ORDINARY DIFFERENTIAL OPERATORS POSSESSING INVARIANT SUBSPACES OF THE POWER TYPE

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Abstract

Ordinary differential operators possessing invariant subspaces spanned by the functions x^i, $i = 0, 1, \ldots, n-1$, are considered. A complete description of operators of the submaximal order $n-2$ is obtained. The dimension C^n_{2n-1} of the linear space of such operators is conjectured to be the upper bound for the operators possessing arbitrary n-dimensional invariant subspace. A general representation for translation-invariant operators is found.

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1. Introduction

There are numerous examples [3, 10, 16] of evolution equations

$$\frac{\partial u}{\partial t} = F\Big(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^p u}{\partial x^p}\Big), \quad u = u(x,t),\ x,\ t \in \mathbf{R}, \tag{1}$$

possessing special solutions in the form of "finite sums"

$$u(x,t) = \sum_{i=1}^{n} \varphi_i(t) f_i(x), \tag{2}$$

where $\{f_i(x)\}$ and $\{\varphi_i(t)\}$ $(i = 1, \dots, n)$ are some linearly independent sets of functions.

It was observed in [2] that the majority of such examples have a common algebraic nature: they are constructed on finite-dimensional linear subspaces

$$W_n = L\{f_1(x), \dots, f_n(x)\} \equiv \Big\{ \sum_{i=1}^{n} C_i f_i(x) \mid C_i \in \mathbf{R} \Big\} \tag{3}$$

(linear spans of functions $f_1(x), \dots, f_n(x)$), invariant with respect to the operator F: $F[W_n] \subseteq W_n$ (see also [15]).

This observation allowed V.A. Galaktionov [2] to considerably increase the number of such examples and obtain a description of different classes of differential operators (evolution equations) possessing invariant subspaces of some special (exponential, trigonometric and power) types.

Some general results concerning this approach can be found in [12, 13].

Following [12], suppose that the subspace (3) is defined as the space of solutions of a linear ordinary differential equation

$$L[y] \equiv a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0, \quad y = y(x), \tag{4}$$

with the fundamental system of solutions $\{f_i(x)\}$.

Let $\mathcal{F}_{n-1}(W_n)$ (or simply $\mathcal{F}_{n-1}$) denote the whole set of differential operators

$$F[y] = F(x, y, y', \dots, y^{(n-1)}) \tag{5}$$

of the order not greater than $n-1$, leaving the subspace (3), i.e. equation (4), invariant. We assume that the higher-order derivatives are excluded by means of equation (4).

The complete description of $\mathcal{F}_{n-1}$ was obtained in [14] and is given by

Theorem 1. *The set $\mathcal{F}_{n-1}(W_n)$ consists of all operators (5) of the form*

$$F = \sum_{i=1}^{n} A^i(I_1, \dots, I_n) f_i(x), \tag{6}$$

where $A^i(I_1, \dots, I_n)$, $i = 1, \dots, n$, are arbitrary functions of the complete set of first integrals of equation (4).

One can use the following set of first integrals,

$$I_i = \alpha_{i,1}(x) y^{(n-1)} + \dots + \alpha_{i,n-1}(x) y' + \alpha_{i,n}(x) y, \quad i = 1, \dots, n,$$

where the coefficients $\alpha_{i,1}, \dots, \alpha_{i,n}$ are defined recursively as

$$\alpha_{i,1} = a_0 z_i, \quad \alpha_{i,2} = a_1 z_i - (\alpha_{i,1})', \quad \dots, \quad \alpha_{i,n} = a_{n-1} z_i - (\alpha_{i,n-1})'$$

via a fundamental system of solutions $\{z_i(x)\}$ of the equation conjugate to the equation (4) ($\{z_i(x)\}$ are constructed in terms of $\{f_i(x)\}$ without any integration) [11].

