CONSTANT MEAN CURVATURE SURFACES IN 3-DIMENSIONAL SPACE FORMS

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Abstract. This paper presents a unified treatment of constant mean curvature (CMC) surfaces in the simply-connected 3-dimensional space forms $\mathbb{R}^3$, S^3 and H^3 in terms of meromorphic loop Lie algebra valued 1-forms. We discuss global issues such as period problems and asymptotic behaviour involved in the construction of CMC surfaces with non-trivial topology. We prove existence of new examples of complete non-simply-connected CMC surfaces in all three space forms, with a special emphasis on the case where the surface is homeomorphic to a thrice punctured sphere. We explicitly compute the extended frame for any associated family of Delaunay surfaces and prove two general asymptotics results which are applied to our new examples.

Introduction. The Gauß map of a constant mean curvature (CMC) surface in three dimensional Euclidean space $\mathbb{R}^3$ is a harmonic map [28] and such a harmonic map can be obtained as a projection of a horizontal holomorphic map from the universal cover of the surface into a certain loop group [36]. There is a Weierstrass type representation for simply-connected CMC surfaces, commonly called the DPW method [11]. It involves solving a holomorphic complex linear 2×2 system of ordinary differential equations (ODE) with values in a loop group, and subsequently Iwasawa decomposing the solution into two factors, one of which is the moving frame of the surface. Even though in general both of these steps are computationally not explicit, and thus obstructive to keeping track of the topology, the DPW method has been successfully implemented in software [29] and has provided new insights in recent years and enabled the discovery of new examples of non-simply-connected CMC surfaces [10], [19], [20]. In particular, the construction of CMC immersions with the topology of the n-punctured Riemann sphere, the so called n-noids, via the DPW method allows us to present new techniques in solving period problems and describing asymptotics.

The problem in constructing surfaces with non-trivial topology lies in manipulating the ODE by varying both the coefficients as well as the initial condition. There remain many open questions as to how the ODE affects the geometry of the resulting surface since the metric of the resulting surface is buried deep within this procedure. On the other hand, this method allows full control over the Hopf differential, thus yielding information of the location and order of umbilic points, which occur naturally on compact CMC surfaces of genus $g \geq 2$ as well as on non-compact CMC surfaces with at least three ends. When the underlying domain is the n–punctured sphere, the punctures correspond to ends of the surface where the coefficient matrix of the ODE has poles. We show how the pole structure at the punctures determines the geometry of the ends of the surface and prove that simple poles generate Delaunay ends.

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To solve period problems one has to study the monodromy representation of the solution. In order for the resulting immersion to be invariant under deck transformations for one specific value of the loop parameter, the monodromy has to take values in the loop group of special unitary matrices and pass through the identity at that specific value. The case $n = 2$ has been studied in [10], [18] and [19], and several classes of CMC cylinders are now well known. Trinoids, the case $n = 3$, have recently been investigated and are understood when the ends have Delaunay asymptotics [6], [12], [30]. We shall build on these results and construct CMC immersions of the thrice punctured sphere into all three space forms.

We specialise the DPW representation of harmonic maps into symmetric spaces [11] by considering the two sphere $S^2 \cong \mathrm{SU}_2/\mathrm{U}_1$ as the target. As in [4] we use 2×2 matrices and thus obtain a unified treatment of CMC surfaces in the three dimensional space forms with constant sectional curvature 0, 1 and -1 ($\mathbb{R}^3$, S^3 and H^3). Since this construction is local, we consider the transformation behaviour of the various maps involved under deck transformations and characterise the period problem involved in each case. One notable difference in our exposition to the existing literature is that we work exclusively with 'untwisted' loop algebras: Consequences of this approach are that the formulae for the immersions simplify and dressing by simple factors can be performed without leaving the category of 'real' surfaces. In the remaining part of this work we move away from the local theory and discuss global issues such as period problems and asymptotic behaviour, and present new examples of non-simply-connected CMC surfaces.

Let us briefly outline the contents: Section 1 is concerned with a discussion of the matrix models we use for the space forms. Section 2 gathers the necessary background from the theory of loop groups. Sections 3, 4 and 5 provide the local theory of CMC surfaces in space forms in terms of meromorphic loop Lie algebra valued 1-forms. In particular, we present immersion formulae in Theorem 4.1 that arise from our untwisted loop algebra approach. Section 6 lays the foundation for globalising the construction of open CMC surfaces by discussing the notion of monodromy on various levels of our construction and characterising the period problems. In Section 7 we interrupt our exposition and describe the simplest examples with non-trivial fundamental group: Round cylinders and Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3; and we derive a necessary inequality on the weight associated to an end of a Delaunay surface.

In Section 8 we explicitly compute the extended unitary frame for any associated family of Delaunay surfaces and use this in Section 9 to investigate the asymptotics of an immersion arising from a holomorphic perturbation of a meromorphic potential at a regular singular point. For the ambient space $\mathbb{R}^3$ this is done in [30]. We extend these results to the other space forms, using the explicit knowledge of the extended unitary frame given in Section 8.

We return to further examples of CMC cylinders in Section 10 and consider the bubbletons [32] via dressing by simple factors as well as perturbed Delaunay surfaces. In Section 11 we turn to a detailed study of trinoids - CMC immersions of the thrice punctured Riemann sphere. The study of trinoids requires the use of a well-known condition for the simultaneous unitarisability of three matrices. This pointwise condition is then used to obtain the loop group valued unitariser. The constructions of the trinoids are explained in Section 11, and the asymptotic behaviour of their ends is established by applying Section 9.

In Section 12 we discuss how the results of [20] carry over to the space forms S^3 and H^3 and present examples of trinoids dressed by simple factors in all space forms. Furthermore, we prove that dressing by simple factors preserves asymptotic behaviour. (This asymptotic behaviour was not proven in [20].) The results here also apply to the dressed Delaunay surfaces in Section 10.

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1. Conformal immersions into three dimensional space forms

1.1. **Preliminaries and notation.** Let M be a Riemann surface and G a matrix Lie group with Lie algebra $(\mathfrak{g}, [\,,\,])$, where $[A, B] = AB - BA$ is the commutator. For $\alpha,\, \beta \in \Omega^1(M, \mathfrak{g})$ smooth 1–forms on M with values in $\mathfrak{g}$, we define the $\mathfrak{g}$–valued 2–form $[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]$, $X,\, Y \in TM$. Let $L_g : h \mapsto gh$ be left multiplication in G, and $\theta : TG \to \mathfrak{g}$, $v_g \mapsto (dL_{g^{-1}})_g v_g$ the (left) Maurer–Cartan form. It satisfies the Maurer–Cartan equation

$$2\, d\theta + [\theta \wedge \theta] = 0. \tag{1.1}$$

For a map $F : M \to G$, the pullback $\alpha = F^*\theta$ also satisfies (1.1). We shall make frequent use of the Maurer–Cartan Lemma, which asserts that if N is a connected and simply connected smooth manifold, then every solution $\alpha \in \Omega^1(N, \mathfrak{g})$ of (1.1) integrates to a smooth map $F : N \to G$ with $\alpha = F^*\theta$.

When the target of a map is a symmetric space $f : M \to G/K$, we shall denote the canonical projection by $\pi : G \to G/K$ and call a map $F : M \to G$ a lift of $f : M \to G/K$ if $\pi \circ F = f$.

We complexify the tangent bundle TM and decompose $TM^{\mathbb{C}} = T'M \oplus T''M$ into $(1, 0)$ and $(0, 1)$ tangent spaces and write $d = \partial + \overline{\partial}$. We may decompose $\omega \in \Omega^1(M, \mathfrak{g})$ into the $(1, 0)$ part ω', the $(0, 1)$ part ω'' and write $\omega = \omega' + \omega''$. We set the $*$–operator on $\Omega^1(M, \mathfrak{sl}_2(\mathbb{C}))$ to $*\omega = -i\omega' + i\omega''$.

We shall study maps $f : M \to \mathcal{M}$ into the space forms $\mathcal{M} = \mathbb{R}^3$, S^3, H^3. Each space form is endowed with a metric of an ambient vector space. If $\langle \cdot\,, \cdot\rangle$ denotes the complex bilinear extension of such a metric to the complexified ambient vector space and $df = df' + df''$ is the differential of f, then an immersion $f : M \to \mathcal{M}$ is conformal if and only if

$$\langle df',\, df'\rangle = 0. \tag{1.2}$$

We fix the following basis of $\mathfrak{sl}_2(\mathbb{C})$ as

$$\epsilon_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\ \epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \epsilon = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{1.3}$$

and will denote by $\langle \cdot , \cdot \rangle$ the bilinear extension of the Ad–invariant inner product of $\mathfrak{su}_2$ to $\mathfrak{su}_2^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. Then

$$\begin{aligned} &\langle \epsilon_-, \epsilon_- \rangle = \langle \epsilon_+, \epsilon_+ \rangle = 0,\ \epsilon_-^* = -\epsilon_+, \\ &[\epsilon, \epsilon_-] = 2i\epsilon_-,\ [\epsilon_+, \epsilon] = 2i\epsilon_+ \text{ and } [\epsilon_-, \epsilon_+] = i\epsilon. \end{aligned} \tag{1.4}$$

1.2. **Euclidean three space.** We identify Euclidean three space $\mathbb{R}^3$ with the matrix Lie algebra $\mathfrak{su}_2$ of skew-hermitian trace-free 2×2 matrices. The double cover of the isometry group under this identification is $\mathrm{SU}_2 \ltimes \mathfrak{su}_2$. Let $\mathbb{T}$ denote the stabiliser of $\epsilon \in \mathfrak{su}_2$ under the adjoint action of SU_2 on $\mathfrak{su}_2$. We shall view the two-sphere as $S^2 = \mathrm{SU}_2/\mathbb{T}$.

Lemma 1.1. *The mean curvature H of a conformal immersion $f : M \to \mathfrak{su}_2$ is given by $2\, d * df = H\,[df \wedge df]$.*

Proof. Let $U \subset M$ be an open simply connected set with coordinate $z : U \to \mathbb{C}$. Writing $df' = f_z dz$ and $df'' = f_{\bar z} d\bar z$, conformality is equivalent to $\langle f_z, f_z \rangle = \langle f_{\bar z}, f_{\bar z} \rangle = 0$ and the existence of a function $v \in C^\infty(U, \mathbb{R}^+)$ such that $2\langle f_z, f_{\bar z} \rangle = v^2$. Let $N : U \to \mathrm{SU}_2/\mathbb{T}$ be the Gauss map with lift $F : U \to \mathrm{SU}_2$ such that $N = F\,\epsilon\,F^{-1}$ and $df = vF(\epsilon_- dz + \epsilon_+ d\bar z)F^{-1}$. The mean curvature is $H = 2v^{-2}\langle f_{z\bar z}, N \rangle$ and the Hopf differential is $Q\,dz^2$ with $Q = \langle f_{zz}, N \rangle$. Hence $[df \wedge df] = 2iv^2 N dz \wedge d\bar z$. Then $\alpha = F^{-1}dF$ is given by

$$\alpha = \tfrac{1}{2v}\left((-v^2 H dz - 2\overline{Q} d\bar z) i\epsilon_- + (2Q dz + v^2 H d\bar z) i\epsilon_+ - (v_z dz - v_{\bar z} d\bar z) i\epsilon \right). \tag{1.5}$$

This allows us to compute $d * df = iv^2 H N dz \wedge d\bar z$ and proves the claim. ☐

The Maurer-Cartan equation $2d\alpha + [\alpha \wedge \alpha] = 0$ decomposes into $\epsilon_\pm$ components

$$v^2 H_{\bar z} = 2\overline{Q}_z,\ v^2 H_z = 2Q_{\bar z}, \tag{1.6}$$

called the Codazzi equations, and into an ϵ component

$$v^{-1} v_{z\bar z} - v^{-2} v_z v_{\bar z} + \tfrac{1}{4} v^2 H^2 - v^{-2}|Q|^2 = 0, \tag{1.7}$$

the Gauss equation. For $u = 2\log v$, (1.7) reads $u_{z\bar z} + \frac{1}{2}e^u H^2 - 2e^{-u}|Q|^2 = 0$.

1.3. **The three sphere.** We identify the three-sphere $S^3 \subset \mathbb{R}^4$ with $S^3 \cong \mathrm{SU}_2$. The double cover of the isometry group $\mathrm{SO}(4)$ is $\mathrm{SU}_2 \times \mathrm{SU}_2$ via the action $X \mapsto FXG^{-1}$. Let $\langle \cdot , \cdot \rangle$ denote the bilinear extension of the Euclidean inner product of $\mathbb{R}^4$ to $\mathbb{C}^4$ under this identification.

Lemma 1.2. *Let $f : M \to \mathrm{SU}_2$ be a conformal immersion and $\omega = f^{-1}df$. The mean curvature H of f is given by $2\, d * \omega = H\,[\omega \wedge \omega]$.*

Proof. Let $U \subset M$ be an open simply connected set with coordinate $z : U \to \mathbb{C}$. Writing $df' = f_z dz$ and $df'' = f_{\bar z} d\bar z$, conformality is equivalent to $\langle f_z, f_z \rangle = \langle f_{\bar z}, f_{\bar z} \rangle = 0$ and the existence of a function $v \in C^\infty(U, \mathbb{R}^+)$ such that $2\langle f_z, f_{\bar z} \rangle = v^2$. By left invariance, $\langle \omega', \omega' \rangle = \langle df', df' \rangle$, so conformality (1.2) is $\langle \omega', \omega' \rangle = 0$. Conformal immersions into S^3 can be described locally on U by a pair of smooth maps $F, G : U \to \mathrm{SU}_2$ which transform the basis $\{\mathrm{Id}, \epsilon_-, \epsilon_+, \epsilon\}$ into the moving frame $\{f, f_z, f_{\bar z}, N\}$.

Thus $f = FG^{-1}$, $df = vF\,(\epsilon_- dz + \epsilon_+ d\bar z)\,G^{-1}$ and $N = F\,\epsilon\,G^{-1}$. Setting $\alpha = F^{-1}dF$, $\beta = G^{-1}dG$, a computation gives

$$
\begin{aligned}
\alpha &= (-\tfrac{1}{2}v(H+i)dz - v^{-1}\overline{Q}d\bar z)i\epsilon_- & \beta &= (-\tfrac{1}{2}v(H-i)dz - v^{-1}\overline{Q}d\bar z)i\epsilon_- \\
&\quad + (v^{-1}Qdz + \tfrac{1}{2}v(H-i)d\bar z)i\epsilon_+ & &\quad + (v^{-1}Qdz + \tfrac{1}{2}v(H+i)d\bar z)i\epsilon_+ \\
&\quad - (\tfrac{1}{2}v^{-1}(v_z dz - v_{\bar z}d\bar z))i\epsilon, & &\quad - (\tfrac{1}{2}v^{-1}(v_z dz - v_{\bar z}d\bar z))i\epsilon.
\end{aligned}
\tag{1.8}
$$

Using $\omega = G(\alpha - \beta)G^{-1}$ we obtain $d*\omega = iv^2 HG\,\epsilon\,G^{-1}dz\wedge d\bar z$. On the other hand, $[\omega\wedge\omega] = 2iv^2 G\,\epsilon\,G^{-1}dz\wedge d\bar z$, proving the claim. □

The Codazzi equations are the same as in (1.6), while the Gauss equation becomes

$$
v^{-1}v_{z\bar z} - v^{-2}v_z v_{\bar z} + \tfrac{1}{4}v^2(H^2+1) - v^{-2}|Q|^2 = 0. \tag{1.9}
$$

1.4. **Hyperbolic three space.** We identify hyperbolic three-space $H^3 \subset \mathbb{R}^{(3,1)}$ as the symmetric space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$ embedded in the real 4-space of Hermitian symmetric matrices as $[g] \hookrightarrow g\,g^*$, where g^* denotes the complex conjugate transpose of g. The double cover of the isometry group $\mathrm{SO}(3,1)$ of H^3 is $\mathrm{SL}_2(\mathbb{C})$ via the action $X \mapsto FXF^*$.

Lemma 1.3. *For a conformal immersion $f : M \to H^3$ and $\omega = f^{-1}df$, the mean curvature H is given by $2\,d*\omega = i\,H\,[\omega\wedge\omega]$.*

Proof. Let $U \subset M$ be an open simply connected set with coordinate $z : U \to \mathbb{C}$. Writing $df' = f_z dz$ and $df'' = f_{\bar z}d\bar z$, conformality is equivalent to $\langle f_z,\, f_z\rangle = \langle f_{\bar z},\, f_{\bar z}\rangle = 0$ and the existence of a function $v \in C^\infty(U,\mathbb{R}^+)$ such that $2\langle f_z,\, f_{\bar z}\rangle = v^2$. Conformal immersions in H^3 can be described locally on U by a smooth map $F : U \to \mathrm{SL}_2(\mathbb{C})$ such that $f = F\,F^*$, $df = vF(\epsilon_- dz - \epsilon_+ d\bar z)F^*$, and $N = -F\,i\epsilon\,F^*$ for the normal. Setting $\alpha = F^{-1}dF$, a computation gives

$$
\begin{aligned}
\alpha = (\tfrac{1}{2}v(H+1)dz + v^{-1}\overline{Q}d\bar z)\epsilon_- + (v^{-1}Qdz + \tfrac{1}{2}v(H-1)d\bar z)\epsilon_+ \\
- (\tfrac{1}{2}v^{-1}(v_z dz - v_{\bar z}d\bar z))i\epsilon.
\end{aligned}
\tag{1.10}
$$

Then $\omega = F^{*-1}(\alpha + \alpha^*)F^*$ together with $\alpha + \alpha^* = v(\epsilon_- dz - \epsilon_+ d\bar z)$ gives $[\omega\wedge\omega] = -2iv^2 F^{*-1}\,\epsilon\,F^* dz\wedge d\bar z$. On the other hand, $d*\omega = v^2 HF^{*-1}\,\epsilon\,F^* dz\wedge d\bar z$. □

The Codazzi equations are the same as in (1.6), while the Gauss equation becomes

$$
v^{-1}v_{z\bar z} - v^{-2}v_z v_{\bar z} + \tfrac{1}{4}v^2(H^2-1) - v^{-2}|Q|^2 = 0. \tag{1.11}
$$

A direct consequence of the Codazzi equation is that when the mean curvature is constant, the Hopf differential is holomorphic. If (v, H, Q) is a solution of a Gauss equation, then so is $(v, H, \lambda^{-1}Q)$ for all $|\lambda| = 1$. Since the invariants (v, H, Q) determine an immersion up to isometries, a CMC surface comes in an S^1–family, the *associated family*. We reconstruct the local theory of CMC surfaces by building in the loop parameter λ, and thus turn our attention to this next ingredient of the DPW recipe.

