Absolute Stability and Integral Control*

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Abstract: Absolute stability results of both circle criterion and Popov type are derived for finite-dimensional linear plants with nonlinearity in the feedback loop. The linear plant contains an integrator (and so is not asymptotically stable). The (possibly time-varying) nonlinearity satisfies a particular sector condition which allows for cases with zero lower gain (such as saturation and deadzone). The conjunction of stable, but not asymptotically stable, linear plants and nonlinearities with possibly zero lower gain is a distinguishing feature of this paper. The absolute stability results are invoked in proving convergence and stability properties of low-gain integral feedback control for tracking of constant reference signals in the context of exponentially stable linear systems subject to input and output nonlinearities.

Keywords: Absolute stability; actuator nonlinearities; circle criterion; integral control; low-gain; Popov criterion; positive-real condition; sensor nonlinearities; tracking.

AMS subject classifications: 93C10, 93C80, 93D05, 93D10, 93D15.

1 Introduction

Absolute stability, and its relation to the concept of positive-real transfer functions, permeates much of the classical and modern control literature (see, for example, [1], [2], [6]-[15]). Criteria for absolute stability generally fall into one of two categories: those of circle-criterion type and those of Popov type, each applicable in the context of a canonical feedback structure with a linear plant Λ in the forward path and a nonlinearity f in the feedback path (see Figure 1).

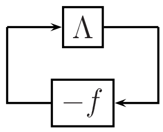

Figure 1: Feedback system with nonlinearity

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In the present paper, we address absolute stability issues in the setting of finite-dimensional, single-input, single-output plants Λ which contain an integrator (and so are not asymptotically stable). We consider both time-varying and time-invariant nonlinearities f satisfying a particular sector condition which, in the time-varying case, posits the existence of constants $t_0 \geq 0$ and $\beta > 0$ such that $\xi f(t,\xi) \geq \beta f^2(t,\xi)$ for all $t \geq t_0$ and all $\xi \in \mathbb{R}$ or, in the time-invariant case, posits the existence of $\beta \geq 0$ such that $\xi f(\xi) \geq \beta f^2(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, the lower gain of f, that is, $\inf\{|f(t,\xi)/\xi| : \ t \geq t_0, \ \xi \neq 0\}$ (or $\inf\{|f(\xi)/\xi| : \ \xi \neq 0\}$), may be zero (as is the case for deadzone nonlinearities and bounded nonlinearities such as saturation). The conjunction of stable, but not asymptotically stable, linear plants Λ and nonlinearities f with possibly zero lower gain is a distinguishing feature of the paper.

In Theorems 2.3 and 2.5, we present a result of circle-criterion type in the context of time-varying nonlinearities and a result of Popov type for time-invariant nonlinearities, respectively. Whilst these results are reminiscent of a number of well-known stability criteria, we stress that (to our knowledge) Theorems 2.3 and 2.5 are unavailable in the literature: we elaborate on this in Remarks 2.4 and 2.6.

The main motivation for developing absolute stability criteria for feedback structures of the form of Figure 1, wherein the linear plant Σ contains an integrator and the nonlinearity has zero lower gain, is their application in proving convergence and stability properties of low-gain integral feedback control for tracking of constant reference signals in the context of exponentially stable linear systems subject to input/actuator and output/sensor nonlinearities. Such applications are developed in Theorem 3.1 (which invokes Theorem 2.5 of Popov type) and in Theorem 3.3 (which invokes Theorem 2.3 of circle-criterion type). Whilst these results have significant overlap with those of [4, 5], we stress that the absolute stability approach adopted in the present paper differs fundamentally from the arguments used in [4, 5] and provides a natural and unified framework for investigations on low-gain integral control.

2 Absolute stability results

We consider real linear single-input single-output systems of the form

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{b}u(t)\,, \quad z(0) = z^0\,; \quad y(t) = \mathbf{c}z(t) \tag{1}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{b}, \mathbf{c}^T \in \mathbb{R}^n$. The following hypothesis remains in force throughout this section.

(H) $\mathbf{A}$ has a simple eigenvalue at zero and every other eigenvalue of $\mathbf{A}$ has negative real part.

It then follows that there exists a similarity transformation T such that

$$T : z \mapsto \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad T\mathbf{A}T^{-1} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad T\mathbf{b} = \begin{bmatrix} b \\ \beta \end{bmatrix}, \quad \mathbf{c}\,T^{-1} = \begin{bmatrix} c & \gamma \end{bmatrix},$$

where $A \in \mathbb{R}^{(n-1)\times(n-1)}$ is Hurwitz (that is, the spectrum of A is contained in the open left half complex plane $\{s \in \mathbb{C} : \operatorname{Re} s < 0\}$). Therefore, system (1) may be expressed in the equivalent form

$$\left.\begin{aligned} &\dot z_1(t) = Az_2(t) + bu(t)\,, \quad z_1(0) = z_1^0\,, \\ &\dot z_2(t) = \beta u(t)\,, \quad z_2(0) = z_2^0\,, \\ &y(t) = cz_1(t) + \gamma z_2(t)\,, \end{aligned}\right\} \tag{2}$$

where

$$\begin{bmatrix} z_1^0 \\ z_2^0 \end{bmatrix} := Tz^0\,.$$

The transfer function $\mathbf{G}$ of (1) (equivalently (2)) is given by

$$\mathbf{G}(s) := \mathbf{c}(sI - \mathbf{A})^{-1}\mathbf{b} = c(sI - A)^{-1}b + \frac{\gamma\beta}{s}$$

with

$$\lim_{s\to 0} s\mathbf{G}(s) = \gamma\beta \in \mathbb{R}. \tag{3}$$

Define the rational function H by $H(s) := c(sI - A)^{-1}b$. Since A is Hurwitz, the function $\omega \mapsto (qi\omega + 1)H(i\omega)$ is bounded for each $q \in \mathbb{R}$; moreover, $(qi\omega + 1)H(i\omega) \to qcb$ as $\omega \to \pm\infty$. Noting that, for all $q \in \mathbb{R}$,

$$\mathrm{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] = q\gamma\beta + \mathrm{Re}[(qi\omega + 1)H(i\omega)]\,, \quad \forall\, \omega \in \mathbb{R}^*,$$

where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ (the punctured real line), we may conclude the following.

Lemma 2.1. $$-\infty < \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] \le q\mathbf{cb} = q(cb + \gamma\beta)\,, \quad \forall\, q \in \mathbb{R}\,.$$

In view of Lemma 2.1, for every $q \in \mathbb{R}$, there exists $\eta \ge 0$ such that

$$\eta + \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] = \eta + q\gamma\beta + \inf_{\omega\in\mathbb{R}} \mathrm{Re}[(qi\omega + 1)H(i\omega)] > 0\,. \tag{4}$$

Lemma 2.2. Let $q \in \mathbb{R}$ and let $\eta \ge 0$ be such that (4) holds. There exists $P \in \mathbb{R}^{(n-1)\times(n-1)}$ such that $P = P^T > 0$ and

$$\begin{bmatrix} PA + A^TP & Pb - (qA^T + I)c^T \\ b^TP - c(qA + I) & -\,2\eta - 2q\mathbf{cb} \end{bmatrix} < 0. \tag{5}$$

Proof. By (4),

$$\eta + q\gamma\beta + \mathrm{Re}[(qi\omega + 1)H(i\omega)] > 0\,, \quad \forall\, \omega \in \mathbb{R} \cup \{\pm\infty\}\,.$$

Writing

$$M = \begin{bmatrix} 0 & -(qA^T + I)c^T \\ -c(qA + I) & -2\eta - 2q\mathbf{cb} \end{bmatrix},$$

we may conclude that, for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$,

$$\begin{aligned} \begin{bmatrix} (i\omega I - A)^{-1}b \\ 1 \end{bmatrix}^* M \begin{bmatrix} (i\omega I - A)^{-1}b \\ 1 \end{bmatrix} &= -2\mathrm{Re}\big(c(qA + I)(i\omega - A)^{-1}b\big) - 2\eta - 2q\mathbf{cb} \\ &= -2\big(\eta + q\gamma\beta + \mathrm{Re}[(qi\omega + 1)H(i\omega)]\big) < 0\,. \end{aligned}$$

By an application of the variant of the Kalman-Yakubovich-Popov Lemma given in Rantzer [12], it follows that there exists $P = P^T > 0$ such that (5) holds. ∎

2.1 A result of circle-criterion type

In this section, we focus on stability properties of the linear system (1) under output feedback with time-varying nonlinearity in the feedback loop (recall Figure 1). A function

$$f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad (t,\xi) \mapsto f(t,\xi)$$

is deemed to be of class $\mathcal{N}$ if it is measurable in t and locally Lipschitz in ξ, uniformly with respect to t on bounded intervals, and there exists a non-negative function $c_f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$|f(t,\xi)| \le c_f(t)\big[1+|\xi|\big], \quad \forall\, t \in \mathbb{R}_+, \ \forall\, \xi \in \mathbb{R}.$$

A function $f \in \mathcal{N}$ is said to be asymptotically autonomous with limit f_a if $f_a : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and, for all $R > 0$ and $\varepsilon > 0$, there exists $\tau > 0$ such that,

$$\text{ess-sup}_{t\ge\tau}|f(t,\xi) - f_a(\xi)| < \varepsilon, \quad \forall\, \xi \in [-R,R].$$

We remark that, for $f \in \mathcal{N}$, the initial-value problem for the feedback system

$$\dot{z}(t) = \mathbf{A}z(t) - \mathbf{b}f(t,\mathbf{c}z(t)), \quad z(0) = z^0, \tag{6}$$

has a unique absolutely continuous solution $z : \mathbb{R}_+ \to \mathbb{R}^n$ which satisfies the differential equation in (6) for almost all $t \in \mathbb{R}_+$.