An invariant subspace of the power type,

$$W_n = L\{1, x, \dots, x^{n-1}\}, \tag{7}$$

is defined by the simplest linear ODE

$$y^{(n)} = 0 \tag{8}$$

with the conjugate equation $z^{(n)} = 0$. In this case $z_i(x) = f_i(x) = x^{i-1}$, $i = 1, \dots, n$, and hence the complete set of first integrals can be written down in the form

$$I_i = \sum_{j=1}^{i} (-1)^{j-1} \frac{(i-1)!}{(i-j)!} x^{i-j} y^{(n-j)}, \quad i = 1, \dots, n, \tag{9}$$

so that the above theorem yields

Theorem 2. *The set $\mathcal{F}_{n-1}$ for the power type subspace (7) (equation (8)) is generated by the operators*

$$F = \sum_{i=1}^{n} A^i(I_1, \dots, I_n) x^{i-1}, \tag{10}$$

where $A^i(I_1, \dots, I_n)$, $i = 1, \dots, n$, are arbitrary functions of the first integrals (9).

Notice that every operator (5) possessing the invariant subspace (3) defines a Lie-Bäcklund (or generalized) symmetry operator (see [1, 4, 8])

$$X = F[y]\frac{\partial}{\partial y}$$

of the corresponding linear ODE (4), and the whole set $\mathcal{F}_{n-1}$ of such operators forms an infinite-dimensional Lie algebra completely described by Theorem 1. Theorem 2 provides then all Lie-Bäcklund symmetries (obtained in [1]) for the simplest equation (8).

Along with the algebra $\mathcal{F}_{n-1}$, consider its linear subspaces $\mathcal{F}_k$ with $k \le n-2$ consisting of the operators (5) of the order not greater than k. The operators of the order not greater than $n-2$ form a linear subspace $\mathcal{F}_{n-2} \subset \mathcal{F}_{n-1}$. We call them operators or symmetries of submaximal order.

One of the purposes of the present paper is to obtain a general representation for the set $\mathcal{F}_{n-2}$ of the operators of submaximal order in the case of equation (8) (subspace (7)) and to prove that they form a finite-dimensional linear space whose dimension is equal to C^n_{2n-1} (Section 2).

For $n = 3$ this has been actually shown by S. Lie [7], who found that the maximal dimension of the algebra of contact symmetries on plane cannot exceed $C^3_5 = 10$, and that this maximal value is reached on the equation $y''' = 0$. Together with the Theorem 3 obtained in the next section it makes realistic the hypothesis:

for n-th order ODEs the maximal dimension of the linear space $\mathcal{F}_{n-2}$ of symmetries of submaximal order is equal to C^n_{2n-1}.

Keeping in mind applications to evolution equations, we also consider the subspace $\widehat{\mathcal{F}}_{n-1} \subset \mathcal{F}_{n-1}$ of translation-invariant operators and give their complete description (Section 3).

In the last Section 4 we find all translation-invariant operators of submaximal order for $n \le 4$, belonging to the intersection of previously considered classes of operators.

2. Operators of submaximal order

For a given linear ODE (4) the symmetry operators (5) are found from the invariance test

$$L[F[y]]\big|_{(4)} \equiv 0. \qquad (11)$$

Following S. Lie we will suppose that operators (5) are defined by analytic functions F. By virtue of the analyticity, the function $F(x, y, y', \dots, y^{(n-1)})$ can be expanded in a convergent power series in variables $y, y', \ldots, y^{(n-1)}$ with coefficients depending on x. Representing this series as an infinite sum of homogeneous polynomials in $y, y', \ldots, y^{(n-1)}$ and using the linearity of the equation under consideration we obtain from (11) that every such polynomial defines a symmetry as well. Hence we can suppose beforehand that operators (5) have the homogeneous form

$$F = \sum_{\substack{0 \le i_0, \ldots, i_{n-1} \le k \\ i_0 + \cdots + i_{n-1} = k}} \beta_{i_0, \ldots, i_{n-1}}(x) y^{i_0} (y')^{i_1} \cdot \ldots \cdot (y^{(n-1)})^{i_{n-1}} \tag{12}$$

where $k = 0, 1, 2, \ldots$. Using (9) we can express the derivatives $y, y', \ldots, y^{(n-1)}$ as linear homogeneous functions of $I_1, \ldots, I_n$ (the corresponding determinant is not zero). The substitution into (12) implies the following representation for F in terms of the first integrals,

$$F = \sum_{\substack{0 \le i_1, \ldots, i_n \le k \\ i_1 + \cdots + i_n = k}} \gamma_{i_1, \ldots, i_n}(x) (I_1)^{i_1} \cdot \ldots \cdot (I_n)^{i_n}, \tag{13}$$

where in accordance with Theorem 1 the coefficients $\gamma_{i_1, \ldots, i_n}(x)$ are linear combinations of the fundamental system $f_1(x), \ldots, f_n(x)$.