2. Loop groups

We introduce various loop groups, state the Iwasawa decomposition and recall the dressing action [8] in our context. For expository accounts consult [1], [15] and [27].

For each real $0 < r \leq 1$, let $C_r = \{\lambda \in \mathbb{C} : |\lambda| = r\}$ and denote the analytic maps of C_r into $\mathrm{SL}_2(\mathbb{C})$ by

$$\Lambda_r \mathrm{SL}_2(\mathbb{C}) = \mathcal{O}(C_r, \mathrm{SL}_2(\mathbb{C})). \tag{2.1}$$

The Lie algebras of these groups are $\Lambda_r \mathfrak{sl}_2(\mathbb{C}) = \mathcal{O}(C_r, \mathfrak{sl}_2(\mathbb{C}))$. For ϵ_+ defined in (1.3) let

$$\mathcal{B} = \left\{ B \in \mathrm{SL}_2(\mathbb{C}) : \mathrm{Ad} B \epsilon_+ = \rho\, \epsilon_+,\ \rho \in \mathbb{R}^*_+ \right\}.$$

We will use the following subgroups of $\Lambda_r \mathrm{SL}_2(\mathbb{C})$: Let $I_r = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and denote by $\mathcal{O}(I_r, \mathrm{SL}_2(\mathbb{C}))$ the analytic maps $B : I_r \to \mathrm{SL}_2(\mathbb{C})$. Define

$$\Lambda_r^+ \mathrm{SL}_2(\mathbb{C}) = \{ B \in \Lambda_r \mathrm{SL}_2(\mathbb{C}) \cap \mathcal{O}(I_r, \mathrm{SL}_2(\mathbb{C})) : B(0) \in \mathcal{B} \}. \tag{2.2}$$

On $A_r = \{\lambda \in \mathbb{C} : r < |\lambda| < 1/r\}$ we define the involution $\varrho : \lambda \mapsto 1/\bar{\lambda}$. For every $F : A_r \to \mathrm{SL}_2(\mathbb{C})$ there is a map $F^* : A_r \to \mathrm{SL}_2(\mathbb{C})$ defined by

$$F^* : \lambda \mapsto \overline{F(1/\bar{\lambda})}^t.$$

Let $\mathcal{O}^*(A_r, \mathrm{SL}_2(\mathbb{C}))$ denote the analytic loops $F : A_r \to \mathrm{SL}_2(\mathbb{C})$ that satisfy

$$F^* = F^{-1}. \tag{2.3}$$

Then $F \in \mathcal{O}^*(A_r, \mathrm{SL}_2(\mathbb{C}))$ implies $F|_{S^1} \in \mathrm{SU}_2$. Define

$$\Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C}) = \Lambda_r \mathrm{SL}_2(\mathbb{C}) \cap \mathcal{O}^*(A_r, \mathrm{SL}_2(\mathbb{C})). \tag{2.4}$$

For $r = 1$ we omit the subscript. Replacing $\mathrm{SL}_2(\mathbb{C})$ by $\mathrm{GL}_2(\mathbb{C})$ in the above definitions we define the analogous loop Lie subgroups of $\Lambda_r \mathrm{GL}_2(\mathbb{C})$. Corresponding to all the above subgroups, we analogously define Lie subalgebras of $\Lambda_r \mathfrak{gl}_2(\mathbb{C})$.

2.1. **Iwasawa decomposition.** Multiplication $\Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C}) \times \Lambda_r^+ \mathrm{SL}_2(\mathbb{C}) \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ is a real-analytic diffeomorphism onto [24] (with respect to the natural smooth manifold structure, as in [25] and Chapter 3 of [27]). The unique splitting of an element $\Phi \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$

$$\Phi = FB,$$

with $F \in \Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C})$ and $B \in \Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$, will be called Iwasawa decomposition. Since $\mathcal{B} \cap \mathrm{SU}_2 = \{\mathrm{Id}\}$, also $\Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C}) \cap \Lambda_r^+ \mathrm{SL}_2(\mathbb{C}) = \{\mathrm{Id}\}$. The normalization $B(0) \in \mathcal{B}$ is a choice to ensure uniqueness of the Iwasawa factorization.

2.2. **Dressing action.** The dressing action of $\Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$ on $\Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C})$ is the composition of left multiplication and Iwasawa decomposition. For $h \in \Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$ and $F \in \Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C})$ let

$$hF = (h \#_r F)\, B \tag{2.5}$$

be the Iwasawa decomposition in $\Lambda_r \mathrm{SL}_2(\mathbb{C})$ with $h \#_r F \in \Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C})$. We say $h \#_r F$ was obtained by r-dressing F by h.

3. Holomorphic potentials

Replacing $Q \mapsto \lambda^{-1}Q$ in the Maurer–Cartan forms (1.5), (1.8) and (1.10) gives $\Lambda\mathfrak{sl}_2(\mathbb{C})$-valued 1-forms of the form

$$\alpha_\lambda = (\alpha_1' + \lambda\alpha_1'')\,\epsilon_- + (\lambda^{-1}\alpha_2' + \alpha_2'')\,\epsilon_+ + (\alpha_3' + \alpha_3'')\,\epsilon. \tag{3.1}$$

Then $2\,d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ decouples into ϵ_-, ϵ_+ and ϵ components

$$\lambda d\alpha_1'' + 2i\lambda\alpha_3' \wedge \alpha_1'' = 2i\alpha_1' \wedge \alpha_3'' - d\alpha_1', \tag{3.2}$$

$$\lambda^{-1}d\alpha_2' + 2i\lambda^{-1}\alpha_2' \wedge \alpha_3'' = 2i\alpha_3' \wedge \alpha_2'' - d\alpha_2'', \tag{3.3}$$

$$d\alpha_3' + i\alpha_1' \wedge \alpha_2'' = i\alpha_2' \wedge \alpha_1'' - d\alpha_3''. \tag{3.4}$$

As the left sides of (3.2) and (3.3) are λ dependent while their right sides are not, both sides of (3.2) and (3.3) must be identically zero. Maps $F_\lambda : M \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ for which $F_\lambda^{-1}dF_\lambda$ is of the form (3.1) will be called r-unitary frames. Define

$$\mathcal{F}_r(M) := \left\{F_\lambda : M \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) : F_\lambda \text{ is an } r\text{-unitary frame}\right\}.$$

If $F_\lambda \in \mathcal{F}_r(M)$ then $\alpha_\lambda = F_\lambda^{-1}dF_\lambda$ is $\mathfrak{su}_2$-valued for $\lambda \in S^1$ and hence $\bar\alpha_1'' = \alpha_2'$, $\bar\alpha_1' = \alpha_2''$ and $\bar\alpha_3' = \alpha_3''$. The crucial idea in [11] is a method that generates r-unitary frames of simply-connected Riemann surfaces as follows. Define

$$\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C}) = \left\{\xi \in \Lambda_r\mathfrak{sl}_2(\mathbb{C}) : \xi = \sum \xi_j\lambda^j,\ j \geq -1,\ \xi_{-1} \in \mathbb{C} \otimes \epsilon_+\right\}$$

and denote the holomorphic 1–forms on M with values in $\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C})$ by

$$\Lambda_r\Omega(M) = \Omega^1(M, \Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C})).$$

Lemma 3.1. [11] *Let $\xi \in \Lambda_r\Omega(M)$ and Φ be the solution of $d\Phi = \Phi\xi$ with initial condition $\Phi_0 \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ at $z_0 \in M$. Then the map F obtained by Iwasawa decomposing $\Phi = FB$ pointwise on M is an r-unitary frame.*

Proof. Expand $B = \sum B_j\lambda^j$, $j \geq 0$ and define $\alpha = F^{-1}dF$. Then $\alpha = B\xi B^{-1} - dBB^{-1}$. Now $\alpha' = B\xi B^{-1} - dB'B^{-1}$ and $\alpha'' = -dB''B^{-1}$, so the λ^{-1} coefficient of α' can only come from $\mathrm{Ad}B_0(\xi_{-1})$. If we set $\xi_{-1} = a\,\epsilon_+$ for $a \in \Omega^1(M, \mathbb{C})$, then $\mathrm{Ad}B_0(\xi_{-1}) = \rho a\epsilon_+$ for some function $\rho : M \to \mathbb{R}_+^*$. So equation (3.1) will hold if α'' has no ϵ_- component. But $\alpha'' = -dB''B^{-1}$ and thus the λ^0 coefficient comes from $-dB_0''B_0^{-1}$, which has no ϵ_- component. This proves the claim. □

4. The Sym–Bobenko Formulas

Given an r-unitary frame, an immersion can be obtained by formulas first found by Sym [33] for pseudo–spherical surfaces in $\mathbb{R}^3$ and extended by Bobenko [3] for CMC immersions in the three space forms. Our formulas differ from these, since we work in untwisted loop groups. Let $\partial_\lambda = \partial/\partial\lambda$.

Theorem 4.1. *Let M be a simply connected Riemann surface and $F_\lambda \in \mathcal{F}_r(M)$ an r-unitary frame for some $r \in (0, 1]$.*
(i) *Let $H \in \mathbb{R}^*$. The map $f : M \times A_r \to \mathfrak{sl}_2(\mathbb{C})$ defined by*

$$f_\lambda = -2i\lambda H^{-1}(\partial_\lambda F_\lambda)F_\lambda^{-1} \tag{4.1}$$

is a possibly branched conformal immersion. For each $\lambda \in C_1$, $f_\lambda : M \to \mathfrak{su}_2 \cong \mathbb{R}^3$ has constant mean curvature H.

(ii) *Let* $\mu \in C_1$. *The map* $f : M \times A_r \to \mathrm{SL}_2(\mathbb{C})$ *defined by*

$$f_\lambda = F_{\mu\lambda} F_\lambda^{-1} \tag{4.2}$$

is a possibly branched conformal immersion and for each $\lambda \in C_1$ *a CMC surface in the three-sphere* $S^3 \cong \mathrm{SU}_2$ *with mean curvature* $H = i(1+\mu)/(1-\mu)$.

(iii) *For* $s \in [r, 1)$ *and any* $\lambda \in C_s$, *the map* $f : M \times C_s \to \mathrm{SL}_2(\mathbb{C})$ *defined by*

$$f_\lambda = F_\lambda \overline{F_\lambda}^t \tag{4.3}$$

is a possibly branched conformal immersion of M *into* $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2 \cong H^3$ *with constant mean curvature* $H = (1+s^2)/(1-s^2)$.

Proof. (i) Expand $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ as in (3.1). Differentiating (4.1) gives

$$df_\lambda = 2iH^{-1} F_\lambda (\lambda^{-1} \alpha_2' \,\epsilon_+ - \lambda \alpha_1'' \,\epsilon_-) F_\lambda^{-1}.$$

Hence $\langle df_\lambda', df_\lambda' \rangle = 0$ by (1.4), proving conformality. Branch points occur when $\mathrm{Ad} F_\lambda (\lambda^{-1} \alpha_2' \,\epsilon_+ - \lambda \alpha_1'' \,\epsilon_-)$ vanishes. Clearly f_λ takes values in $\mathfrak{su}_2$ for $|\lambda| = 1$. Further, $[df_\lambda \wedge df_\lambda] = (8i/H^2)\, \alpha_1'' \wedge \alpha_2' \mathrm{Ad} F_\lambda(\epsilon)$. Using (3.2) and (3.3) we obtain $d * df_\lambda = (4i/H)\, \alpha_1'' \wedge \alpha_2' \mathrm{Ad} F_\lambda(\epsilon)$. By Lemma (1.1), this proves (i).

(ii) Write $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ as in (3.1). Then for $\omega_\lambda = f_\lambda^{-1} df_\lambda$ we obtain

$$\omega_\lambda = \mathrm{Ad} F_\lambda (\alpha_{\mu\lambda} - \alpha_\lambda) = \mathrm{Ad} F_\lambda \left(\lambda^{-1} (\mu^{-1} - 1) \alpha_2' \epsilon_+ + \lambda (\mu - 1) \alpha_1'' \epsilon_- \right).$$

Thus $\langle \omega_\lambda', \omega_\lambda' \rangle = 0$ by (1.4), proving conformality. Further, using (3.2) and (3.3) gives $d * \omega_\lambda = (\mu - \mu^{-1}) \alpha_2' \wedge \alpha_1'' \mathrm{Ad} F_\lambda(\epsilon)$ while $[\omega_\lambda \wedge \omega_\lambda] = -2i(1-\mu^{-1})(1-\mu) \alpha_2' \wedge \alpha_1'' \mathrm{Ad} F_\lambda(\epsilon)$. Using Lemma 1.2 yields the formula for H.

(iii) Let $\omega_\lambda = f_\lambda^{-1} df_\lambda$. Since F_λ satisfies (2.3), we have $\overline{F_\lambda}^t = F_{1/\bar\lambda}^{-1}$ and

$$df_\lambda = F_\lambda (\alpha_\lambda - \alpha_{1/\bar\lambda}) \overline{F_\lambda}^t = (\lambda - \bar\lambda^{-1}) F_\lambda (\alpha_1'' \epsilon_- - \alpha_2' \epsilon_+) \overline{F_\lambda}^t,$$

proving conformality $\langle df_\lambda', df_\lambda' \rangle = 0$ by (1.4). Further $[\omega_\lambda \wedge \omega_\lambda] = 2i(\lambda - \bar\lambda^{-1})(\bar\lambda - \lambda^{-1}) \alpha_2' \wedge \alpha_1'' \mathrm{Ad} F_{1/\bar\lambda}(\epsilon)$ while $d * \omega_\lambda = (\lambda\bar\lambda - \lambda^{-1}\bar\lambda^{-1}) \alpha_2' \wedge \alpha_1'' \mathrm{Ad} F_{1/\bar\lambda}(\epsilon)$. Using Lemma 1.3 yields the formula for H and concludes the proof of the theorem. □

5. The DPW representation

By combining Lemma 3.1 and Theorem 4.1, a conformal CMC immersion from a simply-connected Riemann surface M can be constructed in the following three steps: Let $\xi \in \Lambda_r \Omega(M)$, $z_0 \in M$ and $\Phi_0 \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$.

1. Solve the initial value problem to obtain a unique map $\Phi : M \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$:

$$d\Phi = \Phi \xi, \; \Phi(z_0) = \Phi_0 \,. \tag{5.1}$$

2. Iwasawa decompose $\Phi = F\,B$ pointwise on M to obtain a unique $F \in \mathcal{F}_r(M)$.
3. Insert F into one of the Sym–Bobenko formulas (4.1), (4.2) or (4.3).

An element $\xi \in \Lambda_r \Omega(M)$ has an expansion, with $a,\, b \in \Omega^1(M, \mathbb{C})$, of the form

$$\xi = (b + O(\lambda)) \epsilon_- + (\lambda^{-1} a + O(1)) \epsilon_+ + O(1) \epsilon. \tag{5.2}$$

The metric of the resulting CMC immersions is a nowhere vanishing multiple of $|a|^2$. The Hopf differential of the resulting CMC immersion is a constant multiple of the quadratic differential ab. To avoid branch points in examples we choose $a \in \Omega^1(M, \mathbb{C}^*)$ and prescribe the umbilics as the roots of $b \in \Omega^1(M, \mathbb{C})$.

