STRUCTURE OF BOUNDARY BLOW-UP FOR HIGHER-ORDER QUASILINEAR PARABOLIC EQUATIONS

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ABSTRACT. Singularity formation phenomena for $2m$th order quasilinear parabolic equations are studied by using energy estimates related to Saint-Venant's principle. Propagation of singularities generated by boundary global blow-up regimes are described.

1. Introduction

We study asymptotic properties of solutions to initial-boundary value problems for general $2m$th order quasilinear degenerate parabolic equations in divergence form. Energy estimates on solutions to general classes of equations are an effective and often a unique and exceptional tool to describe their crucial evolution properties, where application of the idea of Saint-Venant's principle leads to special types of such estimates. Typical results and surveys on energy techniques for higher-order PDEs can be found in [15], [4], [5], [18], see also list of references in [8]. In this paper, we present an application of energy estimates related to Saint-Venant's principle for general higher-order quasilinear parabolic equations. We study singularity formation phenomena generated by finite-time blow-up on the boundary. Blow-up singularities are known to play an important role in several applications and can occur in many types of evolution PDEs including first-order conservation laws and Hamilton-Jacobi equations, semilinear and quasilinear heat equations and reaction-diffusion systems, in quasilinear hyperbolic and higher-order PDE and systems. We refer to surveys and books [2], [11], [13], [17], where further references are available. We restrict our attention to the case of *global* blow-up, which was less studied in mathematical literature last decades. Global blow-up phenomena mean that solutions are not localized in space and increase without bound everywhere. The main evolution aspect of such a singularity formation is to describe the law of propagation of such unbounded blow-up waves near blow-up time. We consider three different type of global blow-up including the case of super-exponential growth. Energy estimates proved for a class of $2m$th order quasilinear equations are shown to be optimal for some nonlinear second-order equations.

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1.1. **Statement of the problem.** Main results are stated in a simple geometry and generalizations to arbitrary exterior domains can be done by a suitable domain parameterization. Let $\Omega = \{|x| > 1\}$ be the open complement of the unit ball $B = \{|x| < 1\}$ in $\mathbf{R}^N$. In the cylindrical domain $Q = \Omega \times (0, T)$ we consider the Cauchy-Dirichlet problem for a general quasilinear $2m$th order parabolic equation, $m \geq 1$,

$$(|u|^{q-1}u)_t + \sum_{|\alpha|=m} (-1)^m D_x^\alpha a_\alpha(x, t, u, ..., D_x^m u) = 0, \quad q > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x) \in L_{q+1}(\Omega) \quad \text{in } \Omega, \tag{1.2}$$

$$D_x^\alpha(u - f) = 0 \quad \text{on } \partial\Omega \times (0, T) \text{ for any } |\alpha| \leq m - 1. \tag{1.3}$$

The functions $a_\alpha(x, t, \xi)$ are assumed to be continuous, and the elliptic operator on the left-hand side satisfies the following growth and coercivity conditions:

$$\sum_{|\alpha|=m} a_\alpha(x, t, \xi)\xi_\alpha \geq d_1(\sum_{|\beta|=m} |\xi_\beta|)^{p+1}, \quad d_1 > 0, \ p \geq q, \tag{1.4}$$

$$|a_\alpha(x,t,\xi)| \leq d_2(\sum_{|\beta|=m} |\xi_\beta|)^p \quad \text{for } (x, t, \xi) \in \bar{Q} \times \mathbf{R}^{n(m)}, \ d_2 > 0, \tag{1.5}$$

where $n(m)$ is the number of distinct multiindices of the length not exceeding m. We use the usual notations $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = \alpha_1 + ... + \alpha_N$, $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}$ and $D_x^m u = \{D_x^\alpha u, \ |\alpha| = m\}$. The function $f(x, t)$ determining the boundary values, is assumed to be extended from the lateral boundary $\partial\Omega \times (0, T)$ into the domain Q such that for any $T_0 < T$, there hold

$$f(\cdot, t) \in C([0, T_0]; \ L_{q+1}(\Omega)) \cap L_{p+1}(0, T_0; \ W_{p+1}^m(\Omega)), \tag{1.6}$$

$$f_t(\cdot, t) \in L_1(0, T_0; \ L_{q+1}(\Omega)) \cap L_r(0, T_0; L_r(\Omega)), \ r = (p + 1)/(p + 1 - q). \tag{1.7}$$

We assume that the boundary function $f(\cdot, t)$ *blows up* as $t \to T^-$. The rate of blow-up is characterized by the following function defined for $t \in (0, T)$:

$$F(t) = \sup_{0<\tau<t} \left(\int_\Omega |f(x, \tau)|^{q+1} dx \right) + \int_0^t \int_\Omega |D_x^m f(x, \tau)|^{p+1} dx d\tau$$
$$+ \left(\int_0^t \left(\int_\Omega |f_\tau(x, \tau)|^{q+1} dx \right)^{1/(q+1)} d\tau \right)^{q+1} \to \infty \ \text{ as } \ t \to T^-. \tag{1.8}$$

Without loss of generality, we always suppose that the support of $f(\cdot, t)$ is uniformly bounded for $t \in (0, T)$. Given $S \subset \partial\Omega$, by $W_{p+1}^m(\Omega, S)$ we denote, as usual, the closure in the norm of the Sobolev space $W_{p+1}^m(\Omega)$ of the subset of functions from $C^\infty(\Omega)$ vanishing in a neighbourhood of S. We set $W_{p+1}^m(\Omega, \emptyset) \equiv W_{p+1}^m(\Omega)$ and denote by $\langle \cdot, \cdot \rangle$ the duality product between $\left(W_{p+1}^m(\Omega, \partial\Omega)\right)^*$ and $W_{p+1}^m(\Omega, \partial\Omega)$.

Definition 1. A function $u(x, t)$ is said to be a *generalized energy solution* of the problem (1.1)-(1.3), if for any $T_0 < T$, there hold

(i) $u - f \in L_{p+1}(0, T_0; W_{p+1}^m(\Omega, \partial\Omega)) \cap C([0, T_0]; L_{q+1}(\Omega))$,

(ii) $(|u|^{q-1}u)_t \in L_{(p+1)/p}(0,T_0;(W^m_{p+1}(\Omega,\partial\Omega))^*)$,

(iii) u satisfies the initial condition (1.2) and the following integral identity:

$$\int_0^{T_0} \langle (|u|^{q-1}u)_t, \eta\rangle dt + \int_0^{T_0}\int_\Omega \sum_{|\alpha|=m} a_\alpha(x,t,u(x,t),...,D_x^m u(x,t))D_x^\alpha \eta(x,t)dxdt = 0, \tag{1.9}$$

where $\eta(x,t) \in L_{p+1}(0,T_0;W^m_{p+1}(\Omega,\partial\Omega))$ is an arbitrary function.

Local existence of generalized (weak) solutions of the mixed problem for equations (1.1) are well established. In the case $q=1$, $f=0$ problem (1.1)-(1.3) was studied in detail since the 1950s and 1960s, see [14] and references therein. In the case $m=1$ and bounded domains Ω existence of energy solutions follows from [1] for arbitrary $T_0 < T$ by the second inclusions in (1.6)-(1.7). For $m>1$ and $f=0$ the solvability of problem under consideration was established in [6] in the case where equation (1.1) has a variational structure, in particular, if $a_\alpha(x,t,\xi) \equiv |\xi_\alpha|^{p-1}\xi_\alpha$. The results admit a natural extension to more general higher-order equations and boundary conditions. In the higher-order case, general results on uniqueness are not known.

Let us present some typical concepts and notations related to blow-up solutions. We say that $y \in \overline{\Omega}$ is singular point of the energy solution $u(x,t)$ if for any small $\varepsilon > 0$,

$$\sup_{0<t<T}\left(\int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,t)|^{q+1}dx + \int_0^t\int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,\tau)|^{p+1}dxd\tau\right) = \infty.$$

The subset $\Omega_S \equiv \Omega_S(u)$ of all such singular points is the *singularity* or the *blow-up subset* of solution $u(x,t)$ of problem (1.1)-(1.3). $\Omega_R \equiv \overline{\Omega}\setminus\Omega_S$ is the corresponding *non-singular subset*.

For sufficiently regular solutions, it is expected that $u(x_0,t)$ is uniformly bounded as $t \to T^-$ at any interior point $x_0 \in \Omega_R = \Omega\setminus\Omega_S$, and vise versa, $u(x_0,t)$ is not uniformly bounded as $t\to T^-$ for any $x_0 \in \Omega_S$. For such regular solutions, the blow-up set Ω_S is known as the *localization domain* of the solution (or of the boundary blow-up regime). In the one-dimensional case, $L = \operatorname{meas}\Omega_S$ is called the *localization length* of the blow-up regime, see terminology in Chapt. 3, [17]. Given bounded initial data, we say that the blow-up regime, prescribed by the function $f(x,t)$ on $\partial\Omega$, is *localized* (respectively, *non-localized*) if $\operatorname{meas}\Omega_S < \infty$ (respectively, $\operatorname{meas}\Omega_S = \infty$). Localized blow-up regime is called S-regime if $\operatorname{meas}\Omega_S \in (0,\infty)$ (regional blow-up) and LS-regime if $\operatorname{meas}\Omega_S = 0$ (this includes the case of single point blow-up). A non-localized blow-up regime with $\operatorname{meas}\Omega_S = \infty$ is also called HS-regime (global blow-up).

1.2. **Main results and plan of the paper.** We study the special "resonance" case $p=q$. We consider non-localized energy solutions and establish estimates on the effective length of propagation for such global blow-up. Localized energy solutions are considered in [8], where further references and short surveys on blow-up localization and energy techniques via Saint-Venant's principle are available.

It turns out that global blow-up occurs for exponential boundary data

$$F(t) \le cF_1(t) = c\exp\{w(T-t)^{-1/l}\}, \quad \text{where } l < l_0 = m(p+1)-1. \tag{1.10}$$

There are examples of particular second-order and higher-order PDEs [8] showing that (1.10) is not localized in general and $\Omega_S = \Omega$.