Let us return to equation (8). Symmetries of the order $n-2$ are singled out by the condition

$$\frac{\partial F}{\partial y^{(n-1)}} \equiv \sum_{i=1}^{n} x^{i-1} \frac{\partial F}{\partial I_i} = 0,$$

where we consider F as a function of $x, I_1, \ldots, I_n$. The first integrals of the corresponding characteristic system have the form

$$x, \quad J_i = I_{i+1} - x I_i \quad (i = 1, \ldots, n-1)$$

and therefore the general expression for F is given by

$$F = \widetilde{F}(x, J_1, \ldots, J_{n-1}), \tag{14}$$

where $\widetilde{F}$ is an arbitrary function. Let us pass in (13) from $I_1, \ldots, I_n$ to the new variables

$$J_0 = I_1, \quad J_i = I_{i+1} - x I_i \quad (i = 1, \ldots, n-1). \tag{15}$$

Resolving (15) with respect to $I_1, \ldots, I_n$ we obtain

$$I_1 = J_0, \quad I_{i+1} = \sum_{j=0}^{i} x^{i-j} J_j \quad (i = 1, \ldots, n-1) \tag{16}$$

and therefore (13) takes the form

$$F = \sum_{\substack{0\le i_1,\dots,i_{n-1}\le k\\ i_1+\dots+i_{n-1}=k}} \delta_{i_1,\dots,i_{n-1}}(x)(J_1)^{i_1}\cdot\ldots\cdot(J_{n-1})^{i_{n-1}}, \tag{17}$$

where we took into account that F does not depend on J_0 (see (14)). Note that the coefficients $\delta_{i_1,\dots,i_{n-1}}(x)$ in (17) are some polynomials in x of degrees $s_{i_1,\dots,i_{n-1}}$. Let $s = max\, s_{i_1,\dots,i_{n-1}}$. The expressions $(J_1)^{i_1}\cdot\ldots\cdot(J_{n-1})^{i_{n-1}}$ are also polynomials in x of degree k with coefficients depending on $I_1,\dots,I_n$ and the leading term of x in (17) can be written down in the form

$$x^{s+k} \sum_{\substack{0\le i_1,\dots,i_{n-1}\le k\\ i_1+\dots+i_{n-1}=k}} C_{i_1,\dots,i_{n-1}}(I_1)^{i_1}\cdot\ldots\cdot(I_{n-1})^{i_{n-1}},$$

where $C_{i_1,\dots,i_{n-1}}$ are constants. It follows then from Theorem 2 that $s+k\le n-1$. Hence we obtain that $s_{i_1,\dots,i_{n-1}}\le n-1-k$, $k\le n-1$.

But then every summand in (17) is a polynomial in x of degree not greater then $n-1$ with coefficients of $I_1,\dots,I_n$ and, therefore, by Theorem 2 defines a symmetry of equation (8).

Hence we find that $\mathcal{F}_{n-2}$ can be considered as a linear span of the monomials

$$x^{i_0}\,(J_1)^{i_1}\cdot\ldots\cdot(J_{n-1})^{i_{n-1}}, \tag{18}$$

$$0\le i_0,i_1,\dots,i_{n-1}\le n-1, \quad i_0+i_1+\ldots+i_{n-1}\le n-1.$$

Calculating the number of such monomials, we obtain the following result.

Theorem 3. *The linear space $\mathcal{F}_{n-2}$ of symmetry operators of the submaximal order for equation (8),*

$$y^{(n)} = 0,$$

is spanned by the operators (18). The dimension of this space is given by

$$dim\,\mathcal{F}_{n-2} = C^n_{2n-1}.$$

In the following examples we apply Theorems 2 and 3 to the cases $n = 2,3,4$.

