Uniqueness of the polar factorisation and projection of a vector-valued mapping*

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Abstract

ABSTRACT. - This paper proves some results concerning the polar factorisation of an integrable vector-valued function u into the composition $u = u^{\#} \circ s$, where $u^{\#}$ is equal almost everywhere to the gradient of a convex function, and s is a measure-preserving mapping. It is shown that the factorisation is unique (i.e. the measure-preserving mapping s is unique) precisely when $u^{\#}$ is almost injective. Not every integrable function has a polar factorisation; we introduce a class of counterexamples. It is further shown that if u is square integrable, then measure-preserving mappings s which satisfy $u = u^{\#} \circ s$ are exactly those, if any, which are closest to u in the L^2-norm.

Keywords: Polar factorisation, Monotone rearrangement, Measure-preserving mappings, L^2-projection.

AMS classification: 28A50, 28D05, 46E30

TITRE. Unicité de factorisation polaire et projection des applications à valeurs vectorielles

RÉSUMÉ. - Cet article prouve des résultats au sujet de la factorisation polaire d'une application à valeurs vectorielles sommable u en une composition $u = u^{\#} \circ s$, où $u^{\#}$ est égal presque partout à la dérivée d'une application convexe et s est une application conservant la mesure. On démontre que la factorisation est unique (c'est à dire l'application conservant la mesure s est unique), si et seulement si $u^{\#}$ est presque injectif. Les applications sommables ne possèdent pas toujours les factorisations polaires; on introduit des contre-examples. On prouve aussi que si u est de carré sommable, alors les applications conservant la mesure qui satisfont $u = u^{\#} \circ s$ sont précisement celles qui sont les plus près de u en L^2-norme.

1 Introduction

A vector-valued function has a *polar factorisation* if it can be written as the composition of its *monotone rearrangement*, which is equal almost everywhere to the gradient of a convex function, with a measure-preserving mapping. This concept was introduced by Brenier [2, 3], and may be seen as the extension to vector-valued functions of an idea of Ryff [12], who showed that any real integrable function on a bounded interval could be written as the composition of its increasing rearrangement with a measure-preserving map.

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Brenier proved existence and uniqueness of the monotone rearrangement on sufficiently regular domains, and existence and uniqueness of the polar factorisation subject to a further "nondegeneracy" restriction on the function. The object of the present paper is to investigate the consequences of relaxing his assumptions by studying integrable functions on general sets of finite Lebesgue measure. While the monotone rearrangement continues to exist and be unique, as proved by McCann [9], we give examples where there is no polar factorisation (and show the general existence result by the authors [5] is in a sense sharp). We also give a class of examples where uniqueness fails, which shows that the sufficient condition for uniqueness given by [5] is in fact necessary. Furthermore we show that Brenier's connection between the polar factorisation of an L^2 function u and the measure-preserving maps nearest to u, persists in the more general context.

In this paper, given an integrable function $u : X \to \mathbb{R}^n$, and a set $Y \subset \mathbb{R}^n$ of finite positive Lebesgue measure, we say that u has a *polar factorisation through* Y if $u = u^\# \circ s$, where $u^\#$ is equal to the gradient of a convex function almost everywhere in Y, and $s : X \to Y$ is a measure-preserving mapping. The restriction on X is not severe; we only require that (X, μ) is a complete measure space with the same measure-theoretic structure as an interval of length $\mu(X)$ equipped with Lebesgue measure. (We give precise definitions below.) The existing literature proves existence and uniqueness of $u^\#$ as noted above, but it does not resolve fully the existence and uniqueness of s. Brenier [3] proved existence and uniqueness of s under a *nondegeneracy* hypothesis on u, and subsequently Burton and Douglas [5] proved existence under a weaker hypothesis of *countable degeneracy*, while giving an example of nonuniqueness. It was further shown in [5] that if $u^\#$ is *almost injective* (meaning $u^\#$ is injective off a negligible set, which would be implied by nondegeneracy of u), then s exists and is unique, and the converse result was conjectured. We prove this conjecture in Theorem 1. Our method of proof is to establish that either $u^\#$ is almost injective, or that there exists a nontrivial measure-preserving mapping $s : Y \to Y$ such that $u^\# \circ s = u^\#$. In the latter case, nonuniqueness of the polar factorisation, if one exists, follows easily. The core of our proof is to construct measure-preserving maps leaving invariant a family of line segments; it is noteworthy that in our context no Lipschitz condition is required on the directions of the lines. Our arguments make use of methods developed by Larman [8], in his study of the endpoints of the line-segments on a convex surface.

We remark that if some rearrangement of u is almost injective and has a polar factorisation through Y, then the polar factorisation of u through Y (exists and) is unique, as a consequence of Lemma 2. It is not however necessary for u to be almost injective for u to have a unique polar factorisation.

Having settled the question of uniqueness, we give the first result on nonexistence of polar factorisations. Burton and Douglas [5, Theorem 1.10] proved that if the monotone rearrangement $u^\#$ is almost injective off its level sets of positive measure, then every rearrangement u of $u^\#$ has a polar factorisation $u = u^\# \circ s$ for some measure-preserving mapping s. We prove in Theorem 2 that this result is sharp in the sense that if $u^\#$ is not almost injective on the complement of its level sets of positive measure, then there exists a rearrangement $\hat{u}$ of $u^\#$ such that $\hat{u}$ has no polar factorisation through Y. This class of counterexamples has the property that the functions $\hat{u}$ are almost injective off their level sets of positive measure.