Theorem 2.3. Assume that $\lim_{s\to 0} s\mathbf{G}(s) > 0$. Let $0 < \alpha < \infty$ be such that

$$1/\alpha + \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}\,\mathbf{G}(i\omega) > 0\,. \tag{7}$$

For $z^0 \in \mathbb{R}^n$, denote the solution of (6) by $t \mapsto z(t,z^0)$.

1. If $f \in \mathcal{N}$ is such that, for some $t_0 \ge 0$,

$$\xi f(t,\xi) \ge \frac{1}{\alpha} f^2(t,\xi)\,, \quad \forall\, t \ge t_0\,,\ \forall\, \xi \in \mathbb{R}, \tag{8}$$

then the following statements hold.

(a) There exists $M \ge 1$ such that

$$\|z(t+t_0,z^0)\| \le M\|z(t_0,z^0)\|\,, \quad \forall\,(t,z^0) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

(b) If there exists $\alpha_0 > 0$ such that $\alpha_0\xi^2 \le \xi f(t,\xi)$ for all $t \ge t_0$ and all $\xi \in \mathbb{R}$, then there exist $M \ge 1$ and $\rho > 0$ such that

$$\|z(t+t_0,z^0)\| \le Me^{-\rho t}\|z(t_0,z^0)\|\,, \quad \forall\,(t,z^0) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

(c) For each $z^0 \in \mathbb{R}^n$, the function $t \mapsto y(t) = \mathbf{c}z(t,z^0)$ is such that $y(\cdot)f(\,\cdot\,,y(\cdot)) \in L^1(\mathbb{R}_+)$ and $f(\,\cdot\,,y(\cdot)) \in L^2(\mathbb{R}_+)$.

(d) For each $z^0 \in \mathbb{R}^n$, $z(t, z^0)$ converges as $t \to \infty$ with limit in $\ker \mathbf{A}$, that is,

$$z^\infty := \lim_{t\to\infty} z(t, z^0) \in \ker \mathbf{A}.$$

(e) If f is asymptotically autonomous with limit f_a, then, for each $z^0 \in \mathbb{R}^n$, $\lim_{t\to\infty} z(t, z^0) = z^\infty \in \ker\mathbf{A}$ is such that $\mathbf{c}z^\infty \in f_a^{-1}(0) := \{\xi : f_a(\xi) = 0\}$.

(f) If f is asymptotically autonomous with limit f_a and $f_a^{-1}(0) = \{0\}$, then 0 is a globally attractive equilibrium of the feedback system (6).

2. If $f \in \mathcal{N}$ is of the form $f(t, \xi) = k(t)g(\xi)$, where k is measurable, bounded and non-negative with $\limsup_{t\to\infty} k(t) \le 1$, g is locally Lipschitz with $\liminf_{\xi\to 0} g(\xi)/\xi > 0$ and

$$0 < \xi g(\xi) \le \alpha \xi^2 , \quad \forall\, \xi \in \mathbb{R}^* ,$$

then there exists $N \ge 1$ such that

$$\|z(t, z^0)\| \le N\|z^0\| , \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n .$$

Furthermore, for each $r > 0$, there exist $L \ge 1$ and $\rho > 0$ such that, for all $z^0 \in \mathbb{R}^n$ with $\|z^0\| \le r$,

$$\|z(t, z^0)\| \le L e^{-\rho \int_0^t k(\tau) d\tau} \|z^0\| , \quad \forall\, t \in \mathbb{R}_+ .$$

In particular, if $k \notin L^1(\mathbb{R}_+)$, then $\lim_{t\to\infty} z(t, z^0) = 0$ for all $z^0 \in \mathbb{R}^n$; if $\liminf_{t\to\infty} k(t) > 0$, then system (6) is semi-globally exponentially stable.

Remark 2.4. The above theorem is a stability result of circle-criterion type. Assertion (b) of Statement 1 is well known, see [10, 14]. However, we emphasize that the remaining assertions of Statement 1 and all of Statement 2 consider feedback systems for which (i) the linear part contains an integrator (*i.e.*, we are considering a so-called particular or critical case in the terminology of [2, 9, 11], implying that the linear system is not asymptotically stable) and (ii) the "lower gain" $\inf\{|f(t, \xi)/\xi| : \ t \ge t_0, \ \xi \in \mathbb{R}^*\}$ of the nonlinearity f may be zero (which, for example, is the case for bounded nonlinearities such as saturation and deadzone). In fact, one of the motivations for studying this situation is its importance in the application to the low-gain integral control problem in the presence of input nonlinearities of saturation type (see Section 3). By contrast, in most of the circle-criterion type results available in the literature (such as [7, 10, 11, 13, 14]), the lower gain of the nonlinearity is either assumed to be positive, or, if the lower gain is allowed to be zero, the linear part is assumed to be asymptotically stable (and so does not contain an integrator): an exception is [15], where stability of the origin and Lagrange stability is proved for (6) without assuming that the lower gain of f is positive. However, none of the statements of Theorem 2.3 can be found in [15]. Also note that in contrast to most related results in the literature, Theorem 2.3 does not impose controllability or observability on the linear system. Finally, in [1], a "relaxed" circle criterion is given for the linear time-varying feedback system obtained from (6) by considering functions f of the form $f(t, \xi) = k(t)\xi$ (this is a special case of the setting in Statement 2 of Theorem 2.3). Assuming that the positive real condition (7) holds and $0 \le k(t) \le \alpha$, Theorem 12 in [1] guarantees asymptotic stability of the origin, provided that k satisfies a certain additional condition: the point of interest in the present context is that the latter condition is satisfied by a large class of gain functions k with $\liminf_{t\to\infty} k(t) = 0$.*

Proof of Theorem 2.3. In view of the equivalence of (1) and (2), the feedback system (6) may be expressed in the form

$$\left.\begin{aligned} \dot z_1(t) &= Az_1(t) - bf(t, y(t)), \quad z_1(0) = z_1^0 \\ \dot z_2(t) &= -\beta f(t, y(t)), \quad z_2(0) = z_2^0 \\ y(t) &= cz_1(t) + \gamma z_2(t)\,. \end{aligned}\right\} \tag{9}$$

Let $\alpha^* > \alpha$ be sufficiently close to α so that

$$1/\alpha^* + \inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\mathbf{G}(i\omega) > 0\,.$$

By Lemma 2.2, there exists $P = P^T > 0$ such that

$$Q := \begin{bmatrix} PA + A^TP & Pb - c^T \\ b^TP - c & -2/\alpha^* \end{bmatrix} < 0$$

and so there exists $M_1 > 0$ such that $\langle \xi, Q\xi\rangle \le -M_1\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$. Define the quadratic form $V : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ by

$$V(\xi_1, \xi_2) := \langle \xi_1, P\xi_1\rangle + (\gamma/\beta)\xi_2^2\,,$$

and set $\delta := 1 - (\alpha/\alpha^*) > 0$.

1. By assumption, $\lim_{s\to 0} s\mathbf{G}(s) > 0$ and so, by (3), $\gamma/\beta > 0$, showing that V is positive definite. Let $t \mapsto (z_1(t), z_2(t))$ denote the unique solution of the initial-value problem (9). Observe that

$$\begin{aligned} -\gamma z_2(t)f(t, y(t)) &= cz_1(t)f(t, y(t)) - y(t)f(t, y(t)) \\ &= cz_1(t)f(t, y(t)) - \delta y(t)f(t, y(t)) - (\alpha/\alpha^*)y(t)f(t, y(t)) \\ &\le cz_1(t)f(t, y(t)) - \delta y(t)f(t, y(t)) - f^2(t, y(t))/\alpha^*\,, \quad \text{for a.a. } t \ge t_0\,, \end{aligned} \tag{10}$$

where the last inequality follows from the sector condition (8). Therefore,

$$\begin{aligned} \frac{d}{dt}V(z_1(t), z_2(t)) &= \langle [PA + A^TP]z_1(t), z_1(t)\rangle - 2\langle Pz_1(t), bf(t, y(t))\rangle - 2\gamma z_2(t)f(t, y(t)) \\ &\le \left\langle \begin{bmatrix} z_1(t) \\ -f(t, y(t)) \end{bmatrix}, Q\begin{bmatrix} z_1(t) \\ -f(t, y(t)) \end{bmatrix}\right\rangle - 2\delta y(t)f(t, y(t)) \\ &\le -M_1\big[\|z_1(t)\|^2 + f^2(t, y(t))\big] - 2\delta y(t)f(t, y(t)) \le 0\,, \quad \text{for a.a. } t \ge t_0. \end{aligned} \tag{11}$$

* We mention that the relaxed circle criterion [1, Theorem 12] is not correct in the generality stated in [1], wherein the only regularity assumption explicitly imposed on k is (Lebesgue) measurability: it is not difficult to construct counterexamples with continuous k. However, uniform continuity of k is sufficient for the assertions of Theorem 12 in [1] to hold.

