CRITICAL GLOBAL ASYMPTOTICS IN HIGHER-ORDER SEMILINEAR PARABOLIC EQUATIONS

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Abstract. We consider a higher-order semilinear parabolic equation

$$u_t = -(-\Delta)^m u - g(x,u) \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \ \ m > 1.$$

The nonlinear term is homogeneous: $g(x,su) \equiv |s|^{P-1}s\, g(x,u)$ and $g(sx,u) \equiv |s|^Q\, g(x,u)$ for any $s \in \mathbf{R}$, with exponents $P > 1$, $Q > -2m$. We also assume that g satisfies necessary coercivity and monotonicity conditions for global existence of solutions with sufficiently small initial data. The equation is invariant under a group of scaling transformations. We show that there exists a critical exponent $P = 1 + (2m+Q)/N$ such that the asymptotic behaviour as $t \to \infty$ of a class of global small solutions is not group-invariant and is given by a logarithmic perturbation of the fundamental solution $b(x,t) = t^{-N/2m} f(xt^{-1/2m})$ of the parabolic operator $\partial/\partial t + (-\Delta)^m$, so that for $t \gg 1$,

$$u(x,t) = C_0(\ln t)^{-N/(2m+Q)}[b(x,t) + o(1)],$$

where C_0 is a constant depending on m, N and Q only.

1. Introduction: main results on critical global asymptotics

The main goal of the paper is to present a class of nonlinear higher-order parabolic equations with two homogeneous operators

$$u_t = \mathbf{A}u + \mathbf{G}(u) \quad \text{for } t > 0, \quad u(0) = u_0,$$

which are invariant relative to a group of scaling transformations, but generic global asymptotics of solutions as $t \to \infty$ are not invariants. The basic example is a semilinear $2m$-th order parabolic equation in the critical case, where a special nonlinear interaction between operators produces asymptotics perturbed by logarithmic factors. Such perturbed asymptotics are well known for the second-order ($m = 1$) semilinear and quasilinear heat equations and were studied in detail last two decades.

1.1. Statement of the asymptotic problem with critical nonlinearity.

Consider the Cauchy problem for the $2m$-th order ($m > 1$) semilinear parabolic equation

$$u_t = -(-\Delta)^m u - g(x,u) \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad u(x,0) = u_0(x) \ \text{in } \mathbf{R}^N, \tag{1.1}$$

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where Δ denotes the Laplace operator in $\mathbf{R}^N$ and $u_0 \in X = L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. The nonlinear term $g(x,u)$ on the right-hand side is assumed to be smooth and sufficiently small as $u \to 0$, $g = g'_u = 0$ at $u = 0$. Higher-order semilinear and quasilinear diffusion operators occur in applications in thin film theory, nonlinear diffusion and lubrication theory, flame and wave propagation, phase transition at critical Lifschitz points and bi-stable systems (e.g., the Kuramoto-Sivashinskii equation and the extended Fisher-Kolmogorov equation). See models and references in [30].

We are going to describe a class of equations (1.1) admitting a non-standard, logarithmically perturbed asymptotic behaviour as $t \to \infty$. In order to guarantee the existence of global solutions, without loss of generality of the asymptotic technique to be applied, we assume that the perturbation term g is a sufficiently smooth lower-order operator generated by linear differential or integral (nonlocal) operators and it is well-defined to ensure existence of a local solution obtained by means of application of the continuous semigroup generated by $-(-\Delta)^m$. We impose the sign (coercivity) condition

$$g(x,u)\,u \geq 0 \quad \text{for any } \; u \in \mathbf{R} \; \text{ and } \; x \in \mathbf{R}^N. \tag{1.2}$$

Then the lower-order term is of the same sign as the diffusion operator and plays a role of an absorption-like operator ensuring boundedness of the orbits. Multiplying equation by u and integrating by parts, this guarantees global *a priori* estimates on $u(\cdot,t)$ in $L^2(\mathbf{R}^N)$ and in $L^2(H^m(\mathbf{R}^N) : \mathbf{R}_+)$, and semigroup techniques apply to give global solutions, see [29], [33] and [32], where detailed assumptions on nonlinearities are stated. In typical applications, g is a monotone operator satisfying in its domain $\langle g(u) - g(v), u - v\rangle \geq 0$, where $\langle \cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbf{R}^N)$. Let us introduce the crucial assumption on the *critical nonlinearity* ensuring existence of special non-invariant global asymptotics.
Critical scaling hypothesis. We assume that $g(x,u)$ is homogeneous in both variables: for all $s \in \mathbf{R}$, $x \in \mathbf{R}^N$ and $u \in \mathbf{R}$,

$$g(x,su) \equiv |s|^{P-1}s\, g(x,u) \quad \text{and} \quad g(x,su) \equiv |s|^Q\, g(x,u), \tag{1.3}$$

with exponents $P > 1$ and $Q > -2m$. Then the *critical exponent* is given by

$$P = P_c \equiv 1 + (2m+Q)/N. \tag{1.4}$$

Inequality $P > 1$ implies that $g'_u(x,0) \equiv 0$. In the classical case of the homogeneous algebraic nonlinearity $g(u) = |u|^{p-1}u$ with exponent $p > 1$, where the semilinear equation with absorption takes the form

$$u_t = -(-\Delta)^m u - |u|^{p-1}u \quad \text{in } \; \mathbf{R}^N \times \mathbf{R}_+, \tag{1.5}$$