If the initial condition $\Phi_0 \in \Lambda_r \mathrm{GL}_2(\mathbb{C})$, then $\Phi : M \to \Lambda_r \mathrm{GL}_2(\mathbb{C})$. The corresponding Iwasawa decomposition [27] of $\Lambda_r \mathrm{GL}_2(\mathbb{C})$ yields a unique map $F : M \to \Lambda_r^{\mathbb{R}} \mathrm{GL}_2(\mathbb{C})$

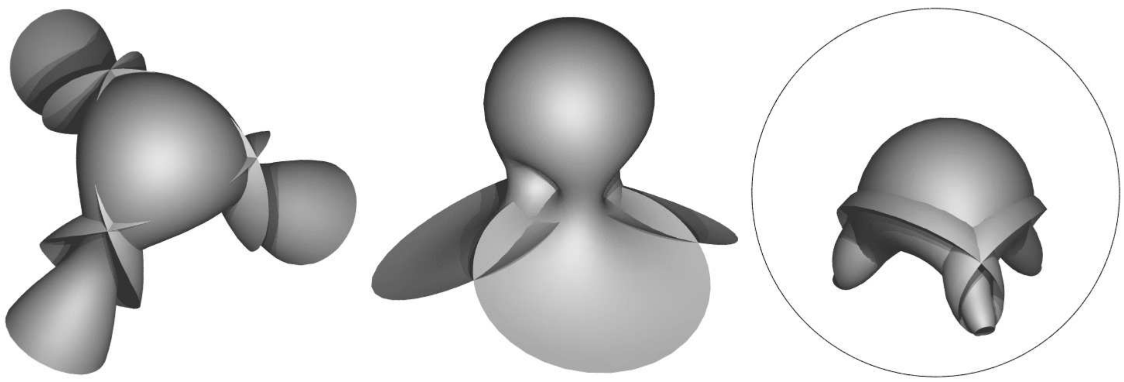

FIGURE 1. Smyth surfaces in $\mathbb{R}^3$, S^3 and H^3.

and the Sym-Bobenko formulas (4.1), (4.2) and (4.3) must be modified and become, respectively,

$$f_\lambda = -2i\lambda H^{-1}\left((\partial_\lambda F_\lambda)F_\lambda^{-1} - \mathrm{tr}((\partial_\lambda F_\lambda)F_\lambda^{-1})\mathrm{Id}\right), \tag{5.3}$$

$$f_\lambda = \sqrt{\det(F_\lambda F_{\mu\lambda}^{-1})}\, F_{\mu\lambda}F_\lambda^{-1} \text{ and} \tag{5.4}$$

$$f_\lambda = |\det F_\lambda|^{-1} F_\lambda \overline{F_\lambda}^t. \tag{5.5}$$

The map $(\xi, \Phi_0, z_0) \mapsto F$ is surjective [11]. Injectivity fails, since the Gauge group

$$\mathcal{G}_r(M) = \{g : M \to \Lambda_r^+\mathrm{SL}_2(\mathbb{C}) \text{ holomorphic }\} \tag{5.6}$$

acts by right multiplication on the fibers of this map: Indeed, on the holomorphic potential level, the gauge action $\Lambda_r\Omega(M) \times \mathcal{G}_r(M) \to \Lambda_r\Omega(M)$ is

$$\xi.g = g^{-1}\xi\, g + g^{-1}dg. \tag{5.7}$$

For $g \in \mathcal{G}_r(M)$, (ξ, Φ_0, z_0) and $(\xi.g, \Phi_0 g(z_0), z_0)$ induce the same r-unitary frame.

By Proposition 2.9 of [8], the dressing action (2.5) descends to $\mathcal{F}_r(M)$ and in the context of the DPW representation is the variation of the initial condition in (5.1) by left multiplication of $\Lambda_r\mathrm{SL}_2(\mathbb{C})$ or $\Lambda_r\mathrm{GL}_2(\mathbb{C})$.

Example 5.1. *The triple* $(\lambda^{-1}\epsilon_+ dz, \mathrm{Id}, 0)$ *yields* $F = (1+z\bar{z})^{-1/2}(\mathrm{Id} + z\lambda^{-1}\epsilon_+ + \bar{z}\lambda\epsilon_-)$, *which inserted into* (4.1), (4.2) *or* (4.3) *yields spheres in* $\mathbb{R}^3$, S^3 *or* H^3.

Example 5.2. *Define* $\xi = (\lambda^{-1}\epsilon_+ - cz^k\epsilon_-)dz$ *on* $M = \mathbb{C}$, *for* $c \in \mathbb{C}^*$ *and* $k \in \mathbb{N} \cup \{0\}$. *Then* $(\xi, \mathrm{Id}, 0)$ *yields surfaces with intrinsic rotational symmetry* [5], *called Smyth surfaces* [31]. *In the* $\mathbb{R}^3$ *case, they are proper and complete* [35], *and their asymptotics are investigated in* [3]. *From the symmetries of* ξ, *it follows that Smyth surfaces in* $\mathbb{R}^3$, S^3 *and* H^3 *have reflective symmetry with respect to* $k+2$ *geodesic planes that meet equiangularly along a geodesic line. When* $k = 0$ *and* $c \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, *the surface is contained in a round cylinder. In Figure 1 we display Smyth surfaces in* $\mathbb{R}^3$, S^3 *and* H^3 *with* $c = \frac{1}{5}$ *and* $k = 1$. *When the target space is* $\mathbb{R}^3$ *(respectively* S^3, H^3*), we chose* $\lambda_0 = 1$ *(respectively* $\mu = \lambda_0^{-2} = -1$, $\lambda_0 = 8/25$*). In all three target spaces,* $\Phi_0 = \mathrm{Id}$ *and* $z_0 = 0$.

Remark 5.1. *For viewing graphics in* S^3, *we stereographically project* S^3 *from its north pole to* $\mathbb{R}^3 \cup \{\infty\}$. *For surfaces in* H^3, *we use the Poincare model, which is stereographic projection of the Minkowski model in Lorentz space from the point*

$(0,0,0,-1)$ *to the* 3*-ball of points* $(0,x,y,z) \in \mathbb{R}^{3,1}$ *such that* $x^2+y^2+z^2<1$, *equivalent to the* 3*-ball of points* $(x,y,z) \in \mathbb{R}^3$ *such that* $x^2+y^2+z^2<1$.

6. Invariant potentials & Monodromy

Let M be a connected Riemann surface with universal cover $\widetilde{M} \to M$ and let Δ denote the group of deck transformations. Let $\xi \in \Lambda_r\Omega(M)$ be a holomorphic potential on M. Then $\gamma^*\xi = \xi$ for all $\gamma \in \Delta$. Let $\Phi : \widetilde{M} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ be a solution of the differential equation $d\Phi = \Phi\xi$. Writing $\gamma^*\Phi = \Phi \circ \gamma$ for $\gamma \in \Delta$, we define $\chi(\gamma) \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ by $\chi(\gamma) = (\gamma^*\Phi)\,\Phi^{-1}$. The matrix $\chi(\gamma)$ is called the *monodromy* matrix of Φ with respect to γ. If $\Psi : \widetilde{M} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ is another solution of $d\Phi = \Phi\xi$ and $\widehat{\chi}(\gamma) = (\gamma^*\Psi)\,\Psi^{-1}$, then there exists a constant $C \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ such that $\Psi = C\Phi$. Hence $\widehat{\chi}(\gamma) = C\chi(\gamma)C^{-1}$ and different solutions give rise to mutually conjugate monodromy matrices.

A choice of base point $\tilde{z}_0 \in \widetilde{M}$ and initial condition $\Phi_0 \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ gives the *monodromy representation* $\chi : \Delta \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ of a holomorphic potential $\xi \in \Lambda_r\Omega(M)$. Henceforth, when we speak of the monodromy representation, or simply monodromy, we tacitly assume that it is induced by an underlying triple $(\xi, \Phi_0, \tilde{z}_0)$. Note that the invariance $\gamma^*\xi = \xi$ for all $\gamma \in \Delta$ is equivalent to $d\chi = 0$, ensuring that the monodromy is z–independent and thus well defined. It is shown in [10] that CMC immersions of open Riemann surfaces M can always be generated by such invariant holomorphic potentials.

If $\Phi = FB$ is the pointwise Iwasawa decomposition of $\Phi : \widetilde{M} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$, then we shall need to study the monodromy of F to control the periodicity of the resulting CMC immersion given by (4.1), (4.2) or (4.3). A priori, we are not assured that the quantity $\mathcal{H}(\gamma) = (\gamma^*F)F^{-1}$ is z–independent for all $\gamma \in \Delta$. In case χ is $\Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$–valued for all $\gamma \in \Delta$ then, by uniqueness of the Iwasawa decomposition, one sees that $\chi = \mathcal{H}$. Assuming $\mathcal{H}$ is z–independent, there are well known closing conditions for the CMC immersions, first formulated in [9] for the target $\mathbb{R}^3$.

Theorem 6.1. *Let* M *be a Riemann surface with universal cover* $\widetilde{M}$, *and let* $F \in \mathcal{F}_r(\widetilde{M})$ *be an* r*-unitary frame on* $\widetilde{M}$ *with monodromy* $\mathcal{H} : \Delta \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$.

(i) *Let* f_λ *be as in* (4.1). *There is a* $\lambda_0 \in C_1$ *such that* $\gamma^* f_{\lambda_0} = f_{\lambda_0}$ *for all* $\gamma \in \Delta$ *if and only if* $\mathcal{H}(\gamma)|_{\lambda_0} = \pm\mathrm{Id}$ *and* $\partial_\lambda \mathcal{H}(\gamma)|_{\lambda_0} = 0$ *for all* $\gamma \in \Delta$.

(ii) *Let* $\mu \in C_1$ *and* f_λ *as in* (4.2). *There is a* $\lambda_0 \in C_1$ *such that* $\gamma^* f_{\lambda_0} = f_{\lambda_0}$ *for all* $\gamma \in \Delta$ *if and only if* $\mathcal{H}(\gamma)|_{\lambda_0} = \mathcal{H}(\gamma)|_{\mu\lambda_0} = \pm\mathrm{Id}$ *for all* $\gamma \in \Delta$.

(iii) *Let* f_λ *be as in* (4.3) *for* $s \in [r,1)$. *There is a* $\lambda_0 \in C_s$ *such that* $\gamma^* f_{\lambda_0} = f_{\lambda_0}$ *for all* $\gamma \in \Delta$ *if and only if* $\mathcal{H}(\gamma)|_{\lambda_0} = \pm\mathrm{Id}$ *for all* $\gamma \in \Delta$.

Proof. The proof of the above *closing conditions* (i) - (iii) are immediate consequences of the Sym-Bobenko formulas (4.1), (4.2) and (4.3). □

The closing conditions in Theorem 6.1 are invariant under conjugation. Furthermore, Theorem 6.1 also holds when (4.1), (4.2) or (4.3) are replaced respectively by (5.3), (5.4) or (5.5), and F and $\mathcal{H}$ take values in $\Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$.

7. Cylinders and Delaunay surfaces

We apply the preceeding ideas and derive triples (ξ, Φ_0, z_0) that generate CMC cylinders in the three dimensional space forms. The domain will be the Riemann

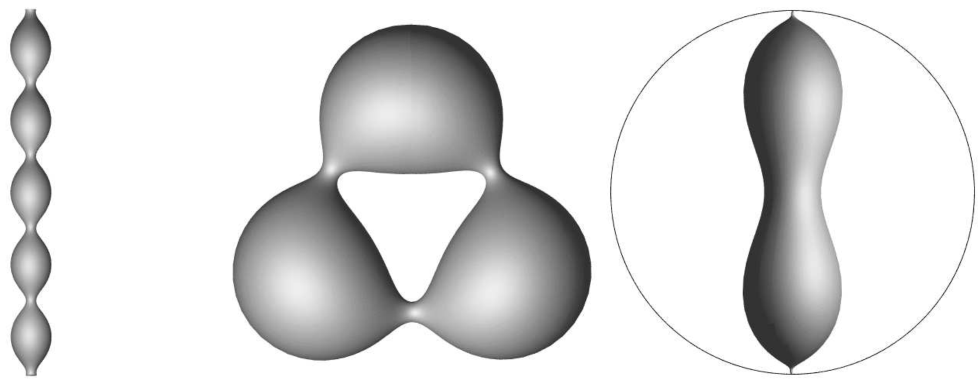

FIGURE 2. Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3.

surface $M = \mathbb{C}^*$. Hence the group of deck transformations is generated by

$$\tau : \log z \mapsto \log z + 2\pi i \,. \tag{7.1}$$

7.1. **Round cylinders.** Define $\xi = (\lambda^{-1} a\epsilon_+ - a\epsilon_-)z^{-1}dz$, $a \in \mathbb{R}$. Then $(\xi, \mathrm{Id}, 1)$ yields the unitary frame $F = \exp\left(\log z(\lambda^{-1}a\epsilon_+ - a\epsilon_-) - \log \bar{z}(a\epsilon_+ - \lambda a\epsilon_-)\right)$ with monodromy $\mathcal{H}(\tau) = \exp\left(2\pi i a((1+\lambda^{-1})\epsilon_+ - (1+\lambda)\epsilon_-)\right)$.

When the target is $\mathbb{R}^3$ we choose $\lambda_0 = 1$ and $a = 1/4$; for S^3 we choose $\lambda_0 = e^{-i\theta}$, $\mu = e^{2i\theta}$ for $\theta \in \mathbb{R}$ and $a = 1/(2\sqrt{2+2\cos\theta})$; for H^3 we choose $\lambda_0 = e^q$ for $q \in \mathbb{R}^*$ and $a = 1/(2\sqrt{2+2\cosh q})$. In each case, the appropriate condition of Theorem 6.1 holds and thus the resulting maps f_{λ_0} given by the corresponding Sym-Bobenko formula are conformal CMC immersions of $\mathbb{C}^*$ and thus homeomorphically cylinders. Inserting the appropriate F into Equations (4.1), (4.2) or (4.3) yields explicit parametrizations of round cylinders. In S^3, each cylinder becomes a covering of a torus, since its geodesic axis in S^3 is closed. Round cylinders are members of the family of Delaunay surfaces to which we now turn our attention.

7.2. **Delaunay surfaces.** Delaunay surfaces in $\mathbb{R}^3$ via the DPW method are described in detail in [18]. We include a brief description, as we will need to refer to properties of Delaunay surfaces in H^3 and S^3. Define for $a, b \in \mathbb{C}$, $c \in \mathbb{R}$

$$A = ic\epsilon + (a\lambda^{-1} + \bar{b})\epsilon_+ - (\bar{a}\lambda + b)\epsilon_-. \tag{7.2}$$

Then $(z^{-1}Adz, \mathrm{Id}, 1)$ generates a unitary frame with monodromy $\exp(2\pi i A)$. The closing conditions (i) - (iii) of Theorem 6.1 are respectively

- For $\mathbb{R}^3$, we choose $\lambda_0 = 1$ and a, b, c so that

$$c^2 + |a + \bar{b}|^2 = 1/4 \text{ and } ab \in \mathbb{R}. \tag{7.3}$$

- For S^3, we choose $\lambda_0 = e^{-i\theta}$, $\mu = e^{2i\theta}$ for $\theta \in (0, \frac{\pi}{2}) \subset \mathbb{R}$ and a, b, c so that

$$c^2 + |a + \bar{b}|^2 - 4ab\sin^2(\theta/2) = 1/4 \text{ and } ab \in \mathbb{R}. \tag{7.4}$$

- For H^3, we choose $\lambda_0 = e^q$ for $q \in \mathbb{R}^*$ and a, b, c so that

$$c^2 + |a + \bar{b}|^2 + 4ab\sinh^2(q/2) = 1/4 \text{ and } ab \in \mathbb{R}. \tag{7.5}$$

Since $\Phi = \exp(A \log z)$ solves $d\Phi = \Phi A z^{-1}dz$ with $\Phi(z_0 = 1) = \mathrm{Id}$, the Iwasawa splitting $\Phi = FB$ satisfies $F = F_1F_2$, where $F_1 = \exp(iA\arg z)$ and F_2 and B depend only on $|z|$ (and λ). Then F_2B is the Iwasawa splitting of $\exp(A \ln |z|)$. Under the rotation of the domain $z \mapsto \exp(i\beta)z$, $\beta \in \mathbb{R}$, B is unchanged and F changes to

$\exp(i\beta A)F$. Then equations (4.1), (4.2) and (4.3) evaluated at λ_0 each change by a rotation of angle β about a fixed geodesic line. Hence we obtain surfaces of revolution and thus Delaunay surfaces in each of the three space forms.

Example 7.1. *Figure 2 depicts Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3. These are obtained from triples $(z^{-1}Adz, \mathrm{Id}, 1)$ with $c = 0$ and the following data: $\mathbb{R}^3$: $a = .4$, $b = .1$ and $\lambda_0 = 1$; S^3: $a = .077$, $b = .442579$ and $\mu = \lambda_0^{-2} = e^{i\pi/2}$; H^3: $a = 1/(2\sqrt{34})$, $b = \sqrt{2/17}$ and $\lambda_0 = 1/4$.*

7.3. **Weights.** Let γ be an oriented loop about an annular end of a CMC H surface in $\mathbb{R}^3$ or S^3 or H^3, and let $\mathcal{Q}$ be an immersed disk with boundary γ. Let η be the unit conormal of the surface along γ and let ν be the unit normal of $\mathcal{Q}$, the signs of both of them determined by the orientation of γ. Then the *flux* of the end with respect to a Killing vector field Y (in $\mathbb{R}^3$ or S^3 or H^3) is

$$w(Y) = \frac{2}{\pi}\left(\int_\gamma < \eta, Y > -2H\int_{\mathcal{Q}} < \nu, Y >\right).$$

When the end is asymptotic to a Delaunay surface with axis ℓ and Y is the Killing vector field associated to unit translation along the direction of ℓ, we abbreviate $w(Y)$ to w and say that w is the *weight* of the end. This weight w changes sign when the orientation of γ is reversed, but otherwise is independent of the choices of γ and $\mathcal{Q}$ and hence a homology invariant [23], [22]. (Since the mean curvature in [23] and [22] is defined as the sum of the principal curvatures, rather than the average, we must replace H by $2H$ in the formulas for the weights there.)