For a detailed characterization of the blow-up propagation in the general problem, we introduce a certain "effective length" of propagation of such non-localized blow-up waves. Given a constant $L \gg 1$, for any $t \approx T$, we define a continuous function $\kappa_L(t)$ as follows:

$$\int_0^t \int_{\{|x| > \kappa_L(t)\}} |u(x,\tau)|^{p+1} dx d\tau = L. \tag{1.11}$$

This length has a natural physical sense. For non-localized HS-regimes, $\kappa_L(t)$ is unbounded as $t \to T$.

Theorem 1.1. *Let $p = q > 0$ and condition (1.10) with $l \le l_0$ hold. Then for a fixed $L \gg 1$, for any energy solution $u(x,t)$ of (1.1) – (1.3), as $t \to T$,*

$$\kappa_L(t) \le c_1 w^{l_0/(l_0+1)} (T-t)^{-(l_0-l)/l(l_0+1)}, \tag{1.12}$$

with a constant $c_1 = c_1(L) > 0$.

Corollary 1.1. *If $l = l_0$, then $\kappa_L(t) \le c$ for all $t \in (0,T)$, i.e., this blow-up regime (1.10), $l = l_0$ is, at most, a localized S-regime.*

The next two theorems show extra capabilities of the energy estimate method. For a more refines scale of boundary HS-regimes, we establish the following estimate on propagation showing how a slow-growing logarithmic factors can affect the length of propagation.

Theorem 1.2. *Let $p = q > 0$ and for $t \approx T$,*

$$F(t) \le cF_2(t) \equiv c \exp\{w(T-t)^{-1/l} |\ln(T-t)|^\beta\}, \quad c > 0, \tag{1.13}$$

where $l \le l_0 = m(p+1) - 1$ and $\beta > 0$. Then for $L \gg 1$, as $t \to T$,

$$\kappa_L(t) \le c_1 (w|\ln(T-t)|^\beta)^{l_0/(l_0+1)} (T-t)^{-(l_0-l)/l(l_0+1)}. \tag{1.14}$$

Finally, we describe global blow-up propagation for super-exponential boundary data.

Theorem 1.3. *Let $p = q > 0$ and for $t \approx T$,*

$$F(t) \le cF_3(t) \equiv c \exp\{w e^{(T-t)^{-1}}\}, \quad c > 0, \ w > 0. \tag{1.15}$$

Then

$$\kappa_k(t) \le c_1 (T-t)^{2/(l_0+1)} e^{D(T-t)^{-1}} \quad \text{as } t \to T, \ \text{where } D = l_0/(l_0+1). \tag{1.16}$$

These three theorems are proved in Sections 4 and 5. In Section 2 we present a number of results for a model one-dimensional second-order parabolic equation $(|u|^{p-1}u)_t = (|u_x|^{p-1}u_x)_x$ with $p = q > 0$. Though this equation does not admit suitable similarity solution, by constructing approximate blow-up self-similar solutions, we show that all the above estimates are sharp and cannot be improved for $m = 1$. In Section 3 we derive the basic system of functional inequalities which is the cornerstone for our energy estimates.

2. **Global blow-up for a quasilinear heat equation with** $p = q$

In this section we present a detailed description of boundary global blow-up for the quasilinear equation with $m = 1$ and $p = q > 0$ in one dimension,

$$(|u|^{p-1}u)_t = \Delta_p u \equiv (|u_x|^{p-1}u_x)_x \quad \text{in } Q = \mathbf{R}_+ \times (0,T), \tag{2.1}$$

$$u(x,t) = f(t) > 0 \quad \text{for } t \in (0,T), \quad u(x,0) = u_0(x) \ge 0 \quad \text{in } \mathbf{R}_+. \tag{2.2}$$

We assume that $f(t)$ blows up as $t \to T$ monotone and is sufficiently smooth for $t < T$. Without loss of generality, we suppose that $f \ge 1$ and bounded initial data satisfy $u_0(x) \ge 1$ in $\mathbf{R}_+$. Hence, $u \ge 1$ in Q by the Maximum Principle. Such quasilinear parabolic equations of the second order are well known in the literature, see the classical book [14] by J.L. Lions, Kalashnikov's survey [12] and E. DiBenedetto's book [7]. By energy solutions we then mean weak ones.

2.1. **Global blow-up via Hamilton-Jacobi equation.**

In order to describe the asymptotic behaviour of main blow-up singularities for this equation, as in the linear case $p = 1$ [16] (see more details in [17], Chapt. 3), we perform the change of the dependent variable by setting $u = e^v$ for $v \ge 0$, which leads to the quasilinear equation with a typical Hamilton-Jacobi operator and the diffusion p-Laplacian term

$$v_t = |v_x|^{p+1} + p^{-1}\Delta_p v \quad (u = e^v). \tag{2.3}$$

The main idea is to show that the asymptotic behaviour of this blow-up singularities is described by the corresponding *Hamilton-Jacobi equation*

$$V_t = |V_x|^{p+1} \quad \text{in } Q. \tag{2.4}$$

More precisely, HS-regimes of global boundary blow-up are given by its similarity solution

$$V_*(x,t) = (T-t)^{-1/l} g(\eta), \quad \eta = x/(T-t)^{\beta}, \ \beta = (l-p)/l(p+1) < 0, \tag{2.5}$$

where $g \ge 0$ solves the first-order ODE

$$\mathbf{B}(g) \equiv |g'|^{p+1} - \beta g'\eta - g/l = 0, \quad \eta > 0; \quad g(0) = 1. \tag{2.6}$$

This ODE is scaling invariant and reduces to an autonomous ODE integrated in quadratures. See a similar analysis of the case $p = 1$ in [17], p. 171. We then obtain that in the case $l < p$, such a monotone profile g exists and vanishes at a finite point $\eta_0 = \text{meas supp}\, g < \infty$. We then set $g(\eta) \equiv 0$ for all $\eta \ge \eta_0$ so that $g(\eta)$ becomes a continuous (but non-smooth) compactly supported rescaled viscosity solution of the Hamilton-Jacobi equation. Comparison of v and V_* in the domain $\{x < \eta_0(T-t)^{\beta}\} \times (0,1)$, where both solutions are sufficiently smooth, is performed as in the case of the S-regime above (see also [17], p. 382) and gives uniform blow-up on any unbounded subset (2.8) with arbitrary $c_0 < \eta_0$. We then arrive at the following conclusion (more details in [8].

Proposition 2.1. *Let in the problem* (2.1), (2.2) *the boundary blow-up be given by the exponential function*

$$f(t) = e^{(T-t)^{-1/l}} \quad \text{with } l \in (0,p). \tag{2.7}$$

Then $u(x,t)$ blows up as $t \to T^-$ uniformly on the expanding subset

$$\{0 \le x \le c_0(T-t)^\beta\}, \quad \beta = (l-p)/l(p+1) < 0, \tag{2.8}$$

with a constant $c_0 = c_0(p,l) > 0$.

A general stability approach to the asymptotic degeneracy of parabolic flows into the Hamilton-Jacobi ones was developed in [9] and [10]. By these results, it is not difficult to show that the self-similar solution (2.5) of the Hamilton-Jacobi equation (2.4) actually describes the asymptotic behaviour of the HS-regime for the parabolic equation (2.3). This is similar to the behaviour in the linear case $p = 1$, [17], p. 381.

2.2. **Slow logarithmic perturbations of exponential blow-up regimes.** In order to study global blow-up for non-exponential blow-up regime, we need to perform some auxiliary manipulations with the rescaled equation (2.3). Setting for convenience

$$f(t) = e^{\psi(t)}, \quad \psi(t) \ge 0, \quad t \in (0,T), \tag{2.9}$$

we obtain that the rescaled solution $\theta(\xi,\tau)$ given by

$$v(x,t) = \psi(t)\theta(\xi,\tau), \quad \xi = x/\phi(t), \quad \phi^{-p-1}(t)\psi^p(t)dt = d\tau, \tag{2.10}$$

with a function $\phi(t) > 0$ to be determined, satisfies the equation

$$\theta_\tau = |\theta_\xi|^2 + r_1(t)\theta_\xi\xi + r_2(t)\theta + \frac{1}{p\psi(t)}\Delta_p\theta, \quad r_1 = \frac{\phi^p\phi'}{\psi^p}, \ r_2 = -\frac{\phi^{p+1}\psi'}{\psi^{p+1}}. \tag{2.11}$$

The rescaled solution satisfies the boundary condition

$$\theta(0,\tau) = 1 \quad \text{for all } \tau \gg 1. \tag{2.12}$$

According to (1.13), we now consider an unbounded but slow growing perturbation of the power HS-regime

$$\psi(t) = (T-t)^{-1/l}\rho(t), \quad l \in (0,p),$$

where $\rho(t) > 0$ is a sufficiently smooth increasing function satisfying the slow growth condition:

$$(\rho(t)/\rho'(t))' \to \infty, \quad t \to T. \tag{2.13}$$

For instance, $\rho(t) = |\ln(T-t)|^\beta$ satisfies (2.13). Then we define still unknown function $\phi(t)$ in (2.10) describing the propagation of global blow-up from the following ODE:

$$\phi^p\phi' = |\beta|\psi^p \implies \phi^{p+1}(t) = |\beta|(p+1)\int \psi^p(t)dt. \tag{2.14}$$

By the L'Hospital rule, (2.13) implies that as $t \to T$,

$$\phi(t) = c_*(T-t)^{-(p-l)/l(p+1)}\rho^{p/(p+1)}(t)(1+o(1)), \quad c_* = [(p-l)/(p+l)]^{1/(p+1)}. \tag{2.15}$$

Obviously, (2.14) means that $r_1 \equiv -\beta$ in equation (2.11). Moreover, by the L'Hospital rule, one can see that under hypothesis (2.13), the second coefficient r_2 in (2.11) satisfies

$$r_2(t) \to -1/l, \quad t \to T.$$

Therefore, for $\tau \gg 1$ the rescaled equation (2.11) takes the form

$$\theta_\tau = \mathbf{B}(\theta) + \mathbf{C}(\theta, \tau), \tag{2.16}$$

where the second-order operator $\mathbf{C}(\theta, \tau)$ plays a role of an asymptotically small perturbation of the Hamilton-Jacobi equation with the operator $\mathbf{B}$ given in (2.6). The passage to the limit $\tau \to \infty$ $(t \to T)$ in the singularly perturbed Hamilton-Jacobi problem (2.16), (2.12) is performed as in [9], [10], see also Chapt. 6 in [17]. This establishes convergence of the rescaled solution to the unique similarity profile g satisfying (2.6),

$$\theta(\xi, \tau) \to g(\eta) \quad \text{as } \tau \to \infty \text{ uniformly.} \tag{2.17}$$

Therefore, the function $\phi(t)$ given by (2.14) or (2.15) actually describes the propagation of global blow-up wave. One can see that for this particular case, the general estimate (1.14) contains the necessary slow-growing logarithmic factor and is sharp.