we have $P = p$, $Q = 0$ and the critical exponent is $P_c = p_c = 1 + 2m/N$.
Remark: the same Fujita exponent for a blow-up problem. The same exponent $p_c = 1 + 2m/N$ is the *critical Fujita* one for the semilinear equation with source

$$u_t = -(-\Delta)^m u + |u|^p \quad \text{in } \; \mathbf{R}^N \times \mathbf{R}_+, \tag{1.6}$$

but in the different sense: any solution $u \not\equiv 0$ with arbitrarily small initial data satisfying $\int u_0 \geq 0$ blows up in finite time [9], [18]. For (1.6), p_c is exactly the critical case, where the trivial stationary solution $u \equiv 0$ loses its stability (for $p > p_c$ it is stable and is unstable for $p \le p_c$). For (1.5), $u \equiv 0$ is stable for any $p > 1$ and bounded solutions are always global, but their asymptotic behaviour is different in the subcritical range $p < p_c$ and in the supercritical one $p > p_c$. It is worth mentioning that for any $p > 1$, equation (1.6) admits blow-up patterns corresponding to the evolution on the centre manifold with a non-invariant logarithmically perturbed behaviour [13].

For the homogeneous function g containing a first-order gradient nonlinearity, we have

$$g(u) = |u|^{p_0-1}u\,|\nabla u|^{p_1},\ \ p_0, p_1 > 1 \implies P = p_0 + p_1,\ Q = -p_1. \tag{1.7}$$

For a general l-th order differential operator

$$g(x,u) = |x|^{\sigma}|u|^{p_0-1}u|D_x u|^{p_1}...|D_x^l u|^{p_l}, \quad \sigma > -N,\ p_k > 1,\ k = 0, ..., l < 2m, \tag{1.8}$$

where $D_x^k u = \{\partial_x^\beta u : \ |\beta| = k\}$, $\beta = (\beta_1, ..., \beta_N)$ is a multiindex, $|\beta| = \beta_1 + ... + \beta_N$,

$$P = \sum_{k=0}^{l} p_k, \quad Q = \sigma - \sum_{k=0}^{l} kp_k. \tag{1.9}$$

As a further example, we put in (1.5) an extra nonlocal multiplier via the norm in $L^q(\mathbf{R}^N)$,

$$g(x,u) = |u|^{p_0-1}u\left(\int_{\mathbf{R}^N} |u|^q dx\right)^{\nu/q},\ \nu > 1,\ q \ge 1 \implies P = p_0 + \nu,\ Q = N\nu/q. \tag{1.10}$$

1.2. **Main result.** We will show that in the critical case $P = P_c$, under certain assumptions of g and initial data, there exist small global solutions with the following asymptotic behaviour as $t \to \infty$:

$$u(x,t) = \pm C_0 t^{-N/2m}(\ln t)^{-N/(2m+Q)}\left[f\left(x/t^{1/2m}\right) + o(1)\right], \tag{1.11}$$

where f is the rescaled kernel of the fundamental solution of the linear parabolic operator (see the next section). It is important that the constant $C_0 \ne 0$ depends on the parameters m, N and Q and is independent of initial data. In the critical case the asymptotic behaviour of global solutions is given by the fundamental solution perturbed by a slow decaying logarithmic factor. This behaviour corresponds to the evolution on the local centre manifold of the nonlinear operator in the rescaled equation to be introduced in Section 4. It is a generic behaviour, though there exist other types of exponentially decaying rescaled patterns on the stable manifold.

The main result for $P = P_c$ is proved in Section 4, where we also discuss possible generic behaviour in the supercritical $P > P_c$ and subcritical $P \in (1, P_c)$ ranges.

The phenomenon of critical non-invariant asymptotics is expected to exist for a wider class of evolution equations including quasilinear higher-order parabolic equations. Nevertheless, it is not easy to specify such reasonable well-posed quasilinear models. Indeed, the invariant scaling hypothesis on the diffusion term implies that the homogeneous quasilinear operator is degenerate, e.g., $-(-\Delta)^m(|u|^\sigma u)$ with $\sigma > 0$. Uniqueness and regularity results for such degenerate equations (necessary for applying centre manifold techniques) as well as a detailed analysis of the fundamental instantaneous source-type solutions of the unperturbed equations and spectral properties of the corresponding linear non self-adjoint operators remain open for higher-order equations with $m > 1$, unlike the second-order equations, which we discuss in a small survey below. We expect that a number of higher-order degenerate quasilinear equations with critical absorption exponents, for which proper solutions can be constructed by regularization, can admit similar non-invariant asymptotics. For instance, the phenomenon of critical non-invariant asymptotic behaviour as $t \to \infty$ is expected to exist for non-negative solutions of the well-posed thin film equation with critical absorption in $\mathbf{R} \times \mathbf{R}_+$

$$u_t = -(u^n u_{xxx})_x - u^p \quad \text{with } n > 4 \text{ and } p_c = n - 3 > 1.$$

1.3. **Critical behaviour in nonlinear heat equations: a short survey.** For the canonical semilinear heat equation with $m = 1$

$$u_t = \Delta u - u^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+ \ (u \geq 0), \tag{1.12}$$

the asymptotic behaviour in the critical exponent $p_c = 1 + 2/N$ was established in [15], [16]. For positive L^1 initial data $u_0(x)$ with exponential decay as $|x| \to \infty$, it was proved to be given by the logarithmically perturbed Gaussian kernel

$$u(x,t) = C_0 (t \ln t)^{-N/2} [e^{-|x|^2/4t} + o(1)] \quad \text{as } t \to \infty \text{ uniformly,} \tag{1.13}$$

where C_0 is calculated explicitly $C_0 = (N/2)^{N/2} (1 + 2/N)^{N^2/2}$. Compactness of the rescaled orbits was proved by constructing suitable super and subsolutions having a structure similar to (1.13) (an estimate from below was earlier obtained in [23]), and the uniqueness of the rescaled limit (uniqueness of the constant C_0) was proved by a special energy analysis of the rescaled equation.