Example 1. In the case $n = 2$ equation (8) takes the form

$$y'' = 0, \tag{19}$$

and for the spaces $\mathcal{F}_1$ and $\mathcal{F}_0$ we obtain:

1) The whole Lie-Bäcklund algebra $\mathcal{F}_1$ of the equation (19) is generated by the operators

$$F = A^1(I_1, I_2) + A^2(I_1, I_2)x, \tag{20}$$

where A^1, A^2 are arbitrary functions of the first integrals

$$I_1 = y', \quad I_2 = xy' - y. \tag{21}$$

2) The set $\mathcal{F}_0$ is 3-dimensional ($C_3^2 = 3$) and is spanned by the operators

$$x^{i_0}(J_1)^{i_1},$$

$$0 \le i_0, i_1 \le 1, \quad i_0 + i_1 \le 1,$$

or, explicitly,

$$J_1, \quad x, \quad 1, \tag{22}$$

where

$$J_1 = I_2 - xI_1 = -y.$$

The operators obtained define a three-dimensional subalgebra of the algebra of Lie point symmetries of equation (19).

Example 2 [12]. Let $n = 3$, then equation (8) takes the form

$$y''' = 0 \tag{23}$$

and for the spaces $\mathcal{F}_2$ and $\mathcal{F}_1$, Theorems 2 and 3 yield:

1) The whole Lie-Bäcklund algebra $\mathcal{F}_2$ of the equation (23) is generated by the operators

$$F = A^1(I_1, I_2, I_3) + A^2(I_1, I_2, I_3)x + A^3(I_1, I_2, I_3)x^2, \tag{24}$$

where A^1, A^2, A^3 are arbitrary functions of the first integrals

$$I_1 = y'', \quad I_2 = xy'' - y', \quad I_3 = x^2y'' - 2xy' + 2y. \tag{25}$$

2) The space $\mathcal{F}_1$ (actually coinciding with the subalgebra of contact symmetries) is 10-dimensional ($C_5^3 = 10$), and is spanned by the operators

$$x^{i_0}(J_1)^{i_1}(J_2)^{i_2},$$

$$0 \le i_0, i_1, i_2 \le 2, \quad i_0 + i_1 + i_2 \le 2,$$

or, explicitly,

$$(J_1)^2, \quad (J_2)^2, \quad J_1J_2, \quad xJ_1, \quad xJ_2, \quad x^2, \quad ,x, \quad 1, \tag{26}$$

where

$$J_1 = I_2 - xI_1 = -y', \quad J_2 = I_3 - xI_2 = 2y - xy'. \tag{27}$$

Substituting the last expressions into (26) yields a basis of the well-known 10-dimensional algebra of contact symmetries of the equation (19) (see [7]):

$$F_1 = (2y - xy')^2, \quad F_2 = (y')^2, \quad F_3 = y'(2y - xy'),$$

$$F_4 = x(2y - xy'), \quad F_5 = 2y - xy', \quad F_6 = xy', \quad F_7 = y',$$

$$F_8 = x^2, \quad F_9 = x, \quad F_{10} = 1$$

(in Lie-Bäcklund canonical form).

Example 3. Let $n = 4$, then equation (8) takes the form

$$y^{IV} = 0. \tag{28}$$

The space $\mathcal{F}_3$ is described as above by Theorem 2 while Theorem 3 yields:

The space $\mathcal{F}_2$ of the operators of submaximal order is 35-dimensional ($C_7^4 = 35$) and is spanned by the operators

$$x^{i_0}(J_1)^{i_1}(J_2)^{i_2}(J_3)^{i_3}, \tag{29}$$

$$0 \leq i_0, i_1, i_2, i_3 \leq 3, \quad i_0 + i_1 + i_2 + i_3 \leq 3,$$

where

$$J_1 = I_2 - xI_1, \quad J_2 = I_3 - xI_2, \quad J_3 = I_4 - xI_3$$

and I_1, I_2, I_3, I_4 are given by

$$I_1 = y''', \quad I_2 = xy''' - y'', \quad I_3 = x^2y''' - 2xy'' + 2y',$$

$$I_4 = x^3y''' - 3x^2y'' + 6xy' - 6y. \tag{30}$$

Substituting (30) into the expressions for J_1, J_2, J_3, we obtain

$$J_1 = -y'', \quad J_2 = -xy'' + 2y', \quad J_3 = -x^2y'' + 4xy' - 6y. \tag{31}$$