2.3. **Approximate self-similarity for super-exponential global blow-up.** We describe another similarity solution of the Hamilton-Jacobi equation (2.4). Namely, we use the following similarity solution defined for all $t \in \mathbf{R}$

$$V_1(x,t) = e^t g(\eta), \quad \eta = x/e^{pt/(p+1)}, \tag{2.18}$$

where $g \geq 0$ satisfies the ODE

$$\mathbf{B}(g) \equiv |g'|^{p+1} + pg'\eta/(p+1) - g = 0, \quad \eta > 0; h(0) = 1. \tag{2.19}$$

This non-autonomous ODE is reduced to an autonomous one which is integrated in quadratures. This is similar to the case $p = 1$ in [17], p. 171.

We prove that this kind of non-blow-up self-similar solution describes super-exponential global blow-up for equation (2.3). We impose the following condition on the blow-up regime $\psi(t)$ which is true for exponential and super-exponential functions:

$$(\psi(t)/\psi'(t))' \to 0, \quad t \to T. \tag{2.20}$$

We next perform the change of variables (2.10), where $\phi(t)$ is now defined from the ODE

$$\phi^p\phi'/\psi^p = p/(p+1) \implies \phi^{p+1}(t) = p \int \psi^p(t)dt. \tag{2.21}$$

Indeed, then $r_1 \equiv p/(p+1)$. By the L'Hospital rule, we conclude from (2.20) that

$$\lim r_2 = -p \lim\Big(\int \psi^p\Big)/(\psi^{p+1}/\psi') = -p \lim \psi^p/(\psi^{p+1}/\psi')' = -p[p + \lim(\psi/\psi')']^{-1} = -1.$$

Therefore, the rescaled solution θ satisfies the perturbed Hamilton-Jacobi equation (2.16) with the operator $\mathbf{B}$ in (2.19) and a perturbation $\mathbf{C}(\theta, \tau) \to 0$ as $t \to T$ on sufficiently smooth functions θ. We then arrive at stabilization (2.17), where the unique profile g is prescribed by the ODE problem (2.19). For the super-exponential boundary blow-up $f(t) = \exp\{e^{(T-t)^{-1}}\}$ we have $\psi(t) = e^{(T-t)^{-1}}$, (2.20) holds and hence by (2.21), the propagation of global blow-up wave is asymptotically estimated as follows:

$$\phi(t) \sim (T-t)^{2/(p+1)} e^{c_*(T-t)^{-1}}, \quad t \to T, \quad c_* = p/(p+1).$$

It is curious that for this case, the general estimate (1.16), where $l_0 = p$, correctly describes both exponential term and the lower-order power factor.

3. **Basic system of functional inequalities**

3.1. **Energy estimates.** Consider the general problem (1.1)–(1.3). We introduce families of the subdomains of Q and for any fixed $s > 1$ and $\delta > 0$, define

$$\Omega(s) = \{|x| > s\}, \quad \Omega(s,\delta) = \Omega(s) \setminus \Omega(s+\delta) \quad (\Omega = \Omega(1)),$$
$$Q_a^b(s,\delta) = \Omega(s,\delta) \times (a,b), \quad 0 \le a \le b \le T; \quad Q_a^b(s) \equiv Q_a^b(s,\infty).$$

The following result is proved in [8], Section 4.

Lemma 3.1. *Let $u(x,t)$ be an energy solution of the problem* (1.1) – (1.3)*. Then for any $0 \le a < b < T$, $s > 1$, $\delta > 0$ and any $\varepsilon > 0$, there holds*

$$\int_{\Omega(s+\delta)} |u(x,b)|^{q+1}dx + \int_a^b \int_{\Omega(s+\delta)} |D_x^m u(x,t)|^{p+1}dxdt \le (1+\varepsilon) \int_{\Omega(s)} |u(x,a)|^{q+1}dx$$
$$+ \ \delta^{-m(p+1)}c(\varepsilon) \int_a^b \int_{\Omega(s,\delta)} |u(x,t)|^{p+1}dxdt. \tag{3.1}$$

The positive constant $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$ depends on parameters of the problem and is independent of $u(x,t)$, s and δ.

3.2. **System of functional inequalities.** We introduce an arbitrary strictly monotone sequence

$$\{t_j\} \to T^-, \quad \Delta_j = t_j - t_{j-1} > 0 \quad \text{for } j = 1,2,..., \ t_0 \in (0,T), \tag{3.2}$$

and define two families of energy functions of the energy solution $u(x,t)$ of problem (1.1)-(1.3) for $s > 1$, $j \ge 1$,

$$I_j(s) = \int_{t_{j-1}}^{t_j} \int_{\Omega(s)} |u(x,t)|^{p+1}dxdt, \quad h_j(s) = \sup_{t\in(t_{j-1},t_j)} \int_{\Omega(s)} |u(x,t)|^{q+1}dx, \tag{3.3}$$
$$I_0(s) = \int_0^{t_0} \int_{\Omega(s)} |u(x,t)|^{p+1}dxdt, \quad h_0(s) = \int_{\Omega(s)} |u(x,t_0)|^{q+1}dx.$$

The following system of functionals inequalities follows from the above energy estimates, see Lemma 4.2 in [8].

Lemma 3.2. *Given an arbitrary energy solution of* (1.1)-(1.3)*, for any $s > 1$ and $\delta > 0$, functions* (3.3) *satisfy the following system of functional inequalities:*

$$h_j(s+\delta) \le (1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\Delta I_j(s)/\delta^{m(p+1)}, \quad \Delta I_j(s) \equiv I_j(s) - I_j(s+\delta), \tag{3.4}$$
$$I_j(s+\delta) \le c_1\Delta_j^{1-\theta_1}\left[(1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\Delta I_j(s)/\delta^{m(p+1)}\right]^{1+\nu}, \quad j = 1,2,\dots, \tag{3.5}$$

where $c_1 > 0$ is a constant,

$$\theta_1 = \frac{n(p-q)}{n(p-q)+m(p+1)(q+1)}, \quad \nu = \frac{(p-q)(1-\theta_1)}{q+1}, \tag{3.6}$$

$\{\varepsilon_j > 0\}$ is an arbitrary sequence and positive constants $c(\varepsilon_j) \to \infty$ as $\varepsilon_j \to 0$.

Corollary 3.1. *For equation (1.1) with $p = q$, the main functional system (3.4), (3.5) reads as follows: for any $s > 1$ and $\delta > 0$,*

$$h_j(s+\delta) \le (1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\Delta I_j(s)/\delta^{m(p+1)}, \tag{3.7}$$

$$I_j(s+\delta) \le c_1\Delta_j(1+\varepsilon_j)h_{j-1}(s) + c_1 c(\varepsilon_j)\Delta_j\Delta I_j(s)/\delta^{m(p+1)}, \quad j = 1, 2, \dots . \tag{3.8}$$

This is our basic discrete system of functional inequalities to be studied for the case of global blow-up singularity posed on the boundary. The partitions in independent variables (s, t) then essentially depend on the blow-up rate and often on the estimates of energy functions. In order to derive optimal blow-up estimates we need to use a countable number of different partitions thus exploiting the functional character of inequalities. These lead to some not difficult but rather technical manipulations with such functional systems forming special mathematical tools for studying their asymptotic properties. It seems that in essentially non-stationary cases of studying blow-up evolution in both independent variables (s, t), such systems of functional inequalities cannot be reduces to simpler partial differential inequalities of lower order. A discussion on relationships between functional and differential inequalities and their asymptotic properties is available in Section 4.3 in [8].

4. Exponential blow-up: proof of Theorems 1.1 and 1.2

Consider system (3.7), (3.8). For convenience, we introduce the rescaled energy functions

$$J_j(s) = \alpha_j I_j(s), \tag{4.1}$$

where $\{\alpha_j > 0\}$ is a sequence to be determined later on. Then iterating the system for J_j leads to the following functional system of inequalities (see Lemma 4.3 in [8]):

$$\begin{aligned} J_j(s) \le\ & \frac{\gamma_j}{1+\gamma_j}\, J_j(s-\delta_j) + \frac{c_1}{1+\gamma_j}\sum_{k=1}^{j-1} c(\varepsilon_k)\Delta_j\beta_j\dots\beta_{k+1}\, J_k\big(s - \sum_{i=k}^{j}\delta_i\big)\, \delta_k^{-m(p+1)} \\ & + \frac{c_1}{1+\gamma_j}\,\beta_j\dots\beta_1\Delta_j\alpha_0 h_0\big(s-\sum_{i=1}^{j}\delta_i\big), \quad \text{for } s \ge 1 + \sum_{i=1}^{j}\delta_i, \end{aligned} \tag{4.2}$$

where $\gamma_j = c_1 c(\varepsilon_j)\Delta_j\delta_j^{-m(p+1)}$, $\beta_j = (1+\varepsilon_j)\alpha_j/\alpha_{j-1}$ and $\{\Delta_j > 0\}$, $\{\delta_j > 0\}$ are arbitrary sequences of displacements obtained by suitable partitions in the variables s and t respectively.

