 + o(1)$ for $t \gg 1$, or, resolving this asymptotically,

$$\frac{\alpha'(t)}{\alpha(t)} \ln \alpha(t) = 2 + o(1).$$

Integrating for $t \gg 1$ yields (1.10). It then follows that $g(\tau) \sim 1/2\tau$ for $\tau \gg 1$.

This completes the proof of Theorem 1.1. □

For $N \geq 4$, our matching asymptotic analysis leads to monotone decreasing profiles $F_0(\zeta)$ without oscillatory behaviour as $\zeta \to \infty$. Therefore, a discrete spectrum of other GUSs changing sign cannot be observed on compact subsets in ζ and needs an additional scaling procedure.

In dimensions $N \geq 4$ the general structure of composite asymptotics of the GUSs is of the same type as the asymptotic extinction behaviour of the fast diffusion equation

$$u_t = \Delta u^m \quad \text{in } \Omega \times (0, T), \quad u = 0 \text{ on } \partial\Omega \times (0, T), \tag{3.38}$$

with bounded positive initial data, in the critical Sobolev case $m = m_s = 1/p_s = (N-2)/(N+2)$. If T is the finite extinction time of the radial solution $u(r,t)$, then its asymptotic extinction behaviour as $t \to T^-$ is given by [16]

$$\|u(\cdot, t)\|_\infty = \gamma_0 (T-t)^{(N+2)/4} |\ln(T-t)|^{(N+2)/2(N-2)}(1 + o(1)), \tag{3.39}$$

where γ_0 is a constant depending on dimension N only. The first dimensional multiplier $(T-t)^{(N+2)/4}$ is exactly the similarity scaling factor obtained by J.G. Berryman and C.J. Holland in 1980 [5] in the subcritical range $m \in (m_s, 1)$. The extra slow growing, but unbounded, logarithmic factor in the right-hand side of (3.39) comes from centre-centre manifolds matching.

4. **On detailed formal matching analysis**

We briefly present here a more detailed formal matching analysis prescribing some important exponents and constants in the asymptotic behaviour of GUSs. A rigorous justification of such calculations lead to more sophisticated asymptotic problems which are not studied here. At the initial stage of the analysis we repeat the main arguments of the asymptotic expansion techniques given above presenting the first terms of the asymptotic behaviour of GUSs.

4.1. **Inner problem:** $\xi = O(1)$. Substituting the inner scalings $u(r,t) = \alpha(t) f(\xi, t)$, $\xi = r\alpha^{2/(N-2)}(t)$ into the radial problem

$$u_t = r^{1-N}(r^{N-1}u_r)_r + u^{(N+2)/(N-2)}, \quad u = 0 \ \text{ on } \ r = 1, \tag{4.1}$$

gives

$$\alpha^{-(N+2)/(N-2)}\alpha'(f + \frac{2}{N-2}\xi f_\xi) = f_{\xi\xi} + \frac{N-1}{\xi} f_\xi + f^{(N+2)/(N-2)}.$$

We require the two-term inner expansion

$$f(\xi, t) \sim \Phi_0(\xi) + \alpha^{-(N+2)/(N-2)}\alpha'\Phi_1(\xi),$$

whereby $\Phi_0(\xi) = S_1(\xi) = [1 + \xi^2/N(N-2)]^{-(N-2)/2}$ and Φ_1 solves

$$\Phi_{1\xi\xi} + \frac{N-1}{\xi}\Phi_{1\xi} + \frac{N+2}{N-2}\Phi_0^{4/(N-2)}\Phi_1 = \Phi_0 + \frac{2}{N-2}\xi\Phi_{0\xi}. \tag{4.2}$$

The homogeneous problem with zero right-hand side has a solution

$$\omega(\xi) = [1 - \xi^2/N(N-2)][1 + \xi^2/N(N-2)]^{-N/2}$$

so that

$$\xi^{N-1}(\omega\Phi_{1\xi} - \Phi_1\omega_\xi) = \int_0^\xi [\zeta^{N-1}\omega(\zeta)\Phi_0(\zeta) + \frac{2}{N-2}\zeta^N\omega(\zeta)\Phi_{0\zeta}(\zeta)]d\zeta. \tag{4.3}$$