Assertion (a) follows immediately. Moreover, we may conclude that $z_1(\cdot) \in L^2(\mathbb{R}_+, \mathbb{R}^{n-1})$ and $f(\cdot, y(\cdot)) \in L^2(\mathbb{R}_+)$. Furthermore, $2\delta \int_{t_0}^t |y(s)f(s, y(s))| ds \le V(z_1(t_0), z_2(t_0))$ for all $t \ge t_0$ and so $t \mapsto y(t)f(t, y(t))$ is in $L^1(\mathbb{R}_+)$. This establishes Assertion (c). Since $f(\cdot, y(\cdot)) \in L^2(\mathbb{R}_+)$ and A is Hurwitz, it follows from (9) that $z_1(t) \to 0$ as $t \to \infty$. By (11), the function $t \mapsto V(z_1(t), z_2(t))$ is non-increasing for $t \ge t_0$. Since V is non-negative, we obtain that $V(z_1(t), z_2(t))$ converges to a finite limit as $t \to \infty$. Combining this with the fact that $z_1(t)$ converges as $t \to \infty$, shows that $z_2^2(t)$ (and hence $z_2(t)$) converges to a finite limit as $t \to \infty$. We have now shown that, for some $z_2^\infty \in \mathbb{R}$, $(z_1(t), z_2(t)) \to (0, z_2^\infty)$ as $t \to \infty$, whence Assertion (d). Clearly, $y(t) = cz_1(t) + \gamma z_2(t) \to y^\infty := \gamma z_2^\infty$ as $t \to \infty$. Assertion (e) follows if we can show that $f_a(y^\infty) = 0$. Suppose otherwise, that is, suppose $|f_a(y^\infty)| = 3\varepsilon$ for some $\varepsilon > 0$. Then there exists $\tau > 0$ such that $|f_a(y(t))| \ge 2\varepsilon$ for all $t \ge \tau$. By boundedness of $y(\cdot)$ and asymptotic autonomy of f, there exists $\tau_1 \ge \tau$ such that $\text{ess-sup}_{t \ge \tau_1} |f(t, y(t)) - f_a(y(t))| < \varepsilon$. Therefore, $|f(t, y(t))| \ge \varepsilon$ for almost all $t \ge \tau_1$ which contradicts the fact that $f(\cdot, y(\cdot))$ is square integrable. This establishes Assertion (e). Assertion (f) is a direct consequence of Assertions (d) and (e). It remains to prove Assertion (b). Assume the existence of $\alpha_0 > 0$ such that $\alpha_0 \xi^2 \le \xi f(t, \xi)$ for all $t \ge t_0$ and all $\xi \in \mathbb{R}$. Then $f^2(t, y(t)) \ge \alpha_0^2 y^2(t)$ for all $t \ge t_0$. By the hypothesis and (3), $\gamma \ne 0$. Therefore, the map

$$Z: \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \mapsto \begin{bmatrix} \zeta_1 \\ \alpha_0 \big(c\zeta_1 + \gamma\zeta_2\big) \end{bmatrix}$$

is invertible and so $Z^T Z$ is positive definite. Therefore, there exists a constant $N_1 > 0$ such that

$$\begin{aligned} \|z_1(t)\|^2 + f^2(t, y(t)) \ge \|z_1(t)\|^2 + \alpha_0^2 y^2(t) &= \begin{bmatrix} z_1^T(t) & z_2^T(t) \end{bmatrix} Z^T Z \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \\ &\ge N_1 V(z_1(t), z_2(t)), \quad \forall\, t \ge t_0, \end{aligned} \tag{12}$$

which, together with (11), establishes Assertion (b).

2. Let $t_0 \ge 0$ be such that $0 \le k(t) \le 1$ for all $t \ge t_0$. Then $0 \le \xi f(t, \xi) \le \alpha \xi^2$ for all $t \ge t_0$. Let $r > 0$ be given, and set $B := \{z \in \mathbb{R}^n : \|z\| \le r\}$. Define

$$\theta(t, z^0) := \begin{cases} \dfrac{g(\mathbf{c}z(t, z^0))}{\mathbf{c}z(t, z^0)}, & \mathbf{c}z(t, z^0) \ne 0 \\ \alpha, & \mathbf{c}z(t, z^0) = 0 \end{cases}, \qquad \forall\, t \ge t_0, \ \forall\, z^0 \in \mathbb{R}^n.$$

A routine application of Gronwall's lemma shows that there exists $N_2 \ge 1$ such that

$$\|z(t, z^0)\| \le N_2 \|z^0\|, \quad \forall\, t \in [0, t_0], \ \forall\, z^0 \in \mathbb{R}^n. \tag{13}$$

Consequently, by Assertion (a) of part 1, there exists $N \ge 1$ such that

$$\|z(t, z^0)\| \le N \|z^0\|, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Therefore it follows from the assumptions on g that there exists $\varepsilon > 0$ (in general depending on the ball B) such that

$$\frac{g(\mathbf{c}z(t, z^0))}{\mathbf{c}z(t, z^0)} \ge \varepsilon, \quad \text{for all } t \ge 0 \text{ with } \mathbf{c}z(t, z^0) \ne 0 \text{ and all } z^0 \in B.$$

Thus,

$$\theta(t, z_0) \in [\varepsilon, \alpha], \quad \forall\, t \geq 0\,,\ \forall\, z^0 \in B\,. \tag{14}$$

Moreover,

$$f(t, y(t)) = k(t)\theta(t, z^0)y(t)\,, \quad \forall\, t \geq 0\,. \tag{15}$$

For $z^0 \in B$, let $t \mapsto (z_1(t), z_2(t))$ denote the solution of (9) with $(z_1^0, z_2^0) = Tz^0$. Set $K(t) := \int_0^t k(\tau)d\tau$ and introduce functions (parameterized by $\rho > 0$) v_ρ, w_ρ and y_ρ defined by

$$v_\rho(t) = \exp(\rho K(t))z_1(t), \quad w_\rho(t) := \exp(\rho K(t))z_2(t), \quad y_\rho(t) := \exp(\rho K(t))y(t); \quad \forall\, t \geq 0.$$

Invoking (9), a straightforward calculation yields

$$\left.\begin{array}{l} \dot v_\rho(t) = Av_\rho(t) + \rho k(t)v_\rho(t) - be^{\rho K(t)}f(t, y(t))\,, \quad v_\rho(0) = z_1^0\,, \\ \dot w_\rho(t) = \rho k(t)w_\rho(t) - \beta e^{\rho K(t)}f(t, y(t))\,, \quad w_\rho(0) = z_2^0\,, \\ y_\rho(t) = cv_\rho(t) + \gamma w_\rho(t)\,. \end{array}\right\} \tag{16}$$

Via a routine calculation, (16) together with (14) and (15) leads to the following counterpart of (11)

$$\begin{aligned} \frac{d}{dt}V(v_\rho(t), w_\rho(t)) \leq -M_1\left[\|v_\rho(t)\|^2 + (k(t)\theta(t, z^0)y_\rho(t))^2\right] - 2\delta k(t)\varepsilon y_\rho^2(t) \\ + 2\rho k(t)\left[\langle v_\rho(t), Pv_\rho(t)\rangle + (\gamma/\beta)w_\rho^2(t)\right], \quad \forall\, t \geq t_0\,, \end{aligned}$$

where M_1 does not depend on z^0. Observing that, for some constant $M_2 > 0$ (not depending on z^0),

$$k(t)\left[\langle v_\rho(t), Pv_\rho(t)\rangle + (\gamma/\beta)w_\rho^2(t)\right] \leq M_2\left[\|v_\rho(t)\|^2 + k(t)y_\rho^2(t)\right], \quad \forall\, t \geq t_0,$$

we may conclude that

$$\frac{d}{dt}V(v_\rho(t), w_\rho(t)) \leq -(M_1 - 2\rho M_2)\|v_\rho(t)\|^2 - 2(\delta\varepsilon - \rho M_2)k(t)y_\rho^2(t)\,, \quad \forall\, t \geq t_0\,.$$

Therefore, choosing ρ such that $0 < M_2\rho \leq \min\{M_1/2\,, \delta\varepsilon\}$, it follows that, for all $t \geq t_0$, $V(v_\rho(t), w_\rho(t)) \leq V(v_\rho(t_0), w_\rho(t_0))$. Hence there exits $M_3 \geq 1$ such that

$$\|z(t, z^0)\| \leq M_3 \exp(-\rho K(t))\|z(t_0, z^0)\|\,, \quad \forall\, t \geq t_0\,,\ \forall\, z^0 \in B\,.$$

By (13), it follows that for sufficiently large $L > 0$,

$$\|z(t, z^0)\| \leq L\exp(-\rho K(t))\|z_0\|\,, \quad \forall\, t \geq 0\,,\ \forall\, z^0 \in B\,.$$

■

2.2 A result of Popov type

In this section, we focus on stability properties of system (1) under output feedback with a time-invariant locally Lipschitz nonlinearity $f : \mathbb{R} \to \mathbb{R}$ in the feedback path. In this case, the feedback system takes the form

$$\dot{z}(t) = \mathbf{A}z(t) - \mathbf{b}f(\mathbf{c}z(t)), \quad z(0) = z^0. \tag{17}$$

As usual, in the statement of the following theorem we adopt the convention $1/\infty := 0$.