First, we define time displacements for $i = 1, 2, \dots$,

$$\Delta_i = D(i+K)^{-l-1}, \quad D = w^l[l/(1+\gamma)^l]^{l+1}, \tag{4.3}$$

where w, l are as given in (1.10), $\gamma > 0$ and $K > 0$ are some constants. Then it is easy to see that for any $j = 1, 2, \dots$,

$$l^{-1}D(j+1+K)^{-l} \le T - t_j \le l^{-1}D(j+K)^{-l}. \tag{4.4}$$

We choose K such that (4.4) holds for $j = 0$ with $t_0 \ge 0$. This yields the following restriction on K:

$$l^{-1} D K^{-l} < T \iff K > wlT^{-1/l}. \tag{4.5}$$

It is convenient for dealing with blow-up regime (1.10) to rewrite down (4.4) in the form

$$l^{-1}(1+\gamma)^l(j+K) \le w(1+\gamma)^{-1}(T-t_j)^{-1/l} \le (1+1/K)l^{-1}(1+\gamma)^l(j+K). \tag{4.6}$$

This makes it possible to specify the weights in (4.1)

$$\alpha_i = \exp\{-l^{-1}(1+\gamma)^l(i+K)\}. \tag{4.7}$$

We fix constant parameters $\{\varepsilon_i\}$ in (4.2), $\varepsilon_i = \varepsilon_0 > 0$ for $i = 1, 2, \dots$.

Let us define the spatial displacements which have to be related to the time ones in a special scaling invariant (self-similar) manner; see more comments below. Given a fixed $j > 0$, we set for any $i \le j$,

$$\delta_i = \delta_j \equiv [c_1 c(\varepsilon_0)\Delta_j]^{1/(l_0+1)} \omega_0^{-1/(l_0+1)}, \quad \omega_0 > 0, \tag{4.8}$$

where Δ_j is given in (4.3). Substituting displacements (4.8), (4.3) into (4.2), we deduce that

$$\begin{aligned} J_j(s) \le {} & \omega_1 J_j(s-\delta_j) + \omega_1 \sum_{k=1}^{j-1} \lambda_0^{j-k} J_k(s-(j-k+1)\delta_j) \\ & + \ c\alpha_0 \lambda_0^j h_0(s-j\delta_j)\Delta_j, \ \ s > 1 + j\delta_j, \end{aligned} \tag{4.9}$$

where $\omega_1 = \omega_0/(1+\omega_0) < 1$, $c = c_1/(1+\omega_0)$. Now we fix $\gamma > \gamma_0(\varepsilon_0)$ so that

$$\lambda_0 = \lambda_0(\gamma, \varepsilon_0) = (1+\varepsilon_0)\alpha_j/\alpha_{j-1} = (1+\varepsilon_0)\exp\{-(1+\gamma)^l/l\} < 1. \tag{4.10}$$

Let us begin the analysis of the functional system (4.9). We iterate inequality (4.9) l_1 times. As a result, we obtain that for any $s > 1 + (l_1 + j - 1)\delta_j$,

$$\begin{aligned} J_j(s) \le {} & \omega_1^{l_1} J_j(s - l_1\delta_j) + \sum_{k_1=1}^{j-1} \lambda_0^{j-k_1} \sum_{i_1=1}^{l_1} \omega_1^{i_1} J_{k_1}(s-(i_1+j-k_1)\delta_j) \\ & + \alpha_0 c \lambda_0^j \Delta_j \sum_{i_1=1}^{l_1} \omega_1^{i_1-1} h_0(s-(i_1+j-1)\delta_j) \equiv A_1^{(1)} + A_1^{(2)} + A_1^{(3)}. \end{aligned} \tag{4.11}$$

Consider $A_1^{(2)}(s)$ in the following form:

$$\begin{aligned} A_1^{(2)} = {} & \lambda_0 \sum_{i_1=1}^{l_1} \omega_1^{i_1} J_{j-1}(s-(i_1+1)\delta_j) \\ & + \sum_{k_1=1}^{j-2} \lambda_0^{j-k_1} \sum_{i_1=1}^{l_1} \omega_1^{i_1} J_{k_1}(s-(i_1+j-k_1)\delta_j) \equiv A_{1,1}^{(2)} + A_{1,2}^{(2)}. \end{aligned}$$

We introduce the new (relative to (4.8)) value of displacements $\delta_{j-1} = [c_1 c(\varepsilon_0)\Delta_{j-1}]^{1/(l_0+1)} \times\omega_0^{-1/(l_0+1)}$ and estimate the terms from $A_{1,1}^{(2)}$ using (4.11) with $j \mapsto j-1$ and $s \mapsto s-(i_1+1)\delta_j$. For any $s > 1+(i_1+1)\delta_j+(l_2+j-2)\delta_{j-1}$, there holds

$$J_{j-1}(s-(i_1+1)\delta_j) \le \omega_1^{l_2} J_{j-1}(s-(i_1+1)\delta_j - l_2\delta_{j-1})$$
$$+\sum_{k_2=1}^{j-2}\lambda_0^{j-1-k_2}\sum_{i_2=1}^{l_2}\omega_1^{i_2} J_{k_2}(s-(i_1+1)\delta_j-(i_2+j-1-k_2)\delta_{j-1})$$
$$+\alpha_0 c\,\lambda_0^{j-1}\Delta_{j-1}\sum_{i_2=1}^{l_2}\omega_1^{i_2-1} h_0(s-(i_1+1)\delta_j-(i_2+j-1-1)\delta_{j-1}). \tag{4.12}$$

Substituting (4.12) into the expression for $A_{1,1}^{(2)}$, by a simple transformation we get

$$J_j(s) \le A_2^{(1)}(s) + A_2^{(2)}(s) + A_2^{(3)}(s), \quad j \ge 1,$$

where

$$A_2^{(1)} = A_1^{(1)} + \lambda_0 \sum_{i_1=1}^{l_1} \omega_1^{i_1+l_2} J_{j-1}(s-(i_1+1)\delta_j - l_2\delta_{j-1}),$$
$$A_2^{(2)} = A_{1,2}^{(2)} + \sum_{k_2=1}^{j-2}\sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2}\omega_1^{i_1+i_2}\lambda_0^{j-k_2} J_{k_2}(s-(i_1+1)\delta_j-(i_2+j-1-k_2)\delta_{j-1}),$$
$$A_2^{(3)} = A_1^{(3)} + \alpha_0 c\lambda_0^{j}\Delta_{j-1}\sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2}\omega_1^{i_1+i_2-1} h_0(s-(i_1+1)\delta_j-(i_2+j-2)\delta_{j-1}).$$

Since $\delta_j \le \delta_{j-1}$, in view of monotonicity of functions $J_k(s)$ it follows that

$$A_2^{(2)} \le \sum_{k_2=1}^{j-2}\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\omega_1^{i_1+i_2}\lambda_0^{j-k_2} J_{k_2}(s-(i_1+1)\delta_j-(i_2+j-1-k_2)\delta_{j-1}). \tag{4.13}$$

Repeating this cycle of calculations b times, we derive b inequalities for each $j \in \mathbb{N}$,

$$J_j(s) \le A_h^{(1)}(s) + A_h^{(2)}(s) + A_h^{(3)}(s), \ \ h = 1, 2, ..., b, \tag{4.14}$$

where

$$A_h^{(1)} \equiv A_h^{(1)}(s) = \omega_1^{l_1} J_j(s-l_1\delta_j) + \lambda_0\sum_{i_1=1}^{l_1}\omega_1^{i_1+l_2} J_{j-1}(s-(i_1+1)\delta_j - l_2\delta_{j-1}) \tag{4.15}$$
$$+\lambda_0^2\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\omega_1^{i_1+i_2+l_3} J_{j-2}(s-(i_1+1)\delta_j-(i_2+1)\delta_{j-1}-l_3\delta_{j-2}) + ... + \lambda_0^{h-1}$$
$$\times\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_{h-1}=0}^{l_{h-1}}\omega_1^{i_1+i_2+...+i_{h-1}+l_h} J_{j-h+1}(s-(i_1+1)\delta_j-...-(i_{h-1}+1)\delta_{j-h+2}-l_h\delta_{j-h+1}),$$

$$(4.16)\quad A_h^{(2)} = \sum_{k_h=1}^{j-h}\sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}...\sum_{i_h=0}^{l_h}\lambda_0^{j-k_h}\omega_1^{i_1+i_2+...+i_h}J_{k_h}(s-(i_1+1)\delta_j-(i_2+1)\delta_{j-1}-...$$
$$-(i_{h-1}+1)\delta_{j-h+2}-(i_h+j-h-k_h+1)\delta_{j-h+1}),$$

$$(4.17)\quad A_h^{(3)} = \alpha_0 c\lambda_0^j[\Delta_j\sum_{i_1=1}^{l_1}\omega_1^{i_1-1}h_0(s-(i_1+j-1)\delta_j)$$

$$+\Delta_{j-1}\sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2}\omega_1^{i_1+i_2-1}h_0(s-(i_1+1)\delta_j-(i_2+j-2)\delta_{j-1})+...+\Delta_{j-h+1}$$
$$\times\sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2}...\sum_{i_h=1}^{l_h}\omega_1^{i_1+i_2+...+i_h-1}h_0(s-(i_1+1)\delta_j-(i_2+1)\delta_{j-1}-...-(i_h+j-h)\delta_{j-h+1})].$$

Here for $k \le j$,

$$(4.18)\quad \delta_k = [c_1c(\varepsilon_0)\Delta_k/\omega_0]^{1/(l_0+1)} = D_1[w^l(k+K)^{-l-1}]^{1/(l_0+1)},$$

where $D_1 = [l^{(l+1)}c_1c(\varepsilon_0)(1+\gamma)^{-l(l+1)}\omega_0^{-1}]^{1/(l_0+1)}$. It follows from the structure of $A_h^{(3)}$ that (4.14) is valid for all s such that

$$(4.19)\quad s \ge 1 + s_h^{(j)} + \sum_{k=1}^{h-1} i_k\delta_{j-k+1} + l_h\delta_{j-h+1}, \quad s_h^{(j)} = \sum_{i=j-h+2}^{j}\delta_i + (j-h)\delta_{j-h+1}.$$

Let us comment on how to choose parameters l_1, ..., l_b. l_1 is free. Next, for arbitrary $i_1 \le l_1$, we find $l_2 = l_2(i_1)$. Then for any $i_1 \le l_1$ and $i_2 \le l_2$ we find $l_3 = l_3(i_1, i_2)$ and so on. Finally, we find $l_h = l_h(i_1, ..., i_{h-1})$.