A principle generalization was obtained in [4], [5], where such logarithmically perturbed asymptotics were justified by a perturbation analysis of linearized second-order self-adjoint operator and, hence, were shown to exist for a wide class of second-order semilinear evolution equations.

Contracted by $\ln t$ asymptotics can occur in (1.12) by means of a different phenomenon. For instance, for $p > p_c$, assuming that the initial function has a critical asymptotics $\sim |x|^{-N}$ as $|x| \to \infty$ [25]. It is a phenomenon of interaction of the Laplacian with initial data (having a critical behaviour at infinity), which follows from the convolution $b(t) * u_0$.

The critical asymptotic behaviour of nonnegative solutions exists for second-order quasilinear heat equations like the porous medium equation (PME) with absorption

$$u_t = \Delta |u|^\sigma u - |u|^{p-1} u \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad \sigma > 0, \ p > 1, \tag{1.14}$$

The critical exponents is $p_c = \sigma + 1 + 2/N$, and for compactly supported initial data u_0, the critical behaviour as $t \to \infty$ takes the form

$$u(x,t) = (t \ln t)^{-\gamma} [f_*(x t^{-\gamma/N} (\ln t)^{\gamma\sigma/2}) + o(1)], \quad \gamma = N/(N\sigma + 2), \tag{1.15}$$

where $f_* \geq 0$ is a uniquely chosen rescaled profile of the famous Zel'dovich-Kompaneetz-Barenblatt similarity solution of the PME $u_t = \Delta |u|^\sigma u$. It has the form

$$f_*(y) = \left[\frac{\gamma\sigma}{2N(\sigma+1)} (a_*^2 - |y|^2)_+ \right]^{1/\sigma}, \tag{1.16}$$

where $(\cdot)_+$ denotes the positive part. The constant a_* playing a role of C_0 in the semilinear case (1.13) is uniquely determined as follows:

$$a_* = \left[\frac{2N(\sigma+1)}{\gamma\sigma}\right]^{1/2} \left[\frac{N\,B(N/2,1+1/\sigma)}{2\,B(N/2,1+p_c/\sigma)}\right]^{\gamma\sigma/2},$$

$B(\cdot,\cdot)$ being Euler's Beta function. Unlike the semilinear case $\sigma = 0$, in (1.15) an unbounded $\ln t$-factor scales also the space variable. Proof of convergence (1.15) was done in [19] for the one-dimensional case $N = 1$, where the rescaled equation was shown to admit an approximate Lyapunov function being "almost" monotone on evolution orbits (compactness of the rescaled orbits via super and subsolutions as for $\sigma = 0$ and the uniqueness of the asymptotic limit was established for any $N \geq 1$). The proof of (1.15) for any $N > 1$ was done in [21] by a general stability approach for perturbed dynamical systems with uniformly stable ω-limit sets. Similar $\ln t$-perturbed asymptotics are available for the p-Laplacian equation with the critical absorption

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) - |u|^{p-1}u, \quad \sigma > 0,\ p_c = \sigma + 1 + (\sigma+2)/N. \tag{1.17}$$

Compactness of the rescaled orbits and the uniqueness of the limit profile were established in [19], and passage to the limit was performed in [21] via a dynamical systems approach. Let us note that linearization techniques similar to those in [4], [5] are not straightforward for quasilinear equations like (1.14) and (1.17). A linearization procedure about compactly supported profiles like (1.16) even for $N = 1$ leads to a singular second-order symmetric ordinary differential operators on bounded intervals (unlike the semilinear case $\sigma = 0$) having singularities at finite end-points. Spectral analysis and completeness of eigenfunctions for suitable self-adjoint extensions of such operators are not straightforward. For $N > 1$ one arrives at complicated problems on self-adjoint extensions of singular elliptic operators. Further results on asymptotics of quasilinear heat equations with absorption can be found in [31], see Chapters 2, 4 and references therein.

Logarithmically perturbed asymptotics of dipole type for the PME with absorption ($u \geq 0$)

$$u_t = (u^m)_{xx} - u^p \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+, \quad u|_{x=0} \equiv 0,$$

$m > 0$, $p_c = m + 1$, were studied in [20]. Critical absorption exponents cannot be calculated explicitly, if the corresponding unperturbed equation admits the generic behaviour described by self-similarity of the *second kind*, which cannot be found via a dimensional analysis. For example, this is true for the 1D dual PME $u_t = |u_{xx}|^{m-1}u_{xx}$, where $m > 1$, see [1]. The critical absorption exponent for the dual PME with absorption

$$u_t = |\Delta u|^{m-1}\Delta u - |u|^{p-1}u \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad m > 1,\ p > 1, \tag{1.18}$$

is calculated but cannot be expressed explicitly via the diffusion exponent m and dimension N [14]. The critical Fujita exponent for the one-dimensional dual PME with the source

$$u_t = |u_{xx}|^{m-1}u_{xx} + u^p, \quad m > 1,\ p > 1.$$

was calculated in [17].