Our third result concerns the connection between polar factorisation of a vector-valued function $u : X \to \mathbb{R}^n$ through $Y \subset \mathbb{R}^n$, and the L^2-projection (i.e. set of nearest points) of u on the set of measure-preserving mappings from $X \to Y$. Assuming $u \in L^p$ and $\mathrm{id}_Y \in L^q$ (where p, q are conjugate), we prove in Theorem 3 that the set of maximisers for the functional $\int_X u(x) \cdot s(x)$ among measure-preserving mappings s from X to Y (which, if $p = 2$, is equal to the set of minimisers for $||u - s||_2$) comprises exactly those s (if any) which satisfy $u = u^\# \circ s$. Brenier [3, Theorem 1.2 (b)] had obtained this result in his setting, of a sufficiently regular (e.g. smooth bounded) domain Y and a nondegenerate function u.

Polar factorisations arise naturally in the Lagrangian formulation of the semigeostrophic equations, a model for weather frontogenesis. At each time t, the geostrophic transformation $\mathbf{X}(t, .)$, from which information about the physical quantities (velocity, temperature and pressure) of the system can be extracted, is equal to the gradient of a convex function. Tracking nondifferentiabilities of these convex functions as time evolves is thought of as weather fronts forming and moving. The flow is incompressible, therefore the trajectory mapping (of the fluid particles) is measure-preserving. It follows that at each time t, the Lagrangian variable $\tilde{\mathbf{X}}(t, .)$ takes the form of a polar factorisation. Numerical schemes (for calculating solutions) have exploited the characterisation of the trajectory mapping in terms of L^2-projections. (See Benamou [1], Brenier [4].)

1.1 Definitions and notation

Definition. Let (X, μ) and (Y, ν) be finite positive measure spaces with $\mu(X) = \nu(Y)$. Two vector-valued functions $f \in L^1(X, \mu, \mathbb{R}^n)$ and $g \in L^1(Y, \nu, \mathbb{R}^n)$ are *rearrangements* of each other (or *equimeasurable*) if

$$\mu(f^{-1}(B)) = \nu(g^{-1}(B)) \text{ for every } B \in \mathcal{B}(\mathbb{R}^n),$$

where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel field of $\mathbb{R}^n$. Equivalent formulations can be found in Douglas [6].

Definitions. A *measure-preserving mapping* from a finite positive measure space (X, μ) to a positive measure space (Y, ν) with $\mu(X) = \nu(Y)$ is a mapping $s : X \to Y$ such that for each ν-measurable set $A \subset Y$, $\mu(s^{-1}(A)) = \nu(A)$.

We will be considering the special case of (X, μ) complete, $Y \subset \mathbb{R}^n$ and ν being n-dimensional Lebesgue measure. The ν-measurable sets will be the Borel-measurable sets; the same measure-preserving properties can then be deduced for the Lebesgue-measurable sets.

Moreover $s : X \to Y$ is a *measure-preserving transformation* if
(i) $s : X \backslash L \to Y \backslash M$ is a bijection, where L and M are some sets of zero (respectively μ and ν) measure; and
(ii) s and s^{-1} are measure-preserving mappings.

A finite complete measure space (X, μ) is a *measure-interval* if there exists a measure-preserving transformation from (X, μ) to $[0, \mu(X)]$ with Lebesgue measure (on the Lebesgue sets). We recall that any complete separable metric space, equipped with a finite nonatomic Borel measure, is a measure interval.

Throughout this paper we will denote n-dimensional Lebesgue measure by λ_n, and the extended real numbers, that is the set $\mathbb{R} \cup \{-\infty, \infty\}$, by $\overline{\mathbb{R}}$.

Definition. Let $u \in L^1(X, \mu, \mathbb{R}^n)$, where (X, μ) is a measure-interval. Let Lebesgue measurable $Y \subset \mathbb{R}^n$ be such that $\lambda_n(Y) = \mu(X)$. The *monotone rearrangement of* u *on* Y is the unique function $u^\# : Y \to \mathbb{R}^n$ that is a rearrangement of u, and satisfies $u^\# = \nabla\psi$

almost everywhere in Y for some proper lower semicontinuous convex function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$. (A $\overline{\mathbb{R}}$-valued function is called *proper* if it is not identically ∞, and nowhere takes the value $-\infty$.)

The existence and uniqueness of the monotone rearrangement follows from the main result of McCann [9]. It is unique in the sense that if $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is another convex function, and $\nabla\varphi$ (as a function defined on Y) is a rearrangement of u, then $\nabla\varphi(y) = \nabla\psi(y)$ for almost every $y \in Y$.

Definition. Let $u \in L^1(X, \mu, \mathbb{R}^n)$ where (X, μ) is a measure-interval. Let Lebesgue measurable $Y \subset \mathbb{R}^n$ be such that $\lambda_n(Y) = \mu(X)$, and let $u^\#$ denote the monotone rearrangement of u on Y. We say u has a *polar factorisation through* Y if there exists a measure-preserving mapping s from (X, μ) to (Y, λ_n) such that $u = u^\# \circ s$ almost everywhere.

Definition. A mapping $s : X \to Y$, where (X, μ) is a finite positive measure space, is *almost injective* if there exists a set $X_0 \subset X$ such that s restricted to X_0 is injective, and $\mu(X \backslash X_0) = 0$.

