We describe a countable subset of blow-up patterns for the fourth-order one-dimensional semilinear parabolic equation

$$u_t = -u_{xxxx} + \beta[(u_x)^3]_x + e^u$$

with a parameter $\beta \geq 0$, which is a model equation from explosion-convection theory. We show that, unlike the classical Frank-Kamenetskii equation $u_t = u_{xx} + e^u$ (a solid fuel model), the generic blow-up in this fourth-order problem is described by a nontrivial similarity solution $u_*(x,t) = -\ln(T-t) + f_1(x/(T-t)^{1/4})$ ($T > 0$ is the blow-up time), with a non-trivial profile $f_1 \not\equiv 0$. The existence of the stable primary, f_1 and unstable secondary, f_2, non-trivial self-similar solutions are explained through the analysis of an associated linearized operator. Numerical solution of the PDE shows convergence to an exactly self-similar solution from a wide variety of initial data. A further countable subset of non-similarity blow-up patterns is constructed by linearization and matching with similarity solutions of a first-order Hamilton-Jacobi equation.

Blow-up in a fourth-order semilinear parabolic equation from explosion-convection theory

V. A. Galaktionov [1] and J. F. Williams [2]

[1] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK and Keldysh Institute of Applied Mathematics, Miusskaya Sq. 4, 125047 Moscow, RUSSIA*
email: vag@maths.bath.ac.uk
[2] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*
email: mapjfw@maths.bath.ac.uk

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1 Introduction

The role of self-similar solutions is well known in second-order reaction-diffusion equations which exhibit finite time blow-up. Because of their importance to applications many canonical equations such as the *semilinear* one-dimensional Frank-Kamenetskii equation [11] (the solid fuel model [34])

$$u_t = u_{xx} + e^u \tag{1.1}$$

and its counterpart with a power nonlinearity

$$u_t = u_{xx} + u^p, \quad p > 1,$$

have been well studied for for the past thirty years. While exact self-similar solutions are known to exist for second-order reaction-diffusion *quasilinear* problems [27], [3]

$$u_t = (|u_x|^\sigma u_x)_x + e^u \quad \text{or} \quad u_t = (u^\sigma u_x)_x + u^p \quad \text{with} \;\; \sigma > 0,$$

it has been shown that none exist for the above semilinear problems, where the asymptotic blow-up behaviour is described by approximate similarity solutions satisfying first-order Hamilton-Jacobi equations. Therefore, it is important to discover if semilinear fourth-order parabolic equations exhibit similar behaviour to their second-order counterparts and not possess exact self-similar solutions due to the *semilinear* structure of both problems.

In this paper we show that this is not the case for a model from explosion-convection theory [22]

$$u_t = -u_{xxxx} - [(2 - (u_x)^2)u_x]_x - \alpha u + q e^{su}, \tag{1.2}$$

where α, q and s are positive constants obtained from physical parameters. This structure is understood by examining a rescaled operator associated with a large solution reduction of this equation, $u_t = -u_{xxxx} - \beta[(u_x)^3]_x + e^u$ of which the fourth-order extension of the

Frank-Kamenetskii equation,

$$u_t = -u_{xxxx} + e^u, \tag{1.3}$$

is a limit case.

In the remainder of this Section we describe some of the structure of our model equation and introduce the relevant mathematical definitions and rescaled equations. In particular we show in what sense our reduction is reasonable. In Section 2 we present the properties of the underlying linearized operator which governs the dynamics of both equations.

In Section 3 we present formal analytical and numerical evidence which indicates that there are two non-trivial self-similar solutions to the reduced equation for different values of the parameter $\beta \geq 0$. Moreover, numerical simulations of both equations show that the "ground state", which is monotone decreasing in $|x|$, is asymptotically stable in the rescaled sense. The solution to the reduced equation captures the blow-up dynamics of the full model equation.

These results are in striking contrast with properties of blow-up for the second-order semilinear equations, where the generic singularity behaviour is only approximately self-similar, and is instead described by similarity solutions of the first-order Hamilton-Jacobi equations and is obtained by a matched asymptotic analysis, see references below.

In Section 4 we construct the rest of the countable subset of different blow-up patterns by using matched asymptotic expansions and spectral properties of non self-adjoint ordinary differential operators to be described in Section 2. Also in Section 4 we present numerical evidence of the existence of two self-similar solutions, and convergence of the PDE onto the ground-state solution.

1.1 The Semenov-Rayleigh-Benard problem

The fourth-order one-dimensional semilinear parabolic equation (1.2) occurs in the Semenov-Rayleigh-Benard problem [22], where the equation is derived in studying the interaction between natural convection and the explosion of an exothermically-reacting fluid confined between two isothermal horizontal plates. This is an evolution equation for the temperature fluctuations in the presence of natural convection, wall losses and chemistry. It can be considered as a formal combination of the equation derived in [18] (see also [5]) for the Rayleigh-Benard problem and of the Semenov-like energy balance [28], [12] showing that natural convection and the explosion mechanism may reinforce each other. In this statement, solutions $u = u(x,t)$ are 2π-periodic in x with bounded periodic initial data $u(x,0) = u_0(x)$ for $x \in I_0 = [0, 2\pi]$. Equation (1.2) can be treated as a higher-order generalization of the Frank-Kamenetskii equation (1.1).