2. **Preliminaries: fundamental solution and semigroup**

We consider classical solutions of the Cauchy problem satisfying the integral equation

$$u(t) = e^{-(-\Delta)^m t} u_0 - \int_0^t e^{-(-\Delta)^m (t-s)} g(u(s)) ds, \quad t > 0. \tag{2.1}$$

Let $p(\omega) = -|\omega|^m$ be the characteristic polynomial of $-(-\Delta)^m$. Then $e^{-(-\Delta)^m t} u_0 = b(t) * u_0$, where the kernel $b(x,t)$ of the integral operator $e^{-(-\Delta)^m t}$ is the fundamental solution of the parabolic operator $\partial/\partial t + (-\Delta)^m$,

$$b(x,t) = \mathcal{F}^{-1}(e^{p(\omega)t}) \equiv (2\pi)^{-N} \int_{\mathbf{R}^N} e^{-|\omega|^m t - i(\omega \cdot x)} d\omega \quad (b(x,0) = \delta(x)).$$

It follows that is takes the standard self-similar form

$$b(x,t) = t^{-N/2m} f(y), \quad y = x/t^{1/2m}. \tag{2.2}$$

Substituting $b(x,t)$ into the linear equation

$$u_t = -(-\Delta)^m u, \tag{2.3}$$

by the uniqueness of the fundamental solution of linear differential operators, the radially symmetric profile $f(y)$ is a unique solution of a linear ordinary differential equation (ODE), which is the radial restriction of the elliptic equation

$$\mathbf{B} f \equiv -(-\Delta_y)^m f + \frac{1}{2m} \nabla_y f \cdot y + \frac{N}{2m} f = 0 \ \text{ in } \mathbf{R}^N, \quad \int_{\mathbf{R}^N} f = 1. \tag{2.4}$$

The operator $\mathbf{B}$ has the divergent representation

$$\mathbf{B} f \equiv -(-\Delta)^m f + \frac{1}{2m} \nabla \cdot (y f). \tag{2.5}$$

For $m > 1$ the rescaled kernel changes sign and $f = f(\xi)$, $\xi = |y|$, is oscillating as $\xi \to \infty$. Estimates of fundamental solutions, their derivatives and other properties are available in [11]. In particular, it is convenient to present an upper estimate of f in the following form: there exist constants $D > 1$ and $d > 0$ depending on m and N such that

$$|f(y)| < D F(y) \equiv D \omega_1 e^{-d|y|^\alpha} \ \text{ in } \mathbf{R}^N, \quad \alpha = 2m/(2m-1) \in (1,2), \tag{2.6}$$

where $\omega_1 > 0$ is a normalization constant such that $\int F = 1$. The positive kernel

$$\bar{b}(x,t) = t^{-N/2m} F(y), \quad \int \bar{b}(x,t) dx \equiv 1, \tag{2.7}$$

is then the majorizing one for b in the sense that $|b(x,t)| \le D\bar{b}(x,t)$ in $\mathbf{R}^N \times \mathbf{R}_+$. Therefore solutions of order-preserving integral equation with the majorizing kernel $\bar{b}$ can be compared with solutions of the original PDEs like (2.1) and this gives global existence of small solutions of (1.6) for $p > 1 + 2m/N$ and estimates on blow-up rates of general solutions, see [18] and [6]. Local and global solvability and regularity properties of classical solutions of

$$u(t) = b(t) * u_0 - \int_0^t b(t-s) * g(u(s)) ds \tag{2.8}$$

are well known, see e.g. [33], Chapter 15 and recent results in [7], [10], [18].

3. Spectral properties of B and of the adjoint operator $\mathbf{B}^*$

We begin with the spectral properties of $\mathbf{B}$ and the corresponding adjoint operator $\mathbf{B}^*$ which will play a key role in further asymptotic analysis of the nonlinear problem.

3.1. **Point spectrum of non self-adjoint operator B.** For $m > 1$, $\mathbf{B}$ is not symmetric and does not admit a self-adjoint extension. We consider $\mathbf{B}$ in the weighted space $L^2_\rho(\mathbf{R}^N)$ with the exponentially growing weight function

$$\rho(y) = e^{a|y|^\alpha} > 0 \quad \text{in } \mathbf{R}^N,$$

where $a \in (0, 2d)$ is a constant. We ascribe to $\mathbf{B}$ the domain $H^{2m}_\rho(\mathbf{R}^N)$. There holds [10], [13].

Lemma 3.1. *(i)* $\mathbf{B} : H^{2m}_\rho(\mathbf{R}^N) \to L^2_\rho(\mathbf{R}^N)$ *is a bounded linear operator with the real point spectrum only*

$$\sigma(\mathbf{B}) = \{\lambda_\beta = -|\beta|/2m, \ |\beta| = 0, 1, 2, ...\}. \tag{3.1}$$

Eigenvalues λ_β *have finite multiplicity with eigenfunctions*

$$\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta f(y). \tag{3.2}$$

(ii) The set of eigenfunctions $\Phi = \{\psi_\beta, \ |\beta| = 0, 1, 2, ...\}$ *is complete in* $L^2_\rho(\mathbf{R}^N)$.
(iii) $\mathbf{B}$ *is sectorial in* l^2_ρ.

The "little" L^2-space $l^2_\rho \subset L^2_\rho(\mathbf{R}^N)$ consists of functions $v = \sum a_\beta \psi_\beta$ with coefficients $\{a_\beta\} \in l^2$, i.e., $\sum a_\beta^2 < \infty$ with the same inner product [10]. In the classical second-order case $m = 1$, $f(y) = (4\pi)^{-N/2} e^{-|y|^2/4}$ is the rescaled positive Gaussian kernel and the eigenfunctions are

$$\psi_\beta(y) = e^{-|y|^2/4} H_\beta(y), \quad H_\beta(y) \equiv H_{\beta_1}(y_1)...H_{\beta_N}(y_N),$$

where H_β are Hermite polynomials in $\mathbf{R}^N$ [2]. Operator $\mathbf{B}$ with the domain $H^2_\rho(\mathbf{R}^N)$ with the weight $\rho = e^{|y|^2/4}$, is self-adjoint and the eigenfunctions form an orthogonal basis in $L^2_\rho(\mathbf{R}^N)$.