1.2 Statements of results

Our main results are Theorems 1, 2 and 3 below, whose proofs are given in Sections 2, 3 and 4 respectively.

Theorem 1. *Suppose that $u \in L^1(X, \mu, \mathbb{R}^n)$ where (X, μ) is a measure-interval. Let Lebesgue measurable $Y \subset \mathbb{R}^n$ satisfy $\lambda_n(Y) = \mu(X)$ and let $u^\#$ denote the monotone rearrangement of u on Y. Then u has a unique polar factorisation through Y if and only if $u^\#$ is almost injective.*

Theorem 2. *Let integrable $u^\# : Y \to \mathbb{R}^n$ be the restriction of the gradient of a proper lower semicontinuous convex function to a set $Y \subset \mathbb{R}^n$ of finite positive Lebesgue measure, and suppose that $u^\#$ restricted to the complement of its level sets of positive measure is not almost injective. Let (X, μ) be a measure-interval satisfying $\mu(X) = \lambda_n(Y)$. Then $u^\#$ has a rearrangement $u : X \to \mathbb{R}^n$ which does not have a polar factorisation through Y.*

Theorem 3. *Let $1 \le p, q \le \infty$ be conjugate exponents, let (X, μ) be a measure interval, let $Y \subset \mathbb{R}^n$ be a Lebesgue measurable set such that $\mu(X) = \lambda_n(Y) \, (< \infty)$, and suppose Y is bounded if $p = 1$, or $\int_Y |y|^q dy < \infty$ if $p > 1$. Let $\mathfrak{S}$ denote the set of all measure-preserving maps from X to Y. Let $u \in L^p(X, \mu, \mathbb{R}^n)$, and let $u^\#$ be the monotone rearrangement of u on Y, and write*

$$I(s) := \int_X u(x) \cdot s(x) \, d\mu(x) \text{ for } s \in \mathfrak{S},$$

$$J(u) := \int_Y u^\#(y) \cdot y \, d\lambda_n(y).$$

Then

$$\sup\{I(s) \mid s \in \mathfrak{S}\} = J(u),$$

and $s \in \mathfrak{S}$ satisfies $I(s) = J(u)$ if and only if $u = u^\# \circ s$ almost everywhere in X.

The following is an immediate consequence of Theorem 3:

Corollary 1. *Let the hypotheses of Theorem 3 be satisfied, with $p = 2$. Then $s \in \mathfrak{S}$ is a nearest point of $\mathfrak{S}$ to u, relative to $||\cdot||_2$ (with the Euclidean norm on $\mathbb{R}^n$), if and only if $u = u^\# \circ s$ almost everywhere in X. In particular, u has a polar factorisation through Y if and only if there is a nearest point of $\mathfrak{S}$ to u.*

Brenier introduced the optimisation problems of Theorem 3 and Corollary 1, and in [3, Theorem 1.2] proved their equivalence to the polar factorisation problem assuming that u is nondegenerate (i.e. the inverse image of any set of zero measure has zero measure), and that Y is open, connected, and has smooth boundary; these assumptions ensure that the polar factorisation exists and is unique.

Corollary 1 implies in particular, that the L^2-projection of u on $\mathfrak{S}$ is a singleton set if and only if $u^\#$ is almost injective. Note that if u is nondegenerate, then $u^\#$ is almost injective, but the converse is false in general (see Burton and Douglas [5, Lemma 2.4 and Sect. 3]).

2 A nonuniqueness result

Theorem 1 will be proved in this section. The key step is to show that either the monotone rearrangement $u^\#$ is almost injective, or there exists a nontrivial measure-preserving mapping s such that $u^\# \circ s = u^\#$; we give this result separately as Theorem 4. Now if u has a polar factorisation and $u^\#$ is not almost injective, nonuniqueness of the polar factorisation follows easily.

Recall that the *effective domain* of a proper function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the set $\{x \in \mathbb{R}^n \mid \psi(x) < \infty\}$. We denote the subdifferential of ψ at y by $\partial\psi(y)$.

Theorem 4. *Let $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function whose effective domain has nonempty interior $\Omega \subset \mathbb{R}^n$. Let $Y \subset \Omega$ be a set of positive Lebesgue measure, and suppose $u^\# : Y \to \overline{\mathbb{R}}$ is Lebesgue measurable and satisfies $u^\#(x) \in \partial\psi(x)$ for almost all $x \in Y$. Then either $u^\#$ is almost injective, or there exists a nontrivial λ_n-measure-preserving map $s : Y \to Y$ such that $u^\# \circ s = u^\#$ almost everywhere in Y.*

The construction of the nontrivial measure-preserving mapping of Theorem 4 is achieved by using the following lemma to exploit a product structure; we prove the existence of a section-preserving measure-preserving mapping which is almost nowhere equal to the identity in a compact set, and leaves everything fixed outside this set.

Lemma 1. *Let Γ be a finite positive Borel measure on a metric space X, let ν be a Borel measure on $X \times \mathbb{R}^k$ satisfying*

$$\nu(S) = \int_S f \, d(\Gamma \times \lambda_k)$$

where $f : X \times \mathbb{R}^k \to [0, \infty)$ is a bounded Borel measurable function, and let $A \subset X \times \mathbb{R}^k$ be a compact subset with $\nu(A) > 0$.