Theorem 2.5. Assume that $\lim_{s\to 0} s\mathbf{G}(s) > 0$. Let $0 < \alpha \leq \infty$ and $q \geq 0$ be such that

$$1/\alpha + \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] > 0\,.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz with

$$\xi f(\xi) \geq \frac{1}{\alpha} f^2(\xi)\,, \quad \forall\, \xi \in \mathbb{R}\,. \tag{18}$$

For each $z^0 \in \mathbb{R}^n$, the unique solution of the feedback system (17), denoted by $t \mapsto z(t, z^0)$, exists on $\mathbb{R}_+$ (no finite escape-time) and the following statements hold.

(a) (Stability in the large.) There exists $M \geq 1$ such that

$$\|z(t, z^0)\| \leq M\|z^0\|\,, \quad \forall\, (t, z^0) \in \mathbb{R}_+ \times \mathbb{R}^n\,.$$

(b) For each $z^0 \in \mathbb{R}^n$, the function $t \mapsto y(t) = \mathbf{c}z(t, z^0)$ is such that $yf(y) \in L^1(\mathbb{R}_+)$ and $f(y) \in L^2(\mathbb{R}_+)$.

(c) For each $z^0 \in \mathbb{R}^n$, $z(t, z^0)$ converges as $t \to \infty$ with limit $z^\infty \in \ker \mathbf{A}$ and, moreover, $y^\infty := \mathbf{c}z^\infty \in f^{-1}(0) := \{\xi : f(\xi) = 0\}$.

(d) If 0 is an isolated point of $f^{-1}(0)$, then 0 is an asymptotically stable equilibrium.

(e) If $f^{-1}(0) = \{0\}$, then 0 is a globally asymptotically stable equilibrium.

(f) If $\liminf_{\xi\to 0} f(\xi)/\xi > 0$, then 0 is an exponentially stable equilibrium.

(g) If there exists $\alpha_0 > 0$ such that $\alpha_0\xi^2 \leq \xi f(\xi)$ for all $\xi \in \mathbb{R}$, then 0 is a semi-globally exponentially stable equilibrium.

(h) If $\alpha < \infty$ and there exists $\alpha_0 > 0$ such that $\alpha_0\xi^2 \leq \xi f(\xi)$ for all $\xi \in \mathbb{R}$, then 0 is a globally exponentially stable equilibrium.

Remark 2.6. The above theorem is a stability result of Popov type which is reminiscent of a number of results which can be found in the literature (see [2, 6, 8, 10, 11, 14, 15]), where it is usually assumed that $f^{-1}(0) = \{0\}$. Consequently, Assertion (e) is well known. However, to the best of our knowledge the remaining assertions of Theorem 2.5 are not available in the published literature. We emphasize the importance of Assertion (c) in the context of the application of Theorem 2.5 to low-gain integral control (see Subsection 3.1, proof of Theorem 3.1). Also note that in contrast to most related results in the literature, Theorem 2.5 does not impose controllability or observability on the linear system. Finally, we mention that Theorem 2.5 is a finite-dimensional counterpart of an infinite-dimensional result in the preprint [3] (see Theorem 3.1 in [3]). In contrast to the familiar Lyapunov techniques used in the proof of Theorem 2.5 of the present paper, the proof of Theorem 3.1 in [3] hinges on possibly less familiar integral equation techniques applied in the context of an input-output approach and involves many infinite-dimensional technicalities.

Proof of Theorem 2.5. In view of the equivalence of (1) and (2), the feedback system (17) may be expressed in the form

$$\left.\begin{aligned} \dot z_1(t) &= Az_1(t) - bf(y(t))\,, \quad z_1(0) = z_1^0\,, \\ \dot z_2(t) &= -\beta f(y(t))\,, \quad z_2(0) = z_2^0\,, \\ y(t) &= cz_1(t) + \gamma z_2(t)\,. \end{aligned}\right\} \tag{19}$$

If $\alpha < \infty$, let $\alpha^* > \alpha$ be sufficiently close to α so that

$$1/\alpha^* + \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}[(qi\omega + 1)\mathbf{G}(i\omega)] > 0\,;$$

if $\alpha = \infty$, let $\alpha^* = \infty$. By Lemma 2.2, there exists $P = P^T > 0$ such that

$$Q := \begin{bmatrix} PA + A^TP & Pb - (qA^T + I)c^T \\ b^TP - c(qA + I) & -2/\alpha^* - 2q\mathbf{cb} \end{bmatrix} < 0$$

and so there exists $M_1 > 0$ such that $\langle \xi, Q\xi\rangle \le -M_1\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$. Define the function $V : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ by

$$V(\xi_1, \xi_2) := \langle \xi_1, P\xi_1\rangle + (\gamma/\beta)\xi_2^2 + 2q\int_0^{(c\xi_1+\gamma\xi_2)} f(\zeta)d\zeta\,. \tag{20}$$

Furthermore, we introduce a number $\delta > 0$ by setting

$$\delta := \begin{cases} 1 - (\alpha/\alpha^*)\,, & \text{if } \alpha < \infty \\ 1\,, & \text{if } \alpha = \infty \end{cases}\,.$$

By assumption, $\lim_{s\to 0} s\mathbf{G}(s) > 0$ and so, by (3), $\gamma/\beta > 0$. Moreover, the fact that $\xi f(\xi) \ge 0$ for all $\xi \in \mathbb{R}$ implies the non-negativity of the integral term on the right-hand side of (20). Hence, we may conclude that V is positive definite.

Let $(z_1^0, z_2^0) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be arbitrary. By a standard result from differential equations there exists a unique maximal solution $(z_1, z_2) : [0, t^*) \to \mathbb{R}^{n-1} \times \mathbb{R}$ of the initial-value problem (17), where $0 < t^* \le \infty$. If $t^* < \infty$, then (z_1, z_2) is unbounded. As in the proof of Theorem 2.3 (see (10)), the sector condition (18) can be used to obtain

$$-\gamma z_2(t)f(y(t)) \le cz_1(t)f(y(t)) - \delta y(t)f(y(t)) - f^2(y(t))/\alpha^*\,, \quad \forall\, t \in [0, t^*)\,. \tag{21}$$

Note that (21) holds for finite as well as infinite α. Using (21), we arrive at a counterpart of inequality (11) in the proof of Theorem 2.3:

$$\begin{aligned}\frac{d}{dt}V(z_1(t),z_2(t)) &= \langle [PA+A^TP]z_1(t),z_1(t)\rangle - 2\langle Pz_1(t),bf(y(t))\rangle - 2\gamma z_2(t)f(y(t)) \\ &\qquad + 2qf(y(t))\big(cAz_1(t)-\mathbf{c}\mathbf{b}f(y(t))\big) \\ &\le \left\langle \begin{bmatrix} z_1(t) \\ -f(y(t))\end{bmatrix}, Q\begin{bmatrix} z_1(t) \\ -f(y(t))\end{bmatrix}\right\rangle - 2\delta y(t)f(y(t)) \\ &\le -M_1\big[\|z_1(t)\|^2+f^2(y(t))\big] - 2\delta y(t)f(y(t)) \le 0\,, \quad \forall\, t\in[0,t^*)\,. \end{aligned} \tag{22}$$

Consequently, the function $t\mapsto V(z_1(t),z_2(t))$ is bounded on $[0,t^*)$. Observing that there exists a constant $M_2>0$ such that $\|(\xi_1,\xi_2)\|\le M_2V(\xi_1,\xi_2)$ for all $(\xi_1,\xi_2)\in\mathbb{R}^{n-1}\times\mathbb{R}$, we may conclude that the function $t\mapsto(z_1(t),z_2(t))$ is bounded on $[0,t^*)$, which in turn implies that $t^*=\infty$. Assertions (a), (b) and (e) of the theorem now follow by the arguments (*mutatis mutandis*) previously adopted in the proof of Theorem 2.3.

To prove Assertion (c) first note that, since $f(y)\in L^2(\mathbb{R}_+)$ and A is Hurwitz, $\lim_{t\to\infty}z_1(t)=0$. Next observe that it follows from the last two equations in (19) that $\gamma z_2\dot z_2=\beta f(y)(cz_1-y)$, which upon integration from 0 to t gives

$$\gamma z_2^2(t)=\gamma z_2^2(0)+2\beta\left(\int_0^t f(y(\tau))[cz_1(\tau)-y(\tau)]d\tau\right)\,,\quad \forall t\in\mathbb{R}_+\,. \tag{23}$$

By (22), $z_1\in L^2(\mathbb{R}_+,\mathbb{R}^{n-1})$ and, since $f(y)\in L^2(\mathbb{R}_+)$, we have that $f(y)cz_1\in L^1(\mathbb{R}_+)$. Furthermore, by Assertion (b), $yf(y)\in L^1(\mathbb{R}_+)$, showing that the integrand in the right-hand side of (23) is in $L^1(\mathbb{R}_+)$. Since $\gamma\neq 0$, (23) shows that $z_2^2(t)$, and hence $z_2(t)$, converges to a finite limit as $t\to\infty$.