The point spectrum of $\mathbf{B}$ is calculated by differentiating the elliptic equation (2.4),

$$D^\beta \mathbf{B} f = \mathbf{B} D^\beta f + \frac{|\beta|}{2m} D^\beta f = 0.$$

Performing the rescaling $u(x, t) = t^{-N/2m} w(y, \tau)$, $y = x/t^{1/2m}$ and $\tau = \ln t \in \mathbf{R}$ of a solution $u(x, t)$ of (2.3) with initial data $u_0 \in H^{2m}_\rho(\mathbf{R}^N)$ yields the parabolic equation

$$w_\tau = \mathbf{B} w \quad \text{for } \tau = \ln t \in \mathbf{R}. \tag{3.3}$$

Rescaling the convolution $u(t) = b(t) * u_0$ leads to the following explicit representation of the semigroup $e^{\mathbf{B}\tau}$:

$$w(y, \tau) = \int_{\mathbf{R}^N} f(y - ze^{-\tau/2m}) u_0(z) dz, \tag{3.4}$$

and further Taylor expansion in the kernel shows that (3.1) is the point spectrum. Completeness of Φ is proved [10] by the Riesz-Fischer theorem similar to completeness of Hermite's or Laguerre's orthogonal polynomials, see [27], p. 431. Completeness is also associated with exact semigroup representation (3.4) (no other eigenfunctions from $L^2_\rho(\mathbf{R}^N)$ can occur in the expansion).

Operator $\tilde{\mathbf{B}} = \mathbf{B} - I$ has the strictly negative point spectrum $\sigma_p(\tilde{\mathbf{B}}) = \{\tilde{\lambda}_\beta = -1 - |\beta|/2m\}$. By the explicit convolution representation (3.4), the descent method of construction of fundamental solutions implies that

$$\tilde{\mathbf{B}}^{-1} g \equiv K * g, \quad g \in L^2_\rho(\mathbf{R}^N), \tag{3.5}$$

with the kernel

$$K(y, \zeta) = - \int_0^1 (1 - z)^{-N/2m} f[(y - \zeta z^{1/2m})(1 - z)^{-1/2m}] dz. \tag{3.6}$$

In view of known oscillatory properties of the exponentially decaying rescaled kernel f, see (2.6), using a transformation $\mathbf{R}^N \to B_1$ in both independent variables in (3.6) (B_1 is the unit ball), we have that K is an L^p-kernel, $p \in (1, 2]$, and (3.5) is a compact operator with a discrete spectrum accumulating at 0. Thus, $\mathbf{B}$ has a point spectrum only, the resolvent $(\mathbf{B} - \lambda I)^{-1}$ in l^2_ρ has a pole $\sim 1/\lambda$ as $\lambda \to 0$ ($\lambda_0 = 0$ has multiplicity one) [24], and $\mathbf{B}$ is sectorial [12].

Lemma 3.1 gives the centre and stable subspaces of $\mathbf{B}$, $E^c = \mathrm{Span}\{\psi_0 = f\}$, $E^s = \mathrm{Span}\{\psi_\beta, |\beta| > 0\}$. By completeness, $L^2_\rho(\mathbf{R}^N) = E^c \oplus E^s$.

3.2. **Spectrum and polynomial eigenfunctions of the adjoint operator $\mathbf{B}^*$.** We now describe eigenfunctions of the adjoint operator

$$\mathbf{B}^* = -(-\Delta)^m - \frac{1}{2m} y \cdot \nabla. \tag{3.7}$$

In the second-order case $m = 1$ it admits a symmetric representation

$$\mathbf{B}^* = \frac{1}{\rho^*} \nabla \cdot (\rho^* \nabla), \quad \rho^*(y) = e^{-|y|^2/4}. \tag{3.8}$$

Then $-\mathbf{B}^* \geq 0$ is semibounded and there exists its unique Friedrichs extension, which is a self-adjoint operator in the weighted Hilbert space $L^2_{\rho^*}(\mathbf{R}^N)$ with the domain $\mathcal{D}(\mathbf{B}^*) = H^2_{\rho^*}(\mathbf{R}^N)$ and a discrete spectrum. The eigenfunctions form an orthonormal basis in $L^2_{\rho^*}(\mathbf{R}^N)$ and the classical Hilbert-Schmidt theory applies [2].

Let $m > 1$ and consider $\mathbf{B}^*$ in $L^2_{\rho^*}(\mathbf{R}^N)$ with the exponentially decaying weight function

$$\rho^*(y) = 1/\rho(y) \equiv e^{-a|y|^\alpha} > 0. \tag{3.9}$$

There holds [10].

Lemma 3.2. *(i)* $\mathbf{B}^* : H^{2m}_{\rho^*}(\mathbf{R}^N) \to L^2_{\rho^*}(\mathbf{R}^N)$ *is a bounded linear operator with spectrum* $\sigma(\mathbf{B}^*)$ *given by* (3.1)*. Eigenfunctions* $\psi^*_\beta(y)$ *are* $|\beta|$*-th order polynomials*

$$\psi^*_\beta(y) = \frac{1}{\sqrt{\beta!}}[y^\beta + \sum_{j=1}^{[|\beta|/2m]} \frac{1}{j!}(-\Delta)^{mj} y^\beta]. \tag{3.10}$$

(ii) The subset $\{\psi^*_\beta\}$ *is complete in* $L^2_{\rho^*}(\mathbf{R}^N)$.
(iii) $\mathbf{B}^*$ *is sectorial in* $l^2_{\rho^*}$.