Then there is a ν-measure-preserving map $\tau : X \times \mathbb{R}^k \to X \times \mathbb{R}^k$ such that
(i) $\tau(x, y) \neq (x, y)$ for almost every $(x, y) \in A$,
(ii) $\tau(\{x\} \times \mathbb{R}^k) \subset \{x\} \times \mathbb{R}^k$ for all $x \in X$, and
(iii) $\tau(z) = z$ for all $z \in X \times \mathbb{R}^k \setminus A$.

Proof. By identifying $X \times \mathbb{R}^k$ with $(X \times \mathbb{R}^{k-1}) \times \mathbb{R}$ we see that it is sufficient to consider the case $k = 1$.

For $x \in X$ let

$$A(x) = \{y \in \mathbb{R} \mid (x, y) \in A\}$$

and for $x \in X$, $y \in \overline{\mathbb{R}}$ define

$$\varphi(x, y) = \int_{-\infty}^{y} f(x, z) 1_{A(x)}(z) dz.$$

Then, for $x \in X$, if $A(x) \neq \emptyset$ then $\varphi(x, \cdot)$ is a continuous increasing (i.e. non-decreasing) map of $A(x)$ onto $[0, \varphi(x, \infty)]$, and is measure-preserving relative to the measure ν_x with density $f(x, \cdot)$ on $A(x)$, and λ_1 on $[0, \varphi(x, \infty)]$. Now for $(x, y) \in A$ let

$$s(x, y) = \min\{t \in A(x) \mid \varphi(x, t) = \varphi(x, \infty) - \varphi(x, y)\},$$

which is well-defined by continuity of $\varphi(x, \cdot)$ and compactness of $A(x)$.

For each x for which $A(x) \neq \emptyset$, $s(x, \cdot)$ is a continuous map of $A(x)$ into $A(x)$; moreover if $\varphi(x, \infty) > 0$ then $s(x, \cdot)$ is a ν_x-preserving map of $A(x)$ into itself. Also $s(x, \cdot)$ fixes only those y satisfying $2\varphi(x, y) = \varphi(x, \infty)$, which form a set of zero ν_x-measure for Γ-almost every x. Define $\tau : A \to A$ by $\tau(x, y) = (x, s(x, y))$. Since τ is Borel measurable, we can now deduce that $\tau : A \to A$ is ν-measure-preserving and fixes $\Gamma \times \lambda_1$-almost no points of A. If we extend the definition by setting

$$\tau(x, y) = (x, y) \;\; \forall (x, y) \in X \times \mathbb{R} \setminus A$$

then we obtain the desired map τ. $\square$

We establish some notation before proceeding with the proof of Theorem 4.

Notation. We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, on any metric space. ∂C and $\mathrm{relint} C$ will denote respectively the boundary and interior of a finite-dimensional convex set C relative to its affine hull.

We say that a measure μ is absolutely continuous with respect to a measure ν, and write $\mu \ll \nu$, if $\mu(E) = 0$ for every ν-measurable set E for which $\nu(E) = 0$. We denote the Radon-Nikodým derivative of μ with respect to ν by $d\mu/d\nu$.

If K is a nonempty compact convex set in $\mathbb{R}^n$, φ is an affine functional on $\mathbb{R}^n$ that is nonconstant on K, and $\varepsilon > 0$, we write

$$K(\varphi, \varepsilon) = \{x \in K \mid \varphi(x) > \sup_K \varphi - \varepsilon\},$$

$$K(\varphi) = \{x \in K \mid \varphi(x) = \sup_K \varphi\};$$

we call $K(\varphi, \varepsilon)$ a *cap* on K.

Proof of Theorem 4.

Let G denote the graph and E the epigraph of ψ, let $\chi(x) = (x, \psi(x))$ for $x \in \Omega$, $\pi(x, \alpha) = x$ for $x \in \Omega$, $\alpha \in \mathbb{R}$. We suppose that $u^\#$ is not almost injective, and proceed to construct the required map $s : Y \to Y$. We can choose a bounded open convex set Ω' with $\overline{\Omega'} \subset \Omega$, such that $\lambda_n(\Omega' \cap Y) > 0$, and such that $u^\#$ is not almost injective when restricted to $\Omega' \cap Y$. Henceforth we shall assume $Y \subset \Omega'$, since any measure-preserving map from $\Omega' \cap Y$ to itself can be extended to the whole of Y by defining it to be the identity on $Y\backslash\Omega'$. By discarding a set of measure zero if necessary, we can assume Y is a Borel set and that $\partial\psi(x) = \{u^{\#}(x)\}$ for every $x \in Y$. Notice that $\chi : \Omega' \to \mathbb{R}^{n+1}$ is Lipschitz, say with constant $\gamma > 0$.

We note that, for each k, the union of the extreme faces of E having dimension at least k, is an F_σ-set (i.e. a countable union of closed sets). If x, y are two points of Y for which $\chi(x), \chi(y)$ lie on the same line segment in G, then $u^{\#}(x) = u^{\#}(y)$. If the images under π of the line-segments in G covered λ_n-almost none of Y, then $u^{\#}$ would be almost injective. We may therefore assume, replacing Y by a Borel subset having positive measure if necessary, that every point of $\chi(Y)$ lies on a line-segment in G. Applying a result of Larman [8, Theorem 1] to the intersections of E with large cubes, we can prove that the union of the relative boundaries of all extreme faces of E of dimensions $1, \dots, n$ is a set of zero $\mathcal{H}^n$-measure. Hence, passing to a subset of smaller positive λ_n-measure if necessary, we may choose k, $1 \le k \le n$, such that $\chi(Y)$ is covered by the relative interiors of the k-dimensional extreme faces of E.