We proceed to prove Assertion (d). Assume that 0 is an isolated point of $f^{-1}(0)$ and so there exists $\varepsilon>0$ such that $|f(\xi)|>0$ for all ξ with $0<|\xi|\le\varepsilon$. Define $\theta:=\varepsilon/(M\|\mathbf{c}\|)$, with $M\ge 1$ as in Assertion (a). For each z^0 with $\|z^0\|\le\theta$, we have $\lim_{t\to\infty}\mathbf{c}z(t,z^0)=: y^\infty\in f^{-1}(0)$ (by Assertion (c)) and $|\mathbf{c}z(t,z^0)|\le\|\mathbf{c}\|\|z(t,z^0)\|\le M\|\mathbf{c}\|\|z^0\|\le\varepsilon$ for all $t\ge 0$: therefore, $y^\infty=0$ and so $z(t,z^0)\to 0$ as $t\to\infty$. This establishes local attractivity of 0 which, together with Assertion (a), yields Assertion (d).

To prove Assertions (f)–(h), note that for any $R>0$ there exists $M_3>0$ such that

$$\|\xi_1\|^2+\|\xi_2\|^2\ge M_3V(\xi_1,\xi_2)\,,\quad \text{for all } (\xi_1,\xi_2)\in\mathbb{R}^{n-1}\times\mathbb{R} \text{ with } \|(\xi_1,\xi_2)\|\le R\,, \tag{24}$$

which can easily be shown using the local Lipschitz condition on f. Consider Assertion (f) and assume that $\liminf_{\xi\to 0}f(\xi)/\xi>0$. Then there exist $\alpha_0>0$ and $\varepsilon>0$ such that $\alpha_0\xi^2\le\xi f(\xi)$ for all $\xi\in\mathbb{R}$ with $|\xi|\le\varepsilon$. As above in the proof of Assertion (d), define $\theta:=\varepsilon/(M\|\mathbf{c}\|)$. Then, for all z^0 with $\|z^0\|\le\theta$, $y(\cdot)=\mathbf{c}z(\cdot,z^0)$ is such that $|y(t)|\le\varepsilon$ for all $t\ge 0$ and so $f^2(y(t))\ge\alpha_0^2y^2(t)$ for all $t\ge 0$. Invoking Assertion (a) and (24), we may conlude as in the proof of Theorem 2.3 (see (12)) that there exists a constant $M_4>0$ such that for every z^0 with $\|z^0\|\le\theta$

$$\|z_1(t)\|^2+f^2(t,y(t))\ge M_4V(z_1(t),z_2(t))\,,\quad \forall\, t\in\mathbb{R}_+\,. \tag{25}$$

Combining this with (22), we may infer the existence of a constant $M_5 > 0$ such that, for all (z_1^0, z_2^0) with $\|(z_1^0, z_2^0)\|$ sufficiently small,

$$\frac{d}{dt}V(z_1(t), z_2(t)) \leq -M_5 V(z_1(t), z_2(t)), \quad \forall\, t \in \mathbb{R}_+ .$$

Assertion (f) now follows. Next consider Assertion (g) and assume that $\alpha_0 > 0$ is such that $\alpha_0 \xi^2 \leq \xi f(\xi)$ for all $\xi \in \mathbb{R}$. The argument used to prove Assertion (f) now applies to any bounded set of initial conditions and hence yields semi-global exponential stability. Finally, the same argument can be used to prove Assertion (h): we only need to realize that if α is finite, then there exists $M_3 > 0$ such that the inequality in (24) holds for all $(\xi_1, \xi_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$. ■

3 Low-gain integral control

In this section we apply Theorem 2.3 and Theorem 2.5 to obtain results on low-gain integral control. In particular, the problem of tracking constant reference signals $r \in \mathbb{R}$ for linear systems with nonlinearity in the input channel will first be analysed (in Subsection 3.1)) in the "Popov" context of Theorem 2.5; linear systems with nonlinearity in both the input and output channels are then considered (in Subsection 3.2) and analysed in the "circle criterion" context of Theorem 2.3.

3.1 Integral control in the presence of input nonlinearities

The problem of tracking constant reference signals $r \in \mathbb{R}$ will be first addressed in the context of a class of finite-dimensional single-input single-output continuous-time real linear systems $\Sigma = (A, b, c, d)$ having a globally Lipschitz nonlinearity φ in the input channel:

$$\dot{x} = Ax + b\varphi(u), \quad x(0) = x^0 \in \mathbb{R}^n; \qquad y = cx + d\varphi(u). \tag{26}$$

In (26), A is assumed to be Hurwitz. Furthermore, the transfer function G, given by $G(s) = c(sI - A)^{-1}b + d$, is assumed to satisfy $G(0) > 0$. To achieve the objective of tracking a constant reference signal $r \in \mathbb{R}$, we consider integral control action

$$u(t) = u^0 + k\int_0^t [r - y(\tau)]\, d\tau ,$$

with constant control gain k, yielding the following nonlinear feedback system (illustrated in Figure 2)

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + b\varphi(u(t)), \quad x(0) = x^0 \in \mathbb{R}^n, \\ \dot{u}(t) &= k[r - (cx(t) + d\varphi(u(t))], \quad u(0) = u^0 \in \mathbb{R}. \end{aligned}\right\} \tag{27}$$

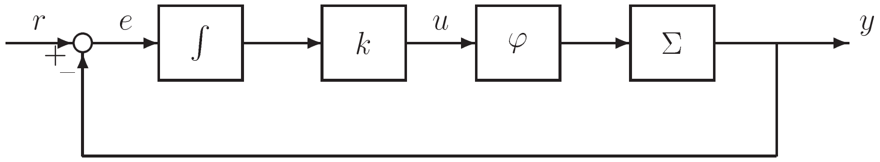

Figure 2: Constant low-gain integral control with input nonlinearity

For later convenience, we introduce the quantity ν_q, parameterized by $q \in \mathbb{R}_+$,

$$\nu_q := \inf_{\omega \in \mathbb{R}^*} \mathrm{Re}[(qi\omega + 1)G(i\omega)/i\omega]. \tag{28}$$

Theorem 3.1. Consider the feedback system (27). Assume that A is Hurwitz, $G(0) = d - cA^{-1}b > 0$, φ is non-decreasing and globally Lipschitz, with Lipschitz constant $\lambda > 0$, and $r \in \mathbb{R}$ is such that $r/G(0) =: \varphi^r \in \operatorname{im} \varphi$. Define

$$\nu := \sup_{q \geq 0} \nu_q \quad \text{and} \quad \kappa := \begin{cases} \dfrac{1}{|\nu|\lambda}, & \nu < 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Then $-\infty < \nu \leq \infty$ and $0 < \kappa \leq \infty$. For each $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}$ and $k \in (0, \kappa)$ there exists a unique solution $t \mapsto (x(t), u(t))$ of (27) defined on $\mathbb{R}_+$ (no finite escape-time). The solution has the following properties.

(a) $\lim_{t\to\infty} x(t) = -A^{-1}b\varphi^r$.

(b) $u^r := \lim_{t\to\infty} u(t)$ exists, is finite and $\varphi(u^r) = \varphi^r$.

(c) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(cx(t) + d\varphi(u(t))) = r$.

Proof. That $-\infty < \nu \leq \infty$ (and hence $0 < \kappa \leq \infty$) is an immediate consequence of Lemma 2.1. Let $(x^0, u^0, k) \in \mathbb{R}^n \times \mathbb{R} \times (0, \kappa)$. It is clear that (27) has a unique solution $t \mapsto (x(t), u(t))$ defined on $\mathbb{R}_+$.

Since $r/G(0) \in \operatorname{im} \varphi$, there exists $\varphi^r \in \operatorname{im} \varphi$ such that $G(0)\varphi^r = r$. Let $u^* \in \mathbb{R}$ be such that $\varphi(u^*) = \varphi^r$ and define

$$f : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \varphi(\xi + u^*) - \varphi^r .$$

Then $f(0) = 0$ and, since φ is non-decreasing and globally Lipschitz (with Lipschitz constant $\lambda > 0$), we have

$$0 \leq \xi f(\xi) \leq \lambda \xi^2 , \quad \forall \xi \in \mathbb{R} .$$

Introduce new variables

$$z_1(t) := x(t) + A^{-1}b\varphi^r , \qquad z_2(t) := u(t) - u^* , \quad \forall t \in \mathbb{R}_+ .$$

By direct calculation, we have

$$\dot z_1(t) = Az_1(t) + bf(z_2(t)), \quad \dot z_2(t) = -kcz_1(t) - kdf(z_2(t))$$

which, on writing

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A & 0 \\ -kc & 0 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} -b \\ kd \end{bmatrix}, \quad \mathbf{c} := \begin{bmatrix} 0 & 1 \end{bmatrix},$$

takes the form of system (17), where $\mathbf{A}$ satisfies hypothesis (H). Moreover, setting $\mathbf{G}(s) = \mathbf{c}(sI - \mathbf{A})^{-1}\mathbf{b}$, we have

$$s\mathbf{G}(s) = k\big[c(sI - A)^{-1}b + d\big] = kG(s),$$

and so $\lim_{s\to 0} s\mathbf{G}(s) = kG(0) > 0$. Define $\varepsilon := (\kappa - k)/(2\lambda\kappa) > 0$ ($\varepsilon = 1/(2\lambda)$ if $\kappa = \infty$) and let $q \geq 0$ be such that $\inf_{\omega\in\mathbb{R}^*} \mathrm{Re}\big[(qi\omega + 1)G(i\omega)/i\omega\big] \geq \nu - (\varepsilon/k)$. Then,

$$\inf_{\omega\in\mathbb{R}^*} \mathrm{Re}\big[(qi\omega + 1)\mathbf{G}(i\omega)\big] = k \inf_{\omega\in\mathbb{R}^*} \mathrm{Re}\big[(qi\omega + 1)G(i\omega)/i\omega\big] \geq k\nu - \varepsilon \geq \varepsilon - \frac{1}{\lambda}.$$