It follows that the asymptotic properties of Φ_1 depend on whether N is greater than, equal to or less than four. For $N \le 4$ the integral in (4.3) diverges and we find that

$$\Phi_1(\xi) \sim -[N(N-2)]^{(N-2)/2}\frac{\xi^{4-N}}{2(4-N)} \quad \text{as } \ \xi \to \infty, \quad 2 < N < 4, \tag{4.4}$$

a result which can also readily be obtained directly from (4.2). On the other hand,

$$\Phi_1(\xi) \sim -4\ln\xi + 6\ln 2 + \frac{20}{3} \quad \text{as } \ \xi \to \infty, \quad N = 4, \tag{4.5}$$

where the coefficient of the $\ln\xi$ term can also be deduced directly from (4.2), but calculation of the constant term requires the first integral, as it does for $N > 4$, when the integral (4.3) converges to give

$$\Phi_1(\xi) \sim -\frac{2N(N+2)\Gamma^2(N/2)}{(N-2)(N-4)\Gamma(N)} \quad \text{as } \ \xi \to \infty, \quad N > 4. \tag{4.6}$$

The differences in the asymptotic forms (4.4)–(4.6) indicate that the nature of the outer expansion will differ with the sign of $N-4$.

4.2. **Outer problem:** $r = O(1)$**. Matching.** Here matching with Φ_0 requires the outer scaling $u(r,t) = \alpha^{-1}(t)\Psi(r,t)$, so that

$$\Psi_{rr} + \frac{N-1}{r}\Psi_r = -\frac{\alpha'}{\alpha}\Psi + \Psi_t - \alpha^{-4/(N-2)}\Psi^{(N+2)/(N-2)}.$$

The first term on the right-hand side may or may not enter at leading order, depending on the sign of $N-4$.

Dimensions $N > 4$**.** This is the simplest case to discuss since it turns out that $(\alpha'/\alpha)(t) = O(t^{-1})$ is negligible as $t \to \infty$, so that

$$\Psi(r,t) \to \Psi_0(r) = [N(N-2)]^{(N-2)/2}(r^{2-N} - 1), \tag{4.7}$$

where we have matched the singular term with Φ_0. The constant term in (4.7) matches with (4.6), so that

$$\frac{1}{\alpha}[N(N-2)]^{(N-2)/2} \sim \frac{2N(N+2)\Gamma^2(N/2)}{(N-2)(N-4)\Gamma(N)}\alpha^{-4/(N-2)}\alpha',$$

i.e. as $t \to \infty$,

$$\alpha^{2(N-4)/(N-2)}(t) \sim [N(N-2)]^{(N-2)/2}\frac{(N-4)^2\Gamma(N)}{N(N+2)\Gamma^2(N/2)}\,t. \tag{4.8}$$

This uniquely determines the constant in (1.12).

Dimensions $N < 4$**.** Since (4.7) cannot match with (4.4), we require

$$(\alpha'/\alpha)(t) \to \mu^2 \quad \text{as } t \to \infty \tag{4.9}$$

for some real constant $\mu > 0$ and hence

$$\Psi_{0rr} + \frac{N-1}{r}\Psi_{0r} + \mu^2\Psi_0 = 0.$$

Writing $\Psi_0(r) = r^{-(N-2)/2}\sigma(r)$ gives

$$r^2\sigma_{rr} + r\sigma_r + [\mu^2 r^2 - \frac{1}{4}(N-2)^2]\sigma = 0. \tag{4.10}$$