We may now choose an $(n-k+1)$-dimensional linear subspace Λ_0 of $\mathbb{R}^{n+1}$ parallel to the x_{n+1}-axis, and points $w_1, \dots, w_{k+1}$ in general position in $\Lambda_0^\perp \subset \mathbb{R}^n \times \{0\}$, such that, writing $\Lambda_i = w_i + \Lambda_0$ for $i = 1, \dots, k+1$ and writing $\mathcal{F}$ for the family of all extreme k-faces of E whose relative interiors intersect all of $\Lambda_1, \dots \Lambda_{k+1}$ at points of $\chi(\Omega')$, $\mathcal{F}$ covers $\chi(Y')$ for some subset $Y' \subset Y$ having positive $\mathcal{H}^n$-measure (countably many such families of faces cover $\chi(Y)$, so such a choice is possible). We may further replace Y' by a compact set with $\lambda_n(Y') > 0$.

We may assume the origin is located so that the centroid of $w_1, \dots, w_{k+1}$ is 0. Let $M = \sup_{\Omega'} \psi$ and let
$$E' = \{(x, \alpha) \in E \mid x \in \overline{\Omega'},\ \alpha \le M + 1\}$$
which is a convex body in $\mathbb{R}^{n+1}$. Let $K_i = E' \cap \Lambda_i$, $i = 0, 1, \dots, k+1$, and let $X = \{F \cap \Lambda_0 \mid F \in \mathcal{F}\}$, which is a subset of the extreme points of K_0. For $x \in X$ let $F(x)$ be the unique element of $\mathcal{F}$ that contains x, and let $z_i(x) = F(x) \cap \Lambda_i \in K_i$ for $i = 1, \dots, k+1$. Let
$$\Sigma = \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{k+1}) \in \mathbb{R}^{k+1} \mid \lambda_1 \ge 0, \dots, \lambda_{k+1} \ge 0, \lambda_1 + \cdots + \lambda_{k+1} = 1\},$$
and for $x \in X$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{k+1}) \in \Sigma$ let
$$T(x, \boldsymbol{\lambda}) = \lambda_1 z_1(x) + \cdots + \lambda_{k+1} z_{k+1}(x) \in \operatorname{relint} F(x).$$
Since $\operatorname{relint} F(x) \cap \operatorname{relint} F(y) = \emptyset$ for distinct $x, y \in X$, it follows that T is injective. The range of T is contained in G. Continuity of T is a consequence of the $F(x)$, $x \in X$, being extreme faces.

Define the measure μ on (Borel subsets of) $G' := \chi(\Omega')$ by $\mu(B) = \lambda_n(\pi(B))$; then $\mu \ll \mathcal{H}^n$ and $\mathcal{H}^n \ll \mu$ with $\gamma^{-n} \le d\mu/d\mathcal{H}^n \le 1$ almost everywhere. Let $\Gamma = \Gamma^{n-k}$ be the Borel measure on the extreme points of K_0 defined by Larman [8], that is
$$\Gamma(S) := \lim_{\delta \downarrow 0} \inf_{\{C_j\}_j} \sum_{j=1}^{\infty} \mathcal{H}^{n-k}(\partial C_j)$$
where the infimum is taken over all countable covers $\{C_j\}_{j=1}^{\infty}$ of S by caps on K_0 having diameter less than δ. Clearly $\mathcal{H}^{n-k}(S) \le \Gamma(S)$; Larman [8, Theorem 2] proved that Γ is a finite measure (which is nontrivial, but crucial to our argument). We will show that the measure ν defined by $\nu(B) = \mu(T(B))$ is absolutely continuous with respect to the product

measure $\Gamma \times \lambda_k$ on $X \times \Sigma$, then apply Lemma 1 to construct a suitable ν-measure-preserving map on $X \times \Sigma$. The proof of absolute continuity is achieved by adapting an argument of Larman [8, Theorem 1].

As a preparatory step, consider a cap C on K_0, say $C = K_0(\varphi, \varepsilon)$ where $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ is linear and nonconstant on Λ_0, and $\varepsilon > 0$, and let $U \subset \Sigma$ be a relatively open subset. If $x \in C \cap X$ then $x = (k+1)^{-1}(z_1(x) + \cdots + z_{k+1}(x))$, hence

$$\varphi(x) = (k+1)^{-1}(\varphi(z_1(x)) + \cdots + \varphi(z_{k+1}(x))),$$

from which we deduce that $z_i(x) \in C_i := K_i(\varphi, (k+1)\varepsilon)$, $i = 1, \ldots, k+1$. Choose $v_i \in K_i(\varphi)$. Then, for $i = 1, \ldots, k+1$ we have $(k+1)^{-1}(v_1 + \cdots + v_{k+1} - v_i + C_i) \subset C$, so $C_i \subset p_i + (k+1)C$ where $p_i = v_i - (v_1 + \cdots + v_{k+1}) \in v_i - \Lambda_0 = \Lambda_i$. So for $x \in C \cap X$ and $\boldsymbol{\lambda} \in \Sigma$ we have

$$T(x, \boldsymbol{\lambda}) \in \lambda_1 C_1 + \cdots + \lambda_{k+1} C_{k+1} \subset \lambda_1 p_1 + \cdots + \lambda_{k+1} p_{k+1} + (k+1)C.$$

Thus $T((C \cap X) \times U) \subset D + (k+1)C$, where $D = \{\lambda_1 p_1 + \cdots + \lambda_{k+1} p_{k+1} \mid \boldsymbol{\lambda} \in U\}$ which lies in an affine k-space skew to $\Lambda_0 \supset (k+1)C$.