As the unitary Delaunay frame has the simple form $F|_{S^1} = \exp(iA\arg z)$ for $z \in S^1$, the image of S^1 under the resulting immersion is a geodesic circle which can be explicitly computed. Also, one can explicitly compute the normal vector to the surface along the image of S^1, allowing us to compute the weights of Delaunay surfaces:

Lemma 7.1. *The weights of Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3 are given respectively by the following quantities:*

$$\begin{aligned} w &= \frac{16ab}{|H|}, \\ w &= \frac{16ab}{\sqrt{H^2+1}}, \\ w &= \frac{16ab}{\sqrt{H^2-1}}. \end{aligned} \tag{7.6}$$

Because $F \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$, we have the following:

Corollary 7.2. *The weights of a Delaunay surfaces in $\mathbb{R}^3$, S^3 or H^3 are subject to the following bounds, respectively:*

$$\begin{aligned} w &\le \frac{1}{|H|}, \\ -2(\sqrt{H^2+1}+|H|) \le w &\le 2(\sqrt{H^2+1}-|H|), \\ w &\le 2(|H|-\sqrt{H^2-1}), \end{aligned} \tag{7.7}$$

An unduloid (respectively nodoid, singular chain of spheres, round cylinder) is produced when $ab > 0$ (respectively $ab < 0$, $ab = 0$, $c = 0$ and $|a| = |b|$).

8. Extended frames of Delaunay surfaces

8.1. **The Delaunay Extended Frame.** The following theorem gives the extended unitary frame F of a Delaunay surface explicitly in terms of elliptic functions and elliptic integrals. We restrict to the case that A is off-diagonal in Theorems 8.1 and 8.4, but return to the more general form for A in Corollary 8.5.

Theorem 8.1. *Let $\Phi : \mathbb{C} \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ be defined by $\Phi = \exp((x+iy)A)$, with A given as in (7.2) where $a,\, b \in \mathbb{R}^*$ and $c = 0$. The r–Iwasawa factorization $\Phi = FB$ for any $r \in (0,\, 1]$ is given by*

$$F = \Phi \exp(-\mathbf{f}A)B_1^{-1}, \quad B = B_1 \exp(\mathbf{f}A),$$

and the functions $v = v(x)$, $\mathbf{f} = \mathbf{f}(x)$ and the matrices B_0, B_1 satisfy

$$\begin{aligned} &v'^2 = -\big(v^2 - 4a^2\big)\big(v^2 - 4b^2\big),\ v(0) = 2b,\\ &\mathbf{f} = \int_0^x \frac{2\,dt}{1 + (4ab\lambda)^{-1}v^2(t)},\\ &B_0 = \begin{pmatrix} 2v(b + a\lambda) & -v' \\ 0 & 4ab\lambda + v^2 \end{pmatrix}, \quad B_1 = (\det B_0)^{-1/2}B_0\,. \end{aligned} \tag{8.1}$$

Proof. Choose $H \in \mathbb{R}^*$ and set $Q = -2abH^{-1}\lambda^{-1}$. Let v be the solution of (8.1) and set $v_1^2 = H^{-2}v^2$. Let $\Theta = \Theta_1 dx + \Theta_2 dy$, where

$$\begin{aligned} \Theta_1 &= \begin{pmatrix} 0 & -v_1^{-1}Q - \frac{1}{2}v_1 H \\ v_1^{-1}Q^* + \frac{1}{2}v_1 H & 0 \end{pmatrix}\\ \Theta_2 &= i\begin{pmatrix} -\frac{1}{2}v_1^{-1}v_1' & -v_1^{-1}Q + \frac{1}{2}v_1 H \\ -v_1^{-1}Q^* + \frac{1}{2}v_1 H & \frac{1}{2}v_1^{-1}v_1' \end{pmatrix}. \end{aligned}$$

A calculation shows that B satisfies the gauge equation

$$dB + \Theta B = BA(dx + idy), \quad B(0,\, \lambda) = \mathrm{Id}, \tag{8.2}$$

or equivalently,

$$\Theta_2 B - iBA = 0, \quad B' + (\Theta_1 + i\Theta_2)B = 0, \quad B(0,\, \lambda) = \mathrm{Id}.$$

Since $\Theta_1 + i\Theta_2$ is smooth on $\mathbb{C}$ with holomorphic parameter λ on $\mathbb{C}$, the same is true of B. Since $\Theta_1 + i\Theta_2$ is tracefree and $B(0,\, \lambda) = \mathrm{Id}$, then $\det B = 1$. Also,

$$B(x,\, 0) = \begin{pmatrix} (\frac{2b}{v})^{1/2} & -\frac{v'}{v^2}(\frac{2b}{v})^{-1/2} \\ 0 & (\frac{2b}{v})^{-1/2} \end{pmatrix},$$

which is upper-triangular with diagonal elements in $\mathbb{R}^+$. Hence $B : \mathbb{C} \to \Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$ is smooth. From $\Phi = FB$ it follows that $F : \mathbb{C} \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ is smooth. Equivalently to equation (8.2), F satisfies the Maurer-Cartan equation

$$F^{-1}dF = \Theta, \quad F(0,\, \lambda) = \mathrm{Id}.$$

The symmetry $\Theta^* = -\Theta$ together with $F(0,\, \lambda) = \mathrm{Id}$ imply that F satisfies $F^* = F^{-1}$. Hence $F : \mathbb{C} \to \Lambda_r^{\mathbb{R}} \mathrm{SL}_2(\mathbb{C})$ is smooth. The resulting associated family of surfaces has constant mean curvature H, $\sqrt{H^2 - 1}$, $\sqrt{H^2 + 1}$ respectively in $\mathbb{R}^3$, S^3 and H^3. One can verify that v_1^2 is the conformal factor and $Q\,dz^2$ the Hopf differential using Equations (1.5), (1.8) and (1.10) □

Remark 8.2. *The functions v and $\mathbf{f}$ have explicit forms in terms of elliptic functions (we follow the notational conventions of the Mathematica programming language here). For example, when $0 < |b| < |a|$, we have*

$$v(x) = 2b \cdot \operatorname{sn}\left(2ia\left(x - \frac{iK}{2a}\right), \frac{b^2}{a^2}\right) = \frac{2b}{\operatorname{dn}(2ax,\, 1 - b^2/a^2)}, \tag{8.3}$$

where sn, dn *are Jacobi elliptic functions and $K = K(b^2/a^2)$ is the quarter-period of* $\operatorname{sn}(x, b^2/a^2)$.

Corollary 8.3. *The unitary frame of any Delaunay surface up to isometry can be written as $F_{\mathrm{D}} : \mathbb{C} \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ with*

$$F_{\mathrm{D}} = \exp\left((x + iy - \mathbf{f}(x))\begin{pmatrix} 0 & a\lambda^{-1} + b \\ b + a\lambda & 0 \end{pmatrix}\right) B_1^{-1} \tag{8.4}$$

for suitable a, $b \in \mathbb{R}$ and $\mathbf{f}$ and B_1 as in (8.1).

Proof. A Delaunay surface is determined up to rigid motion by its weight w. There exist a, $b \in \mathbb{R}$ satisfying (7.6) and (7.3), (7.4) or (7.5) with $c = 0$. The claim now follows from Theorem 8.1. $\square$

8.2. **Delaunay Asymptotics.** The following results, which compute the growth rate of the gauge B that gauges the Delaunay Maurer-Cartan form to the Delaunay potential, will be used to show that the CMC immersion resulting from a perturbation of a Delaunay potential is asymptotic to the base Delaunay immersion.

Let $X(\lambda) = (X_{ij}(\lambda)) : E \to \mathfrak{gl}_2(\mathbb{C})$ for $E \subset \mathbb{C}$ and define the operator norm $||X||_E$ by

$$||X||_E := \sup_{\lambda \in E} |X| \,, \quad \text{where}$$

$$|X| := \max_{u,v \in \mathbb{C}; |u|^2+|v|^2=1} \sqrt{|X_{11}u + X_{12}v|^2 + |X_{21}u + X_{22}v|^2}\,.$$

This is a multiplicative norm and is equivalent to the Euclidean norm on matrices in $\mathfrak{gl}_2(\mathbb{C})$. We will use the following estimate on $\exp(\mathbf{f}A)$ for $A \in \mathfrak{sl}_2(\mathbb{C})$ as in Theorem 8.1, with $\mu = \mu(\lambda)$ an eigenvalue of A:

$$|\exp(\mathbf{f}A)| \le \max(|\mathrm{Id} \pm \mu^{-1}A|)\exp(|\operatorname{Re}\mathbf{f}\mu|), \tag{8.5}$$

which is computable using $\exp(\mathbf{f}A) = \frac{1}{2}\exp(\mathbf{f}\mu)(\mathrm{Id} + \mu^{-1}A) + \frac{1}{2}\exp(-\mathbf{f}\mu)(\mathrm{Id} - \mu^{-1}A)$.

Theorem 8.4. *With B and A as in Theorem 8.1, let $\mu(\lambda)$ be an eigenvalue of A and let*

$$c = \max_{\lambda \in C_1} |\mu(\lambda)|. \tag{8.6}$$

Then there exists an $\mathcal{X} \in \mathbb{R}^+$ and $c_0 : \{0 < |\lambda| \le 1\} \to \mathbb{R}^+$ such that for all $|x| > \mathcal{X}$ and all $\lambda \in \{0 < |\lambda| \le 1\}$, we have

$$|B(x,\, \lambda)| \le c_0(\lambda)\exp(c|x|). \tag{8.7}$$

Proof. We compute the growth rate of B using its periodicity properties. Let $\rho \in \mathbb{R}^+$ be the period of the function v of Theorem 8.1. For any $x \in \mathbb{R}$ there exist $x_0 \in [0,\, \rho)$ and $n \in \mathbb{Z}$ such that $x = x_0 + n\rho$. Then $v(x_0 + n\rho) = v(x_0)$, $\mathbf{f}(x_0 + n\rho) = \mathbf{f}(x_0) + n\mathbf{f}(\rho)$ and $B(x_0 + n\rho) = B(x_0)\exp(n\mathbf{f}\rho A)$ and consequently

$$B(x) = B(x_0)\exp((x - x_0)\mathbf{f}(\rho)/\rho A). \tag{8.8}$$

Let d be the point in the set $\{-a/b,\, -b/a\} \cap \overline{I_1}$, and assume first that $\lambda \in \overline{I_1} \setminus \{0,\, d\}$. Equation (8.8) and formula (8.5) yield the estimate

$$\text{(8.9)} \qquad |B(x)| \le |B(x_0)| \exp(|\operatorname{Re} \mu \mathbf{f}(\rho)/\rho|\, |x - x_0|) \le c_0 \exp(|\operatorname{Re} \mu \mathbf{f}(\rho)/\rho|\, |x|),$$

where $c_0 = c_1 c_2 c_3$, $c_1 = \sup_{x_0 \in [0,\, \rho)} |B(x_0)|$, $c_2 = \max |\mathrm{Id} \pm \mu^{-1} A|$ and $c_3 = e^{|\operatorname{Re} \mu \mathbf{f}(\rho)|}$.

Next we compute the upper bound of $|\operatorname{Re} \mu \mathbf{f}(\rho)/\rho|$ on $\overline{I_1}$. Since $M = \exp(\mathbf{f}(\rho) A)$, then $\mathbf{f}(\rho) A$ is defined on $\overline{I_1}$ modulo the additive quantity $\pi i \mu^{-1} A$. Let $J = [0,\, d]$ be the closed interval. Then μ is single-valued on $\overline{I_1} \setminus J$, so $\mu \mathbf{f}(\rho)$ is defined on $\overline{I_1} \setminus J$ modulo the additive quantity πi. Hence $h = \operatorname{Re} \mu \mathbf{f}(\rho)/\rho$ is well-defined on $\overline{I_1} \setminus J$. Moreover, h is a harmonic function, since it is locally the real part of a holomorphic function. A calculation shows that $h = \mu$ on $|\lambda_0| = 1$, and $h = 0$ on J. The harmonicity of h then implies that $|h| \le c$ for all $\lambda \in I_1$. This bound, together with the estimate (8.9), yields the estimate (8.7) for all $x \in \mathbb{R}$ and all $\lambda \in \overline{I_1} \setminus \{0,\, d\}$.

In the exceptional case $\lambda_0 \in \{0,\, d\}$, $\mathbf{f} A$ is nilpotent at λ_0, so $B(x,\, \lambda_0)$ has linear growth in $|x|$. Define $c_1(\lambda_0) = 1$. Since $c > 0$, and because exponential growth eventually supersedes linear growth, there exists an $\mathcal{X} \in \mathbb{R}^+$ such that (8.7) holds at λ_0 for all $|x| > \mathcal{X}$. $\square$

This next result extends Theorems 8.1 and 8.4 to the case when $c \neq 0$ in (7.2).

Corollary 8.5. *Let $\Phi : \mathbb{C} \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ be defined by $\Phi = \exp((x + iy)A)$, with*

$$A = \begin{pmatrix} \tilde{c} & a\lambda^{-1} + \bar{b} \\ b + \bar{a}\lambda & -\tilde{c} \end{pmatrix} \quad \textit{where } a,\, b \in \mathbb{C}^* \textit{ and } ab,\, \tilde{c} \in \mathbb{R}.$$

Let $\Phi = FB$ be the Iwasawa factorization of Φ, and let $\mu(\lambda)$ be an eigenvalue of A and c as in (8.6). Then there exists an $\mathcal{X} \in \mathbb{R}^+$ and $c_0 : \{0 < |\lambda| \le 1\} \to \mathbb{R}^+$ such that for all $|x| > \mathcal{X}$ and all $\lambda \in \{0 < |\lambda| \le 1\}$ the inequality (8.7) holds.

Proof. To prove the inequality, we first show that there exists a $\theta \in \mathbb{R}$ and a g such that the following two properties hold:

(i) $g\, D \in \Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$ with $D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, and g and D are independent of $x + iy$.

(ii) $g^{-1} A g = \tilde{A}$, where $\tilde{A} := \begin{pmatrix} 0 & \tilde{a}\lambda^{-1} + \tilde{b} \\ \tilde{b} + \tilde{a}\lambda & 0 \end{pmatrix}$ for some $\tilde{a},\, \tilde{b} \in \mathbb{R}^*$.

Choose $\tilde{a},\, \tilde{b} \in \mathbb{R}^*$ such that $\tilde{c}^2 + |a|^2 + |b|^2 = \tilde{a}^2 + \tilde{b}^2$ and $ab = \tilde{a}\tilde{b}$ and so that $|\tilde{a}/\tilde{b}| \ge 1$ if $|a/b| \ge 1$, respectively $|\tilde{a}/\tilde{b}| \le 1$ if $|a/b| \le 1$. If $|a/b| \ge 1$, define g by

$$\text{(8.10)} \qquad g := \frac{1}{\sqrt{(\tilde{a} + \tilde{b}\lambda)(a + \bar{b}\lambda)}} \begin{pmatrix} a + \bar{b}\lambda & 0 \\ -\tilde{c}\lambda & \tilde{a} + \tilde{b}\lambda \end{pmatrix}.$$

If $|a/b| \le 1$, define g by

$$\text{(8.11)} \qquad g := \frac{1}{\sqrt{(\tilde{a}\lambda + \tilde{b})(\bar{a}\lambda + b)}} \begin{pmatrix} \tilde{a}\lambda + \tilde{b} & \tilde{c} \\ 0 & \bar{a}\lambda + b \end{pmatrix}.$$

With the proper choice of θ, we have the two above properties.

Let $\tilde{\Phi} = \exp((x+iy)\tilde{A})$, and let $\tilde{\Phi} = \tilde{F}\tilde{B}$ be its Iwasawa decomposition. By Theorem 8.4, $\tilde{B}$ satisfies the growth rate (8.7). We will use $\tilde{B}$ to show that B also satisfies (8.7).