Therefore, the hypotheses of Theorem 2.5 hold with $\alpha = \lambda$ and so, by Assertion (c) of that theorem, $z(t) \to z^\infty \in \ker \mathbf{A}$ as $t \to \infty$. It follows that $\lim_{t\to\infty} z_1(t) = 0$ and $\lim_{t\to\infty} z_2(t) = z_2^\infty \in f^{-1}(0)$. Hence, Assertion (a) of the present theorem follows, and, furthermore, the limit $\lim_{t\to\infty} u(t) =: u^r$ exists and is finite. Since $f(z_2^\infty) = 0$, we may conclude that

$$\varphi(u^r) - \varphi^r = f(u^r - u^*) = f(z_2^\infty) = 0\,,$$

whence Assertions (b) of the present theorem. Finally, to establish Assertion (c), simply observe that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \big(cx(t) + d\varphi(u(t))\big) = (-cA^{-1}b + d)\varphi^r = G(0)\varphi^r = r\,.$$

■

3.2 Integral control in the presence of input and output nonlinearities

The problem of tracking constant reference signals $r \in \mathbb{R}$ will be addressed in the context of a class of finite-dimensional single-input single-output continuous-time real linear systems $\Sigma = (A, b, c, d)$ having a continuous nonlinearity in both the input and output channels:

$$\dot x = Ax + b\varphi(u)\,, \quad x(0) = x^0 \in \mathbb{R}^n\,; \qquad y = \psi(cx + d\varphi(u))\,. \tag{29}$$

In (29), A is assumed to be Hurwitz. Furthermore, the transfer function G, given by $G(s) = c(sI - A)^{-1}b + d$, is assumed to satisfy $G(0) > 0$. To achieve the objective of tracking a constant reference signal $r \in \mathbb{R}$, we consider integral control action

$$u(t) = u^0 + \int_0^t k(\tau)[r - \psi(cx(\tau) + d\varphi(u(\tau))]\,d\tau = u^0 + \int_0^t k(\tau)[r - y(\tau)]\,d\tau\,,$$

with control gain function k (possibly constant), yielding the following nonlinear feedback system (illustrated in in Figure 3)

$$\left.\begin{aligned} &\dot x(t) = Ax(t) + b\varphi(u(t))\,, \quad x(0) = x^0 \in \mathbb{R}^n\,, \\ &\dot u(t) = k(t)[r - \psi(cx(t) + d\varphi(u(t)))]\,, \quad u(0) = u^0 \in \mathbb{R}\,. \end{aligned}\right\} \tag{30}$$

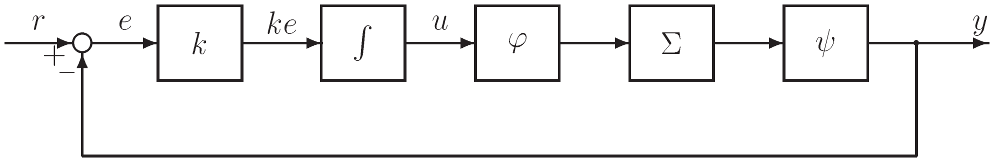

Figure 3: Time-varying low-gain integral control with input and output nonlinearities

The objective in this subsection is to determine a gain function k such that the tracking error

$$e(t) := r - y(t) = r - \psi(cx(t) + d\varphi(u(t))$$

converges to 0 as $t \to \infty$. We introduce the set of feasible reference values

$$\mathcal{R} := \{\psi(G(0)v) : v \in \overline{\operatorname{im}}\,\varphi\}\,.$$

By continuity of φ and ψ, it is clear that $\mathcal{R}$ is an interval. The motivation for the introduction of $\mathcal{R}$ is as follows. If asymptotic tracking occurs for a given $r \in \mathbb{R}$, we would expect that $\varphi^r := \lim_{t\to\infty} \varphi(u(t))$ exists and is finite. Then, using the final value theorem, we may conclude that $\lim_{t\to\infty} y(t) = \psi(G(0)\varphi^r)$, and so, $r = \psi(G(0)\varphi^r)$, which in turn implies that $r \in \mathcal{R}$. In fact, it can been shown that if ψ is continuous and monotone, then $r \in \mathcal{R}$ is close to being a necessary condition for asymptotic tracking insofar as, if asymptotic tracking of r is achievable, whilst maintaining boundedness of $\varphi \circ u$, then $r \in \mathcal{R}$.

The following lemma will be used later. The proof is straightforward and is therefore omitted.

Lemma 3.2. Let $g : \mathbb{R} \to \mathbb{R}$ be piecewise continuously differentiable and non-decreasing. Define the function $g^\nabla : \mathbb{R} \to \mathbb{R}$ by $g^\nabla(\xi) := \min\{g'_-(\xi), g'_+(\xi)\}$, where $g'_-(\xi)$ and $g'_+(\xi)$ denote the left and right derivative of g at ξ, respectively. Then g^∇ is Borel measurable, non-negative and $g^\nabla \in L^\infty_{\text{loc}}(\mathbb{R})$. If $v : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous, then $g \circ v$ is absolutely continuous and

$$\frac{d}{dt}(g \circ v)(t) = g^\nabla(v(t))\dot{v}(t)\,, \quad \text{for a.a. } t \in \mathbb{R}_+\,.$$

Under the extra assumption that g is globally Lipschitz continuous with Lipschitz constant $\lambda \geq 0$ we have $0 \leq g^\nabla(\xi) \leq \lambda$ for all $\xi \in \mathbb{R}$.

If $g : \mathbb{R} \to \mathbb{R}$ is a piecewise continuously differentiable and non-decreasing function, then a point $w \in \mathbb{R}$ is said to be a *critical value* of g if there exists $\xi \in \mathbb{R}$ such that $g(\xi) = w$ and $g^\nabla(\xi) = 0$: a critical value w is said to be *strong* if $g^{-1}(w)$ is not a singleton.

We are now in the position to formulate the main result of this subsection.

Theorem 3.3. Consider the feedback system (30). Assume that A is Hurwitz, $G(0) = d - cA^{-1}b > 0$, $r \in \mathcal{R}$, $k \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, φ and ψ are piecewise continuously differentiable and non-decreasing, φ is globally Lipschitz and there exists $\sigma > 0$ such that

$$0 \le \xi(\psi(\xi) - \psi(0)) \le \sigma\xi^2\,, \quad \forall\, \xi \in \mathbb{R}. \tag{31}$$

Then, for each $(x^0, u^0, k) \in \mathbb{R}^n \times \mathbb{R} \times L^\infty_{\text{loc}}(\mathbb{R}_+)$, there exists a unique solution $t \mapsto (x(t), u(t))$ of (30) on $\mathbb{R}_+$ (no finite escape-time). Moreover, there exists $\kappa_0 > 0$ such that, for each $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}$ and every non-negative-valued function $k \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ with

$$\limsup_{t\to\infty} k(t) < \kappa_0,$$

the unique solution $t \mapsto (x(t), u(t))$ of (30) on $\mathbb{R}_+$ has the following properties:

(a) the limit $\varphi^r := \lim_{t\to\infty} \varphi(u(t))$ exists and is finite;

(b) $\lim_{t\to\infty} x(t) = -A^{-1}b\varphi^r$;

furthermore, if $k \notin L^1(\mathbb{R}_+)$, then

(c) $\psi(G(0)\varphi^r) = r$;

(d) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \psi(cx(t) + d\varphi(u(t))) = r$;

(e) if r is an interior point of $\mathcal{R}$, then u is bounded;

(f) if r is an interior point of $\mathcal{R}$ and $\psi(G(0)w) \ne r$ for all strong critical values w of φ, then $u^r := \lim_{t\to\infty} u(t)$ exists, is finite and $\psi(G(0)\varphi(u^r)) = r$;

(g) if r is an interior point of $\mathcal{R}$ and is not a critical value of ψ, and $\psi(G(0)w) \ne r$ for all critical values w of φ, then the convergence in (a), (b), (d) and (f) is of order $\exp(-\rho\int_0^t k(\tau)d\tau)$ for some $\rho > 0$.

Proof. Let $(x^0, u^0, k) \in \mathbb{R}^n \times \mathbb{R} \times L^\infty_{\text{loc}}(\mathbb{R}_+)$. It follows from standard results in differential equations that (30) has a unique solution $t \mapsto (x(t), u(t))$ defined on $\mathbb{R}_+$.