Due to the monotonicity of sequence $\{\delta_k\}$, the sequence $\{s_h^{(j)}\}$ is monotone increasing relative to the index h. Therefore, a sufficient condition for (4.19) to be valid is as follows

$$(4.20)\quad s \ge 1 + s_b^{(j)} + \sum_{k=1}^{h-1} i_k\delta_{j-k+1} + l_h\delta_{j-h+1}, \quad h = 2, ..., b.$$

Now we are ready to define the numbers l_1, l_2,..., l_b. Namely, fix a new parameter ξ and let

$$(4.21)\quad b = \xi j, \quad \text{where } \xi \in (0,1).$$

Then, fix another free parameter $\zeta > 0$ and set

$$(4.22)\quad l_h = [(\zeta s_{\xi j}^{(j)} - \sum_{k=1}^{h-1} i_k\delta_{j-k+1})/\delta_{j-h+1}], \quad h = 2, ..., \xi j, \quad l_1 = [\zeta s_{\xi j}^{(j)}\delta_j^{-1}],$$

where $[\cdot]$ denotes the entire part. Due to definition (4.22) of the number of iteration l_h, inequality (4.20) with $b = \xi j$ is valid for

$$(4.23)\quad s > 1 + (1+\zeta)s_{\xi j}^{(j)}.$$

By definition (4.22) we have $i_1 + ... + i_k + l_{k+1} \geq \zeta s^{(j)}_{\xi j} \delta^{-1}_{j-k} + i_1(1 - \delta_j/\delta_{j-k}) + i_2(1 - \delta_{j-1}/\delta_{j-k}) + ... + i_k(1 - \delta_{j-k+1}/\delta_{j-k}) - 1$. Therefore

$$\sigma_k \equiv \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} ... \sum_{i_k=0}^{l_k} \omega_1^{i_1+...+i_k+l_{k+1}} \leq \omega_1^{-1} e^{\ln \omega_1 \zeta s^{(j)}_{\xi j} \delta^{-1}_{j-k}} \left[\prod_{i=0}^{k-1} (1 - \omega_1^{1-\delta_{j-i}/\delta_{j-k}}) \right]^{-1}. \tag{4.24}$$

We now estimate σ_k from above. Due to the concavity of the function $g_1(t) = 1 - \omega_1^t$ on $t \in (0, 1)$, we deduce that for any $i < k$,

$$1 - \omega_1^{1-\delta_{j-i}/\delta_{j-k}} \geq (1 - \omega_1) \left[1 - \left(\frac{j + K - k}{j + K - i} \right)^{\beta_1} \right], \quad \beta_1 = \frac{l + 1}{l_0 + 1} \leq 1.$$

By the convexity of $g_2(t) = 1 - (1 - t)^{\beta_1}$ on $(0, 1)$, there holds $1 - \left(\frac{j+K-k}{j+K-i}\right)^{\beta_1} \geq \beta_1 \frac{k-i}{j+K-i}$. Therefore, the following inequality is true:

$$1 - \omega_1^{1-\delta_{j-i}/\delta_{j-k}} \geq \beta_1 (1 - \omega_1)(k - i)/(j + K - i), \quad i < k. \tag{4.25}$$

As a consequence, denoting the binomial coefficients $C^k_{j+K} = (j + K)!/(j + K - k)!k!$, we derive

$$\sigma_k \leq \omega_1^{-1} \exp\{\ln \omega_1 \zeta s^{(j)}_{\xi j} \delta^{-1}_{j-k}\} \beta_1^{-k} (1 - \omega_1)^{-k} C^k_{j+K}. \tag{4.26}$$

It follows from (4.19) that

$$s^{(j)}_{\xi j} = D_1 w^{l/(l_0+1)} [R_1(l) + (1 - \xi) j (j(1 - \xi) + 1 + K)^{-\beta_1}], \tag{4.27}$$

where $R_1(l) = \sum_{i=(1-\xi)j+2}^{j} (i + K)^{-\beta_1}$ and $\beta_1 = (l + 1)/(l_0 + 1)$. It is easy to see that

$$R_1(l) \geq (1 - \beta_1)^{-1} [(j + K + 1)^{1-\beta_1} - (j(1 - \xi) + K + 2)^{1-\beta_1}] \quad \text{for any } l < l_0,$$

$$R_1(l_0) \geq \ln \left[\frac{j + K + 1}{(1 - \xi)[j + (2 + K)(1 - \xi)^{-1}]} \right].$$

Therefore, for any $j \geq 1$, $k \leq \xi j$, $0 < \xi < 1$, there holds

$$s^{(j)}_{\xi j} / \delta_{j-k} \geq d_4 j, \tag{4.28}$$

where constant $d_4 > 0$ does not depend on ξ. Due to (4.28), it follows from (4.26) that

$$\sigma_k \leq [\beta_1 (1 - \omega_1)]^{-k} \omega_1^{\zeta d_4 j} C^k_{j+K}, \quad k \leq \xi j. \tag{4.29}$$

Then, returning to the representation of $A^{(1)}_h$ given by (4.14), we obtain

$$A^{(1)}_{\xi j}(s) \leq \omega_1^{l_1} J_j(1) + \sum_{i=1}^{\xi j-1} \sigma_i \lambda_0^i J_{j-i}(1) \leq \omega_1^{\zeta d_4 j} \sum_{i=0}^{\xi j-1} \lambda_0^i C^i_{j+K} [\beta_1 (1 - \omega_1)]^{-i} J_{j-i}(1) \tag{4.30}$$

for any $s > 1 + (1 + \zeta) s^{(j)}_{\xi j}$ and $\xi \in (0, 1)$. It is easy to check (see Lemma 4.4 in [8]) the following global estimate:

$$\int_{\Omega(1)} |u(x, t)|^{p+1} dx \leq d_3 F(t), \quad t \in (t_0, T). \tag{4.31}$$

By estimate (4.31), definition (4.7) and properties (4.4), (4.6) we have

$$J_{j-i}(1) = I_{j-i}(1)\alpha_{j-i} \le d_3 F(t_{j-i})\alpha_{j-i} \le d_3 \exp\{w(T-t_{j-i})^{-1/l}\}$$
$$\times \exp\{-(1+\gamma)^l (j-i+K)/l\} \le d_3 \exp\{w(1-K/(K+1)(1+\gamma))(T-t_{j-i})^{-1/l}\}$$
$$\le d_3 \exp\{a_1(j-i+K)\}, \tag{4.32}$$

where $a_1 = (Kl)^{-1}(K+1)(1+\gamma)^{l+1}[1-K/(K+1)(1+\gamma)]$. Therefore, inequality (4.30) yields

$$A^{(1)}_{\xi j} \le d_3 \exp\{j(a_1 - \ln \omega_1^{-1}\zeta d_4 + a_1 K)\} \sum_{i=0}^{\xi j-1} C^i_{j+K} R^i$$
$$\le d_3 \exp\{j(a_1 - \ln\omega_1^{-1}\zeta d_4 + \ln(1+R)) + K(a_1 + \ln(1+R))\}, \tag{4.33}$$

where $R = \lambda_0/\exp a_1 \beta_1 (1-\omega_1)$. Let us fix the free parameter ζ such that

$$\zeta d_4 \ln \omega_1^{-1} - a_1 - \ln(1+R) = 1 \implies \zeta = \frac{1 + a_1 + \ln(1+R)}{d_1 \ln \omega_1^{-1}}. \tag{4.34}$$

Then from (4.33) we deduce the following estimate:

$$A^{(1)}_{\xi j}(s) \le d_3 C_1 \equiv d_3 \exp\{K(a_1 + \ln(1+R)) - 1\},\ j \ge 1,\ \xi \in (0,1), \tag{4.35}$$

for $s > 1 + (1+\zeta)s^{(j)}_{\xi j}$. Now we estimate the term $A^{(2)}_{\xi j}$. It follows from (4.32) that

$$A^{(2)}_{\xi j} \le \sum_{k=1}^{j(1-\xi)} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \dots \sum_{i_{\xi j}=0}^{l_{\xi j}} \lambda_0^{j-k} \omega_1^{i_1+i_2+\dots+i_{\xi j}} J_k(1) \le \omega_1 (1-\omega_1)^{-\xi j} \sum_{k=1}^{j(1-\xi)} \lambda_0^{j-k} J_k(1)$$
$$\le d_3 \omega_1 \exp\{a_1 K\}(1-\omega_1)^{-\xi j} \sum_{k=1}^{j(1-\xi)} \lambda_0^{j-k} \exp\{a_1 k\}$$
$$= d_3\omega_1 \exp\{a_1 K + a_1(1-\xi)j\}\lambda_0^{\xi j}(1-\omega_1)^{-\xi j} \sum_{k=1}^{j(1-\xi)} (\lambda_0 \exp\{-a_1\})^{j(1-\xi)-k}$$
$$\le d_3 \omega_1 \exp\{a_1(K+1)\}(\exp a_1 - \lambda_0)^{-1} \exp\{-j[\xi \ln((1-\omega_1)/\lambda_0) - a_1(1-\xi)]\}. \tag{4.36}$$