With this definition of the adjoint eigenfunctions the orthonormality condition holds

$$\langle \psi_\beta, \psi^*_\gamma \rangle = \delta_{\beta,\gamma}.$$

We use the expansion analysis of the explicit convolution representation. In order to get the adjoint operator $\mathbf{B}^*$, we introduce different rescaled variables corresponding to blow-up as $t \to 1^-$, $u(x,t) = w(y,\tau)$, $y = x/(1-t)^{1/2m}$ and $\tau = -\ln(1-t)$ $(0 < t < 1)$, and then w solves the problem

$$w_\tau = \mathbf{B}^* w \quad \text{for } \tau > 0, \quad w(0) = u_0. \tag{3.11}$$

Rescaling the convolution $u(t) = b(t) * u_0$ yields the explicit representation of the semi-group with the infinitesimal generator $\mathbf{B}^*$

$$w(y,\tau) = \int_{\mathbf{R}^N} f(\zeta - \nu) u_0(\zeta t^{1/2m}) d\zeta, \quad \nu = y[(1-t)/t]^{1/2m}. \tag{3.12}$$

The asymptotic expansions in (3.12) as $\tau \to \infty$ $(t \to 1^-)$ gives a complete point spectrum in $L^2_{\rho^*}(\mathbf{R}^N)$, see [13]. Completeness follows from (3.10) and the well-known fact that polynomials $\{y^\beta\}$ are complete in L^p-spaces with any suitable positive weights. Regardless the pure polynomial structure of eigenfunctions, completeness properties in weighted spaces can be seen from the continuity of the corresponding uniformly parabolic flow (3.11). For $m = 1$, both (3.1) and (3.10) are well-known properties of the separable Hermite polynomials generated by a self-adjoint Sturm-Liouville problem [2].

Taking the operator $\mathbf{B}^* - I$ with uniformly negative point spectrum and using the fact that operations $(\cdot)^*$ and $(\cdot)^{-1}$ commute for operators in Banach spaces and adjoint operator of a compact operator is compact, we have that $(\mathbf{B}^* - I)^{-1}$ is compact with the point spectrum only.

4. **Centre manifold behaviour: the main result**

It is convenient to state the main result in terms of rescaled variables generated by the similarity structure of the fundamental solution. We perform the change of the dependent and independent variables $(u, x, t) \mapsto (v, y, \tau)$, where

$$u(x,t) = (1+t)^{-N/2m} v(y,\tau), \quad y = x/(1+t)^{1/2m}, \quad \tau = \ln(1+t) : \mathbf{R}_+ \to \mathbf{R}_+. \tag{4.1}$$

The critical exponent P_c has been chosen in such a way that, under the scaling invariance condition (1.3) with $P = P_c$, the scaling group admitted by the full equation (1.1) is the same as the group of the linear equation (2.3). Therefore, in terms of the new rescaled variables (4.1) we arrive at an autonomous (time-independent) parabolic equation

$$v_\tau = \mathbf{A}(v) \equiv \mathbf{B}v - g(y,v) \quad \text{for } \tau > 0, \quad v(0) = u_0. \tag{4.2}$$

We consider sufficiently small initial data $u_0 \in H^{2m}_\rho(\mathbf{R}^N)$ satisfying $|u_0(y)| \le ce^{-b|y|^\alpha}$ in $\mathbf{R}^N$, where $c > 0$ is small and $b \ge d$ is large enough.

Sectorial operator $\mathbf{B}$ generates a strong continuous analytic semigroup $\{e^{\mathbf{B}\tau}, \tau \ge 0\}$, [12]. The asymptotic behaviour with a finite-dimensional local centre manifold is covered by the invariant manifold theory, [28], Chapter 6, using interpolation spaces $E_i = D_{\mathbf{B}}(\theta + i, \infty)$ for $i = 0, 1$, $\theta \in (0,1)$. The main assumption on the spectral set $\sigma_+(\mathbf{B}) = \{\lambda \in \sigma(\mathbf{B}) : \operatorname{Re}\lambda \ge 0\}$ is valid and moreover $\sigma_+(\mathbf{B})$ consists of a unique zero simple eigenvalue $\lambda_0 = 0$ with the eigenfunction $\psi_0 = f$ (no unstable subspace is available). Setting $\sigma_-(\mathbf{B}) = \{\lambda \in \sigma(\mathbf{B}) : \operatorname{Re}\lambda < 0\}$, we observe a positive gap

$$\omega_- = -\sup\{\operatorname{Re}\lambda : \lambda \in \sigma_-(\mathbf{B})\} = 1/2m > 0. \tag{4.3}$$

Using projection P associated with the spectral set $\sigma_+(\mathbf{B})$, $P(E_0) \subset E_1$, leads to a one-dimensional equation for $X(\tau) = Pv(\tau)$,

$$X' = \mathbf{B}_+ X - Pg(X+Y), \quad \tau \ge 0, \tag{4.4}$$

where $\mathbf{B}_+ = \mathbf{B}|_{P(E_0)}$ is the null operator (since $\lambda_0 = 0$) and $Y(\tau) = (I-P)v(\tau)$. Necessary assumptions on the nonlinear term g are valid for several kinds of such lower-order operators, see conditions [28], Sect. 9.2. Various projectivity methods for non self-adjoint cases can be found in [32]. It then follows from Theorem 9.2.2 in [28] that there exists a one-dimensional invariant local centre manifold $W^c(0)$ of the origin, which is the graph of a Lipschitz continuous function $\gamma : P(E_0) \to (I-P)(E_1)$. Moreover, it follows from (4.3) that it is exponentially attractive provided that g is twice continuously differentiable, see Theorem 9.2.3. Thus, we state the following condition on g:

$$\text{there exists a one-dimensional } W^c_{\rm loc}(0). \tag{4.5}$$

Under the above hypotheses, we have the following result.