Since $r \in \mathcal{R}$, there exists $\varphi^* \in \overline{\text{im}}\,\varphi$ such that $\psi(G(0)\varphi^*) = r$. Defining

$$\tilde{\psi} : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \psi(\xi + G(0)\varphi^*) - r\,, \tag{32}$$

it is clear that $\tilde{\psi}(0) = 0$. Using the sector condition imposed on the function $\xi \mapsto \psi(\xi) - \psi(0)$, it is straightforward to show that there exists $\tilde{\sigma} > 0$ such that

$$0 \le \xi\tilde{\psi}(\xi) \le \tilde{\sigma}\xi^2\,, \quad \forall\, \xi \in \mathbb{R}. \tag{33}$$

Therefore, defining $\theta : \mathbb{R} \to \mathbb{R}$ by $\theta(0) := 0$ and $\theta(\xi) := \tilde{\psi}(\xi)/\xi$ for $\xi \ne 0$, it is obvious that

$$0 \le \theta(\xi) \le \tilde{\sigma} \qquad \text{and} \qquad \tilde{\psi}(\xi) = \theta(\xi)\xi\,; \quad \forall\, \xi \in \mathbb{R}.$$

Introduce new variables

$$z_1(t) := x(t) + A^{-1}b\varphi(u(t)), \qquad z_2(t) := \varphi(u(t)) - \varphi^*; \quad \forall t \in \mathbb{R}_+,$$

and notice that

$$cx + d\varphi(u) = cz_1 + G(0)z_2 + G(0)\varphi^*.$$

By Lemma 3.2, $\dot z_2(t) = \varphi^\nabla(u(t))\dot u(t)$. Therefore,

$$\dot z_1(t) = Az_1(t) - k(t)\varphi^\nabla(u(t))A^{-1}b\tilde\psi(w(t)), \quad \dot z_2(t) = -k(t)\varphi^\nabla(u(t))\,\tilde\psi(w(t)); \quad \text{a.a. } t \geq 0, \tag{34}$$

where $w := cz_1 + G(0)z_2$. Setting

$$f(t,\xi) := k(t)\varphi^\nabla(u(t))\theta(w(t))\xi, \quad \forall\,(t,\xi) \in \mathbb{R}_+ \times \mathbb{R},$$

(34) can be written in the form

$$\dot z(t) = \mathbf{A}z(t) - \mathbf{b}f(t, \mathbf{c}z(t)), \tag{35}$$

where

$$z(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} A^{-1}b \\ 1 \end{bmatrix}, \quad \mathbf{c} := (c,\, G(0)).$$

Clearly $\mathbf{A}$ is such that Hypothesis (H) holds. Let $\mathbf{G}$ be the system transfer function given by $\mathbf{G}(s) := \mathbf{c}(sI - \mathbf{A})^{-1}\mathbf{b}$. Noting that $\mathbf{G}(s) = G(s)/s$, invoking Lemma 2.1 and recalling (28), we have

$$-\infty < \nu_0 = \inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\mathbf{G}(i\omega) \leq 0. \tag{36}$$

Moreover, with a view to applying Theorem 2.3 to (35), observe that

$$\lim_{s\to 0} s\mathbf{G}(s) = G(0) > 0.$$

Let $\lambda_1 > 0$ be a Lipschitz constant for φ and define

$$\kappa_0 := \begin{cases} 1/(\lambda_1\tilde\sigma|\nu_0|), & \nu_0 \neq 0 \\ \infty, & \nu_0 = 0. \end{cases} \tag{37}$$

Assume that $k \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ is non-negative-valued with $\limsup_{t\to\infty} k(t) < \kappa_0$. Then there exist $t_0 \geq 0$ and $\alpha \in (0, \kappa_0\lambda_1\tilde\sigma)$ (in which case, $|\nu_0| < 1/\alpha$) such that

$$k(t) := \frac{k(t)\varphi^\nabla(u(t))\theta(w(t))}{\alpha} \in [0,1], \quad \forall\, t \geq t_0. \tag{38}$$

Note that the time-varying nonlinearity f can be written in the form $f(t,\xi) = k(t)\alpha\xi$. Since $|\nu_0| < 1/\alpha$, we have that

$$1/\alpha + \inf_{\omega\in\mathbb{R}^*} \operatorname{Re}\mathbf{G}(i\omega) > 0.$$

Proof of (a)–(b): An application of Theorem 2.3 to (35) now shows that $z^\infty := \lim_{t\to\infty} z(t) \in \ker\mathbf{A}$. Therefore, $\lim_{t\to\infty} z_1(t) = 0$ and $\lim_{t\to\infty} \varphi(u(t)) =: \varphi^r$ exists and is finite, whence Assertions (a) and (b).

Now assume that $k \notin L^1(\mathbb{R}_+)$.

Proof of (c)–(d): It suffices to show that $\psi(G(0)\varphi^r) = r$. Seeking a contradiction, suppose that $\psi(G(0)\varphi^r) \neq r$. This implies that $\delta := (r - \psi(G(0)\varphi^r))/2 \neq 0$. Using continuity of ψ, we obtain for sufficiently large $\tau > 0$

$$|y(t) - \psi(G(0)\varphi^r)| \leq |\delta|\,, \quad \forall\, t \geq \tau\,.$$

As a consequence, and noticing that $\dot{u}(t) = k(t)[r - y(t)] = k(t)[2\delta - y(t) + \psi(G(0)\varphi^r)]$, we have

$$-|\delta| k(t) \leq \dot{u}(t) - 2\delta k(t) \leq |\delta| k(t)\,, \quad \forall\, t \geq \tau\,.$$

Since $\delta \neq 0$, either $\delta > 0$ or $\delta < 0$. Assume $\delta > 0$. Then $\dot{u}(t) \geq \delta k(t)$ for all $t \geq \tau$ which, on integration, yields $u(t) - u(\tau) \geq \delta \int_\tau^t k(s)ds$ for all $t \geq \tau$. Since $\int_\tau^t k(s)ds \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$, and hence, since φ is non-decreasing,

$$\sup \overline{\mathrm{im}}\, \varphi = \lim_{t\to\infty} \varphi(u(t)) = \varphi^r\,. \tag{39}$$

Let $\xi \in \overline{\mathrm{im}}\, \varphi$ with $\psi(G(0)\xi) = r$. Since ψ is non-decreasing, $G(0) > 0$ and $\delta > 0$, it follows that $\xi > \varphi^r$, contradicting (39). A similar argument shows that the assumption $\delta < 0$ also leads to a contradiction.

Proof of (e) and (f): To prove statements (e) and (f), assume that r is an interior point of $\mathcal{R}$. Using monotonicity of ψ and Assertion (a), we obtain that $\varphi^r = \lim_{t\to\infty} \varphi(u(t))$ is an interior point of $\mathrm{im}\, \varphi$. The monotonicity of φ then implies that u is bounded, yielding Assertion (e). Under the additional assumption that $\psi(G(0)w) \neq r$ for all strong critical values w of φ, it follows that φ^r is not a strong critical value of φ (since $\psi(G(0)\varphi^r) = r$ by Assertion (a)). Combined with the monotonicity of φ, this shows that $\varphi^{-1}(\varphi^r)$ is a singleton, that is $\varphi^{-1}(\varphi^r) = \{u^r\}$ for some $u^r \in \mathbb{R}$. By Assertion (a), $\lim_{t\to\infty} \varphi(u(t)) = \varphi^r$, and so, $\lim_{t\to\infty} u(t) = u^r$.

Proof of (g): It follows from an application of statement (2) of Theorem 2.3 to (35) that there exist $L \geq 1$ and $\tilde{\rho} > 0$ such that

$$\|z(t)\| \leq L \exp\left(-\tilde{\rho} \int_0^t k(\tau)d\tau\right)\,, \quad \forall\, t \in \mathbb{R}_+\,,$$

where k is given by (38). Consequently, in order to show that the convergence of $\varphi(u(t))$, $x(t)$ and $y(t)$ is of order $\exp(-\rho \int_0^t k(\tau)d\tau)$ for some $\rho > 0$ as $t \to \infty$, it is sufficient to prove that $\liminf_{t\to\infty} \varphi^\nabla(u(t)) > 0$ and $\liminf_{t\to\infty} \theta(w(t)) > 0$. By Assertion (f), $\psi(G(0)\varphi(u^r)) = r$, and hence, by hypothesis, $\varphi(u^r)$ is not a critical value of φ, showing that $\varphi^\nabla(u^r) > 0$. Since φ is piecewise continuously differentiable it follows that

$$\liminf_{t\to\infty} \varphi^\nabla(u(t)) \geq \varphi^\nabla(u^r) > 0\,.$$

Furthermore, using that $\psi(G(0)\varphi^*) = \psi(G(0)\varphi^r) = r$ combined with the monotonicity of ψ and the assumption that r is not a critical value of ψ, we may conclude that $\varphi^* = \varphi^r$. It therefore follows that $w(t) = cz_1(t) + G(0)z_2(t) \to 0$ as $t \to \infty$. Consequently,

$$\liminf_{t\to\infty} \theta(w(t)) \geq \liminf_{\xi\to 0} \frac{\tilde{\psi}(\xi)}{\xi} = \tilde{\psi}^\nabla(0) = \psi^\nabla(G(0)\varphi^r) > 0\,,$$

where the last inequality follows from the fact that $\psi(G(0)\varphi^r) = r$ combined with the hypothesis that r is not a critical value of ψ.