In view of definition of numbers ω_1 and λ_0 (see (4.9) and (4.10)) we have $(1-\omega_1)/\lambda_0 \equiv \exp\{l^{-1}(1+\gamma)^l\}/(1+\omega_0)(1+\varepsilon_0)$. Therefore, we can fix γ so large that in addition to condition (4.10), the following inequality holds:

$$(1-\omega_1)/\lambda_0 = 1 + R_2, \quad \text{with } R_2 > 0. \tag{4.37}$$

We now define the free parameter ξ such that

$$\xi \ln(1+R_2) - a_1(1-\xi) > 0 \implies 1 > \xi > a_1/[a_1 + \ln(1+R_2)]. \tag{4.38}$$

In this case we obtain the uniform estimate

$$A^{(2)}_{\xi j}(s) \le d_3 C_2 \equiv d_3 \omega_1 \exp\{a_1(K+1)\}/(\exp a_1 - \lambda_0),\ \ j \ge 1, \tag{4.39}$$

where ξ is from (4.38) and ζ is as given in (4.34). It is easy to estimate $A^{(3)}_{\xi j}(s)$ from above. Since $\omega_1/(1-\omega_1) = \omega_0 < 1$ and $\omega_1(1+\omega_0) = \omega_0$, we obtain using a priori estimate (4.31) that

$$A^{(3)}_{\xi j}(s) \leq c_1\alpha_0\lambda_0^j[\omega_1(1+\omega_0)]^{-1}h_0(1)\sum_{i=1}^{\xi j}\Delta_{j-i+1}\omega_0^i \leq c_1\alpha_0 h_0(1)(1-\omega_0)^{-1}$$

$$\times\,\Delta_{j(1-\xi)+1}\lambda_0^j \leq d_3c_1\alpha_0F(t_0)\Delta_1(1-\omega_0)^{-1} \equiv d_3C_3. \tag{4.40}$$

Combining estimates (4.35), (4.39), (4.40), we deduce from (4.14) that

$$J_j(s) \leq d_3C_4 \equiv d_3(C_1+C_2+C_3), \quad j\geq 1,\ s > 1+(1+\zeta)s^{(j)}_{\xi j}. \tag{4.41}$$

In view of definition of $J_j(s)$ and (4.7), it follows that

$$I_j(s) \leq d_3C_4\alpha_j^{-1} = d_3C_4\exp\{(1+\gamma)^l(j+K)/l\}.$$

Therefore, by (4.6) we get

$$I^{(t_j)}(s) = \int_0^{t_j}\int_{\Omega(s)}|u(x,t)|^{p+1}dxdt \leq d_3C_4\sum_{i=0}^{j}\exp\{(1+\gamma)^l(i+K)/l\}$$

$$\leq d_3C_5\exp\{(1+\gamma)^l(j+K)/l\} \leq d_3C_5\exp\{[w/(1+\gamma)](T-t_j)^{-1/l}\}, \tag{4.42}$$

where $C_5 = C_4[1-\exp\{-(1+\gamma)^l/l]^{-1}$. Let us estimate $s^{(j)}_{\xi j}$ from above. Due to (4.27) it follows that

$$s^{(j)}_{\xi j} \leq D_1w^{l/(l_0+1)}[(1-\beta_1)^{-1}((j+K)^{1-\beta_1}-((1-\xi)j+K)^{1-\beta_1})+((1-\xi)j(j(1-\xi)+K))^{-\beta_1}].$$

After simple computations we then arrive at the estimate

$$s^{(j)}_{\xi j} \leq D_2w^{l/(l_0+1)}(j+K)^{1-\beta_1}, \quad j\geq 1;\ \ D_2 = D_1(1+\ln(1-\xi)^{-1}), \tag{4.43}$$

which implies due to (4.4) that $s^{(j)}_{\xi j} \leq D_3w^{l_0/(l_0+1)}(T-t_j)^{-(1-\beta_1)/l}$, with $D_3 = D_2l^{1-\beta_1}(1+\gamma)^{-(1-\beta_1)(l+1)}$, and hence

$$s^{(j)}_{\xi j} \leq D_4w^{l_0/(l_0+1)}(T-t_{j-1})^{-(1-\beta_1)/l}, \quad D_4 = D_3[K/(K-1)]^{1-\beta_1}. \tag{4.44}$$

By monotonicity of $I^{(t)}(s)$, (4.42), (4.4) and (4.44), we get for arbitrary $t\in(t_{j-1},t_j)$,

$$I^{(t)}(s) \leq d_3C_6\exp\{(\gamma+1)^l(j-1+K)/l\} \leq d_3C_6\exp\{[w/(1+\gamma)](T-t)^{-1/l}\}$$

for $s > 1+(1+\zeta)D_4w^{l_0/(l_0+1)}(T-t)^{-(1-\beta_1)/l}|$, where $C_6 = C_5\exp\{(\gamma+1)^l/l\}$. Finally, denoting $w_1 = w/(1+\gamma)$, we obtain the following first iteration of the boundary estimate (4.31):

$$I^{(t)}(s) \leq d_3C_6\exp\{w_1(T-t)^{-1/l}\},\ s > 1+(1+\zeta)D_4w^{l_0/(l_0+1)}(T-t)^{-(1-\beta_1)/l}. \tag{4.45}$$

Recall that the initial boundary estimate took the form

$$I^{(t)}(1) \leq d_3\exp\{w(T-t)^{-1/l}\}, \quad t\in(0,T). \tag{4.46}$$

Fix an arbitrary $\tilde{t} \in (0, T)$ and denote

$$s_1 = 1 + (\zeta + 1) D_4 w^{l_0/(l_0+1)} (T - \tilde{t})^{-(1-\beta_1)/l}. \tag{4.47}$$

For the second cycle of calculations we use the following initial estimate, which is a consequence of (4.46):

$$I^{(t)}(s_1) \le d_3 C_6 \exp\{w_1 (T - t)^{-1/l}\}, \quad t \in (0, \tilde{t}]. \tag{4.48}$$

Using this estimate instead of (4.31) and repeating all calculations began from (4.2) with the new partition

$$\Delta_i = [w_1/(1+\gamma)^{l+1}]^l [l/(i+K)]^{l+1}, \tag{4.49}$$

instead of (4.3), we arrive at the estimate

$$I^{(t)}(s_2) \le d_3 C_6^2 \exp\{w_2 (T - t)^{-1/l}\}, \quad t \in (0, \tilde{t}], \tag{4.50}$$

where $s_2 = s_1 + (1+\zeta) D_4 w_1^{l_0/(l_0+1)} (T - \tilde{t})^{-(1-\beta_1/l}$, $w_2 = w_1/(1+\gamma)$. Estimate (4.50) will be used for the next cycle of calculations similar to the previous two. After M cycles we obtain

$$I^{(t)}(s_M) \le d_3 C_6^M \exp\{w_M (T - t)^{-1/l}\}, \quad t \in (0, \tilde{t}],$$

$$\begin{aligned} s_M = 1 &+ (1+\zeta) D_4 (T - \tilde{t})^{-(1-\beta_1)/l} w^{l_0/(l_0+1)} \sum_{i=1}^{M} (1+\gamma)^{-(i-1)l_0/(l_0+1)} \\ &\le 1 + (1+\zeta) D_5 (T - \tilde{t})^{-(1-\beta_1)/l} w^{l_0/(l_0+1)} \equiv s_\infty(\tilde{t}), \beta_1 = (l+1)/(l_0+1), \end{aligned} \tag{4.51}$$

where $w_m = w(1+\gamma)^{-M}$ and $D_5 = D_4[1 - (1+\gamma)^{-l_0/(l_0+1)}]^{-1}$. As a consequence, we have

$$I^{(t)}(s_\infty(\tilde{t})) \le d_3 C_6^M \exp\{w_M (T - t)^{-1/l}\} \quad \text{for } t \in (0, \tilde{t}] \text{ and any } M \in \mathbb{N}. \tag{4.52}$$

We now optimize estimate (4.52) with respect to the free parameter $M \in \mathbb{N}$. Namely, let us take M from the condition

$$(1+\gamma)^M = w(T - \tilde{t})^{-1/l} \implies M = [\ln w + l^{-1} \ln(T - \tilde{t})]/\ln(1+\gamma).$$

Then $C_6^M = w^{\ln C_6/\ln(1+\gamma)} (T - \tilde{t})^{-\ln C_6/l \ln(1+\gamma)}$, and therefore it follows form (4.52) that

$$I^{(t)}(s_\infty) \le d_3 w^{\ln C_6/\ln(1+\gamma)} (T - \tilde{t})^{-\ln C_6/l \ln(1+\gamma)}, \quad t \in (0, \tilde{t}], \tag{4.53}$$

where $s_\infty(\tilde{t})$ is as given in (4.51). Since $\tilde{t}$ is arbitrary, we conclude from (4.53) that

$$I^{(t)}(s_\infty(t)) \le d_3 w^{\ln C_6/\ln(1+\gamma)} (T - t)^{-\ln C_6/l \ln(1+\gamma)}, \quad t \in (0, T). \tag{4.54}$$

Fixing an arbitrary $\bar{t} \in (0, T)$, we obtain from (4.54)

$$I^{(t)}(s_\infty(\bar{t})) \le d_3 w^{\ln C_6/\ln(1+\gamma)} (T - t)^{-\ln C_6/l \ln(1+\gamma)}, \quad t \in (0, \bar{t}].$$

Such blow-up regimes have been studied in [8], see Theorem 1.1 there, and it was established that these are LS-blow-up regimes localized in the domain $\{(x,t) : |x| \ge s_\infty(\bar{t}),\, t \in (0, \bar{t}]\}$. Therefore, we derive the following estimate: for any $\varepsilon > 0$,

$$I^{(\bar{t})}(s_\infty(\bar{t}) + \varepsilon) \le C = C(\varepsilon, d_3, w, C_6) \quad \text{for any } \bar{t} \in (0, T), \tag{4.55}$$

which yields necessary estimate (1.12). $\square$

Proof of Theorem 1.2. It repeats the proof of the previous one with some technical changes. Let us point out the main new ingredients of the proof. We will begin from the system of functional inequalities (4.2) with the displacements $\{\Delta_i\}$ given in (4.3) containing an extra logarithmic factor,

$$\Delta_i = D[\ln(i+K)]^{\beta l}(i+K)^{-l-1}, \quad D = w^l[l/(1+\gamma)^l]^{l+1}. \tag{4.56}$$