Equation (1.2) is uniformly parabolic with all spatial differential operators appearing in divergence form. It admits a unique classical local in time solution, see the classical parabolic theory in [9], [13] and [30]. The operator on the right-hand side of (1.2)

$$\mathbf{N}(u) \equiv -u_{xxxx} - [(2-(u_x)^2)u_x]_x - \alpha u + q e^{su}$$

is potential and the equation admits the Lyapunov function

$$L[u](t) = \int_{I_0} \left[\frac{1}{2}(u_{xx})^2 - (u_x)^2 + \frac{1}{4}(u_x)^4 + \frac{\alpha}{2}u^2 - \frac{q}{s}e^{su} \right] dx,$$

which is monotone on bounded orbits,

$$\frac{d}{dt}L[u](t) = -\int_{I_0} (u_t)^2 dx \le 0.$$

For such smooth gradient systems, the ω-limit set of any bounded orbit $\{u(\cdot,t),\, t>0\}$,

$$\omega(u_0) = \{f \in C(I_0):\ \exists\ \{t_k\} \to \infty \text{ such that } u(\cdot,t_k) \to f \text{ uniformly}\},$$

is known to consist of stationary solutions: $\mathbf{N}(f) = 0$ in I_0 for any $f \in \omega(u_0)$. Some results on bifurcation of stationary solutions were obtained in [22]. If the subset of stationary solutions consists of isolated equilibria, the asymptotic behaviour of uniformly bounded orbits does not essentially differ from that for the second-order parabolic equation (1.1) where any bounded orbits are known to approach a stationary profile as $t \to \infty$.

1.2 Finite time blow-up solutions and similarity variables

One of the central concepts to reaction-diffusion equations is finite time blow-up (meaning explosion in combustion theory [34]), where the solution becomes unbounded at some time, $T > 0$ in the sense that $u(x,t)$ exists and is classical on any time-interval $[0,T']$ with $T' \in (0,T)$ and

$$\sup_x |u(x,t)| \to \infty \quad \text{as } t \to T^-. \tag{1.4}$$

The fact that solutions of higher-order parabolic equations may blow-up is well known, see the surveys [24], [17] and more recent results in [8], [6] for equations $u_t = -(-\Delta)^m u + f(u)$, where the critical Fujita exponent and estimates on blow-up rates were established for any $m > 1$. The proof of blow-up for the periodic initial value problem for (1.2) is straightforward.

Proposition 1.1 *Let $u(x,t)$ be a solution of* (1.2) *in $I_0 \times \mathbf{R}_+$ with periodic boundary conditions and bounded continuous initial data $u_0(x)$. (i) If*

$$\alpha < sqe^{1/2\pi} \tag{1.5}$$

then the solution blows up in finite time. (ii) If (1.5) *does not hold then the solution blows up if initial data is sufficiently large in the mean sense.*

Proof Denoting by $\bar{u}(t) = \int u(x,t)dx$ the mean of the solution on I_0 and integrating equation (1.2) over I_0, we obtain

$$\bar{u}' = q\int e^{su}dx - \alpha\bar{u}.$$

By Jensen's inequality for convex functions

$$\int e^{su}dx = 2\pi\int e^{su}\frac{1}{2\pi}dx \ge 2\pi e^{s\bar{u}/2\pi},$$

and we arrive at an ordinary differential inequality

$$\bar{u}' \ge F(\bar{u}) \equiv 2\pi q e^{s\bar{u}/2\pi} - \alpha\bar{u}. \tag{1.6}$$

If (1.5) holds, then $F>0$ in $\mathbf{R}$. Hence, $\bar{u}'>0$, $\bar{u}(t)>\bar{u}_0$ for $t>0$ and (1.6) implies that $\bar{u}(t)$ (and $u(x,t)$) blows up at

$$T \le T_0 = \int_{\bar{u}_0}^{\infty} ds/F(s).$$

(ii) If $\alpha \ge sqe^{1/2\pi}$, then similarly we have that blow-up occurs if $\bar{u}_0 > s_+$, where s_+ is the maximal root of equation $F(s)=0$. $\square$

Blow-up is an essential feature of explosion-convection problems and the corresponding parabolic equations under consideration. As for the solid fuel model (1.1), the structure of such a blow-up singularity formation is of importance in the present higher-order model. For the Frank-Kamenetskii equation (1.1), as well as for its counterpart with a power nonlinearity, the single point formation of blow-up singularities was known from the 1970's [21] and was rigorously established in the 1990's; see [2], [10], [20], [26], [31], [32] and references in [17].

Finite time blow-up involves a delicate balance between the spatial and temporal derivatives and the reaction terms driving the blow-up. This balance is made naturally apparent by considering the scaling invariance of the underlying PDE. This scaling structure is also important for the numerical methods employed in integrating the full PDE, see Section 4.

For convenience, we rescale $su \mapsto u$, $(qs)^{1/4}x \mapsto x$ and $qst \mapsto t$ to obtain the equation

$$u_t = \mathbf{A}(u) - \gamma u_{xx} - \delta u, \quad \gamma = 2/\sqrt{qs}, \;\; \delta = \alpha/qs, \tag{1.7}$$

where $\mathbf{A}$ is the operator

$$\mathbf{A}(u) = -u_{xxxx} + \beta[(u_x)^3]_x + e^u, \quad \beta = 1/s^2.$$

Here β is an essential parameter which cannot be removed by scaling. The physically admissible range of the parameter is $\beta \in (0,\infty)$, but we also include the limit case $\beta=0$ which formally corresponds to $s=\infty$ and leads to the fourth-order extended Frank-Kamenetskii equation (1.3).