Matching with the inner expansion taking into account both Φ_0 and Φ_1 demands that

$$\sigma(r) \sim [N(N-2)]^{(N-2)/2}[r^{-(N-2)/2} - \frac{\alpha'}{\alpha}\frac{1}{2(4-N)}r^{(6-N)/2}] \quad \text{as } r \to 0, \tag{4.11}$$

thereby excluding the $J_{(N-2)/2}(\mu r)$ solution to (4.10), we obtain

$$\sigma(r) = [N(N-2)]^{(N-2)/2}2^{-(N-2)/2}\mu^{(N-2)/2}\Gamma(N/2)J_{-(N-2)/2}(\mu r), \tag{4.12}$$

whose Taylor expansion about $r = 0$ automatically reproduces (in view of (4.9)) the second term in (4.11). Finally, μ^2 is generically given as the first eigenvalue of the singular Sturm-Liouville problem for equation (4.10) with the boundary condition $\sigma(1) = 0$, so that $J_{-(N-2)/2}(\mu) = 0$. For $N = 3$, $J_{-1/2}(\mu r) = \sqrt{2/\pi\mu r}\cos(\mu r)$, so that $\mu = \pi/2$.

Inspection of the size of terms omitted from the above expansions suggests that (4.9) implies that

$$\alpha(t) \sim \alpha_\infty e^{\mu^2 t} \quad \text{as } t \to \infty \tag{4.13}$$

for some constant α_∞ which depends on initial data u_0.
Critical dimension $N = 4$. As usual, the critical case is the most delicate one, requiring calculation of the first correction term Ψ_1 in the outer expansion

$$\Psi(r,t) \sim \Psi_0(r) + \frac{\alpha'(t)}{\alpha(t)}\Psi_1(r) \quad \text{as } t \to \infty.$$

whereby

$$\Psi_0(r) = 8(r^{-2} - 1) \tag{4.14}$$

(as in (4.7)) and

$$\Psi_{1rr} + \frac{3}{r}\Psi_{1r} = -8(r^{-2} - 1).$$

Since matching with the two-term inner expansion requires that Ψ_1 have no term of $O(r^{-2})$ as $r \to 0$, it follows that

$$\Psi_1(r) = -4\ln r + r^2 - 1. \tag{4.15}$$

Since $\xi = r\alpha(t)$ in this case, the outer limit of the inner expansion gives (using (4.5)) that

$$\Psi(r,t) \sim 8r^{-2} + \frac{\alpha'}{\alpha}(-4\ln r - 4\ln\alpha + 6\ln 2 + \frac{20}{3}), \tag{4.16}$$

so matching in the overlap regime using (4.14)–(4.15) requires

$$\frac{\alpha'}{\alpha}(-4\ln\alpha + 6\ln 2 + \frac{20}{3}) \sim -8 - \frac{\alpha'}{\alpha}.$$

Hence, $\ln^2\alpha(t) - (3\ln 2 + 23/6)\ln\alpha(t) \sim 4t$ as $t \to \infty$, and therefore

$$\alpha(t) \sim 2^{3/2}e^{23/12}e^{2\sqrt{t}} \quad \text{as } t \to \infty. \tag{4.17}$$

As in (4.8) (but in contrast to (4.13)), this expression contains no arbitrary constants, with the leading order dependence of the large-time behaviour on the initial conditions presumably corresponding (as in (4.13)) to translations in t.

4.3. **Higher modes.** We consider equation (1.1) with the reaction term $|u|^{p-1}u$. For $N < 4$, subsequent zeros of the Bessel function, $J_{-(N-2)/2}(\mu) = 0$, in (4.12) – (4.13) dictate the behaviour of (non-generic) sign-changing GUSs. For $N > 4$, the term (4.7) dominates the outer limit of the inner expansion (so that $u = O(1/\alpha)$ no longer holds in the outer region) and hence

$$u \sim \sigma_\infty \Gamma(N/2) e^{\mu^2 t}(\mu r/2)^{-(N-2)/2} J_{(N-2)/2}(\mu r) \tag{4.18}$$

holds as $t \to \infty$ in the outer region $r = O(1)$ for some positive constant σ_∞ which depends on the initial data. Matching with (4.7) implies that