The gauge g can be explicitely decomposed into $g = UR$, where $U \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ and $R : C_r \to \mathfrak{gl}_2(\mathbb{C})$ and $[R, \tilde{A}] = 0$. In fact, when g takes the form in (8.10), then

$$U = \mathfrak{T}\begin{pmatrix} (\tilde{a}+\tilde{b}\lambda^{-1})\mathfrak{R} + (a+\bar{b}\lambda)\mathfrak{R}^* & \tilde{c}\mathfrak{R}\lambda^{-1} \\ -\tilde{c}\mathfrak{R}^*\lambda & (\tilde{a}+\tilde{b}\lambda)\mathfrak{R}^* + (\bar{a}+b\lambda^{-1})\mathfrak{R} \end{pmatrix}$$

and

$$R = \frac{(\mathfrak{R}\mathfrak{R}^* + \mathfrak{S})\mathrm{Id} - \tilde{c}\tilde{A}}{\sqrt{2\mathfrak{S}(\mathfrak{R}\mathfrak{R}^* + \mathfrak{S} - \tilde{c}^2)}} = \exp\left(\mathfrak{S}^{-1/2}\log(\mathfrak{F})\tilde{A}\right),$$

where

$$\begin{aligned} \mathfrak{R} &= \sqrt{(\tilde{a}+\tilde{b}\lambda)(a+\bar{b}\lambda)}, \\ \mathfrak{S} &= (\tilde{a}\lambda^{-1}+\tilde{b})(\tilde{a}\lambda+\tilde{b}), \\ \mathfrak{T} &= (2\mathfrak{R}\mathfrak{R}^*(\mathfrak{R}\mathfrak{R}^*+\mathfrak{S}))^{-1/2} \text{ and} \\ \mathfrak{F} &= \frac{\sqrt{\mathfrak{S}} + \sqrt{(a+\bar{b}\lambda)(\bar{a}+b\lambda^{-1})} - \tilde{c}}{\sqrt{2(\mathfrak{S}+\mathfrak{R}\mathfrak{R}^*-\tilde{c}^2)}}. \end{aligned}$$

Furthermore, $\mathfrak{S}^{-1/2}\log\mathfrak{F}$ can be algebraically decomposed into

$$\mathfrak{S}^{-1/2}\log\mathfrak{F} = (\mathfrak{S}^{-1/2}\log\mathfrak{F})_u + (\mathfrak{S}^{-1/2}\log\mathfrak{F})_0 + (\mathfrak{S}^{-1/2}\log\mathfrak{F})_+,$$

where $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u$ satisfies $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u = -(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u^*$ and $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_+|_{\lambda=0}$ is zero and $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_0$ is independent of λ. Although R is single-valued in λ, $\mathfrak{S}^{-1/2}\log\mathfrak{F}$ can be multiple valued, with values in $\mathbb{C}\cup\{\infty\}/\Omega$, where

$$\Omega = \{2\pi ik\mathfrak{S}^{-1/2} \mid k \in \mathbb{Z}\}.$$

In other words, different values of $\mathfrak{S}^{-1/2}\log\mathfrak{F}$ can differ by terms of the form $2\pi ik\mathfrak{S}^{-1/2}$ for $k \in \mathbb{Z}$. Since $2\pi ik\mathfrak{S}^{-1/2} = -(2\pi ik\mathfrak{S}^{-1/2})^*$, we conclude that $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_0$ and $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_+$ are single-valued, and that $\exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_u\tilde{A})$ is single-valued although $(\mathfrak{S}^{-1/2}\log\mathfrak{F})_u$ might not be. Thus we have the following single-valued decomposition of R:

$$\begin{aligned} R &= R_uR_0R_+, \text{ where} \\ R_u &= \exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_u\tilde{A}) \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}), \\ R_0 &= \exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_0\tilde{A}) \text{ is constant, and} \\ R_+ &= \exp((\mathfrak{S}^{-1/2}\log\mathfrak{F})_+\tilde{A}) \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C}), \end{aligned}$$

and R_u, R_0 and R_+ all commute with $\tilde{A}$. Noting that

$$R_0\tilde{\Phi} = \exp((x+\hat{x}+iy)\tilde{A}) = \tilde{F}|_{x\to x+\hat{x}}\cdot\tilde{B}|_{x\to x+\hat{x}}$$

with $\hat{x} = (\mathfrak{S}^{-1/2}\log\mathfrak{F})_0$, we have

$$\Phi = (UR_u\tilde{F}|_{x\to x+\hat{x}}D)(D^{-1}\tilde{B}|_{x\to x+\hat{x}}D)(D^{-1}R_+D)(gD)^{-1},$$

where $UR_u\tilde{F}|_{x\to x+\hat{x}}D \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ and $D^{-1}\tilde{B}|_{x\to x+\hat{x}}D, D^{-1}R_+D, gD \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$. Thus

$$B = D^{-1}\tilde{B}|_{x\to x+\hat{x}}R_+g^{-1},$$

and so B satisfies the same growth rate (8.7) as $\tilde{B}$, since D and R_+ and g are all independent of $x+iy$. When g takes the form in (8.11), the argument is similar and concludes our proof of Corollary 8.5. □

9. Asymptotics for Delaunay Ends

We now explain a Delaunay asymptotics result in $\mathbb{R}^3$ from [12] and [30], and we prove extensions of this result to the other two space forms S^3 and H^3. The following lemma is a well-known result from the theory of regular singular points. We write $z^X = \exp(X \log z)$.

Lemma 9.1. *Let U be an open neighbourhood of $z_e \in \mathbb{C}$, $U^* = U \setminus \{z_e\}$ and $R \in (0,1)$. Suppose that $\xi \in \Lambda_r\Omega(U^*)$ for any $r \in (R,1)$ has a simple pole at z_e and the following expansion in z at z_e:*

$$\xi = A\,(z-z_e)^{-1}dz + \sum_{j\geq 0} \xi_j(\lambda)(z-z_e)^j dz\,,$$

where $A = A(\lambda)$ and λ_0 are as in Subsection 7.2 and either (7.3) or (7.4) or (7.5) holds. Then there exists an $r_0 \in (R,1)$ and an open set V with $z_e \in V \subset U$ and a matrix $P : V \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ for all $r \in (r_0,1)$ such that $P = I + \mathrm{O}((z-z_e)^1)$ at z_e and

$$(A\,(z-z_e)^{-1}dz).P = \xi$$

and P is defined and holomorphic for all λ in A_{r_0} except at a finite number of points in C_1. Furthermore, in the case that $\xi_0(\lambda) \equiv 0$ and $R < |\lambda_0|$ and that λ_0 and A are chosen so that (7.5) holds, then r_0 can be chosen strictly less than $\min(|\lambda_0|^{\pm 1})$.

Proof. As this result is well known (see [12], for example), we merely outline the proof. Setting $P = \sum_{k\geq 0} P_k (z-z_e)^k$ yields the recursive equations

$$kP_k - [P_k, \xi_{-1}] = \sum_{j=0}^{k-1} P_j \xi_{k-1-j}. \tag{9.1}$$

The solvability for all P_k follows from the fact that the difference of the two eigenvalues of A over C_1 is an integer at only isolated points. The series for P converges by standard ODE arguments, see [16], [12].

In fact, P is defined and holomorphic in λ and $\det P = 1$, for any nonzero $\lambda \neq \lambda_{k,\pm}$, where

$$\lambda_{k,\pm} = \frac{1}{2}\left(\tfrac{k^2-1}{4ab} - 4\Upsilon + 2 \pm \sqrt{\left(\tfrac{k^2-1}{4ab} - 4\Upsilon + 2\right)^2 - 4}\right),\ k \in \mathbb{N},$$

with $\Upsilon = 0$ or $\Upsilon = \sin^2(\theta/2)$ or $\Upsilon = -\sinh^2(q/2)$ for $\mathbb{R}^3$ or S^3 or H^3, respectively. Defining $\mathcal{P} = \{\lambda_{k,\pm} : |\lambda_{k,\pm}| < 1\}$, then r_0 can be chosen to be the maximum norm of elements in $\mathcal{P}$.

When $\xi_0 = 0$, then, with $k = 1$ in Equation (9.1), we find that $P_1 = 0$, and hence P is nonsingular at $\lambda_{1,\pm}$. Since λ_0 is in the set $\{\lambda_{1,+},\ \lambda_{1,-}\}$, the final part of the lemma follows. □

9.1. **Dressings that preserve Delaunay surfaces.** In preparation to proving asymptotics results we state two results from [30] which show that, under suitable conditions, a dressed Delaunay immersion is again a Delaunay immersion with the same weight. These are stated only for the target space $\mathbb{R}^3$ in loc.cit., but the proofs there apply to all three target spaces $\mathbb{R}^3$, S^3 and H^3.

Lemma 9.2. [30] *Let A be as in (7.2) satisfying $ab \neq 0$. Let λ_0 (and also μ in the case of S^3) be as in Subsection 7.2, and suppose that (7.3) or (7.4) or (7.5) is satisfied for the respective target space $\mathbb{R}^3$ or S^3 or H^3.*

Let $s \in (0, |\lambda_0|)$ and let $C : A_s \to \mathfrak{gl}_2(\mathbb{C})$ be an analytic map, where $\det C = 0$ only at a finite number of points $p_1, ..., p_k \in C_1$, and suppose $C \exp(2\pi i A)C^{-1}$ satisfies the reality condition (2.3).

Then there exists an $R \in (0, 1)$ such that for every $r \in (R, 1)$, the CMC immersion from $\mathbb{C} \setminus \{z_e\}$ to $\mathbb{R}^3$ or S^3 or H^3 produced by (5.3) or (5.4) or (5.5) respectively from the unitary part of the r-Iwasawa splitting of $C(z - z_e)^A$ is a Delaunay surface with the same weight as the Delaunay surface produced from $(z - z_e)^A$.

Lemma 9.3. [30] *Let A, λ_0, s and C (and μ in the case of S^3) be as in Lemma 9.2. Then there exists an $R \in (0, 1)$ such that, for every $r \in (R, 1)$, there exist a $U \in \Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ and an $R : A_r \to \mathfrak{gl}_2(\mathbb{C})$ so that $C = UR$ and $[R, A] = 0$. Furthermore, $\hat{A} := UAU^{-1}$ is also of the Delaunay potential form in (7.2), and has the same weight as A, and satisfies (7.3) or (7.4) or (7.5) at λ_0 if A does.*

9.2. **Asymptotics result in $\mathbb{R}^3$.** We begin with an asymptotics result in $\mathbb{R}^3$, which has been proven in [30]. A similar result can also be found in [12].

Theorem 9.4. *Let U be an open neighborhood of $z_e \in \mathbb{C}$ with universal cover $\tilde{U}$. Let $\xi_{-1} = A(z-z_e)^{-1}dz$ with A as in (7.2) satisfying $ab \neq 0$ and (7.3), and choose $\lambda_0 = 1$. Suppose that the $\mathbb{R}^3$ weight (7.6) associated to A satisfies*

$$w > \frac{-3}{|H|} \, . \tag{9.2}$$

Let $\xi = \xi_{-1} + \eta$ with $\eta = \sum_{j\geq 0} \xi_j(z - z_e)^j dz \in \Lambda_r\Omega(U)$. Let $\Psi : \tilde{U} \times A_r \to \mathfrak{gl}_2(\mathbb{C})$ be a solution of $d\Psi = \Psi\xi$ so that $\det \Psi = 0$ in A_r only at isolated points of C_1. With P defined on V and $z_e \in V \subset U$ as in Lemma 9.1, define $\Phi := \Psi P^{-1}$ on the universal cover of $V^* := V \setminus \{z_e\}$. Let $s \in (0, 1)$ be such that P is nonsingular on A_s minus a finite number of points in C_1. Let $\Phi = FB_D$ and $\Psi = GB$ be r-Iwasawa splittings, and suppose that the monodromy representations of F and G on V^* both satisfy the closing condition (i) of Theorem 6.1. Let f and g be the CMC immersions produced by (5.3) at λ_0 using F and G, respectively.

Then there exists an $R \geq s$ so that for all $r \in (R, 1)$,

$$\lim_{z\to z_e} ||F^{-1}G - \mathrm{Id}||_{A_r} = 0 \tag{9.3}$$

and

$$\lim_{z\to z_e} ||g - f||_{\{\lambda_0\}} = 0 \, . \tag{9.4}$$

Remark 9.5. *In* [30] *and* [12], *it is shown in proving this result that one can assume $\xi_0 = 0$, hence P is nonsingular at $\lambda_{1,\pm}$. Furthermore, condition* (9.2) *implies that P has no singularities on C_1. Thus P is nonsingular on A_s. Since $\Phi = C(z - z_e)^A$*

for some $C = C(\lambda)$, and since C is finite wherever P is nonsingular, C is finite on A_s. Then Lemma 9.2 implies that f is a portion (containing one end) of a Delaunay surface in $\mathbb{R}^3$ with the same weight w. Therefore, g has an asymptotically Delaunay end at z_e of weight w, in the sense of (9.4).

9.3. **Asymptotics result in S^3.** The following asymptotics result in S^3 is not proven in [30] nor [12]. However, the proof is entirely analogous to the proof of Theorem 9.4, since λ_0 and μ still lie in C_1 in the case of S^3.

Theorem 9.6. *Let ξ, $\Phi = F B_D$, $\Psi = G B$, P, s and $ab \neq 0$ all be as in Theorem 9.4. However, suppose that A now satisfies* (7.4) *and $\lambda_0^{-2} = \mu = e^{2i\theta}$ for some $\theta \in \mathbb{R}$. Suppose that the S^3 weight* (7.6) *associated to A and the S^3 mean curvature $H = -\cot\theta$ satisfy*

$$w > 6(H - \sqrt{H^2 + 1})\ . \tag{9.5}$$

Suppose that the monodromy representations of F and G on V^* both satisfy the closing condition (ii) of Theorem 6.1. Let f (respectively g) be the surface produced by (5.4) *using F (respectively G) and λ_0 and μ. Then there exists an $R > s$ so that, for all $r \in (R, 1)$, equations* (9.3) *and* (9.4) *hold.*

Remark 9.7. *Just like in the $\mathbb{R}^3$ case, using condition* (9.5) *and Lemma 9.2, f is an end of a Delaunay surface in S^3 with weight w, and g has an asymptotically Delaunay end at z_e in the sense of* (9.4) *of weight w.*

Proof. There are only two parts of the proof in [30] of the $\mathbb{R}^3$ Delaunay asymptotics theorem that require minor modifications in order to prove this theorem:

(i) The weight condition in equation (9.5) must be computed by taking into account that A now satisfies (7.4) rather than (7.3).

(ii) It follows from (9.3) that (9.4) holds. When the target space is S^3 rather than $\mathbb{R}^3$, a slightly different computation is required: Given $\lambda, \mu \in C_1$, both $G_\lambda^{-1} F_\lambda$ and $G_{\mu\lambda}^{-1} F_{\mu\lambda}$ converge uniformly to Id as $z \to 0$. Hence both $\alpha = \det(F_\lambda G_\lambda^{-1})$ and $\beta = \det(G_{\mu\lambda} F_{\mu\lambda}^{-1})$ converge uniformly to 1 as $z \to 0$. Then $g \to f$ as $z \to 0$ follows from (5.4) by rewriting

$$g_\lambda - f_\lambda = G_{\mu\lambda}\left(G_\lambda^{-1} F_\lambda - \sqrt{\alpha\beta}\, G_{\mu\lambda}^{-1} F_{\mu\lambda}\right) F_\lambda^{-1} \sqrt{\det(G_\lambda G_{\mu\lambda}^{-1})}.$$

□

9.4. **Asymptotics result in H^3.** The case of Delaunay asymptotics in H^3 requires a significantly different proof, using Corollary 8.5, because now $\lambda_0 \notin C_1$.

Theorem 9.8. *Let ξ, $\Phi = F B_D$, $\Psi = G B$, P and $ab \neq 0$ all be as in Theorem 9.4. However, suppose that A now satisfies* (7.5) *and $\lambda_0 = e^q$ for some $q < 0$. Suppose that the H^3 weight* (7.6) *associated to A and the H^3 mean curvature $H = -\coth q$ satisfy*

$$w > 6(\sqrt{H^2 - 1} - H)\ . \tag{9.6}$$

Suppose that the monodromy representations of F and G on V^* both satisfy the closing condition (iii) of Theorem 6.1. Suppose further that for all $\lambda \in A_r$,

$$[A, \eta_0] = 0. \tag{9.7}$$

Let f (respectively g) be the surface produced by (5.5) using F (respectively G) and λ_0. Then F and G are nonsingular at λ_0, and so f and g are well-defined. Furthermore, there exists an R so that, for all $r \in (R, 1)$, equation (9.4) holds.

Remark 9.9. *The condition (9.7) is included in this theorem for the following technical reason: In the proofs of Delaunay asymptotics for $\mathbb{R}^3$ in [30] and [12], first a gauge transformation and a conformal change of parameter z fixing z_e is applied to achieve $\xi_0 = 0$, and hence $\xi = A(z - z_e)^{-1}dz + \mathrm{O}((z - z_e)^1)dz$. To accomplish this gauging, the condition that $\det A \neq -1/4$ in $I_r^* := I_r \setminus \{0\}$ for $r < 1$ and r close to 1 is used. However, $\det A = -1/4$ does occur in I_r^* when $\lambda_0 \notin C_1$. To avoid this problem, we include the condition (9.7), and then a mere conformal change of parameter z fixing z_e can force $\xi_0 = 0$, as in [30].*

Proof. By Remark 9.9, we may assume $\xi_0 = P_1 = 0$. So P is defined at $\lambda_0 = \lambda_{1,+}$ or $\lambda_{1,-}$, and there exists a positive $s < |\lambda_0|$ such that P is nonsingular on $\overline{A_s}$. G is nonsingular at λ_0, because Ψ is nonsingular. F is nonsingular at λ_0, because P is nonsingular at λ_0. So f and g are well-defined.

As in the asymptotics theorem in [30], there exists an $R > s$ such that equation (9.3) holds for all $r \in (R, 1)$. We need to strengthen (9.3) to

$$\lim_{z \to z_e} ||F^{-1}G - \mathrm{Id}||_{\mathcal{C}} = 0\ , \tag{9.8}$$

where $\mathcal{C} = \{\lambda \,|\, s < |\lambda| < 1\}$. Then (9.4) will hold at $\lambda_0 = e^q$, by an argument analogous to the second item in the proof of Theorem 9.6.

Since $d\Phi = \Phi A(z - z_e)^{-1}dz$, we have $\Phi = C(z - z_e)^A$ for some $C = C(\lambda)$, where C is singular at the same points as P. Condition (9.6) and $\xi_0 = 0$ imply that P is nonsingular on A_s. As C is finite wherever P is nonsingular, C is finite on A_s.