Now $T((C \cap X) \times U) \subset E' \cap G$. Moreover, a line parallel to the x_{n+1}-axis through any point q of $T((C \cap X) \times U)$ must intersect $p + (k+1)\partial C$ for some $p \in D$; since such a line intersects G in only the point q, we conclude that $T((C \cap X) \times U) \subset \chi(\pi(D + (k+1)\partial C))$. The map $\chi \circ \pi$ has Lipschitz constant γ, so

$$\begin{aligned}
\mathcal{H}^n(T((C \cap X) \times U)) &\leq \gamma^n \mathcal{H}^n(D + (k+1)\partial C) \\
&\leq \gamma^n \rho^n \mathcal{H}^n(D_0 + (k+1)\partial C) \\
&= \gamma^n \rho^n (k+1)^{n-k} \mathcal{H}^k(D_0) \mathcal{H}^{n-k}(\partial C) \\
&\leq \gamma^n \rho^n \omega^k (k+1)^{n-k} \lambda_k(U) \mathcal{H}^{n-k}(\partial C)
\end{aligned}$$

where $D_0 = \{\lambda_1 w_1 + \cdots + \lambda_{k+1} w_{k+1} \mid \boldsymbol{\lambda} \in U\}$, ρ is the Lipschitz constant of the map $\lambda_1 w_1 + \cdots + \lambda_{k+1} w_{k+1} \mapsto \lambda_1 p_1 + \cdots + \lambda_{k+1} p_{k+1}$ of D_0 onto D; and ω is the Lipschitz constant of the map $(\lambda_1, \cdots, \lambda_{k+1}) \mapsto \lambda_1 w_1 + \cdots + \lambda_{k+1} w_{k+1}$ of Σ into $\Lambda_0^{\perp}$. In particular, $\rho \leq R/r$ where r is the least distance of any w_i from the opposite face of D_0 and R is the diameter of E'; it is important to notice that r, R and ω are independent of the choice of C.

To prove $\nu \ll \Gamma \times \lambda_k$, let us first recall that the Carathéodory construction yields

$$\Gamma \times \lambda_k(S) = \inf_{\{W_j \times U_j\}_j} \sum_{j=1}^{\infty} \Gamma(W_j) \lambda_k(U_j) \tag{1}$$

where the infimum is taken over all countable covers $\{W_j \times U_j\}_{j=1}^{\infty}$ of the Borel set S by measurable rectangles. Consider a rectangle $Z \times U$ where $Z \subset X$ and $U \subset \Sigma$ are Borel sets. Then given $\varepsilon > 0$, for each $\delta > 0$ we can choose a cover $\{C_i\}_{i=1}^{\infty}$ of Z by caps on K_0 of diameter less than δ such that

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-k}(\partial C_i) < \Gamma(Z) + \varepsilon;$$

consequently,

$$\begin{aligned}
\nu(Z \times U) &= \mu(T(Z \times U)) \\
&\leq \mathcal{H}^n(T(Z \times U)) \\
&\leq (\gamma R/r)^n \omega^k (k+1)^{n-k} \lambda_k(U) \sum_{i=1}^{\infty} \mathcal{H}^{n-k}(\partial C_i) \\
&\leq L\lambda_k(U)(\Gamma(Z)+\varepsilon)
\end{aligned}$$

where $L = (\gamma R/r)^n \omega^k (k+1)^{n-k}$, and therefore

$$\nu(Z \times U) \leq L\Gamma(Z)\lambda_k(U).$$

Applying this result to (1) yields

$$\nu(S) \leq L(\Gamma \times \lambda_k)(S)$$

and consequently $\nu \ll \Gamma \times \lambda_k$, with $d\nu/d(\Gamma \times \lambda_k) \leq L$ almost everywhere. We can assume $d\nu/d(\Gamma \times \lambda_k)$ is Borel-measurable and bounded.

Lemma 1 now enables us to construct a ν-measure-preserving map $\tau : X \times \Sigma \to X \times \Sigma$ which satisfies
(i) $\tau(z) \neq z$ for almost every $z \in T^{-1}(\chi(Y'))$,
(ii) $\tau(\{x\} \times \Sigma) \subset \{x\} \times \Sigma$ for each $x \in X$, and
(iii) $\tau(z) = z$ for all $z \in X \times \Sigma \setminus T^{-1}\chi(Y')$.

Define $s : Y \to Y$ by $s := \pi \circ T \circ \tau \circ T^{-1} \circ \chi$. It is routine to verify that s is λ_n-measure-preserving. Moreover it follows from (i) that s is nontrivial. It remains to show that $u^\# \circ s = u^\#$. Notice that, if $x \in X$ then $s(\pi(F(x)) \subset \pi(F(x))$ (by (ii)), and since, as we observed at the beginning of the proof, $u^\#$ is constant on $Y' \cap \pi(F(x))$, we deduce that $u^\# \circ s(y) = u^\#(y)$ for $y \in Y' \cap \pi(F(x))$. Now for $y \in Y'$, there exists $x \in X$ such that $y \in \pi(F(x))$. Hence $u^\# \circ s = u^\#$ on Y as required. □

Proof of Theorem 1.
If $u^\#$ is almost injective, Burton and Douglas [5, Theorem 1.8] yields that u has a unique polar factorisation through Y.