Theorem 4.1. *Let* (1.3) *be valid with the critical exponent $P = P_c \ge 2$ given in* (1.4). *Let twice continuously differentiable function $g(\cdot, v)$ be such that* (4.5) *holds and*

$$R_* \equiv \langle g(\cdot, f), \psi_0^* \rangle > 0, \tag{4.6}$$

where $\psi_0^* \equiv c_0^* > 0$ is the first eigenfunction of the adjoint operator $\mathbf{B}^*$. Then any small solution $v(\cdot,\tau)$, which does not decay exponentially fast, has the following asymptotic behaviour as $t \to \infty$:

$$v(y,\tau) = \pm C_0 \tau^{-N/(2m+Q)}[f(y) + o(1)], \quad \textit{where } C_0 = [R_*(2m+Q)/N]^{-N/(2m+Q)}. \tag{4.7}$$

Hence, (4.7) implies that the null solution is asymptotically stable in E_1, [28], p. 371.

Proof. The projection is $Pv = \langle v, \psi_0^* \rangle \psi_0$ with $\psi_0 = f$ and $\psi_0^* \equiv c_0^*$. The behaviour of the local centre manifold is given by the one-dimensional equation ([28], pp. 365-371) $z'(\tau) = Pg(z(\tau) + \gamma(z(\tau)))$ for $\tau \ge 0$, where by the regularity assumptions on the nonlinearity $\gamma'(0) = 0$. Setting $z(\tau) = a_0(\tau)\psi_0$, we have

$$a_0' = -\langle g(a_0\psi_0 + o(a_0)), \psi_0^* \rangle.$$

Using the second homogenuity hypothesis in (1.3), we finally derive the evolution equation on the local centre manifold

$$a_0' = -|a_0|^{P-1} a_0 \langle g(\psi_0), \psi_0^* \rangle + o(|a_0|^P) \equiv -R_* |a_0|^{P-1} a_0 + o(|a_0|^P). \tag{4.8}$$

In the derivation we have used that $\psi_0(y)$ is exponentially decaying as $|y| \to \infty$ and $g(\cdot, v) = O(|v|^P)$ as $v \to 0$. Equation (4.8) can be integrated asymptotically as a standard ODE and admits globally decaying orbits (4.7) only. □

Remark 1: unstable centre manifold behaviour. The sign restriction (4.6) is essential. If $R_* < 0$, then the asymptotic ODE (4.8) implies unstability of the origin via centre manifold evolution. In the case of equation (1.6) with nonnegative non-monotone perturbation, this is exactly the case: any solution with initially positive first Fourier coefficient, $\int u_0 > 0$, blows up in finite time; see different proofs in [9] (by a test-function method) and in [18] (by a modification of Kaplan's eigenfunction method).

Remark 2: exponentially decaying patterns on the stable manifold. Concerning another assumption of the theorem, we note that in general equation (4.2) admits orbits on the infinite-dimensional stable manifold of the origin, which follows from the eigenfunctions expansion of solutions. Under natural hypotheses on nonlinear term g, $v(y, \tau)$ for $\tau \geq 0$ is sufficiently smooth by the parabolic regularity theory [11], [12]. In view of completeness and orthonormality of eigenfunctions of $\mathbf{B}$, for smooth initial data $v_0 \in H^{2m}_\rho(\mathbf{R}^N)$ we use the uniformly converging eigenfunctions expansion of the solution

$$v(\tau) = \sum_\beta a_\beta(\tau)\psi_\beta = a_0(\tau)\psi_0 + \sum_{|\beta| \geq 1} a_\beta(\tau)\psi_\beta \equiv X(\tau) + Y(\tau), \tag{4.9}$$

where $X(\tau) \equiv Pv(\tau) \in E^c$ and $Y(\tau) \in E^s$ for all $\tau > 0$ are the corresponding projections. The expansion coefficients satisfy the dynamical system

$$a_\beta' = \lambda_\beta a_\beta - \langle g(\cdot, v), \psi_\beta^* \rangle \quad \text{for any } \ \beta, \tag{4.10}$$

where the first one with $|\beta| = 0$ gives the evolution equation on the one-dimensional local centre manifold. The diagonal structure of the system (4.10) shows that if the nonlinear term g forms an exponentially decaying perturbation as $\tau \to \infty$, then there exist patterns with exponential decay as $\tau \to \infty$

$$v(y, \tau) = Ce^{\lambda_\beta \tau}(\psi_\beta(y) + o(1)), \quad C = C(u_0) \neq 0, \tag{4.11}$$

where ψ_β is a suitable eigenfunction with $\lambda_\beta < 0$ for $|\beta| > 0$. Indeed, asymptotically these are exponentially decaying solutions of the linear equation (3.3). Such results are well known in the linear perturbation theory, [8], [12].