Let $C = C_u C_+$ be the r-Iwasawa decomposition of C. By Lemma 9.3, there exists an $R > s$ such that for all $r \in (R, 1)$, there exist a $U \in \Lambda_r^{\mathbb{R}}\mathrm{GL}_2(\mathbb{C})$ and an $R : A_r \to \mathfrak{gl}_2(\mathbb{C})$ commuting with A such that $C_+ = UR$ and so that $\hat{A} := UAU^{-1}$ has the form in (7.2). A and $\hat{A}$ have the same weight w, and $\hat{A}$ satisfies (7.5) at λ_0. It follows that

$$\Phi = C_u (z - z_e)^{\hat{A}} C_+.$$

Consider the r-Iwasawa decomposition $(z - z_e)^{\hat{A}} = F_0 B_0$ for $r \in (R, 1)$. It follows that

$$C_u F_0 = F\ , \quad B_0 C_+ = B_D.$$

We now show that

$$\lim_{z \to z_e} ||B_D P B_D^{-1} - \mathrm{Id}||_{\mathcal{C}} = 0. \tag{9.9}$$

By Corollary 8.5, there exists an $\mathcal{X} \in \mathbb{R}^+$ and a function $c_0(\lambda)$ such that $\log|z - z_e| < -\mathcal{X}$ implies

$$|B_0(z, \lambda)| \leq c_0(\lambda)\, |z - z_e|^{-c}$$

for all $\lambda \in \overline{I_1}$, where $c = \max_{\lambda \in C_1} |\mu(\lambda)|$ with $\mu(\lambda)$ an eigenvalue of A and $\hat{A}$.

Since $C_+ \in \Lambda_r^+\mathrm{GL}_2(\mathbb{C})$ is independent of z, we have that $|C_+|$ is universally bounded on I_r, and so B_0 and B_D have the same growth rate as $z \to z_e$. Thus

$$||B_D P B_D^{-1} - \mathrm{Id}||_{\mathcal{C}} \leq ||B_D||_{\mathcal{C}}\, ||P - \mathrm{Id}||_{\mathcal{C}}\, ||B_D^{-1}||_{\mathcal{C}} \approx |z - z_e|^{2-2c}$$

asymptotically. The weight conditions in equation (9.6) imply that $c < 1$, proving (9.9).

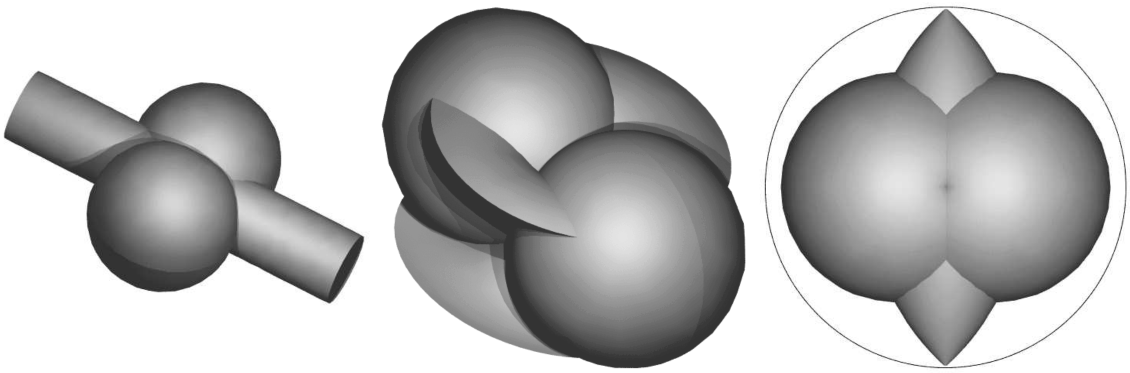

FIGURE 3. Bubbletons in $\mathbb{R}^3$, S^3 and H^3.

As $B_D P B_D^{-1}$ is holomorphic in λ, shifting s to be slightly larger but still in $(0, |\lambda_0|)$ and applying the Cauchy integral formula, we also have that

$$\lim_{z \to z_e} ||\partial_\lambda (B_D P B_D^{-1})||_{\mathcal{C} \cap I_R} = 0.$$

This C^1 convergence of $B_D P B_D^{-1}$ to Id with respect to λ suffices to conclude that the s-unitary part of $B_D P B_D^{-1}$ converges to Id on $C_s \cup C_{1/s}$ (see [27]). Then the maximum modulus principle for holomorphic functions implies that the s-unitary part of $B_D P B_D^{-1}$ converges to Id on $\overline{A_s}$, and in particular on $\mathcal{C}$.

Since $B_D P B_D^{-1}$ is nonsingular on $\overline{A_s} \cap I_1$, ℓ-Iwasawa decomposition is the same for all $\ell \in [s, 1)$. Thus the s-unitary part of $B_D P B_D^{-1}$ is equal to the r-unitary part $F^{-1}G$ of $B_D P B_D^{-1}$, and thus (9.8) holds. □

Remark 9.10. *Just like in the $\mathbb{R}^3$ and S^3 cases, because C is finite in A_s, Lemma 9.2 implies that f is an end of a Delaunay surface in H^3 with weight w, and g has an asymptotically Delaunay end at z_e of weight w in the sense of* (9.4).

10. Further Examples

10.1. **Simple factors.** Let $\pi_{\mathrm{L}} : \mathbb{C}^2 \to \mathrm{L}$ be the hermitian projection onto a line $\mathrm{L} \in \mathbb{CP}^1$. For $\alpha \in \mathbb{C}$, simple factors [34] are loops of the form

$$\psi_{\mathrm{L},\alpha} = \pi_{\mathrm{L}} + \frac{\alpha - \lambda}{1 - \overline{\alpha}\lambda} \pi_{\mathrm{L}}^{\perp}. \tag{10.1}$$

To obtain elements of $\Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$, we write

$$(\det \psi_{\mathrm{L},\alpha}(0))^{-1/2} \, \psi_{\mathrm{L},\alpha}(0) = Q \, R \tag{10.2}$$

with $Q \in \mathrm{SU}_2$, $R \in \mathcal{B}$. A simple factor of $\Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$ with $r < |\alpha|$ is a loop of the form

$$h_{\mathrm{L},\alpha} = (\det \psi_{\mathrm{L},\alpha})^{-1/2} \, Q^{-1} \psi_{\mathrm{L},\alpha} \tag{10.3}$$

with $\psi_{\mathrm{L},\alpha}$ and Q defined as in (10.1) respectively (10.2). By construction

$$h_{\mathrm{L},\alpha} \in \Lambda_s \mathrm{SU}_2 \tag{10.4}$$

for $s > |\alpha|$. By Proposition 4.2 in [34] dressing by simple factors is explicit: In fact, for $F(z, \lambda) \in \mathcal{F}_r(\widetilde{M})$ and $r \in (0, 1)$ and $h_{\mathrm{L},\alpha}$ a simple factor with $\alpha \in \mathbb{C}$ and $r < |\alpha| < 1$, we have

$$h_{\mathrm{L},\alpha} \,\#\, F = h_{\mathrm{L},\alpha} \, F \, h_{\mathrm{L}',\alpha}^{-1} \text{ with } \mathrm{L}' = \overline{F(z,\alpha)}^t \mathrm{L}. \tag{10.5}$$

The proof of this may be found in [20], where furthermore, the following result is obtained:

Theorem 10.1. [20] *Let $F \in \mathcal{F}_r(\widetilde{M})$ with monodromy $\mathcal{H}(\lambda) : \Delta \to \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$. Assume there exists $\alpha \in \mathbb{C}$, $r < |\alpha| < 1$ such that $\mathcal{H}(\alpha)$ is reducible and let $\mathrm{L} \in \mathbb{CP}^1$ such that*

$$\overline{\mathcal{H}(\alpha)}^t \mathrm{L} = \mathrm{L} \tag{10.6}$$

for all $\tau \in \Delta$. Then $h_{\mathrm{L},\alpha}\#F$ has monodromy

$$h_{\mathrm{L},\alpha}\mathcal{H}\, h_{\mathrm{L},\alpha}^{-1}. \tag{10.7}$$

10.2. **Bubbletons.** Let $\Phi = z^A$ with A as in (7.2) be the holomorphic frame of a Delaunay surface and λ_0 (and μ in the S^3 case) such that (7.3) or (7.4) or (7.5) hold. As the monodromy of Φ is unitary, it coincides with the monodromy χ of the unitary Delaunay frame F with respect to the deck transformation (7.1) and is given by $\mathcal{H} = \exp(2\pi i A)$. Now let $\alpha \in \mathbb{C}$ be a solution of

$$2\alpha - 2 - \delta + \sqrt{(2+\delta)^2 - 4} = 0,$$

where $\delta = (ab)^{-1}(k^2/4 - c^2 - |a+\bar{b}|^2)$, for $k \in \mathbb{Z}$ such that $\alpha \notin C_1$, and using the closing conditions (7.3), (7.4) and (7.5) for Delaunay surfaces, we see that $\delta = (k^2-1)(4ab)^{-1} + \Upsilon$, where $\Upsilon = 0$ or $\Upsilon = -4\sin^2(\theta/2)$ or $\Upsilon = 4\sinh^2(q/2)$ for the target space $\mathbb{R}^3$ or S^3 or H^3, respectively. Then $\exp(2\pi i\, A(\alpha)) = \pm\mathrm{Id}$, so for the line $\mathrm{L} = [1:0] \in \mathbb{CP}^1$ and the simple factor $h_{\mathrm{L},\alpha}$, the dressed frame $\hat{F} = h_{\mathrm{L},\alpha}\#_r F$ with $r < |\alpha|$ has monodromy $h_{\mathrm{L},\alpha}\exp(2\pi i A)\, h_{\mathrm{L},\alpha}^{-1}$ and satisfies the conditions (i), (ii) and (iii) of Theorem 6.1 since $\exp(2\pi i A)$ does, so the resulting surfaces are again CMC cylinders. A computation shows that $|\alpha| < \lambda_0 < |\alpha^{-1}|$ when the target space is H^3, so $\lambda_0 \in A_r$ in the H^3 case, and so $\det F_{\lambda_0} \neq 0$, allowing us to apply the generalized Sym-Bobenko formula (5.5).

These circles of ideas where used in [18] to construct bubbletons in $\mathbb{R}^3$ and have recently been utilised in [20] to dress CMC n-noids. In [32], bubbletons in $\mathbb{R}^3$ with ends asymptotic to round cylinders were explicitly constructed. In [38] and [26] less explicit constructions of bubbletons in $\mathbb{R}^3$ with Delaunay ends where obtained. One can easily show that for the surfaces in [32] the ends are asymptotic to round cylinders, but it was unproven that dressing a Delaunay frame with an appropriate $h_{\mathrm{L},\alpha}$ gives a surface with asymptotically Delaunay ends. We prove this in Section 12. (Note that one cannot apply Theorem 9.4, since $r < |\alpha|$ may not be sufficiently close to 1.)

Remark 10.2. *Let $\mathrm{L} = [1:0]$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $\alpha_i \neq \alpha_j$ for $i \neq j$ as above, and define $h = \prod_{j=1}^n h_{\mathrm{L},\alpha_j}$. By similar arguments as above, dressing with h produces multibubbletons.*

When the initial surface is a round cylinder, one can find explicit parametrisations for the resulting bubbletons (and multibubbletons) in all three space forms [21]. When the target space is $\mathbb{R}^3$, these explicit parametrisations coincide with those in [32], which were produced by Bianchi's Bäcklund transformation of a round cylinder. This can also be seen by combining results of [17] and [7].

Example 10.1. *The examples in Figure 3 are generated by triples $(Az^{-1}dz, h_{\mathrm{L},\alpha}, 1)$ where A is as in (7.2) with $c = 0$, $\mathrm{L} = [1:0]$ and the following data: R^3: $a = b = 1/4$, $\lambda_0 = 1$, $\alpha = 7 + 2\sqrt{12}$; S^3: $a = b = 1/3$, $\mu = \lambda_0^{-2} = \exp(2i\tan^{-1}(3\sqrt{7}))$, $\alpha = (7+3\sqrt{5})/2$; H^3: $a = b = 1/6$, $\lambda_0 = (7-3\sqrt{5})/2$, $\alpha = 17 - 12\sqrt{2}$.*

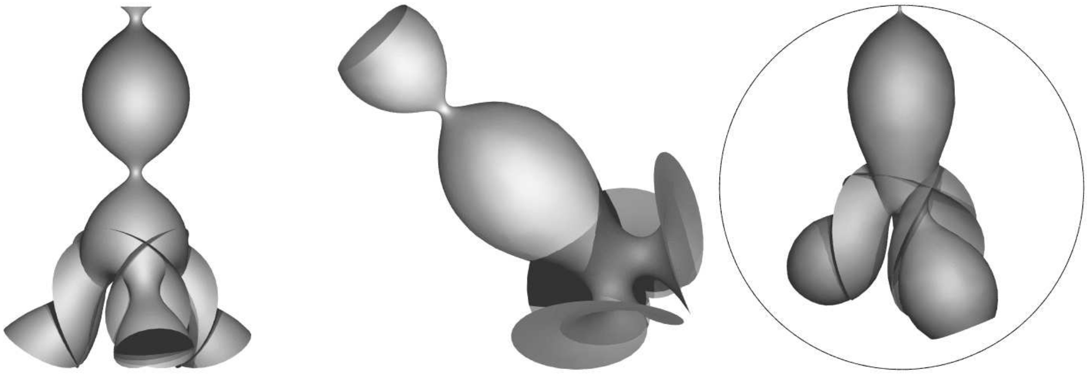

FIGURE 4. Perturbed Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3.

10.3. **Perturbed Delaunay surfaces.** We now provide examples for which Lemma 9.1 and Theorems 9.4 or 9.6 or 9.8 apply. Let $M = \mathbb{C}^*$ and $\xi = Az^{-1}dz$ with A as in (7.2) with A and λ_0 satisfying (7.3) or (7.4) or (7.5) for the respective target. Let $\eta \in \Lambda_r\Omega(\mathbb{C})$. By Theorem 9.1, there is a solution of $d\Phi = \Phi(\xi + \eta)$ of the form $\Phi = z^A P$ with entire $P : \mathbb{C} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ for some $r \in (0, 1)$ arbitrarily close to 1. By (9.1), we may assume $P_0 = \mathrm{Id}$. For τ as in (7.1), the monodromy of Φ is $\chi(\tau) = \exp(2\pi i A) \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$, because $\tau^*P = P$. So the monodromy of the unitary part F of Φ under r-Iwasawa decomposition satisfies $\mathcal{H}(\tau) = \chi(\tau)$, and, by (7.3) (7.4) (7.5), the closing conditions (i), (ii) and (iii) of Theorem 6.1 are satisfied. This yields the existence of many types of CMC cylinders in $\mathbb{R}^3$, H^3, and S^3. When the target is H^3, we must further ensure that F is nonsingular at $\lambda_0 = e^q = \lambda_{1,+}$ or $\lambda_{1,-}$. Choosing $C = \mathrm{Id}$ and $z_e = 0$ in Theorems 9.4, 9.6 and 9.8 respectively, we arrive at the following:

Corollary 10.3. *When A satisfies* (9.2) *or* (9.5) *and the target space is $\mathbb{R}^3$ or S^3 respectively, the CMC surface produced from $\Phi = \exp(A \log z)P$ has an asymptotically Delaunay end at $z = 0$. If A satisfies* (9.6) *and additionally $[A, \eta_0] = 0$, then this is true when the target space is H^3 as well.*

One example of such a CMC surface, looking like a Smyth surface with a Delaunay end added to its head, can be produced with $\eta = dz^k\epsilon_-dz$, for any $d \in \mathbb{C}^*$ and $k \in \mathbb{N}$. See Figure 4 (see also [18] and [19] when the target space is $\mathbb{R}^3$). We note that $\xi + dz^k\epsilon_-dz$ is asymptotically the same as the potential for the Smyth surfaces at $z = \infty$. Furthermore, $\eta_0 = 0$ and so Corollary 10.3 is applicable for all three target space forms, implying that all these surfaces have an asymptotically Delaunay end.

Example 10.2. *Figure 4 displays perturbed Delaunay surfaces in $\mathbb{R}^3$, S^3 and H^3. These surfaces have three umbilics and are generated by triples $(Az^{-1}dz + dz^k\epsilon_-dz, P(1), 1)$ with $d = -1/5$, $c = 0$, $k = 2$ and in $\mathbb{R}^3$: $a = 9/20$, $b = 1/20$ and $\lambda_0 = 1$; in S^3: $a = 0.077$, $b = 0.442579$ and $\mu = \lambda_0^{-2} = e^{i\pi/2}$; in H^3: $a = 1/(2\sqrt{34})$, $b = \sqrt{2/17}$ and $\lambda_0 = 1/4$. The initial condition $P(1)$ is obtained by solving the corresponding equation* (9.1).

11. Trinoids in $\mathbb{R}^3$, S^3 and H^3

To construct trinoids, we find a family of potentials defined on the thrice-punctured Riemann sphere and dressing elements that unitarize their monodromy representations. We do this in two steps. First we choose a potential ξ so that its monodromy representation is pointwise unitarizable on C_1, and then apply Theorem 11.4 to construct a dressing that closes the resulting CMC surface. Secondly, we show that the conditions of Theorem 9.4 are satisfied, and thus the ends of the surface are asymptotic to Delaunay ends.

This family of trinoids is a three-parameter family of surfaces for each choice of target space and each choice of constant mean curvature, parametrized by the three end-weights.

Normalization: We make the following normalizations, for each of the target spaces:

$$\begin{aligned} \mathbb{R}^3:&\quad \lambda_0 = 1\\ S^3:&\quad \lambda_0 \in C_1 \cap \{-\tfrac{\pi}{2} \le \operatorname{arc}(\lambda) < 0\},\ \mu = \lambda_0^{-2}\\ H^3:&\quad \lambda_0 \in (0,1) \text{ in } \mathbb{R}^+. \end{aligned}$$

For S^3 and H^3, H is determined as in Theorem 4.1. For $\mathbb{R}^3$, H can be any positive real number. We next extend a result of [30].