For the converse, suppose that $u \in L^1(X, \mu, \mathbb{R}^n)$ has a polar factorisation through Y, $u = u^\# \circ \sigma$ say, where $\sigma : X \to Y$ is a measure-preserving mapping, but that $u^\#$ is not almost injective. Theorem 4 yields the existence of a nontrivial measure-preserving mapping $s : Y \to Y$ with $u^\# \circ s = u^\#$. Now $u^\# \circ s \circ \sigma(x) = u^\# \circ \sigma(x) = u(x)$ for μ almost every x, and $s \circ \sigma$ is a measure-preserving mapping which differs from σ on a set of positive measure. Thus the polar factorisation of u through Y is not unique. □

3 A nonexistence result

In this section we prove Theorem 2 which demonstrates nonexistence of a polar factorisation for a class of functions. Burton and Douglas [5, Theorem 1.10] proved that if $u^\#$ is a monotone rearrangement which is almost injective on the complement of its level sets of positive measure, then for any rearrangement u of $u^\#$, a polar factorisation exists. We show that this is the best general existence result that can be proved; if $u^\#$ does not have the above property, we can find a rearrangement u of $u^\#$ such that no measure-preserving mapping s such that $u = u^\# \circ s$ exists.

We establish two preliminary results. Firstly we prove that if $u = v \circ s$ for some almost injective u and measure-preserving mapping s, then v is almost injective. Our second result is that every integrable v has a rearrangement u that is almost injective off its level sets of positive measure. Then we apply these results to a monotone rearrangement $u^{\#}$ satisfying the hypotheses of Theorem 2 and show that it has a rearrangement u, almost injective off its level sets of positive measure, which does not have a polar factorisation. Note that the two-dimensional example of [5, Section 3], given there as a monotone function which has a nonunique factorisation, fits this framework.

The following lemma is of interest in its own right, as well as being integral to the proof of Theorem 2. It may be applied to polar factorisations to obtain the following generalisation of Theorem 1; an integrable function u has a rearrangement $\hat{u}$, where $\hat{u}$ is almost injective and has a polar factorisation through Y, if and only if u has a unique polar factorisation through Y.

Lemma 2. *Let X, μ, u, Y be as in Theorem 1. Suppose u is almost injective, and that $u = v \circ s$ for some integrable function $v : Y \to \mathbb{R}^n$, and measure-preserving mapping $s : X \to Y$. Then v is almost injective.*

Proof. Choose $X_0 \subset X$ such that $\mu(X \backslash X_0) = 0$, $u(x) = v \circ s(x)$ for every $x \in X_0$, and u restricted to X_0 is injective. Now for $x, y \in X_0$ such that $s(x) = s(y)$, we have $u(x) = v \circ s(x) = v \circ s(y) = u(y)$, from which it follows that $x = y$. Moreover [5, Lemma 2.5] yields that $\lambda_n(s(X_0)) = \lambda_n(Y)$. Writing $Y_0 = s(X_0)$, we have that $s : X_0 \to Y_0$ is bijective. Now $v(y) = u \circ s^{-1}(y)$ for every $y \in Y_0$; it follows that v is injective on Y_0. $\square$

Lemma 3. *Let $v : Y \to \mathbb{R}^n$ be integrable, where $Y \subset \mathbb{R}^n$ has finite positive Lebesgue measure. Suppose (X, μ) is a measure-interval satisfying $\mu(X) = \lambda_n(Y)$. Then v has a rearrangement $u : X \to \mathbb{R}^n$ that is almost injective on the complement of its level sets of positive measure.*

Proof. Initially we restrict attention to finding $\hat{u} : Y \to \mathbb{R}^n$, a rearrangement of v that is almost injective on the complement of its level sets of positive measure. Let $Y_i = v^{-1}(\alpha_i)$ for $i \in I$ be the level sets of v that have positive measure, where I is a countable index set, and write $Y_0 = Y \backslash \bigcup_{i \in I} Y_i$. By adding and subtracting sets of measure zero, we can suppose Y_0 is a G_δ-set (i.e. a countable intersection of open sets). Define a Borel measure μ on $\mathbb{R}^n$ by $\mu(B) = \lambda_n(v^{-1}(B))$ for Borel sets $B \subset \mathbb{R}^n$. Now $\{\alpha_i \mid i \in I\}$ is the set of atoms of μ. Let μ_0 be the nonatomic part of μ. Then $(Y_0, \lambda_n|_{Y_0})$ and $(\mathbb{R}^n \backslash \{\alpha_i \mid i \in I\}, \mu_0)$ are finite nonatomic Borel measures on separable completely metrisable spaces with the same total measure; it follows that they are isomorphic by, for example, [11, p. 164, Proposition 33 and p. 409, Theorem 16]. Choose a measure-preserving bijection $u_0 : Y_0 \to \mathbb{R}^n \backslash \{\alpha_i \mid i \in I\}$. Then u_0 is a rearrangement of $v_0 = v|_{Y_0}$ for if $B \subset \mathbb{R}^n$ is a Borel set then

$$\lambda_n(u_0^{-1}(B)) = \mu_0(B) = \lambda_n(v_0^{-1}(B)).$$

Define $\hat{u} = u_0$ on Y_0 and $\hat{u} = \alpha_i$ on Y_i for each $i \in I$. Then $\hat{u} : Y \to \mathbb{R}^n$ is a rearrangement of v having the desired properties.