To prove that the convergence of $u(t)$ is of order $\exp(-\rho \int_0^t k(\tau)d\tau)$ as $t \to \infty$, choose $\varepsilon > 0$ such that $\varphi^\nabla(\xi) > 0$ for all $\xi \in [u^r - \varepsilon, u^r + \varepsilon]$ (this is possible since $\varphi^\nabla(u^r) > 0$). Denoting the restriction of φ to $[u^r - \varepsilon, u^r + \varepsilon]$ by φ_ε, it is clear that the inverse function of φ_ε satisfies a Lipschitz condition, that is there exists $l > 0$ such that

$$|\varphi_\varepsilon^{-1}(\xi_1) - \varphi_\varepsilon^{-1}(\xi_2)| \leq l|\xi_1 - \xi_2|\,, \quad \forall\, \xi_1, \xi_2 \in \operatorname{im}\varphi_\varepsilon\,.$$

Choosing $\tau \geq 0$ sufficiently large, so that $u(t) \in [u^r - \varepsilon, u^r + \varepsilon]$ for all $t \geq \tau$, we obtain

$$|u(t) - u^r| = |\varphi_\varepsilon^{-1}(\varphi_\varepsilon(u(t))) - \varphi_\varepsilon^{-1}(\varphi_\varepsilon(u^r))| \leq l|\varphi(u(t)) - \varphi^r|\,, \quad \forall\, t \geq \tau\,.$$

Since we have already proved that the convergence of $\varphi(u(t))$ is of order $\exp(-\rho \int_0^t k(\tau)d\tau)$ as $t \to \infty$, the above inequality shows that the same is true for the convergence of $u(t)$.

■

Corollary 3.4. Assume that the assumptions of Theorem 3.3 are satisfied. Let $\lambda_1 > 0$ be a Lipschitz constant for the globally Lipschitz function φ. Additionally assume that ψ is globally Lipschitz, with Lipschitz constant $\lambda_2 > 0$. Then

$$-\infty < \nu_0 = \inf_{\omega \in \mathbb{R}^*} \operatorname{Re}[G(i\omega)/i\omega] \leq 0$$

and, for each $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}$ and every non-negative-valued function $k \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ satisfying

$$\limsup_{t \to \infty} k(t) < \kappa_0 := 1/(\lambda_1 \lambda_2 |\nu_0|) \qquad (\text{with the convention } 1/0 := \infty)\,, \tag{40}$$

the unique solution $t \mapsto (x(t), u(t))$ of (30) on $\mathbb{R}_+$ has the properties (a)–(g) in Theorem 3.3.

Proof. By Lemma 2.1, $-\infty < \nu_0 \leq 0$ and so, $0 < \kappa_0 \leq \infty$. Assume that the assumptions of Theorem 3.3 hold and that ψ is globally Lipschitz, with Lipschitz constant $\lambda_2 > 0$. Then (33) holds with $\tilde{\sigma}$ replaced by λ_2. Formulae (37) and (40) for κ_0 then coincide and hence the claim follows from Theorem 3.3. ■

Remark 3.5. If sufficient plant information is available *a priori* to compute a suitable $\kappa_0 > 0$, then the tracking objective is achievable by a constant gain integral control

$$u(t) = u^0 + k \int_0^t [r - y(\tau)]d\tau, \quad u^0 \in \mathbb{R}, \quad k \in (0, \kappa_0).$$

In the case of globally Lipschitz output nonlinearity ψ, Corollary 3.4 identifies the following as sufficient *a priori* information for computation of κ_0: knowledge of Lipschitz constants λ_1 and λ_2 for the nonlinearities φ and ψ, together with computability of the quantity $\nu_0 = \inf_{\omega \in \mathbb{R}^*} \operatorname{Re}[G(i\omega)/i\omega]$.

If insufficient plant information is available *a priori*, then Theorem 3.3 implies that the tracking objective is achievable by integral control with time-varying gain

$$u(t) = u^0 + \int_0^t k(\tau)[r - y(\tau)]d\tau, \quad u^0 \in \mathbb{R},$$

provided that $k \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ is non-negative-valued with $k \notin L^1(\mathbb{R}_+)$ and $k(t) \to 0$ as $t \to \infty$ (a canonical choice being $k : t \mapsto 1/(1+t)$).

3.3 Constant gain integral control in the presence of input nonlinearities

In the specific case of constant gain integral control in the presence of time-invariant non-decreasing globally Lipschitz input nonlinearities, Theorem 3.1 and Corollary 3.4 are both applicable. However, since $\kappa \geq \kappa_0$, Theorem 3.1 gives a potentially larger range of gains for which asymptotic tracking is guaranteed. Consequently, in the context of constant gain integral control in the presence of input nonlinearities, it is therefore advantageous to use Theorem 3.1 rather than Corollary 3.4. To illustrate this point, we present two classes of examples where κ is strictly greater than κ_0.

Example 3.6. Consider the transfer function G given by $G(s) = g_1/(s + g_2)$, where $g_1, g_2 > 0$. A routine calculation shows that $\nu_0 = -g_1/g_2^2$ and $\nu = 0$, so that $\kappa/\kappa_0 = |\nu_0|/|\nu| = \infty$.

Example 3.7. Consider the transfer function G given by $G(s) = g_1/(s^2 + g_2 s + g_3)$, where $g_1, g_2, g_3 > 0$. Defining

$$F(q, \omega) := \mathrm{Re}\,\frac{(qi\omega + 1)G(i\omega)}{i\omega} = g_1 \frac{q(g_3 - \omega^2) - g_2}{(g_3 - \omega^2)^2 + g_2^2\omega^2}\,,$$

we have that, for all $q \geq 0$, $F(q, \pm\sqrt{g_3}) = -g_1/(g_2 g_3)$. Hence,

$$\nu = \sup_{q \geq 0} \nu_q = \sup_{q \geq 0} \left(\inf_{\omega \in \mathbb{R}} F(q, \omega) \right) \leq -\frac{g_1}{g_2 g_3}\,.$$

A routine argument shows that $\inf_{\omega \in \mathbb{R}} F(g_2/g_3, \omega) = -g_1/(g_2 g_3)$ and therefore we may conclude that

$$\nu = -\frac{g_1}{g_2 g_3}\,.$$

Next we compute ν_0. An elementary calculation shows that $(\partial F/\partial\omega)(0, \omega) = 0$ if and only if $\omega(2\omega^2 - 2g_3 + g_2^2) = 0$. We consider two cases.

Case 1: $2g_3 > g_2^2$.
Write $F(0, \omega)$ in the form

$$F(0, \omega) = -\frac{g_1 g_2}{g_3^2 + (g_2^2 - 2g_3 + \omega^2)\omega^2}\,.$$

Since the term $(g_2^2 - 2g_3 + \omega^2)\omega^2$ is negative for all sufficiently small $|\omega|$, $\omega \neq 0$, it follows that the function $\omega \mapsto F(0,\omega)$ has a local maximum at $\omega = 0$ and a global minimum at $\omega = \pm\sqrt{(2g_3 - g_2^2)/2}$. Thus,

$$\nu_0 = F\left(0, \pm\sqrt{(2g_3 - g_2^2)/2}\right) = -\frac{g_1}{g_2(g_3 - g_2^2/4)},$$

and consequently,

$$\frac{\kappa}{\kappa_0} = \frac{|\nu_0|}{|\nu|} = \frac{\nu_0}{\nu} = \frac{g_3}{g_3 - g_2^2/4} \in (1,2).$$

Case 2: $2g_3 \leq g_2^2$.
In this case, the function $\omega \mapsto F(0,\omega)$ has a global minimum at $\omega = 0$ and $\nu_0 = F(0,0) = -g_1 g_2/g_3^2$. Consequently,

$$\frac{\kappa}{\kappa_0} = \frac{|\nu_0|}{|\nu|} = \frac{\nu_0}{\nu} = \frac{g_2^2}{g_3} \in [2,\infty).$$

Finally, let $\lambda > 0$ and let $(A,b,c,0)$ be an asymptotically stable realization of $G(s)$. Then the conclusions of Theorem 3.1 are valid for all non-decreasing globally Lipschitz nonlinearities φ with Lipschitz constant λ, all reference values $r \in \mathbb{R}$ with $rg_3/g_1 \in \operatorname{im}\varphi$ and all initial values, provided that the gain k is in the interval $(0,\kappa)$, where $\kappa = 1/(|\nu|\lambda) = g_2 g_3/(g_1\lambda)$. We claim that κ is the largest number with this property. To this end consider the (linear) function φ given by $\varphi(\xi) = \lambda\xi$ and the reference value $r = 0$. An application of the Routh-Hurwitz criterion then shows that the corresponding (linear) closed-loop system is asymptotically stable if and only if $k \in (0,\kappa)$, implying in particular that for every $k \geq \kappa$, there exists an initial condition for which Assertion (a) of Theorem 3.1 is not valid. In this context it is interesting to note that Theorem 2.5, together with the Routh-Hurwitz criterion, yields that the Aizerman conjecture holds for any stabilizable and detectable realization of the transfer function $\mathbf{G}(s) = G(s)/s = g_1/(s^3 + g_2 s^2 + g_3 s)$.

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