3. Translation-invariant operators

In the context of application to evolution equations, it is interesting to obtain a description of symmetry operators for equation (8), which do not explicitly depend on x. Let us use the notation $\widehat{F}$ for such operators and denote by $\widehat{\mathcal{F}}_k$, $k \leq n-1$, the linear space of translation-invariant operators of the orders not greater than k. Using the general representation (10) for operators $F \in \mathcal{F}_{n-1}$ given in Theorem 2, we single out translation-invariant operators by the condition

$$\frac{\partial F}{\partial x} \equiv 0. \tag{32}$$

Example 5. Let us start with the case $n = 2$, i.e. the equation

$$y'' = 0.$$

Applying condition (32) to the operator (20), we obtain the identity

$$I_1 A^1_2 + A^2 + x I_1 A^2_2 = 0,$$

where we denote $A^k_i = \partial A^k / \partial I_i$. Equating to zero coefficients at different powers of x yields the system

$$I_1 A^1_2 = -A^2, \quad A^2_2 = 0,$$

possessing the solution

$$A^2 = I_1 A(I_1), \quad A^1 = -I_2 A(I_1) + B(I_1),$$

where $A(I_1)$ and $B(I_1)$ are arbitrary functions. Substituting these expressions into (20) we obtain

$$\widehat{F} = A(I_1)(x I_1 - I_2) + B(I_1)$$

or

$$\widehat{F} = y A(y') + B(y').$$

Example 6 [12]. Acting in a similar way in the case $n = 3$,

$$y''' = 0,$$

we obtain a general description of operators $\widehat{F}$ belonging to $\widehat{\mathcal{F}}_2$:

$$\widehat{F} = (y')^2 A(\hat{I}_1, \hat{I}_2) + y' A(\hat{I}_1, \hat{I}_2) + C(\hat{I}_1, \hat{I}_2),$$

where A, B, C are arbitrary functions of the expressions

$$\hat{I}_1 = I_1, \quad \hat{I}_2 = I_1 I_3 - I_2^2$$

in terms of the first integrals (25). Using (25) we find that

$$\hat{I}_1 = y'', \quad \hat{I}_2 \sim 2yy'' - (y')^2.$$

Generalizing these examples we arrive at the following result.

Theorem 4. *The set $\widehat{\mathcal{F}}_{n-1}$ of translation-invariant symmetry operators of the equation (8),*

$$y^{(n)} = 0,$$

is generated by the operators

$$\widehat{F} = \sum_{i=0}^{n-1} (y^{(n-2)})^i \alpha_i(\hat{I}_1, \ldots, \hat{I}_{n-1}),$$

where α_i are arbitrary functions of the homogeneous polynomials

$$\hat{I}_1 = I_1, \quad \hat{I}_2 = I_1 I_3 - I_2^2,$$

$$\hat{I}_k = I_1^{k-1} I_{k+1} - I_2^k - \sum_{i=1}^{k-2} C_k^i \hat{I}_{k-i} I_2^i, \quad 3 \leq k \leq n-1,$$

in the first integrals I_1, ... I_n (9).

Note that $\widehat{\mathcal{F}}_{n-1}$ is a subalgebra of the symmetry algebra $\mathcal{F}_{n-1}$. Applying this Theorem to the case $n = 4$, we obtain

Example 7. For equation (28),

$$y^{IV} = 0,$$

the set $\widehat{\mathcal{F}}_3$ consists of the operators

$$\widehat{F} = \sum_{i=0}^{3} (y'')^i \alpha_i(\hat{I}_1, \hat{I}_2, \hat{I}_3),$$

where α_i are arbitrary functions of the homogeneous polynomials

$$\hat{I}_1 = I_1, \quad \hat{I}_2 = I_1 I_3 - I_2^2,$$

$$\hat{I}_3 = I_1^2 I_4 - I_2^3 - 3\widehat{I}_2 I_2 = I_1^2 I_4 - 3I_1 I_2 I_3 + 2I_2^3.$$