We begin our similarity analysis with the following unperturbed equation:

$$u_t = \mathbf{A}(u) \equiv -u_{xxxx} + \beta[(u_x)^3]_x + e^u, \tag{1.8}$$

which is a reasonable simplification of (1.7) for large solutions as only the lowest order terms have been neglected. Equation (1.8) is invariant under the group of transformations

$$t \mapsto \lambda t, \quad x \mapsto \lambda^{1/4}x, \quad u \mapsto u - \ln\lambda, \quad \lambda > 0.$$

Without loss of generality, we assume that the solution $u(x,t)$ blows up at finite time $t=T$ in the sense of (1.4) and the blow-up set

$$B[u_0] = \{x \in I: \text{ there exist } \{x_k\} \to x, \; \{t_k\} \to T^- \text{ such that } u(x_k,t_k) \to \infty\}$$

contains the origin, $0 \in B[u_0]$. Motivated by this assumption, the scaling invariance of the problem and the notion of finite-time blow-up we introduce the following self-similar variables: $y = x/(T-t)^{1/4} : I_0 \to \mathbf{R}$ is the new spatial variable and $t = -\ln(T-t)$:

$(0,T) \to (\tau_0, \infty)$ with $\tau_0 = -\ln T$ is the new time variable and rescale (1.8) by setting

$$u(x,t) = -\tau + \theta(y,\tau), \tag{1.9}$$

giving

$$\theta_\tau = -\theta_{yyyy} + \beta[(\theta_y)^3]_y - \frac{1}{4} y\theta_y + e^\theta - 1 \equiv \mathbf{A}_1(\theta). \tag{1.10}$$

Using these new variables for the full equation (1.2) gives that the rescaled function θ satisfies the following perturbed parabolic equation:

$$\theta_\tau = \mathbf{A}_1(\theta) + \mathbf{C}(\theta,\tau), \tag{1.11}$$

where $\mathbf{A}_1$ is the autonomous operator above and $\mathbf{C}$ is a non-autonomous perturbation,

$$\mathbf{C}(\theta,\tau) = -\gamma e^{-\tau/2}\theta_{yy} - \delta e^{-\tau}(\tau+\theta),$$

which is exponentially small as $\tau \to \infty$ (i.e., as $t \to T^-$) on bounded orbits. It follows from the scaling variable y in (1.9) that for arbitrarily small fixed $|x| > 0$, the corresponding $|y| \to \infty$ as $\tau \to \infty$ which, in general, corresponds to formation of a single point singularity. Therefore, it is natural to consider the Cauchy problem for equation (1.11) with initial data at $\tau = \tau_0$

$$\theta_0(y) = u_0(T^{1/4}y) - \tau_0 \quad \text{in } \mathbf{R},$$

which are assumed to be bounded.

The perturbed equation (1.11) suggests that we consider first the unperturbed rescaled equation

$$g_\tau = \mathbf{A}_1(g) \quad \text{in } Q_0 = \mathbf{R} \times (\tau_0, \infty), \quad g(y,\tau_0) = g_0(y) \quad \text{in } \mathbf{R}. \tag{1.12}$$

According to (1.9), $g(y,\tau)$ is simply the rescaled solution of the following semilinear parabolic equation:

$$u_t = \mathbf{A}(u) \equiv -u_{xxxx} + \beta[(u_x)^3]_x + e^u.$$

This is a reasonable simplification of (1.7) for large solutions as only the lowest order terms have been neglected.

Any solution to (1.8) may be written as a solution to (1.10). However for only exactly (rather than asymptotically) self-similar solutions have θ independent of τ. Hence we begin with construction of blow-up self-similar solutions of (1.8) of the form

$$u_*(x,t) = -\ln(T-t) + f(y), \quad y = x/(T-t)^{1/4}, \tag{1.13}$$

where f satisfies the following ODE:

$$\mathbf{A}_1(f) = 0 \quad \text{in } \mathbf{R}_+ \tag{1.14}$$

supplemented with the symmetry conditions

$$f'(0) = 0, \quad f'''(0) = 0$$

at the origin. The far field behaviour is determined from the PDE by requiring that the limit profile $u_*(x,T^-)$ is bounded in a deleted neighbourhood of the origin $0 \in B[u_0]$. Specifically we have that for any $x > 0$ there exists a finite limit

$$\lim_{t\to T} u(x,t) = \lim_{t \to T}\left[-\ln(T-t) + f\left(\frac{x}{(T-t)^{1/4}}\right)\right].$$

Clearly this demands

$$f(y) = -4 \ln |y| + O(1) \qquad \text{as } y \to \infty.$$

As is typical for blow-up problems, the symmetry condition for stable (generic) blow-up profiles is connected with the idea of infinite time symmetrization (as $\tau \to \infty$) in parabolic equations like (1.10) and (1.11).

2 On spectral properties of the linearized operator

Consider the linearization of operator (1.10)

$$\mathbf{A}_1'(0) = \mathbf{B}^* + I, \quad \text{where } \mathbf{B}^* = -\frac{d^4}{dy^4} - \frac{1}{4} y \frac{d}{dy}, \tag{2.1}$$

I denotes the identity operator and $'$ is the Frechet derivative. In this section we will describe necessary spectral properties of the linear operator $\mathbf{B}^*$ and its formal adjoint differential expression

$$\mathbf{B} = -\frac{d^4}{dy^4} + \frac{1}{4} y \frac{d}{dy} + \frac{1}{4} I. \tag{2.2}$$

Note that both operators are not symmetric and do not admit a self-adjoint extension. We present some results from [8] and [14].

2.1 Fundamental solution

We begin with the fundamental solution of the corresponding linear fourth-order parabolic operator. Consider the linear equation

$$u_t = -u_{xxxx}. \tag{2.3}$$

The fundamental solution has the standard self-similar form

$$b(x,t) = t^{-1/4} F(\eta), \quad \eta = x/t^{1/4}.$$

Substituting $b(x,t)$ into (2.3) and taking the Fourier transform yields that the radially symmetric profile $F(\eta)$ is the unique even solution of the linear ODE

$$\mathbf{B} F = 0 \ \text{ in } \mathbf{R}.$$

Hence,

$$F(\eta) = \alpha \int_0^\infty e^{-s^4} \cos(s\eta) ds, \tag{2.4}$$

where α is chosen to normalize $\int F = 1$,

$$\alpha = \left(\int_0^\infty \int_0^\infty e^{-s^4} \cos(s\eta) \, ds \, d\eta \right)^{-1}.$$

The rescaled kernel $F(\eta)$ satisfies a standard pointwise estimate [9]

$$|F(\eta)| \le D e^{-d|\eta|^{4/3}} \quad \text{in } \mathbf{R},$$

where D and d are positive constants.