4.1. **Asymptotic behaviour in the supercritical range.** Assuming that $P > P_c$ and performing rescaling (4.1), we arrive at a perturbed equation

$$v_\tau = \mathbf{B}v - e^{\gamma\tau} g(y, v), \quad \text{where } \ \gamma = N(P_c - P)/2m \tag{4.12}$$

($\gamma = 0$ for $P = P_c$ leading to the autonomous equation (4.2)), so that $\gamma < 0$ if $P > P_c$ and the nonlinear term forms an exponentially small perturbation of the linear equation (3.3). This implies existence of global small solutions regardless the sign of the nonlinear term g, see [10] and [18], cf. a general semigroup approach in [7].

The asymptotic behaviour is then expected to be "almost" the same as for the linear equation (3.3) (see comments below on special critical cases). For $g(v) = -|v|^p$ with $p > p_c = 1 + 2m/N$ the generic stable behaviour $v(y, \tau) = C_0\psi_0(y) + o(1)$ as $\tau \to \infty$ was established in [10]. A dynamical system approach there admits extensions to more general equations.

4.2. **On stable similarity solutions in the subcritical range.** Let $P \in (1, P_c)$. In view of (1.3) we perform the rescaling corresponding to the invariant group of transformations

$$u = (1+t)^{\mu} v, \quad y = x/(1+t)^{1/2m}, \quad \tau = \ln(1+t) : \mathbf{R}_+ \to \mathbf{R}_+, \tag{4.13}$$

with the negative exponent $\mu = -(1 + Q/2m)/(P-1)$ for $Q > -2m$. This gives the autonomous equation for the rescaled solution

$$v_\tau = \mathbf{B}_* v - g(y, v) \quad \text{for } \tau > 0, \tag{4.14}$$

where the linearized operator takes the form

$$\mathbf{B}_* = \mathbf{B} + c_* I, \;\; c_* = N(P_c - P)/2m(P-1) \tag{4.15}$$

with spectrum $\sigma(\mathbf{B}_*) = \{c_* - |\beta|/2m\}$. Hence, $c_* > 0$ for $P \in (1, P_c)$, operator $\mathbf{B}_*$ in (4.14) has finite positive Morse index and $v \equiv 0$ is unstable stationary solution (unlike the case $P \geq P_c$). This suggests to look for a nontrivial similarity profiles $V = V(y)$ satisfying the stationary elliptic equation

$$\mathbf{B}_* V - g(y, V) = 0 \quad \text{in } \mathbf{R}^N \tag{4.16}$$

with exponential decay as $y \to \infty$. Such *very singular similarity solutions* (VSS) describing the generic asymptotic behaviour as $t \to \infty$ are known from the 1980's for the second-order ($m = 1$) semilinear equations with $g(v) = v^p$; see first results on existence, uniqueness and stability of a similarity profile $V > 0$ in [3], [15], [26]. For higher-order equations with $m > 1$, these interesting problems remain open. For a particular equation (1.5) with $1 < p < 1 + 2m/N$, some analytical and numerical evidence of existence of a *finite number* of VSS's (the first one stable) is presented in [22].

4.3. **On countable subset of critical exponents.** It follows from (4.15) that there exists a countable subset of exponents $\{P_k\}$ with integer $k = |\beta|$ such that $\mathbf{B}_*$ has a nontrivial centre subspace,

$$c_* - k/2m = 0 \implies P_k = 1 + (2m + Q)/(k + N), \quad k = 0, 1, 2, ..., \tag{4.17}$$

and hence P_c is the first one P_0 with $k = 0$. In the radial setting, assuming that $g = g(|x|, u)$, for arbitrary even $k = 2, 4, ...$ operator $\mathbf{B}_*$ has eigenvalue 0 with a one-dimensional centre subspace $E^c = \text{Span}\{\psi_k(|y|)\}$ and a finite number of isolated positive eigenvalues. The rest of construction is quite similar to that for $k = 0$ in Theorem 4.1.

Assuming that a centre manifold analysis applies in this case (though it is more delicate) and looking for a solution of (4.14) in the form $v(\tau) = a_k(\tau)\psi_k(y) + \ldots$, we arrive at the asymptotic ODE $a_k' = -c_k|a_k|^{P-1}a_k + \ldots$ with the coefficient $c_k = \langle g(\cdot, \psi_k), \psi_k^* \rangle$ assumed to be positive. For $c_k < 0$ the centre manifold behaviour is unstable. This gives the orbit $a_k(\tau) = \pm C_k \tau^{-1/(P-1)} + \ldots$ as $\tau \to \infty$, where $C_k = [c_k(2m+Q)/(k+N)]^{-(k+N)/(2m+Q)}$. Finally, returning back to the original variables via (4.13), we derive the following asymptotic patterns in the critical cases $P = P_k$ for even $k = 2, 4, \ldots$ (cf. (1.11)):

$$u(x,t) = \pm C_k t^{-(k+N)/2m} (\ln t)^{-(k+N)/(2m+Q)} \psi_k(|x|/t^{1/2m}) + \ldots \quad \text{for} \quad t \gg 1. \tag{4.18}$$

The first term of such asymptotic behaviour does not reveal any trace of initial data. We again obtain $\ln t$-perturbed asymptotic patterns at the countable subset of critical exponents $P = P_k$. In the case of nonlinearity $g(x,v) = -|v|^p$ spectra of asymptotically exponentially decaying patterns for equation (1.6) including the critical cases were studied in [10]. A countable subset of logarithmically perturbed patterns can be constructed for nonlinear reaction-absorption equations (1.14) and (1.18), [14].

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