$$\frac{2N(N+2)\Gamma^2(N/2)}{(N-2)(N-4)\Gamma(N)}\,\alpha^{-4/(N-2)}\alpha' \sim \sigma_\infty e^{-\mu^2 t}$$

so that

$$\alpha^{(6-N)/(N-2)}(t) \sim \frac{2N(N+2)\Gamma^2(N/2)}{(N-4)(N-6)\Gamma(N)\mu^2\sigma_\infty}\, e^{\mu^2 t} \tag{4.19}$$

as $t \to \infty$. In (4.18) – (4.19), the possible growth rates are determined by the boundary condition $u = 0$ on $r = 1$, so that

$$J_{(N-2)/2}(\mu) = 0; \tag{4.20}$$

the first zero of (4.20) corresponds to a solution with a single sign change, as indicated by the dominance of (4.7) over Φ_0 in the intermediate region. Finally, for $N = 4$ one has (because the $\alpha' \ln \alpha/\alpha$ term dominates in (4.16))

$$\ln \alpha(t) \sim \mu^2 t \quad \text{as } \; t \to \infty \tag{4.21}$$

with $J_1(\mu) = 0$, cf. (4.20); however, the calculation of the pre-exponential terms in $\alpha(t)$ would require very careful consideration of extra terms in the two regions, which we do not pursue here.

5. On GUSs of the nonlinear parabolic equations

In general, taking into account key principles of construction, there is not much difference between GUSs for the above critical semilinear heat equation and quasilinear and fully nonlinear ones with typical scaling invariant operators. The main ideas and techniques of linearization, extension and matching stay the same, though some specific additional technical difficulties may occur.

5.1. **Porous medium equation with source.** As a natural quasilinear generalization, let us first discuss the porous medium equation with source

$$u_t = \Delta u^m + u^p, \quad \text{where } \; m > 1, \; p = p_s = m(N+2)/(N-2), \;\; N \geq 3, \tag{5.1}$$

for which the existence of GUSs in the critical Sobolev case $p = p_s$ was already established in [41]. The steps of the inner asymptotic analysis are quite similar. In this case it is convenient to perform the transformation $u^m = v$ leading to the equation with the same semilinear elliptic operator on the right-hand side

$$(v^{1/m})_t = \mathbf{B}(v) \equiv \Delta v + v^p, \quad \text{where now } \; p = (N+2)/(N-2),$$

and hence with the same family (2.2) of stationary solutions. Hence, we perform exactly the same scaling

$$v(r,t) = \alpha(t) f(\xi, \tau), \quad \xi = r\alpha^{2/(N-2)}(t),$$

to derive a perturbed rescaled equation of the same type

$$(f^{1/m})_\tau = \mathbf{B}(f) + g(\tau)\mathbf{C}(f), \tag{5.2}$$

where the time variable is define by $\alpha^{p-1/m}(t) = d\tau/dt$, and

$$g(\tau) = (\alpha^{-p}(\alpha^{1/m})')(t), \quad \mathbf{C}(f) = -\frac{2m}{N-2}\,(f^{1/m})_\xi \xi - f^{1/m}.$$

As the first step, using similar estimates and methods, we arrive at a result similar to Theorem 2.1 in Section 2. As the second step, we perform the linearization (2.37) and arrive at the corresponding linear semigroup generated by the same operator (2.39) in a weighted space, where the weight function $\rho = m S_1^{1/m-1}(\xi)$ ($\rho \equiv 1$ in the semilinear case $m = 1$) comes from the nonlinearity on the left-hand side of equation (5.2). This implies no essential novelties in the analysis. In particular, the critical dimension is now $N = 2m + 2$ ($N = 4$ if $m = 1$) for which $R(\xi)$ has a logarithmic growth as $\xi \to \infty$. The rest is similar and the stabilization technique uses the strong Maximum Principle, which is true in the positivity domain for such degenerate equations; see Kalashnikov's survey [29]. This analysis partially coincides with that in [16], where composite solution structures of the (quasilinear) fast diffusion equation were studied; see further comments below.