2.2 Discrete real spectrum and eigenfunctions of B

We study the spectrum $\sigma(\mathbf{B})$ in the weighted space $L^2_\rho(\mathbf{R})$ with the exponential weight

$$\rho(y) = e^{a|y|^{4/3}} > 0 \quad \text{in } \mathbf{R},$$

where $a \in (0, 2d)$ is a sufficiently small positive constant. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the corresponding inner product and the induced norm respectively.

We next introduce a Hilbert space of functions $H^4_\rho(\mathbf{R})$ with the inner product

$$\langle v, w\rangle_\rho = \int_{\mathbf{R}} \rho(y) \sum_{k=0}^{4} D^k v(y)\, \overline{D^k w(y)}\, dy,$$

where $D = d/dy$, and the norm

$$\|v\|^2_\rho = \int_{\mathbf{R}} \rho(y) \sum_{k=0}^{4} |D^k v(y)|^2\, dy.$$

Then $H^4_\rho(\mathbf{R}) \subset L^2_\rho(\mathbf{R}) \subset L^2(\mathbf{R})$ and $\mathbf{B}$ is a bounded linear operator from $H^4_\rho(\mathbf{R})$ to $L^2_\rho(\mathbf{R})$. Let us present its spectral properties.

Lemma 2.1 *The spectrum of* $\mathbf{B}$ *consists of real simple eigenvalues only*

$$\sigma(\mathbf{B}) = \{\lambda_k = -k/4,\ k = 0, 1, 2, ...\}. \tag{2.5}$$

The eigenfunctions

$$\psi_k(y) = \frac{(-1)^k}{\sqrt{k!}} D^k F(y) \tag{2.6}$$

form a complete subset in $L^2(\mathbf{R})$ *and in* $L^2_\rho(\mathbf{R})$. With F as defined in (2.4).

2.3 Polynomial eigenfunctions of the adjoint operator B*

We consider the formally adjoint operator (2.1) in the weighted space $L^2_{\rho^*}(\mathbf{R})$ ($(\cdot,\cdot)_*$ and $\|\cdot\|_*$ are the inner product and the norm) with the exponentially decaying weight function

$$\rho^*(y) \equiv \frac{1}{\rho(y)} = e^{-a|y|^{4/3}} > 0,$$

and ascribe to $\mathbf{B}^*$ the domain $H^4_{\rho^*}(\mathbf{R})$ dense in $L^2_{\rho^*}(\mathbf{R})$. Then

$$\mathbf{B}^* :\ H^4_{\rho^*}(\mathbf{R}) \to L^2_{\rho^*}(\mathbf{R})$$

is a bounded linear operator and $\mathbf{B}^*$ is adjoint to $\mathbf{B}$,

$$\langle \mathbf{B}v, w\rangle = \langle v, \mathbf{B}^* w\rangle \quad \text{for any } v \in H^4_\rho(\mathbf{R}),\ w \in H^4_{\rho^*}(\mathbf{R}).$$

Therefore, $\sigma(\mathbf{B}) = \sigma(\mathbf{B}^*)$ [23].

Lemma 2.2 *The eigenfunctions* $\psi^*_k(y)$ *of* $\mathbf{B}^*$ *are polynomials of order* k,

$$\psi^*_k(y) = \frac{1}{\sqrt{k!}} \sum_{j=0}^{\lfloor -\lambda_k \rfloor} \frac{1}{j!} D^{4j} y^k, \quad k = 0, 1, 2, ..., \tag{2.7}$$

and form a complete subset in $L^2_{\rho^*}(\mathbf{R})$. (Here $\lfloor \cdot \rfloor$ denotes the integer part.)

Integrating by parts, we have that the orthonormality condition holds

$$\langle \psi_k, \psi_l^* \rangle = \delta_{k,l} \quad \text{for any } \ k, l \geq 0, \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ is the duality product in $L^2(\mathbf{R})$ and $\delta_{\beta,\gamma}$ is the Kronecker delta. Operators $\mathbf{B}$ and $\mathbf{B}^*$ have zero Morse index (no eigenvalues with positive real parts are available).

3 Self-similar profiles and generic blow-up

Consider the unperturbed rescaled equation (1.12). We begin with the linearized stability analysis and describe invariant subspaces.

3.1 Invariant eigenspaces

Let us write down equation (1.12) in terms of the linearized operator (2.1),

$$g_\tau = (\mathbf{B}^* + I)g + \mathbf{D}(g) \tag{3.1}$$

with the nonlinear operator

$$\mathbf{D}(g) = \mathbf{A}_1(g) - (\mathbf{B}^* + I)g \equiv \beta[(g_y)^3]_y + e^g - (1+g).$$

On solutions $g(\cdot, \tau) \in H^4_{\rho^*}(\mathbf{R}^N)$ the perturbation is quadratic,

$$\mathbf{D}(g) = \frac{1}{2} g^2 + O(\|g\|^3_{\rho^*}) \quad \text{as } \ \|g\|_{\rho^*} \to 0. \tag{3.2}$$

In what follows we restrict our attention to symmetric in x solutions $u = u(|x|, t)$ and hence to symmetric in y rescaled solutions $\theta = \theta(|y|, \tau)$ and $g = g(|y|, \tau)$. In the space $L^2_{0,\rho^*}(\mathbf{R})$ of symmetric functions, in view of (2.5),

$$\sigma(\mathbf{B}^* + I) = \{\tilde{\lambda}_k = 1 - k/4, \ k = 0, 2, 4, ...\}. \tag{3.3}$$