We then obtain (cf. (4.4))

$$D\int_{j+1}^{\infty}[\ln(s+K)]^{\beta l}(s+K)^{-l-1}ds \le T - t_j \le D\int_{j}^{\infty}[\ln(s+K)]^{\beta l}(s+K)^{-l-1}ds$$

and hence

$$\frac{D}{l}\frac{[\ln(1+j+K)]^{\beta l}}{(j+1+K)^l} \le T - t_j \le \frac{D}{l}\left(1-\frac{\beta}{\ln(j+K)}\right)^{-1}\frac{[\ln(j+K)]^{\beta l}}{(j+K)^l}. \tag{4.57}$$

Setting $j=0$ in (4.57) yields the following restriction on the constant K:

$$K \ge K_0(T), \text{ where } K_0 \text{ solves } \frac{D}{l}\left(1-\frac{\beta}{\ln K_0}\right)^{-1}\frac{(\ln K_0)^{\beta l}}{K_0^l} = T. \tag{4.58}$$

We then arrive at the following estimates similar to (4.6) in Section 4:

$$\frac{(1+\gamma)^l}{l^{1-\beta}}(1-\frac{\beta}{\ln K})^{1/l}(j+K) \le w(T-t_j)^{-1/l}[\ln(T-t_j)^{-1}]^{\beta}(1+\gamma)^{-1}, \tag{4.59}$$

$$w(T-t_j)^{-1/l}[\ln(T-t_j)^{-1}]^{\beta}(1+\gamma)^{-1} \le \frac{(1+\gamma)^l}{l^{1-\beta}}\frac{K+1}{K}(j+K). \tag{4.60}$$

Define now the weights α_i as follows:

$$\alpha_i = \exp\{-(1+\gamma)^l(i+K)/l^{1-\beta}\}, \quad i \in \mathbb{N}, \tag{4.61}$$

Let us fix arbitrary $j \in \mathbb{N}$ and introduce the displacements δ_i for all $i \le j$ as follows,

$$\delta_i = [c_1 c(\varepsilon_0)\Delta_j\omega_0^{-1}]^{1/(l_0+1)} = [l^{l+1}c_1 c(\varepsilon_0)w^l(1+\gamma)^{-l(l+1)}(j+K)^{-l-1}\ln^{\beta l}(j+K)]^{1/(l_0+1)}. \tag{4.62}$$

Substituting these Δ_i, δ_i and α_i for $i \le j$ into (4.2), we obtain system (4.9) for energy functions $J_i(s) = \alpha_i I_i(s)$, $i \le j$, with α_i as in (4.61) and $\lambda_0 = (1+\varepsilon_0)\alpha_i/\alpha_{i-1} = (1+\varepsilon_0)\exp\{-(1+\gamma)^l l^{\beta-1}\} < 1$. Iterating this system as in the proof of Theorem 1.1 by using $b < j$ cycles, we derive system (4.14) with necessary changes of the parameters. In particular, in this new system, instead of (4.18), we have

$$\delta_k = [c_1 c(\varepsilon_0)\Delta_k\omega_0^{-1}]^{1/(l_0+1)} = [l^{l+1}c_1 c(\varepsilon_0)w^l(1+\gamma)^{-l(l+1)}(k+K)^{-l-1}\ln^{\beta l}(k+K)]^{1/(l_0+1)}$$
$$\equiv D_1[w^l(k+K)^{-l-1}\ln^{\beta l}(k+K)]^{1/(l_0+1)}. \tag{4.63}$$

Moreover, system (4.14) remains valid for all s satisfying (4.19) with δ_k given by (4.63) instead of (4.18). Fix a $\xi \in (0,1)$ and define the number b of cycles by (4.21). Next, we define $s_h^{(j)}$ for $h = 1,2,...,j\xi$ by (4.19) with δ_k from (4.63) and the numbers of iteration

l_h by (4.22). As in the proof of Theorem 1.1, we need to estimate from below the term $1-\delta_{j-h}/\delta_{j-k}$ for any $j \in \mathbb{N}$ and any $h < k$. We have

$$(4.64)\quad 1-\delta_{j-h}/\delta_{j-k} \equiv 1-g(j-k+K)/g(j-h+K) = 1-g(s-\tau)/g(s) \equiv \Psi(\tau) \geq 0,$$

where $s = j-h+K > K$, $\tau = k-h \geq 1$, $s-\tau = j-k+K \geq K$ and $g(v) = v^{\beta_1}\ln^{-a_2} v$ for $v > 1$, with $\beta_1 = (l+1)/(l_0+1)$, $a_2 = \beta l/(l_0+1)$. It is easy to see that $\Psi'(w) > 0$ for $w \in [0, s-K]$, $s > K$, $K > K_0^{(1)} = \exp(a_2/\beta_1)$ and $\Psi''(w) > 0$ for $w \in [0, s-K]$, $s > K$, $K > K_0^{(2)} = \exp z_0$, where $z_0 > 0$ solves the equation

$$\beta_1(1-\beta_1)z_0^2 + a_2(2\beta_1-1)z_0 - a_2(a_2+1) = 0.$$

Therefore, since $\Psi(0) = 0$, we have

$$(4.65)\qquad 1-\frac{\delta_{j-h}}{\delta_{j-k}} \geq \Psi'(w)|_{w=0}\tau > (\beta_1 - \frac{a_2}{\ln(j-h+K)})\frac{k-h}{j-h+K} \geq \frac{\beta_1}{2}\frac{(k-h)}{(j-h+K)}$$

for arbitrary $j > k > h$ and $K > \max\{(K_0^{(1)})^2, K_0^{(2)}\}$. Instead of (4.26) we then obtain

$$(4.66)\qquad \sigma_k \leq \omega_1^{-1}\exp\{\ln\omega_1\zeta s_{\xi j}^{(j)}\delta_{j-k}^{-1}\}C_{j+K}^k[2/\beta_1(1-\omega_1)]^k$$

for K from (4.65) in addition satisfying (4.58). Here,

$$s_{\xi j}^{(j)} = D_1 w^{l/(l_0+1)}\{\sum_{i=(1-\xi)j+2}^{j}(i+K)^{-\beta_1}\ln^{a_2}(i+K)$$
$$(4.67)\qquad +(1-\xi)j[j(1-\xi)+K+1]^{-\beta_1}\ln^{a_2}[j(1-\xi)+K+1]\}.$$

Since

$$\sum_{i=i_0}^{j}(i+K)^{-\beta_1}\ln^{a_2}(i+K) \geq \int_{i_0}^{j}(\tau+K)^{-\beta_1}\ln^{a_2}(\tau+K)d\tau$$
$$\geq b_1[(j+K)^{1-\beta_1}\ln^{a_2}(j+K) - (i_0+K)^{1-\beta_1}\ln^{a_2}(i_0+K)],$$

where $b_1 = (1-\beta_1)/[1+a_2/\ln K(1-\beta_1)]$, we obtain an estimate similar to (4.28),

$$(4.68)\qquad s_{\xi j}^{(j)}/\delta_{j-k} \geq \tilde{d}_4 j, \quad \forall j \in \mathbb{N}, \quad \forall k < \xi j, \quad \tilde{d}_4 > 0.$$

It then follows from (4.66) that (cf. (4.29))

$$(4.69)\qquad \sigma_k \leq [2/\beta_1(1-\omega_1)]^k\omega_1^{\zeta\tilde{d}_4 j}C_{j+K}^k, \quad k < \xi j.$$

Instead of (4.32), due to (4.59) and (4.60) we have

$$J_{j-i}(1) = I_{j-i}(1)\alpha_{j-i} \leq d_3F(t_{j-i})\alpha_{j-i} \leq d_3\exp\{w(T-t_{j-i})^{-1/l}|\ln(T-t_j)|^\beta\}$$
$$\times\exp\{-(1+\gamma)^l(j-i+K)/l^{1-\beta}\} \leq d_3\exp\{w(1-K/(K+1)(1+\gamma))(T-t_{j-i})^{-1/l}$$
$$(4.70)\qquad \times\ln(T-t_j)|^\beta\} \leq d_3\exp\{\tilde{a}_1(j-i+K)\},$$

where $\tilde{a}_1 = K^{-1}l^{\beta-1}(K+1)(1+\gamma)^{l+1}[1 - K/(K+1)(1+\gamma)]$. Then, similar to (4.30) we conclude that

$$A_{\xi j}^{(1)}(s) \le \omega_1^{\zeta \tilde{d}_4 j} \sum_{i=0}^{\xi j-1} \lambda_0^i C_{j+K}^i [2(l_0+1)/(l+1)(1-\omega_1)]^{-i} J_{j-i}(1)$$

$$\le d_3 \exp\{j[\tilde{a}_1 - \ln \omega_1^{-1} \zeta \tilde{d}_4 + \ln(1+\tilde{R})] + K(\tilde{a}_1 + \ln(1+\tilde{R}))\}, \tag{4.71}$$

for $s > 1 + (1+\zeta)s_{\xi j}^{(j)}$, where $s_{\xi j}^{(j)}$ is given in (4.67) and $\tilde{R} = 2\lambda_0/e^{\tilde{a}_1}\beta_1(1-\omega_1)$. Choosing ζ from the equality

$$\zeta \tilde{d}_4 \ln \omega_1^{-1} - \tilde{a}_1 - \ln(1+\tilde{R}) = 1 \implies \zeta = [1 + \tilde{a}_1 + \ln(1+\tilde{R})]/\tilde{d}_4 \ln \omega_1^{-1},$$

we obtain estimate similar to (4.35). Fixing now $\gamma \gg 1$ such that

$$(1-\omega_1)/\lambda_0 \equiv \omega_0 \exp\{l^{\beta-1}(1+\gamma)^l\}/(1+\omega_0)(1+\varepsilon_0) = 1 + R_2, \quad \text{with } R_2 > 0,$$

and choosing ξ satisfying

$$\xi \ln(1+R_2) - \tilde{a}_1(1-\xi) > 0 \implies \tilde{a}_1/[\tilde{a}_1 + \ln(1+R_2)] = \xi_0 < \xi < 1,$$

we deduce an upper estimate for $A_{\xi j}^{(2)}(s)$ similar to (4.39). Next, in a similar way we derive an estimate on $A_{\xi j}^{(3)}(s)$ of the type (4.40). Finally, we arrive at (4.41) and (cf. (4.42))

$$I^{(t_j)}(s) = \sum_{i=0}^{j} I_i(s) \le \sum_{i=0}^{j} J_i(s)\alpha_i^{-1} \le d_3 C \sum_{i=0}^{j} \exp\{(1+\gamma)^l (i+K)/l^{1-\beta}\}$$

$$\le d_3 C_1 \exp\{(1+\gamma)^l (j+K)/l^{1-\beta}\}, \quad s > 1 + (\zeta+1)s_{\xi j}^{(j)}. \tag{4.72}$$