5.2. **Dual porous medium equation with source.** Consider now a fully nonlinear equation with the dual porous medium operator,

$$u_t = |\Delta u|^{m-1} \Delta u + u^p, \quad m > 1, \tag{5.3}$$

where the critical Sobolev exponent $p = p_s$ is the same as in (5.1). The stationary family (2.2) satisfies the elliptic equation

$$\mathbf{B}(S) \equiv |\Delta S|^{m-1} \Delta S + S^p = 0 \quad \text{in } \mathbf{R}^N.$$

Though the existence of GUSs is not known for this equation (we expect that crucial ideas from [41] can be applied), we are going to describe the composite structure of those solutions which are evolving on the family $\{S_\lambda\}$ in the inner region such that $u(r,t) \sim S_{\alpha(t)}(r)$ as $t \to \infty$. The inner scaling transformation takes a form similar to (2.20),

$$u(r,t) = \alpha(t) f(\xi, \tau), \quad \xi = r\alpha^\gamma(t), \tag{5.4}$$

where $\gamma = (p-m)/2m$, and we obtain the perturbed equation (2.21) with

$$\mathbf{C}f = -\gamma f_\xi \xi - f, \quad g(\tau) = (\alpha^{-p}\alpha')(t), \quad \text{where } d\tau/dt = \alpha^{p-1}(t). \tag{5.5}$$

A result similar to Theorem 2.1 in section 2 follows from similar estimates by using the Sturm theorem, which by approximation applies to sufficiently smooth (minimal) solutions of degenerate parabolic equations with the dual porous medium operator.

The linearization (2.37) leads to a perturbed equation of the type (2.38), with the function $Z = \mathbf{C}S_1$ in the leading perturbation and the linear operator

$$\mathbf{A}Y = m|\Delta S_1|^{m-1} \Delta Y + p S_1^{p-1} Y,$$

which is a perturbed radial Laplacian in a weighted space. It follows that, at least for $p > m$, the perturbing second term is asymptotically vanishing as $\xi \to \infty$, and the further refined inner expansion needs no essential novelties. Finally, we arrive at the expansion (2.50), where R solves the ODE (2.47). Then the rescaled extension analysis, and hence the approximation of $\alpha(t)$ as $t \to \infty$, depends on an expansion of $R(\xi)$ for $\xi \gg 1$ as in Proposition 2.11. At this stage we face a similar stabilization problem for a perturbed evolution equation which can be solved by using the strong Maximum Principle.

5.3. **Gradient diffusivity equations with source.** Let us discuss GUSs of the p-Laplacian equation with source

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^p, \quad \sigma > 0, \;\; p = p_s = [N(\sigma+1) + \sigma + 2]/[N - (\sigma+2)], \tag{5.6}$$

where $N > \sigma + 2$. First of all, the stationary equation

$$\mathbf{B}(S) \equiv \nabla \cdot (|\nabla S|^\sigma \nabla S) + S^p = 0 \quad \text{in } \mathbf{R}^N,$$

admits a suitable family of positive radial solutions (see [17] and [24])

$$S_\lambda(r) = \lambda \left[B(B + \lambda^\beta r^\gamma)^{-1} \right]^\mu, \quad \mu = [N - (\sigma+2)]/(\sigma+2), \;\; \gamma = (\sigma+2)/(\sigma+1),$$

where $\beta = \gamma/\mu$ and $B = \mu\gamma N^{1/(\sigma+1)}$. In this critical p-Laplacian equation, exact solutions occur via invariance of the corresponding variational functional (Noether's Theorem) [48], unlike the semilinear critical case with $\sigma = 0$ where the conformal invariance of the elliptic equation itself is available (see further comments below on such higher-order equations).