By completeness in Lemma 2.2, we have that

$$L^2_{0,\rho^*}(\mathbf{R}) = E^u(0) \oplus E^c(0) \oplus E^s(0),$$

where $E^u(0)$, $E^c(0)$ and $E^s(0)$ are the unstable, centre and stable subspaces of $\mathbf{B}^* + I$,

$$E^u(0) = \text{Span}\{\psi_0^*, \psi_2^*\}, \quad E^c(0) = \text{Span}\{\psi_4^*\}, \quad E^s(0) = \text{Span}\{\psi_6^*, \, \psi_8^*, ...\}.$$

Consider two one-dimensional unstable subspaces corresponding to positive eigenvalues

$$\tilde{\lambda}_0 = 1, \ \psi_0^*(y) = 1 \quad \text{and} \quad \tilde{\lambda}_2 = 1/2, \ \psi_2^*(y) = y^2/\sqrt{2}.$$

As usual in blow-up problems, the first unstable mode with $k = 0$ corresponds to the instability of blow-up behaviour with respect to perturbations of the blow-up time T. The second mode with $k = 2$ then describes an actual instability of the trivial solution $g \equiv 0$ in the space of rescaled solutions having the same fixed blow-up time T. We expect such orbits are uniformly bounded in Q_0, i.e., equation (1.12) does not admit symmetric global solutions which become unbounded as $\tau \to \infty$.

Consider now the centre subspace $E^c(0)$ corresponding to

$$\tilde{\lambda}_4 = 0 \quad \text{with} \ \psi_4^*(y) = \frac{1}{\sqrt{24}}(y^4 + 24). \tag{3.4}$$

Let us present a simple calculation showing that the behaviour on the centre manifold is semistable. In view of known spectral and sectorial properties of operators $\mathbf{B}$, $\mathbf{B}^*$ [8], [14], we expect that the centre (and stable, see the next section) manifold behaviour can be justified by the stable invariant manifold theory in interpolation spaces; see [25], Chapter 9.

Proposition 3.1 *Let $g(\cdot,\tau) \in H^4_{0,\rho^*}(\mathbf{R})$ exhibit the centre subspace dominance,*

$$g(\cdot,\tau) = a_4(\tau)\psi_4^*(\cdot) + w(\cdot,\tau) \quad \text{for } \tau \gg 1, \tag{3.5}$$

where $w(\cdot,\tau) = o(\|g(\cdot,\tau)\|_{\rho^*}) = o(|a_4(\tau)|)$ as $\tau \to \infty$. Then

$$a_4(\tau) = \frac{1}{\gamma_0 \tau}(1 + o(1)) > 0 \quad \text{as } \tau \to \infty, \ \text{where } \gamma_0 = -\frac{1}{2}\langle(\psi_4^*)^2, \psi_4\rangle = 68\sqrt{6}. \tag{3.6}$$

It follows from (3.6) that $a_4(\tau)$ cannot be negative in any neighbourhood of $\tau = \infty$ meaning a one-sided (from below) instability of the centre manifold behaviour.

Proof Looking for a solution of (3.1) in the form of eigenfunction expansion

$$g(\cdot,\tau) = \sum a_k(\tau)\psi_k^*(\cdot),$$

substituting it into equation (3.1) and multiplying by ψ_k in $L^2(\mathbf{R})$, we arrive at the dynamical system on the coefficients

$$\dot{a}_k = \tilde{\lambda}_k a_k + \langle \mathbf{D}(g), \psi_k \rangle, \quad k = 0, 2, \ldots .$$

Consider the equation for the coefficient a_4 with $\tilde{\lambda}_4 = 0$. In view of assumption (3.5) and (3.2), assuming that $|a_4| \ll 1$, we obtain that

$$\dot{a}_4 = -(\gamma_0 + o(1))a_4^2 \quad \text{for } \tau \gg 1. \tag{3.7}$$

We calculate $\gamma_0 > 0$ by using $\psi_4 = 24^{-1/2}D^4F$ and the adjoint eigenfunction in (3.4). Integrating (3.7) as a standard ODE, we obtain that any small solution for $\tau \gg 1$ has the asymptotic behaviour (3.6). □

It follows from the quadratic "ODE" (3.7) that the centre manifold behaviour exhibits a typical semistable "saddle-node" structure.

3.2 Two self-similar profiles

In Figures 1 and 2 we present the results of numerical solution of the ODE (1.14) for different values of the parameter β. They show that there exist exactly two different solutions $f_1(y)$ and $f_2(y)$ including the case $\beta = 0$, i.e., the fourth-order parabolic equation (1.3) admitting two non-trivial blow-up similarity solutions (1.13). Figure 1 shows the structure local to the origin while Figure 2 shows the expected far field behaviour. The solutions for were obtained using a collocation code which guarantees a residual tolerance [29]. Symmetry conditions were imposed at the origin and minimal growth was enforced at the far field.