Theorem 11.1. *Let $M = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$. Let H and λ_0 be as in the normalization above. For $k \in \{0, 1, \infty\}$, let $v_k \in \mathbb{R} \setminus \{0\}$ satisfy $v_k(\lambda_0 + \lambda_0^{-1} \pm 2) \le 4$ and define*

$$\nu_k(\lambda) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{v_k}{4}\left((\lambda_0 + \lambda_0^{-1}) - (\lambda + \lambda^{-1})\right)}\,. \tag{11.1}$$

With $m_k = \nu_k(+1)$ and $n_k = \nu_k(-1)$ and $\{i, j, k\} = \{0, 1, \infty\}$, suppose that

$$\begin{aligned} &|n_0| + |n_1| + |n_\infty| \le 1\\ &|n_i| \le |n_j| + |n_k|\\ &\left.\begin{aligned} |m_0| + |m_1| + |m_\infty| \le 1\\ |m_i| \le |m_j| + |m_k| \end{aligned}\right\} \text{ for the ambient spaces } S^3,\ H^3\\ &|v_i| \le |v_j| + |v_k| \text{ for the ambient space } \mathbb{R}^3. \end{aligned} \tag{11.2}$$

Let

$$\xi = \begin{pmatrix} 0 & \lambda^{-1}dz \\ \lambda(\lambda + \lambda^{-1} - \lambda_0 - \lambda_0^{-1})Q/dz & 0 \end{pmatrix},$$

with

$$Q = \frac{v_\infty z^2 + (v_1 - v_0 - v_\infty)z + v_0}{16z^2(z-1)^2}dz^2.$$

Then there exists a conformal CMC immersion f, constructed via DPW from ξ, of M to $\mathbb{R}^3$ (respectively S^3, H^3) with three asymptotically Delaunay ends having weights $w_0 = cv_0$, $w_1 = cv_1$, $w_\infty = cv_\infty$, where c is respectively $1/H$, $1/\sqrt{H^2+1}$, $1/\sqrt{H^2-1}$.

Remark 11.2. *We make a few comments on the assumptions in Theorem 11.1:*

(i) *Note that Q is proportional to the Hopf differential (by a z-independent constant).*
(ii) *The condition $v_k(\lambda_0 + \lambda_0^{-1} - 2) \le 4$ is redundant when the target space is $\mathbb{R}^3$ or H^3.*

(iii) *The condition* $|m_0| + |m_1| + |m_\infty| \le 1$ *is redundant when the target space is* H^3*, as we will see later.*

Example 11.1. *Examples of equilateral trinoids in* $\mathbb{R}^3$*,* S^3 *and* H^3 *are displayed in Figure 5. The values (end-weight,* H*,* λ_0*) are respectively* $(\frac{8}{9}, 1, 1)$*,* $(\frac{16}{9}, 0, -i)$ *and* $(\frac{2\sqrt{7}}{9}, \frac{4}{\sqrt{7}}, \frac{1}{3}(4-\sqrt{7}))$*. For these graphics, we computed appropriate dressing matrices according to Theorem 11.4 to close the surfaces.*

The following two lemmas are required for proving Theorem 11.1. The first lemma shows that ξ can be gauged to a form with a simple pole that is asymptotically equal to a potential for a Delaunay surface, at any given end. Furthermore, the constant term of the potential becomes zero, which is needed for showing the asymptotic behaviour.

Lemma 11.3. *Let* ξ *be a trinoid potential as in Theorem 11.1. Then for each end* $p \in \{0, 1, \infty\}$ *there exists a gauge on a punctured neighborhood* $V \setminus \{p\}$ *of* p*, which is nonsingular on* I_1*, so that in some conformal coordinate* ζ *with* $\zeta(p) = 0$*, the expansion of* $\xi.g$ *is*

$$\xi.g = \begin{pmatrix} 0 & a\lambda^{-1} + b \\ b + a\lambda & 0 \end{pmatrix} \frac{d\zeta}{\zeta} + O(\zeta)d\zeta;\ a,\, b \in \mathbb{R}, \tag{11.3}$$

where $O(\zeta^1)$ *denotes a holomorphic term that is zero at* $\zeta = 0$*.*

Proof. We perform the calculation at the end $z = 0$; the calculation at the other ends is analogous. Let $\mu = \frac{1}{2} - \nu_0(\lambda)$. There exist $a,\, b \in \mathbb{R}$ with $|a| \ge |b|$ satisfying $pp^* = \mu^2$, where $p = a\lambda^{-1} + b$. With

$$k = \frac{v_0 + v_1 - v_\infty}{2v_0},$$

let

$$g = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\lambda & \lambda p \end{pmatrix} \left(\mathrm{Id} + \frac{k}{2} \begin{pmatrix} -1 & 0 \\ p^{-1} & 1 \end{pmatrix} z \right). \tag{11.4}$$

On a small punctured neighborhood $V \setminus \{0\}$ of $z = 0$, $g : (V \setminus \{0\}) \times I_1 \to \mathrm{SL}_2(\mathbb{C})$ is a nonsingular analytic map such that $g(z,\, 0)$ is upper triangular. (This is sufficient to guarantee that gauging by g has no effect in the Sym-Bobenko formulas.) A calculation shows that the expansion of $\xi.g$ at $z = 0$ is

$$\xi.g = A\, z^{-1}dz + kA dz + O(z)dz, \quad A = \begin{pmatrix} 0 & p \\ p^* & 0 \end{pmatrix}, \tag{11.5}$$

so in the coordinate $\zeta = z - kz^2$, the expansion of $\xi.g$ at $\zeta = 0$ is

$$\xi.g = A\, \zeta^{-1}d\zeta + O(\zeta)d\zeta. \tag{11.6}$$

□

11.1. **Pointwise unitarisation.** Let us denote the ends by $(z_0, z_1, z_\infty) = (0, 1, \infty)$. Let $\Phi : \widetilde{M} \times \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ be a holomorphic solution of $d\Phi = \Phi\xi$ for some potential ξ. Let $\mathcal{H}$ be the monodromy representation of Φ. Let τ_j be the deck transformation associated to a once-wrapped counterclockwise loop about the end z_j, and let $\mathcal{H}_j = \mathcal{H}(\tau_j) \in \mathrm{SL}_2(\mathbb{C})$ be the monodromy of τ_j. Note that $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3 = \mathrm{Id}$.

Assume that each $\mathcal{H}_j$ is individually conjugate to a matrix in SU_2, that is, either $-1 < t_j < 1$ or $\mathcal{H}_j = \pm\mathrm{Id}$, where $2t_j = \mathrm{tr}(\mathcal{H}_j)$. The three matrices $\mathcal{H}_j$ are simultaneously unitarisable, that is, there exists a $C \in \mathrm{GL}_2(\mathbb{C})$ so that $C\mathcal{H}_jC^{-1} \in \mathrm{SU}_2$ for $j = 1, 2, 3$, if and only if, see e.g [14], [37],

$$1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3 \geq 0 \ .$$

Let T_0 be the set with tetrahedral boundary defined by

$$\nu_1 + \nu_2 + \nu_3 \leq 1 \quad \text{and} \quad \nu_i \leq \nu_j + \nu_k \tag{11.7}$$

for all distinct $\{i, j, k\} = \{1, 2, 3\}$ and $\nu_j \in \mathbb{R}$. Let T be the orbit of T_0 by the action of the group generated by the transformations $\nu_k \to \nu_k + 1$ and $\nu_k \to -\nu_k$.

The inequalities (11.7) are the case $n = 3$ of Biswas' inequalities for n punctures [2], and are the spherical triangle inequalities in a 2-sphere of radius $1/(2\pi)$. For a triangle with side lengths ν_j in such a sphere, $\nu_1 + \nu_2 + \nu_3 \leq 1$ means that the triangle is never bigger than the maximal triangle, and $\nu_i \leq \nu_j + \nu_k$ is the triangle inequality.

Define ν_j by $t_j = \cos(2\pi\nu_j)$ for $\nu_j \in \mathbb{R}$. The matrices $\mathcal{H}_j$ are simultaneously unitarisable if and only if, see e.g [2], [30] or [37],

$$(\nu_1, \nu_2, \nu_3) \in T \ . \tag{11.8}$$

11.2. **Global unitarisation.** We will need a result, which implies that once the monodromies of a trinoid are simultaneously pointwise unitarised, they can be simultaneously unitarised globally by a loop element. A similar result can be found in [13].

Theorem 11.4. [30] *(Gluing theorem) Let $\mathcal{H}_1,\ldots,\mathcal{H}_n : \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ and S be a finite set of points in C_1 such that*

(i) *there exist a j and k so that $[\mathcal{H}_j, \mathcal{H}_k] \neq 0$ on $C_1 \setminus S$, and*
(ii) *$\mathcal{H}_1,\ldots,\mathcal{H}_n$ are simultaneously unitarisable pointwise on $C_1 \setminus S$.*

Then there exists an $r \in (0, 1)$ and a $C : I^*_{1/r} \to M_{2\times 2}(\mathbb{C})$ so that

(i) *$\{\det(C) = 0\} \cap A_r$ is a finite subset of C_1,*
(ii) *$C \in \Lambda^+_s\mathrm{GL}_2(\mathbb{C})$ for all $s \in (0, 1)$,*
(iii) *$C\mathcal{H}_jC^{-1} \in \Lambda^{\mathbb{R}}_r\mathrm{GL}_2(\mathbb{C})$ for all $j = 1, \ldots, n$.*

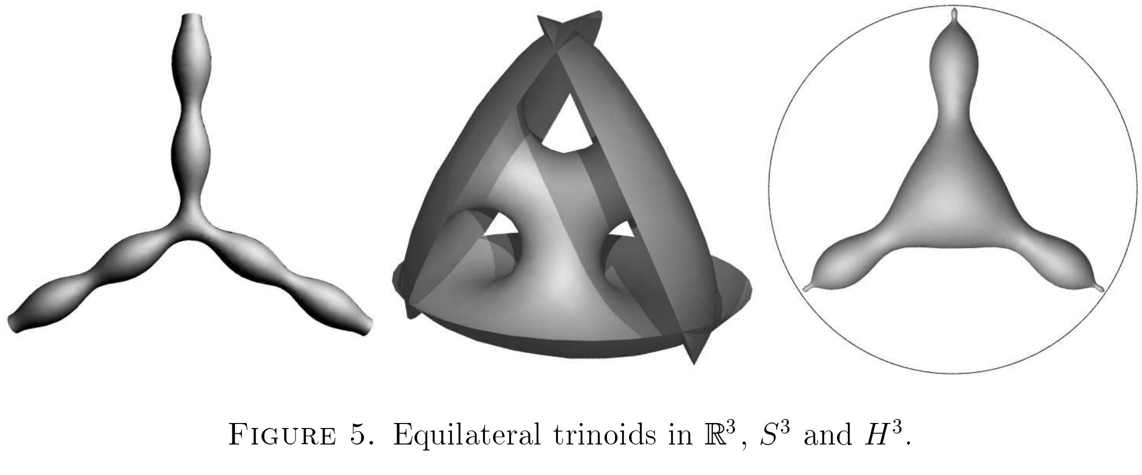

FIGURE 5. Equilateral trinoids in $\mathbb{R}^3$, S^3 and H^3.

Lemma 11.5. *Let ξ be as in Theorem 11.1. Then the monodromy representation $\mathcal{H}$ above is unitarizable on C_1 except possibly at finitely many points. Furthermore, the inequalities* (11.2) *imply that the lower bounds on the weights* (9.2) *and* (9.5) *and* (9.6) *hold.*

Proof. We first compute the eigenvalues of $\mathcal{H}_k$. For the gauge g and new coordinate ζ in Lemma 11.3 at the end z_k, one solution of $d\hat{\Phi} = \hat{\Phi}(\xi.g)$ is $\hat{\Phi} = \Phi g$ with monodromies $-\mathcal{H}_k$ for each τ_k. Since $\xi.g$ has a simple pole at z_k, Lemma 9.1 implies that $-\mathcal{H}_k$ is conjugate to $\exp(2\pi i A_k)$, where $A_k = \operatorname{Res}_{z_k} \xi.g$. Hence the eigenvalues of $\mathcal{H}_k$ are $\exp(\pm 2\pi i \nu_k)$.

We now show that the curve

$$(\nu_0(\lambda),\, \nu_1(\lambda),\, \nu_\infty(\lambda)) \in T \cap [-1/2, 1/2]^3 \subset \mathbb{R}^3$$

for all $\lambda \in C_1$ if and only if the inequalities (11.2) are satisfied. One direction is clear. To prove the other direction, assume equations (11.2) are satisfied. Define

$$\begin{aligned} \rho_k &= \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - v_k x}, \quad k \in \{0,\, 1,\, \infty\} \\ f &= |\rho_0| + |\rho_1| + |\rho_\infty| \\ f_i &= -|\rho_i| + |\rho_j| + |\rho_k|, \quad \{i,\, j,\, k\} = \{0,\, 1,\, \infty\}\ , \end{aligned}$$

for $x \in \mathcal{I} := [\frac{\lambda_0+\lambda_0^{-1}-2}{4}, \frac{\lambda_0+\lambda_0^{-1}+2}{4}]$. All three terms of f are increasing when $x \geq 0$ and decreasing when $x \leq 0$, so the first two inequalities in (11.2) imply that $f \leq 1$ on $\mathcal{I}$. Hence $|\nu_1| + |\nu_2| + |\nu_3| \leq 1$ on C_1.

In the case $0 < v_0 \leq v_1$ or $v_1 \leq v_0 < 0$, f_0 is increasing, so $|n_0| \leq |n_1| + |n_\infty|$ implies f_1 is non-negative on $\mathcal{I}$. Hence $|\nu_0| \leq |\nu_1| + |\nu_\infty|$ on C_1.

We require the following fact: the function ρ_1/ρ_0 extends to a smooth function at $x = 0$, and, if $v_1 > v_0$, then $|\rho_1/\rho_0|$ is strictly increasing.

In the case $v_0 \geq v_1$, the above fact implies that that $f_0/|\rho_0|$ is non-increasing. $|n_0| \leq |n_1| + |n_\infty|$ implies that $f_0/|\rho_0|$ is non-negative at the upper endpoint of $\mathcal{I}$, so $f_0/|\rho_0|$, and hence f_0, is non-negative on $\mathcal{I}$. Hence $|\nu_0| \leq |\nu_1| + |\nu_\infty|$ on C_1.

In the case $v_0 \leq v_1$, the above fact implies that that $f_0/|\rho_0|$ is non-decreasing. But $(f_0/|\rho_0|)(0) = -1 + |v_2/v_1| + |v_3/v_1| \geq 0$, so $f_0/|\rho_0|$ is non-negative on $\mathcal{I}$. Hence $|\nu_1| \leq |\nu_2| + |\nu_3|$ on C_1.

Symmetric arguments for the other cases imply that $(\nu_0,\, \nu_1,\, \nu_\infty) \in T \cap [-1/2, 1/2]^3$ for all $\lambda \in C_1$.

To prove the first part of the lemma, suppose the inequalities (11.2) hold. Because $(\nu_0,\, \nu_1,\, \nu_\infty) \in T \cap [-1/2, 1/2]^3$, Equation (11.7) holds, and so $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_\infty$ are simultaneously unitarizable on $C_1 \setminus S$, where S is the finite set $\{\lambda \in C_1$ such that $\nu \in \partial T\}$.