Using (4.67), we deduce the following estimate similar to (4.43):

$$s_{\xi j}^{(j)} \le \tilde{D}_2 w^{l/(l_0+1)} (j+K)^{1-\beta_1} \ln^{a_2}(j+K). \tag{4.73}$$

Then due to (4.59) we get an estimate similar to (4.44)

$$s_{\xi j}^{(j)} \le \tilde{D}_4 w^{l_0/(l_0+1)} (T - t_{j-1})^{-(1-\beta_1)/l} (\ln(T - t_{j-1})^{-1})^{\beta l_0/(l_0+1)}. \tag{4.74}$$

By the monotonicity of $I^{(t)}(s)$ and by (4.59), we deduce from (4.72) that

$$I^{(t)}(s) \le d_3 C_2 \exp\{w_1 (T-t)^{-1/l} (\ln(T-t)^{-1})^{\beta}\}, \quad t \in (0,T), \tag{4.75}$$

where $w_1 = w/(1+\gamma)(1-\beta/\ln K)$ and $K \gg 1$ is such that $(1+\gamma)(1-\beta/\ln K) \ge 1 + \gamma/2 \equiv 1 + \gamma_1$. Then by using (4.74) we obtain (4.75) with $w_1 = w/(1+\gamma_1)$ for all

$$s \ge 1 + (1+\zeta)\tilde{D}_5 w^{l_0/(l_0+1)} (T-t)^{-(1-\beta_1)/l} (\ln(T-t)^{-1})^{\beta l_0/(l_0+1)} = s_1(t). \tag{4.76}$$

Repeating the described cycle of calculations, in a way similar to deriving (4.52) from (4.45), we obtain the estimate for any $M \in \mathbb{N}$ and $t \in (0, \tilde{t}]$,

$$I^{(t)}(s_\infty) \le d_3 C_2^M \exp\{w(1+\gamma_1)^{-M} (T-t)^{-(1-\beta_1)/l} (\ln(T-t)^{-1})^{\beta}\},$$

$$s_\infty(t) = 1 + (1+\zeta)\tilde{D}_5 [1 - (1+\gamma_1)^{-l_0/(l_0+1)}]^{-1} w^{l_0/(l_0+1)} (T-t)^{-(1-\beta_1)/l}$$

$$\times (\ln(T-t)^{-1})^{\beta l_0/(l_0+1)}. \tag{4.77}$$

As in the proof of Theorem 1.1, we obtain (4.55) completing the proof of Theorem 1.2. $\square$

5. Super-exponential blow-up: proof of Theorem 1.3

Proof of Theorem 1.3. We return to system (4.2) and choose the time-partition

$$\Delta_j = w(j+K)^{-1}\ln^{-2}[(j+K)/w], \quad j=1,2,..., \tag{5.1}$$

the constant w being defined in (1.15). Then

$$\ln^{-1}[(j+K+1)/w] \le (T-t_j) = \sum_{i=j+1}^{\infty} \Delta_i \le \ln^{-1}[(j+K)/w], \tag{5.2}$$

and hence,

$$e^{j+K} \le F_3(t_j) \equiv \exp\{we^{(T-t_j)^{-1}}\} \le e^{j+K+1}. \tag{5.3}$$

The sequence of weights $\{\alpha_i\}$ is now defined as follows:

$$\alpha_j = \exp\{-(j+K)/(1+\gamma)\}, \quad \gamma > 0, \tag{5.4}$$

and we put $\varepsilon_i \equiv \varepsilon$ satisfying the condition

$$(1+\varepsilon)\alpha_j/\alpha_{j-1} = (1+\varepsilon)\exp\{-1/(1+\gamma)\} = \lambda_0 < 1. \tag{5.5}$$

Finally, let us fix $j \in \mathbb{N}$ and define the space-partition by (4.8) for any $i \le j$ so that

$$\delta_i = \delta_j = D_1\{w\omega_0^{-1}(j+K)^{-1}\ln^{-2}[(j+K)/w]\}^{1/(l_0+1)}, \quad D_1 = (c_1 c(\varepsilon))^{1/(l_0+1)}. \tag{5.6}$$

Substituting these partitions into (4.2), we obtain the corresponding system (4.9). Iterating it as in the previous section, we derive estimate (4.14) with the three summands given by (4.15) - (4.17) for $h \le \xi j$, where

$$\delta_k = D_1\{w\omega_0^{-1}(k+K)^{-1}\ln^{-2}[(k+K)/w]\}^{1/(l_0+1)}, \quad k \in \mathbb{N}, \tag{5.7}$$

and s satisfies (4.19) which is guaranteed by (4.20). We fix the parameter $\zeta > 0$ and define l_h by (4.22). Then we obtain the same formula (4.24) for σ_k, (4.25) with $\beta_1 = 1/(l_0+1)$ and (4.26). Let us estimate $s_{\xi j}^{(j)}$ using (5.7),

$$\begin{aligned} s_{\xi j}^{(j)} = D_1(\omega_0^{-1}w)^{1/(l_0+1)}\{ &\sum_{i=(1-\xi)j+2}^{j} [(i+K)^{-1}\ln^{-2}((i+K)/w)]^{1/(l_0+1)} \\ &+ [(1-\xi)j(j(1-\xi)+1+K)^{-1}\ln^{-2}((j(1-\xi)+1+K)/w)]^{1/(l_0+1)}. \end{aligned}$$

Then it is easy to conclude that

$$d_4 j\delta_j \le s_{\xi j}^{(j)} \le d_5 j\delta_j, \quad j \in \mathbb{N}, \tag{5.8}$$

where positive constants d_4 and d_5 do not depend on ξ and j. By the same computations we arrive at estimate (4.41) and consequently

$$I_j(s) \le d_3 C_4 \alpha_j^{-1} = d_3 C_4 \exp\{(j+K)/(1+\gamma)\}, \quad j \ge 1, \ s > 1 + (1+\zeta)s_{\xi j}^{(j)}.$$

Hence, $I^{(t_j)}(s) \le d_3 C_5 \exp\{(j+K)/(1+\gamma)\}$, where $C_5 = C_4/[1-\exp\{-1/\gamma\}]$. Since $j + K \le \ln F_3(t_j) \equiv w \exp\{(T-t_j)^{-1}\}$ by (5.3), we deduce that $I^{(t_j)}(s) \le d_3 C_5 \exp\{w \exp(T - t_j)^{-1}/(1+\gamma)\} = d_3 C_5 [F_3(t_j)]^{1/(1+\gamma)}$, and this estimate can be extended on $(0,T)$ to give

$$I^{(t)}(s) \le d_3 C_6 [F_3(t)]^{1/(1+\gamma)}, \quad t \in (0,T). \tag{5.9}$$

After simple calculations, which involve expression (5.7) for δ_j and relationship (5.2), we deduce from (5.8) that

$$s_{\xi j}^{(j)} \le d_6 w (T-t_j)^{2/(l_0+1)} \exp\{D(T-t_j)^{-1}\}, \quad D = l_0/(l_0+1)$$

Thus, after the first cycle of manipulations we obtain (5.9) for $s \ge s_1(t) \equiv 1 + (1+\zeta) d_6 w (T-t)^{2/(l_0+1)} \exp\{D(T-t)^{-1}\}$. Starting with this estimate, performing the second iteration as in the previous section, we get $I^{(t)}(s) \le d_3 C_6^2 [F_3(t)]^{1/(1+\gamma)^2}$ for $s \ge s_2(t) \equiv s_1(t) + (1+\zeta) d_6 w (1+\gamma)^{-1} (T-t)^{2/(l_0+1)} \exp\{D(T-t)^{-1}\}$. Similarly, M iterations yield

$$I^{(t)}(s) \le d_3 C_6^M \exp\{w(1+\gamma)^{-M} e^{(T-t)^{-1}}\}$$

for all $s \ge s_\infty(t) \equiv 1 + (1+\gamma)\gamma^{-1} d_6 w (1+\zeta)(T-t)^{2/(l_0+1)} \exp\{D(T-t)^{-1}\}$. Given a $t \approx T$, we choose the optimal number of iterations,

$$M_{\rm opt}(t) \sim [\ln w + \ln\ln(1+\gamma) + (T-t)^{-1} - \ln\ln C_6]/\ln(1+\gamma),$$

and obtain the estimate

$$I^{(t)}(s) \le C_7 \exp\{\Lambda (T-t)^{-1}\} \quad \text{for } \ s \ge s_\infty(t), \tag{5.10}$$

where constants C_7 and Λ are independent of t. Exponential estimates of the type (5.10) corresponds to the blow-up boundary regime which was analyzed in the Theorem 1.1. Application of this theorem to the boundary regime (5.10) completes the proof of Theorem 1.3. $\square$

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