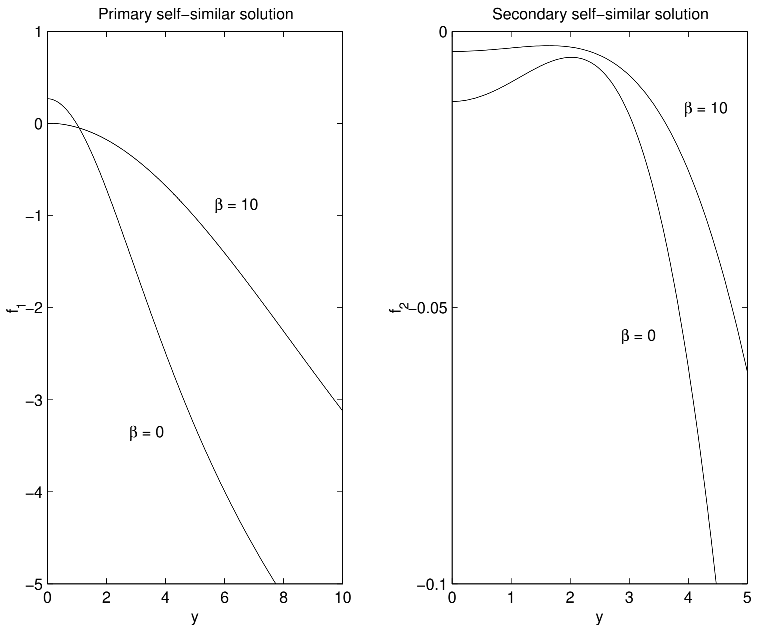

FIGURE 1. Solutions to the ODE describing exact self-similar solutions

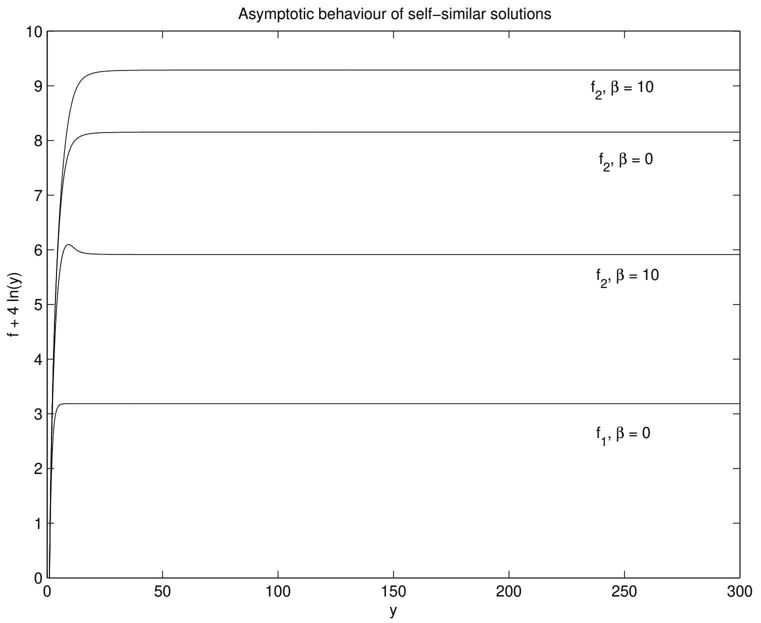

FIGURE 2. Far-field behaviour of solutions to the self-similar ODE

Existence of two similarity profiles is related to the above analysis of unstable and centre manifold behaviour. We expect that in the evolution equation (1.10) there exists an evolution equilibrium connection between the trivial stationary frofile $f \equiv 0$ and f_1 via the second unstable mode with $k = 2$ in (3.3), $\psi_2(y) = y^2/\sqrt{2}$. Another equilibrium connection from $f \equiv 0$ and the second profile f_2 is expected to be governed by the unstable centre manifold behaviour (see Proposition 3.1) corresponding to $k = 4$ and the eigenfunction in (3.4).

Basic properties of connecting equilibria and transversality of intersections of the corresponding stable and unstable manifolds are known for the one-dimensional second-order parabolic equations

$$u_t = u_{xx} + f(x,u) \quad \text{in } (0,1) \times \mathbf{R}_+, \quad u = 0 \text{ at } x = 0, 1 \text{ for } t > 0.$$

See [19], [1] and [7], where the results are obtained by using Sturm's Theorem for the non-increase of the number of zeros (intersection) of solutions to linear second-order parabolic equations. This Sturmian property is not true for the fourth and higher-order parabolic equations and in general the structure of connecting orbits remains an important open problem.

Let us state the following conjecture motivated by a number of numerical experiments.

Conjecture 3.1. *For any $\beta \geq 0$, the problem* (1.14), (1.10) *has precisely two solutions f_1 and f_2.*

It is worth mentioning that the fourth-order semilinear parabolic equation with a power nonlinearity

$$u_t = -u_{xxxx} + |u|^{p-1}u, \quad p > 1,$$

also admits two nontrivial self-similar blow-up solutions [4].

3.3 Generic blow-up

We conjecture that the first similarity profile $f_1(y)$ describes the stable generic blow-up for general solutions of unperturbed equation (1.8) and the perturbed original one (1.7). In Figure 3 we show typical profiles of the evolution, showing the convergence to profiles localized at the origin. Because the blow-up time T is not known a-priori we cannot immediately compare the solutions to the expected asymptotic profiles. To reconstruct the convergence onto the self-similar solution from time integrations of the PDEs (1.8) and (1.2) we rescale under (1.9) in the following way. At each time t we define $\lambda = \exp(u(0,t))$ and set

$$\theta(y,\tau) = -\ln\lambda + u(x\lambda^{1/4}, t) \quad \text{where } \tau = \frac{1}{\lambda}.$$

Figure 4 presents a typical example of convergence of the rescaled solution $\theta(y,\tau)$ (shown with solid lines) to the first similarity profile $f_1(y)$ (shown with a dashed line).

The numerical simulations of the PDEs were done using collocation on a moving grid, see [33] and the references therein. This method moves the grid points to track the solution so as to follow any scalings in the problem. This allows integration of problems whose scale changes dramatically over the time period of interest. In this problem, while the computed solution is never large, it is localized at the origin and care must be taken