Having such a stationary subset $\{S_\lambda\}$, we can describe GUSs evolving in the inner region according to $u(\cdot,t) \sim S_{\alpha(t)}(\cdot)$ as $t \to \infty$. Here the scaling transformation takes the form (5.4) with the exponent $\gamma = [p - (\sigma+1)]/(\sigma+2)$. The linearized equation for Y is (2.38) with the following second-order ordinary differential operator

$$\mathbf{A}Y = (\sigma+1)\nabla \cdot (|\nabla S_1|^\sigma \nabla Y) + p S_1^{p-1} Y,$$

which is a first-order perturbation of the radial Laplacian in a weighted space. The extension of the inner expansion and matching with the zero Dirichlet boundary condition at $r = 1$ via suitable new rescaled variables is based on a result similar to Proposition 2.11, where $R(\xi)$ is obtained from the ODE (2.47) with the same inhomogeneous term $Z = \mathbf{C}S_1$. As usual, a key role in the classification of $\alpha(t)$ is played by the behaviour of $R(\xi)$ for $\xi \gg 1$, which defines new scaling variables matching the boundary condition.

5.4. **On higher-order semilinear equations.** Our general scheme of extension and matching can be formally applied to higher-order parabolic equations, though we do not have sufficient *a priori* estimates, auxiliary bounds and properties of stationary families to complete a rigorous proof. The key example is the $2m$-th order semilinear parabolic equation

$$u_t = -(-\Delta)^m u + |u|^{p-1}u, \quad m > 1, \;\; p = p_s = (N+2m)/(N-2m), \;\; N > 2m, \tag{5.7}$$

which is a natural higher-order generalization of the second-order one.

Consider the fourth-order equation, (5.7) with $m = 2$,

$$u_t = -\Delta^2 u + |u|^{p-1}u \quad \text{in } \Omega \times \mathbf{R}_+, \quad p_s = (N+4)/(N-4), \;\; N \ge 5. \tag{5.8}$$

In the case of the critical Sobolev exponent, the corresponding elliptic equation

$$\mathbf{B}(S) \equiv -\Delta^2 S + |S|^{p-1}S = 0 \quad \text{in } \mathbf{R}^N, \tag{5.9}$$

admits the one-parameter family $\{S_\lambda\}$ (see [18] and [42])

$$(5.10) \quad S_\lambda(r) = \lambda \left[B(B + \lambda^{4/(N-4)} r^2)^{-1} \right]^{(N-4)/2}, \quad B = \sqrt{(N+2)N(N-2)(N-4)}.$$

Solutions (5.10) are unique positive solutions of (5.9) [35], [52]. The concentration-compactness principle [36], pp. 154-165, shows that the best constant C_2 in the embedding

$$\|u\|_{2N/(N-4)} \le C_2 \|D^2 u\|_2$$

is attained on radial solutions of (5.9) and hence by uniqueness is given by functions (5.10) and can be calculated explicitly [42].

A similar exact family is available for the $2m$-th order semilinear elliptic equation $-(-\Delta)^m S + |S|^{p-1} S = 0$ in $\mathbf{R}^N$, with the critical exponent as in (5.7). The family has the form

$$S_\lambda(r) = \lambda \left[B(B + \lambda^{4/(N-2m)} r^2)^{-1} \right]^{(N-2m)/2}, \quad B^m = (N + 2(m-1))!!/(N - 2(m+1))!!,$$

which was constructed [48], [49] by the conformal invariance (Bateman's transformation [2] was also useful).