The substitution of expressions (30) for I_1, I_2, I_3 yields

$$\hat{I}_1 = y''', \quad \hat{I}_2 = 2y'y''' - (y'')^2,$$

$$\hat{I}_3 = -6y(y''')^2 + 6y'y''y''' - 2(y'')^3.$$

4. Translation-invariant operators of submaximal order

In the conclusion we consider the spaces $\widehat{\mathcal{F}}_{n-2}$ of translation-invariant operators of the submaximal order for equations (8) with $n = 2, 3, 4$. Actually such operators belong to the intersection of the classes considered above.

Using the previous Examples 1-3, where the sets $\mathcal{F}_{n-2}$ were found, and applying the condition (32) we obtain the following description of the sets $\widehat{\mathcal{F}}_{n-2}$.

1) *Case* $n = 2$*:* $y'' = 0$. The linear space $\widehat{\mathcal{F}}_0$ is 2-dimensional, spanned by the operators

$$\widehat{F}_1 = 1, \quad \text{and} \quad \widehat{F}_2 = y$$

(it follows directly from (22)).

2) *Case* $n = 3$*:* $y''' = 0$. In this case the linear space $\widehat{\mathcal{F}}_1$ is 4-dimensional and is spanned by the operators

$$\widehat{F}_1 = 1, \quad \widehat{F}_2 = xJ_1 - J_2 \sim y, \quad \widehat{F}_3 = J_1 \sim y', \quad \widehat{F}_4 = J_1^2 = y'^2$$

(where J_1, J_2 are given by (27)).

3) *Case* $n = 4$*:* $y^{IV} = 0$. The linear space $\widehat{\mathcal{F}}_2$ is 9-dimensional and is spanned by the operators

$$\widehat{F}_1 = 1, \quad \widehat{F}_2 = x(xJ_1 - J_2) - (xJ_2 - J_3) \sim y,$$

$$\widehat{F}_3 = xJ_1 - J_2 \sim y', \quad \widehat{F}_4 = J_1 \sim y'',$$

$$\widehat{F}_5 = J_1^2 = y''^2, \quad \widehat{F}_6 = J_1(xJ_1 - J_2) \sim y'y'', \quad \widehat{F}_7 = J_2^2 - J_1 J_3 \sim yy'' - \frac{2}{3}y'^2,$$

$$\widehat{F}_8 = J_1^3 \sim y''^3, \quad \widehat{F}_9 = J_1(J_2^2 - J_1 J_3) \sim y''(yy'' - \frac{2}{3}y'^2)$$

(see (31)).

Notice that the operator $\widehat{F}_7$ from the Case 3 was first found in [6] where for the evolution equation

$$u_t = uu_{xx} - \frac{2}{3}(u_x)^2$$

(connected via the transformation $u = v^{-3/2}$ with the equation $v_t = (v^{-3/2}v_x)_x)$) the explicit solutions of the form

$$u = C_0(t) + C_1(t)x + C_2(t)x^2 + C_3(t)x^3$$

were constructed. A complete analysis of the translation-invariant quadratic operators of the second order possessing polynomial (and some other) invariant subspaces was done in [2].

Final remarks. In the present paper we deal with the following problem concerning nonlinear differential operators and their invariant subspaces:

given a finite-dimensional functional subspace, construct all operators (satisfying some additional conditions) leaving the subspace invariant.

For subspaces spanned by functions of one variable, a general representation of the whole set of operators was found in [14] ([12]) and is given by Theorem 1 (Section 1). An important generalization of this problem to finite-dimensional invariant subspaces of analytic functions of *several variables* was considered in [5], where a complete description of the set of operators possessing a given invariant subspace was obtained. Theorem 1 is included in this approach as a special case.

A survey of some results connected with application of invariant subspaces to evolution equations can be found in [13]. Exhaustive surveys of this and other methods of reduction of differential equations are given in [5, 9]

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