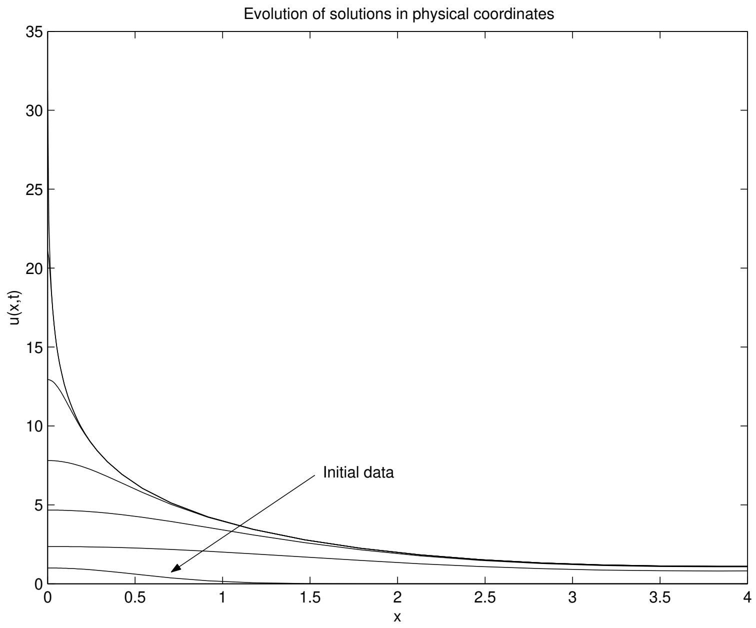

FIGURE 3. Finite time blow-up

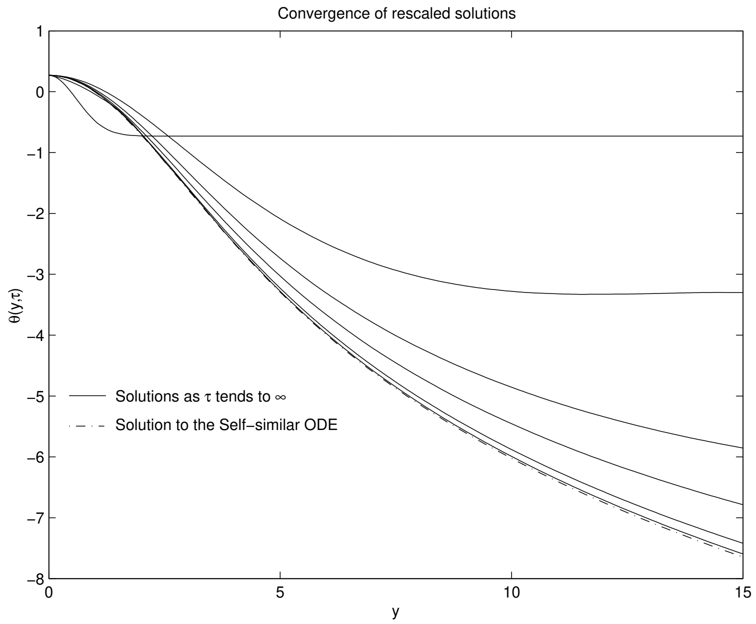

FIGURE 4. Convergence of rescaled profiles

to resolve this fine structure. In the final blow-up limit $(u(0,T^-) = 32, e^u \sim 10^{13})$ the grid spacing varies over five orders of magnitude using only 20 grid points.

4 The countable spectrum of blow-up patterns

We continue our study of blow-up phenomena in equations (1.8) and (1.7) by constructing the rest of countable subset of blow-up patterns generated by the centre and stable subspaces of the linearized operator $\mathbf{B}^* + I$ in (3.1). This analysis is similar to that performed in [14] for the semilinear equation

$$u_t = (-1)^{m+1} D_x^{2m} u + |u|^{p-1} u \quad \text{in } \mathbf{R} \times (0,T), \quad p > 1,$$

for arbitrary order $2m \geq 4$.

4.1 Centre manifold pattern: solutions taking infinite values on moving boundaries

By virtue of Proposition 3.1 and assuming the asymptotic behaviour (3.6), on any compact subset $\{|y| \leq c\}$, $c > 0$, the rescaled solution has the form

$$g(y,\tau) = \frac{1}{\gamma_0 \tau} \frac{1}{\sqrt{24}} (y^4 + 24) + o(\frac{1}{\tau}) > 0 \quad \text{as } \tau \to \infty. \tag{4.1}$$

We then introduce the new rescaled variable

$$\zeta = y/\tau^{1/4} \equiv x/[(T-t)|\ln(T-t)|]^{1/4}, \tag{4.2}$$

so that (4.1) implies that as $\tau \to \infty$,

$$g = \gamma_1 \zeta^{2m} (1 + o(1)) > 0 \quad \text{for small } \zeta > 0, \quad \gamma_1 = 1/\sqrt{24}\gamma_0 = 1/816. \tag{4.3}$$

This extra rescaling forms a remarkable spatial variable (4.2) with an additional logarithmic factor. For the second-order heat equations like (1.1) a similar variable occurs with the exponent 1/2 instead of 1/4. The idea of such non scaling invariant hot-spot variable goes back to the beginning of the 1970's [21], where the equation $u_t = u_{xx} + u^3$ was studied.

Applying the scaling (4.2) in equation (1.12), we deduce that $g = g(\zeta, \tau)$ satisfies the following perturbed equation:

$$g_\tau = \mathbf{H}_4(g) + \frac{1}{\tau} \mathbf{F}(g) \quad \text{for } \tau > \tau_0, \tag{4.4}$$

with the first-order operator

$$\mathbf{H}_4(g) = -\frac{1}{4} \zeta g_\zeta + e^g - 1, \quad \text{and} \quad \mathbf{F}(g) = -g_{\zeta\zeta\zeta\zeta} + \beta[(g_\zeta)^3]_\zeta - \frac{1}{4} \zeta g_\zeta.$$

As $\tau \to \infty$, (4.4) is an asymptotically small singular perturbation of order $O(1/\tau)$ (it is of crucial importance that the perturbation is not integrable, $1/\tau \notin L^1((1,\infty))$) of the first-order Hamilton-Jacobi equation

$$h_\tau = \mathbf{H}_4(h). \tag{4.5}$$

Such singularly perturbed dynamical systems occur in several reaction-diffusion equations, see [15], where a general stability theorem is available. The main hypothesis of this stability approach is the uniform Lyapunov stability of the ω-limit set of the unperturbed equation (4.5) established in [16], Sect. 3, for general equations like (4.5) in a

Banach space $C_\rho(\mathbf{R}_+)$ with a singular weight. For such higher-order parabolic problems compactness of rescaled orbits is a difficult problem and the analysis below is formal.