Let us indicate some similarities of composite structures of GUSs which are available for any $m \ge 1$. The conclusions below are formal and we do not have key estimates, which are necessary to justify the analysis in the higher-order case. Having the family (5.10), we need to assume that the two boundary conditions at $r = 1$ are such that no nontrivial stationary solution in B_1 (with a certain positivity dominance) exists. This can be checked by higher-order Pohozaev's identities [46]. This is proposed to be the first necessary condition for GUSs to exist. Next, we need that the inner scaling $u(r,t) = \alpha(t) f(\xi, \tau)$, $\xi = r\alpha^{2/(N-4)}(t)$ and $d\tau/dt = \alpha^{8/(N-4)}(t)$ leads to stabilization to the stationary profile $S_1(\xi)$ in the inner region, i.e. the chosen boundary conditions imply a slow evolution on the continuous family, $u(\cdot, t) \sim S_{\alpha(t)}(\cdot)$, on compact subsets. The linearization for $t \gg 1$ in the corresponding rescaled equation (2.21) gives the perturbed equation of the type (2.38) governed by the linear operator $\mathbf{A}Y = -\Delta^2 Y + p S_1^{p-1} Y$ which is a perturbation of the biharmonic operator generating a continuous analytic semigroup, see [38]. Here we need an estimate $O(\tau^{-N/4})$ for $\tau \gg 1$ of $\{e^{\mathbf{A}\tau}\}$ in $L^\infty(\mathbf{R}^N)$ coinciding with that for the semigroup $e^{-\Delta^2 \tau}$ [11]. As the next step, once we can state and uniquely solve a fourth-order ODE like (2.47) with suitable conditions, we can arrive at a refined inner expansion similar to (2.50), which is a crucial step towards the extension and matching with the boundary.

The last step is crucial. Using typical uniqueness results for higher-order parabolic equations (see [11] and [38]), by an extra rescaling like (3.4), (3.6), we need to prove the convergence $F(\zeta, s) \to F_0(\zeta)$ as $s \to \infty$, where F_0 is the unique suitable solution of the corresponding linear stationary equations with singular boundary conditions given close to the origin $\zeta = 0$ which are prescribed by the inner expansion. Then if $F_0(\zeta)$ vanishes at some point ζ_0, i.e. $1 - r = +o(1)$), then this would mean that near the boundary $r = 1$, $u(r,t) = G(t) F_0(r) + \ldots$, where the decaying function $G(t) \to 0$ as $t \to \infty$ is given by these two scalings. This makes it possible to predict the boundary condition generating such a GUSs behaviour, namely, $u = 0$ and $\mu u_{rr} - \nu u_r = 0$ at $r = 1$, where $\mu = F_0'(1)$,

$\nu = F_0''(1)$, and we assume the nondegeneracy condition $\mu^2 + \nu^2 \neq 0$ as for classical regular Sturm-Liouville problems.

It is not easy to construct suitable well-posed quasilinear higher-order models with GUSs (unlike the second-order ones). For instance, consider the quasilinear equation

$$u_t = -\Delta(|\Delta u|^\sigma \Delta u) + |u|^{p-1}u, \quad u = \Delta u = 0 \quad \text{on } \partial\Omega \times \mathbf{R}_+, \tag{5.11}$$

where the critical Sobolev exponent for the compact embedding $W_0^{2,2}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is

$$p = p_s = [N(\sigma + 1) + 2(\sigma + 2)][N - 2(\sigma + 2)]^{-1}, \quad \sigma > 0, \ \ N > 2(\sigma + 2);$$

see e.g. [32]. By the corresponding higher-order Pohozaev's identity [46] the stationary problem in the subcritical range $p \in (1, p_s)$

$$\mathbf{B}(S) \equiv -\Delta(|\Delta S|^\sigma \Delta S) + |S|^{p-1}S = 0 \quad \text{in } \Omega, \quad S = \Delta S = 0 \quad \text{on } \partial\Omega.$$

does not admit a nontrivial nonnegative solution for any star-shaped domain Ω, see computations in [18] and on p. 457 in [47]. On the other hand, it is not known if the stationary equation in $\mathbf{R}^N$ admits a suitable smooth family $\{S_\lambda\}$ of positive solutions. Moreover, there is a problem of uniqueness of solutions for degenerate parabolic equation like (5.11) and extended limit semigroups (like those for the second-order equations [22]) are currently unknown.

Acknowledgement. The authors have been partially supported by TMR network EBR FMRX CT98-0201.

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