Finally we note that (X, μ) and (Y, λ_n) are isomorphic, so we can choose a measure-preserving transformation $\tau : X \to Y$. Now $u : X \to \mathbb{R}^n$ defined by $u = \hat{u} \circ \tau$ satisfies the required conditions. $\square$

Proof of Theorem 2.
Write Y_0 for the complement (with respect to Y) of the level sets of positive measure of $u^\#$, and write $u_0^\#$ for $u^\#$ restricted to Y_0. Applying Lemma 3 to $u^\#$, we can choose a rearrangement $u : X \to \mathbb{R}^n$ that is almost injective on the complement of its level sets of positive measure; denote this set X_0, and write u_0 for u restricted to X_0. We note that $u^\#$ is the monotone rearrangement of u on Y, $u_0^\#$ the monotone rearrangement of u_0 on Y_0. Suppose that u has a polar factorisation through Y, $u = u^\# \circ s$ say, where $s : X \to Y$ is a measure-preserving mapping. Modifying s on a set of measure zero if necessary, we have $u_0 = u_0^\# \circ s$ where $s : X_0 \to Y_0$ is measure-preserving; noting u_0 is almost injective, Lemma 2 yields that $u_0^\#$ is almost injective, which is a contradiction. □

4 The projection problem

Here we give the proof of Theorem 3, which is achieved by means of elementary convex analysis. We first establish some notation.
NOTATION. If $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$, then $\psi^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ denotes the (Legendre-Fenchel) *conjugate convex function* of ψ, defined by

$$\psi^*(x) = \sup\{x \cdot y - \psi(y) \mid y \in \mathbb{R}^n\}.$$

Proof of Theorem 3.
Let the proper lower semicontinuous convex function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a potential for $u^\#$, and let ψ^* be the conjugate of ψ, which is also a proper lower-semicontinuous convex function. Standard convex analysis (see for example [10, Theorem 23.5]) gives

$$u^\#(y) \cdot y = \psi^*(u^\#(y)) + \psi(y) \tag{2}$$

for λ_n-almost every $y \in Y$. Since ψ and ψ^* are proper, lower semicontinuous and convex, it follows that each is bounded below by an affine functional (see for example Ekeland and Temam [7, Proposition 3.1, page 14]); since $\int_Y |y| d\lambda_n(y) < \infty$ we now deduce that $\int_Y \psi d\lambda_n$ and $\int_Y \psi^* \circ u^\# d\lambda_n$ both exist in $\mathbb{R} \cup \{\infty\}$. Thus we may integrate (2) over Y to obtain

$$J(u) = \int_Y \psi(y)dy + \int_Y \psi^*(u^\#(y))dy. \tag{3}$$

Since $J(u)$ is finite by Hölder's inequality, we deduce that both integrals on the right-hand side of (3) are finite.

Let $s \in \mathfrak{S}$. The inequality between $I(s)$ and $J(u)$, and the condition for equality, are obtained by making full use of the standard ideas in the polar factorisation literature, as follows. Since $\psi \circ s$ is a rearrangement of $\psi|_Y$, and $\psi^* \circ u$ is a rearrangement of $\psi^* \circ u^\#$, from (3) we now obtain

$$J(u) = \int_X \psi(s(x))dx + \int_X \psi^*(u(x))dx. \tag{4}$$

We have

$$\int_X (\psi(s(x)) + \psi^*(u(x)) - u(x) \cdot s(x))dx \geq 0, \tag{5}$$

because the integrand is everywhere non-negative. From (4) and (5) we deduce $I(s) \leq J(u)$.

An element $s \in \mathfrak{S}$ satisfies $I(s) = J(u)$ if and only if equality holds in (5), which occurs if and only if $\psi(s(x)) + \psi^*(u(x)) - u(x) \cdot s(x) = 0$ for almost every $x \in X$, which occurs if and only if $u(x) \in \partial\psi(s(x))$ for almost every $x \in X$. Thus $I(s) = J(u)$ if and only if $u = u^\# \circ s$ almost everywhere in X.

However in our situation the upper bound $J(u)$ need not be attained, so it still remains to prove that $J(u)$ is approached arbitrarily closely. Let $\varepsilon > 0$, choose a partition $\{Z_m\}_{m=1}^\infty$ of $\mathbb{R}^n$ into countably many Borel sets of diameter less than ε, and for each $m \in \mathbb{N}$ let $X_m = u^{-1}(Z_m)$ and $Y_m = (u^\#)^{-1}(Z_m)$. Now choose a measure-preserving bijection $s : X \to Y$ such that $s(X_m) = Y_m$ for each $m \in \mathbb{N}$, which is possible since X_m and Y_m are measure-intervals of equal measure (we allow s to remain undefined on any X_m that have zero measure). Then $|u \circ s^{-1} - u^\#| < \varepsilon$ almost everywhere in Y, and we deduce

$$\int_X u(x) \cdot s(x) dx = \int_Y u \circ s^{-1}(y) \cdot y dy > \int_Y u^\#(y) \cdot y dy - \varepsilon \int_Y |y| dy,$$

that is, $I(s) > J(u) - \varepsilon ||\mathrm{id}_Y||_1$. Hence $\sup I(\mathfrak{S}) = J(u)$. □

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