Assuming that we can pass to the limit in the singularly perturbed equation (4.4), we have that the orbit approaches the stationary profiles satisfying

$$\mathbf{H}_4(f) = 0 \quad \text{for} \ \zeta > 0, \quad f(0) = 0.$$

Integrating this ODE in the class of nonnegative functions $f \geq 0$, which is one of the matching conditions due to the positivity of expansions (4.1) and (4.3), we obtain a one-parameter family of the limit profiles,

$$f(\zeta) = -\ln(1 - A\zeta^4) \quad \text{with a parameter} \ A \geq 0. \tag{4.6}$$

Since $f(\zeta) = A\zeta^4 + O(\zeta^8)$ as $\zeta \to 0$, comparing with (4.3) for the intermediate values of ζ yields the unique stable profile (4.6), $f_*(\zeta)$, with the constant $A = \gamma_1$.

Thus, in terms of the new rescaled variable (4.2) with an extra logarithmic factor, the centre manifold behaviour is governed by a unique stationary solution of the Hamilton-Jacobi equation (4.5):

$$f_*(\zeta) = -\ln(1 - \gamma_1\zeta^4).$$

This means that in the original variables (x, t), the corresponding blow-up pattern takes the asymptotic form

$$u_4(x,t) = -\ln(T-t) - \ln(1 - \frac{1}{816}\zeta^4) + ..., \quad \zeta = x/[(T-t)|\ln(T-t)|]^{1/4},$$

i.e., it blows up as $t \to T$ in a shrinking domain $\{|x| \leq 4\sqrt{3}[(T-t)|\ln(T-t)|]^{1/4}\}$ and $u_* = +\infty$ on its lateral boundary. Such an infinite Dirichlet condition on moving boundaries is a typical feature for fourth-order PDEs like (1.2), (1.7) or (1.3) (unlike the second-order heat equation (1.1) where such initial-boundary value problems are not locally solvable in general in natural functional classes). Note that the ODE (1.14) admits solutions $f(y)$ blowing up as $y \to y_0 \pm 0$ for any finite y_0 with the singularity given in the first approximation by the equation $f'''' = e^f$.

4.2 Stable manifold patterns

Let us describe the rest of blow-up patterns associated with stable subspaces of the linearized operator. The patterns on the stable manifold tangent to $E^s(0)$ are generated by the corresponding eigenfunctions with the following asymptotic behaviour of solutions of (3.1) for $\tau \gg 1$: on compact subsets

$$g(y,\tau) = Ce^{(1-k/4)\tau}\psi_k^*(y) + ..., \quad k = 6, 8, ... \quad (\text{hence} \ \tilde{\lambda}_k = 1 - k/4 < 0), \tag{4.7}$$

where $C \neq 0$ depends on the initial data. The adjoint eigenfunctions ψ_k^* given by (2.7), do not change sign. Therefore, as we will show, positive C in (4.7) correspond to blow-up patterns in shrinking domains as $t \to T$, while negative values of C generate blow-up patterns which are well-defined in $\mathbf{R} \times (0, T)$.

We concentrate on negative C and set $C \mapsto -C$. Then (4.7) yields for large y as $\tau \to \infty$,

$$g(y,\tau) = -Be^{(1-k/4)\tau}y^k(1 + o(1)) + ... \equiv -B\zeta^k + ..., \ |\zeta| \ll 1, \ B = C/\sqrt{k!} > 0, \tag{4.8}$$

where ζ is the new spatial variable

$$\zeta = ye^{-\tau(k-4)/4k} \equiv |x|/(T-t)^{1/k}, \quad k = 6, 8, \dots .$$

Then $g = g(\zeta, \tau)$ satisfies the exponentially perturbed equation

$$g_\tau = \mathbf{H}_k(g) - e^{-(1-4/k)\tau} \left\{ -g_{\zeta\zeta\zeta\zeta} + \beta[(g_\zeta)^3]_\zeta \right\}, \tag{4.9}$$

with the Hamilton-Jacobi operator

$$\mathbf{H}_k(g) = -\frac{1}{k}\,\zeta g_\zeta + e^g - 1.$$

Using the same stability arguments, similar to the above centre case $k = 4$, we conclude that, under necessary compactness hypotheses, the orbit $\{g(\cdot, \tau)\}$ approaches the stationary subset of the unperturbed equation equation (4.9) corresponding to $\tau = \infty$. Solving the ODE $\mathbf{H}_k(f) = 0$ in the class of nonpositive solutions, we obtain the family

$$f(\zeta) = -\ln(1 + A\zeta^k) \quad \text{with a parameter} \quad A \geq 0.$$

Matching expansion $f(\zeta) = -A\zeta^k + O(\zeta^{2k})$ as $\zeta \to 0$ with (4.8), we obtain

$$A = B = C/\sqrt{k!}\,.$$

We thus obtain the following approximate representation of the stable manifold pattern on compact subsets in ζ:

$$u_k(x,t) = -\ln[(T-t)(1 + A\zeta^k)] + \dots, \quad \zeta = |x|/(T-t)^{1/k}.$$

Passing to the limit $t \to T$, we derive a countable subset of final-time profile created at $t = T$ via such a blow-up evolution

$$u_k(x, T^-) = -k\ln|x| - \ln A + \dots \quad \text{as } x \to 0, \quad k = 6, 8, \dots .$$

5 Conclusions

In this paper we have described the complete spectrum of blow-up patterns in a model from convection-explosion theory. We have found that contrary to results for a number of similar second-order semilinear problems there exist two self-similar solutions to a reduced model. Moreover, the primary self-similar solution is stable in a rescaled sense in the full equation as well. The existence of these solutions is understood through close inspection of an associated linear operator. Lastly we note that this phenomenon is robust, occurring in many higher-order equations [4].

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