To prove the final sentence of the lemma, we note that $n_k \geq -1/2$ and $m_k \geq -1/2$ imply that $v_k \geq -12/(\lambda_0 + \lambda_0^{-1} + 2)$ in all three space forms, and that $v_k \geq -12/(\lambda_0 + \lambda_0^{-1} - 2)$ in S^3. Thus the weights w_k satisfy

$$\begin{aligned} & w_k \geq -3/H \text{ in } \mathbb{R}^3\ , \\ (11.9)\qquad & 6(\sqrt{H^2+1} + H) \geq w_k \geq -6(\sqrt{H^2+1} - H) \text{ in } S^3\ , \\ & w_k \geq -6(H - \sqrt{H^2-1}) \text{ in } H^3\ , \end{aligned}$$

where c is defined as in Theorem 11.1. We must show that the inequalities in (11.9) are strict. If v_0 attains its lower bound in (11.9), then $|n_1| + |n_\infty| = 1/2$ and $v_0 < v_1$

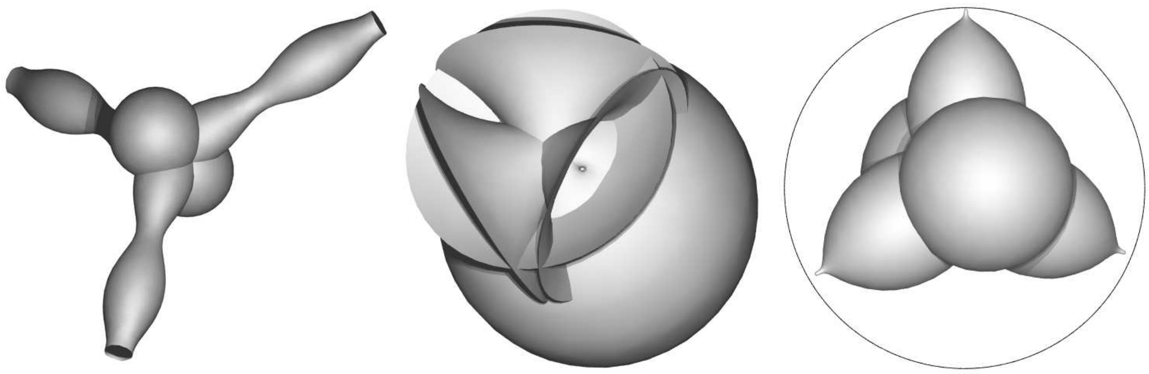

FIGURE 6. Dressed equilateral trinoids in $\mathbb{R}^3$, S^3 and H^3.

and $v_0 < v_\infty$. Therefore $f_0/|\rho_0|$ is an increasing function which is zero at the upper endpoint of $\mathcal{I}$, and so $|m_0| > |m_1| + |m_\infty|$, contradicting one of the inequalities in (11.2). All other cases of equality in (11.9) can be argued similarly. We conclude that (7.6) holds. □

Remark 11.6. *When the target space is S^3, there exists a $\lambda \in C_1$ so that $\nu_k(\lambda) = 0$ for all $k = 0, 1, \infty$ and thus $|v_i| < |v_j| + |v_k|$ automatically holds in S^3, if all the other inequalities in* (11.2) *hold. However, the relation $|v_i| \le |v_j| + |v_k|$ will not hold in general when the target is H^3. For H^3, there exist trinoids that satisfy* (11.2) *but not $|v_i| \le |v_j| + |v_k|$: for example, $\lambda_0 \in (0, 1)$ with $\lambda_0 + \lambda_0^{-1} = 10$ and $v_0 = -3/5$ and $v_1 = v_\infty = 1/4$ in Theorem 11.1 is such a counterexample.*

We still owe the reader a proof of Theorem 11.1:

Proof. By Lemma 11.5, the holonomy representation $\mathcal{H}$ is pointwise unitarizable on C_1 minus a finite number of points. Then, by the gluing theorem 11.4, there exists an r-dressing matrix C which unitarizes the monodromy representation. We have that $\nu_k(\lambda_0) = 0$ for $k = 0,\ 1,\ \infty$, which is equivalent to the conditions in Theorem 6.1 for $\mathbb{R}^3$, S^3 and H^3 respectively, for the dressed monodromy representation. (In the case of $\mathbb{R}^3$, one must also use that $\partial_\lambda \nu_k|_{\lambda_0} = 0$.)

In the case of H^3, with F and B the r-Iwasawa factors of Φ, F extends holomorphically to λ_0 and $\det F_{\lambda_0} \neq 0$. This is because $F = C\Phi B^{-1}$ and C, Φ, B are all nonsingular at λ_0. Thus we are able to apply the Sym-Bobenko formula (5.5) *in the case of H^3. In the other space forms, $\lambda_0 \in C_1$, so F is clearly nonsingular at λ_0.*

We conclude that the resulting CMC immersion satisfies $\tau^* f_{\lambda_0} = f_{\lambda_0}$ for all τ in the group of deck transformations of $\widetilde{M}$, for all three space forms.

Finally, to show that each end z_j is asymptotic to an end of a Delaunay surface, we show that the conditions of Theorems 9.4 and 9.6 and 9.8 are satisfied for $\xi.g$ in the coordinate ζ, where g and ζ are as in Lemma 11.3. All the needed conditions are satisfied: λ_0 equals either $\lambda_{1,+}$ or $\lambda_{1,-}$, and the commutativity condition (9.7) *holds by the choice of g, and the lower bounds on the weights* (9.2) *and* (9.5) *and* (9.6) *hold by Lemma 11.5.* □

12. Asymptotics of dressed n-noids

In [20] it was shown that trinoids and symmetric n-noids in $\mathbb{R}^3$ can be dressed by simple factors. The potentials we use in Theorem 11.1 for constructing trinoids in $\mathbb{R}^3$, H^3 and S^3 are in a slightly more general form than in [20], and are identical when the target space is $\mathbb{R}^3$. In the more general form, the only difference in the potentials is the change in the constant $\lambda_0 + \lambda_0^{-1}$ ($= 2$, < 2 and > 2 for $\mathbb{R}^3$, S^3 and H^3, respectively.). This change has little effect on the computations in [20], and only deforms the infinite discrete set of values of λ where the monodromy is reducible. Then Lemma 3.2 of [20] can be applied verbatim, and we conclude existence of trinoids dressed by simple factors in S^3 and H^3 as well.

Example 12.1. *In Figure 6 we display three equilateral trinoids dressed by suitable simple factors. For the $\mathbb{R}^3$ model, the potential in Theorem 11.1 has entries $v_0 = v_1 = v_\infty = 8/9$ and $\lambda_0 = 1$. The S^3 model is displayed with the front bubble cut away to show the trinoidal structure. The potential data is $v_0 = v_1 = v_\infty = 16/9$ and $\lambda_0 = i$. For the H^3 surface, in the potential we take $v_0 = v_1 = v_\infty = 1/2$ and $\lambda_0 = 1/4$. For each picture, the dressing was computed to close the surface, and then the resulting surface was further dressed by a simple factor, having the effect of adding two bubbles to each surface at its center.*

However, it was not shown in [20] that the ends of the trinoids in $\mathbb{R}^3$ dressed by simple factors are asymptotically Delaunay. Here we prove that any asymptotically Delaunay end, which is dressed by simple factors to another closed end, gives again an asymptotically Delaunay end. This result applies to the examples (dressed trinoids and dressed n-noids) in $\mathbb{R}^3$ in [20], and to the bubbletons in Section 10.2 and dressed trinoids in all three target space forms $\mathbb{R}^3$, S^3 and H^3. We state the result with respect to a single end, so that it applies to further examples of simple type dressings of CMC surfaces with Delaunay ends.

12.1. **Asymptotics of dressed CMC surfaces.** For the line $L = [a_0 : b_0] \in \mathbb{CP}^1$, the projections π_L and $\pi_L^\perp$ have the following matrix forms:

$$\pi_L = \frac{1}{|a_0|^2 + |b_0|^2} \begin{pmatrix} |a_0|^2 & a_0\bar{b}_0 \\ \bar{a}_0 b_0 & |b_0|^2 \end{pmatrix} \tag{12.1}$$

$$\pi_L^\perp = \frac{1}{|a_0|^2 + |b_0|^2} \begin{pmatrix} |b_0|^2 & -a_0\bar{b}_0 \\ -\bar{a}_0 b_0 & |a_0|^2 \end{pmatrix}. \tag{12.2}$$

We set

$$\mathfrak{A} = \{\alpha \in I_1^* : |\alpha|^2 < \tfrac{b^2}{a^2} \text{ and } 1/4 - \Upsilon + \alpha^{-1} ab(1-\alpha)^2 > 0\}\ , \tag{12.3}$$

where a, $b \in \mathbb{R}$ with $|a| \geq |b| > 0$ satisfy equations (7.3) or (7.4) or (7.5) with $c = 0$, and $\Upsilon = 0$ or $\Upsilon = -4ab\sin^2(\theta)$ or $\Upsilon = 4ab\sinh^2(q/2)$ for the target space $\mathbb{R}^3$ or S^3 or H^3. (Note that by switching a and b if necessary, we may assume without loss of generality that $|b| \leq |a|$.) In the following theorem, we denote by

$$\mathcal{U}_\varepsilon = \{z : 0 < |z| < \varepsilon\} \tag{12.4}$$

an open punctured disk in a Riemann surface M, where z is a centered coordinate about a point $p \in M$. Let $\gamma : [0, 1] \to \mathcal{U}_\varepsilon$ be a simple closed curve and denote by δ the corresponding deck transformation of the universal cover of $\mathcal{U}_\varepsilon$.

Defining convergence via the norm defined in Section 8.2, we have the following:

Theorem 12.1. *Let F be an extended frame on $\mathcal{U}_\varepsilon$ that converges to UF_{D} as $z \to 0$, with F_{D} as in* (8.4) *and $U \in \Lambda_r^{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ depending only on λ. Assume that F has reducible monodromy with respect to δ at some $\alpha \in \mathfrak{A}$ with invariant subspace $\mathrm{L} \in \mathbb{CP}^1$ and let $h_{\mathrm{L},\alpha}$ be the corresponding simple factor.*

Let $\tilde{f}$ be the CMC surface obtained from (4.1) *or* (4.2) *or* (4.3) *using the dressed frame $h_{\mathrm{L},\alpha} F h_{\mathrm{L}',\alpha}^{-1}$ as in equation* (10.5) *with $\alpha \in \mathfrak{A}$ as in equation* (12.3). *Then $\tilde{f}$ converges to an end of a Delaunay surface as $z \to 0$.*

Proof. Let f_{D} be the immersion obtained from the frame $h_{L,\alpha} U F_{\mathrm{D}} h_\infty^{-1}$, where h_∞ is defined below. We will show below that f_{D} is a Delaunay surface. Define x, y by $\log z = -(x+iy)$ and let λ_0 be as in Section 7.2.

We need to show that $\|\tilde{f} - f_{\mathrm{D}}\|_{\lambda_0} \to 0$ as $z \to 0$. As in the proofs of Theorems 9.4 and 9.6 and 9.8, it suffices to show that $\|\mathrm{Id} - (h_{\mathrm{L}',\alpha} F^{-1} h_{\mathrm{L},\alpha}^{-1})(h_{L,\alpha} U F_{\mathrm{D}} h_\infty^{-1})\|_{B_\varepsilon(\lambda_0)} \to 0$ for some sufficiently small $\varepsilon > 0$, where $B_\varepsilon(\lambda_0)$ is the open ball of radius ε about λ_0 in $\mathbb{C}$. We recall that $\mathrm{L}' = \overline{F(z,\alpha)}^t \mathrm{L}$, and setting $\mathrm{L}'' = \overline{U(\alpha) F_{\mathrm{D}}(z,\alpha)}^t \mathrm{L}$, we thus consider

$$\begin{aligned}|\mathrm{Id} - (h_{\mathrm{L}',\alpha} F^{-1} h_{\mathrm{L},\alpha}^{-1})(h_{L,\alpha} U F_{\mathrm{D}} h_\infty^{-1})| \leq{}& |h_{\mathrm{L}',\alpha}|\,|\mathrm{Id} - F^{-1} U F_{\mathrm{D}}|\,|h_{\mathrm{L}',\alpha}^{-1}| \\ &+ |h_{\mathrm{L}',\alpha} F^{-1} U F_{\mathrm{D}} h_{\mathrm{L}',\alpha}^{-1}|\,|\mathrm{Id} - h_{\mathrm{L}',\alpha} h_{\mathrm{L}'',\alpha}^{-1}| \\ &+ |h_{\mathrm{L}',\alpha} F^{-1} U F_{\mathrm{D}} h_{\mathrm{L}'',\alpha}^{-1}|\,|\mathrm{Id} - h_{\mathrm{L}'',\alpha} h_\infty^{-1}|.\end{aligned}$$

By assumption, $\lim_{x\to\infty} |\mathrm{Id} - F^{-1} U F_{\mathrm{D}}| = 0$. Writing

$$\mathrm{L}'' = \overline{F^{-1}\, U\, F_{\mathrm{D}}}^t \mathrm{L}',$$

we conclude $\lim_{x\to\infty} |\mathrm{Id} - h_{\mathrm{L}',\alpha} h_{\mathrm{L}'',\alpha}^{-1}| = 0$. It thus only remains to show that $h_{\mathrm{L}'',\alpha}$ converges to h_∞. Using equation 8.4, we write

$$F_{\mathrm{D}} = \frac{1}{\mathcal{L}} \begin{pmatrix} 2v(b+a\lambda)C & -v'C + (4ab\lambda + v^2)(a\lambda^{-1} + b)X^{-1}S \\ 2v(b+a\lambda)(a\lambda+b)X^{-1}S & -v'(a\lambda+b)X^{-1}S + (4ab\lambda+v^2)C \end{pmatrix}, \tag{12.5}$$

where $\mathcal{L} = \sqrt{2v(b+a\lambda)(4ab\lambda+v^2)}$, $C = \cosh((x+iy-\mathbf{f})X)$, $S = \sinh((x+iy-\mathbf{f})X)$, $X = \sqrt{1/4 - \Upsilon + \lambda^{-1}ab(1-\lambda)^2}$, and a, b are defined in Equation (7.3) or (7.4) or (7.5), and $\Upsilon = 0$ or $\Upsilon = -4ab\sin^2(\theta)$ or $\Upsilon = 4ab\sinh^2(q/2)$ for the target spaces $\mathbb{R}^3$, S^3 or H^3 respectively. We set

$$F_{\mathrm{D}}|_{\lambda=\alpha} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & \mathcal{A}^* \end{pmatrix}. \tag{12.6}$$

The upper-left entry of F_{D} defined from equation (12.5) with $\lambda = \alpha$ has no zeroes near $x = \infty$, so $\mathcal{A}$ has no zeroes near $x = \infty$. Thus from equations (12.1) and (12.2), the entries of $\psi_{\mathrm{L}'',\alpha}$ (equation (10.1)) can be written in terms of rational functions in $\mathcal{B}/\mathcal{A}$, $\mathcal{B}^*/\mathcal{A}$, $\mathcal{A}^*/\mathcal{A}$ and their complex conjugates with coefficients independent of z. Thus the entries of $h_{\mathrm{L}'',\alpha}$ are rational in terms of $\mathcal{B}/\mathcal{A}$, $\mathcal{B}^*/\mathcal{A}$, $\mathcal{A}^*/\mathcal{A}$ and their complex conjugates with coefficients independent of z . We show that $\mathcal{B}/\mathcal{A}$ respectively $\mathcal{B}^*/\mathcal{A}$, $\mathcal{A}^*/\mathcal{A}$ converge to the following periodic functions P_1 respectively P_2, P_3 as $x \to \infty$:

$$P_1 = \frac{-v' + (4ab\alpha + v^2)(a\alpha^{-1} + b)X_\alpha^{-1}}{2v(b+a\alpha)}, \tag{12.7}$$

$$P_2 = -\frac{2v(b+a\alpha)(a\alpha+b)X_\alpha^{-1}}{2v(b+a\alpha)}, \tag{12.8}$$

$$P_3 = -\frac{v'(a\alpha+b)X_\alpha^{-1} + 4ab\alpha + v^2}{2v(b+a\alpha)} \ , \tag{12.9}$$

where X_α denotes the value of X at $\lambda = \alpha$. We show only that $\mathcal{B}/\mathcal{A}$ converges to P_1. (To show convergence of $\mathcal{B}^*/\mathcal{A}$ and $\mathcal{A}^*/\mathcal{A}$ to P_2 respectively P_3 is similar.) We have

$$\begin{aligned}\frac{\mathcal{B}}{\mathcal{A}} &= \frac{-v' + (4ab\alpha+v^2)(a\alpha^{-1}+b)X_\alpha^{-1}S_\alpha/C_\alpha}{2v(b+a\alpha)} \\ &= \frac{-v' + (4ab\alpha+v^2)(a\alpha^{-1}+b)X_\alpha^{-1}\frac{1-\exp(-2(x+iy-\mathbf{f}_\alpha)X_\alpha)}{1+\exp(-2(x+iy-\mathbf{f}_\alpha)X_\alpha)}}{2v(b+a\alpha)} \ ,\end{aligned}$$

where C_α, S_α and $\mathbf{f}_\alpha$ denote the values of C, S respectively $\mathbf{f}$ at $\lambda = \alpha$. Thus the following claim proves that $\mathcal{B}/\mathcal{A}$ converges to P_1:

$$\lim_{x\to\infty} \mathrm{Re}((x+iy-\mathbf{f}_\alpha)X_\alpha) = \infty \ . \tag{12.10}$$

When $|a| = |b|$, then $v = 2b$ is constant and (12.10) is clear. When $|a| > |b|$, because $\alpha \in \mathfrak{A}$, $\mathrm{Re}((x+iy-\mathbf{f}_\alpha)X_\alpha) = (x-\mathbf{f}_\alpha)X_\alpha$. Since v is as in (8.3) and $|\alpha| < |b/a|$ and $\mathrm{dn}(2ax, 1-b^2/a^2) \in (0,\,1]$, it follows that $\partial_x(x-\mathbf{f}_\alpha)X_\alpha$ is positive and uniformly bounded away from zero for all $x \in \mathbb{R}$. Thus (12.10) holds.

We set h_∞ to be $h_{\mathrm{L}'',\alpha}$ with $\mathcal{B}/\mathcal{A}$ (respectively $\mathcal{B}^*/\mathcal{A}, \mathcal{A}^*/\mathcal{A}$) replaced by P_1 (respectively P_2, P_3) in equation (12.7) (respectively (12.8), (12.9)). Thus $\lim_{x\to\infty} |\mathrm{Id} - h_{\mathrm{L}'',\alpha} h_\infty^{-1}| = 0$. Therefore

$$\lim_{x\to\infty} \|\tilde{f} - f_{\mathrm{D}}\|_{\lambda_0} = 0.$$

Finally, we show that f_{D} is a Delaunay surface. (We show only the $\mathbb{R}^3$ case, as the arguments for S^3 and H^3 are similar.) The immersion f_{D} has frame $h\,U\,F_{\mathrm{D}}h_\infty^{-1}$ and $h\,U$ affects a rigid motion of the surface generated by the frame $F_{\mathrm{D}}h_\infty^{-1}$, and h_∞ on the right is independent of y, hence vertical translation of the domain is equivalent to rotating the surface about a fixed line. Thus f_{D} is a Delaunay surface. More rigorously, we consider the translation of the domain

$$\mathcal{T}_\theta : x+iy \mapsto x + i(y+\theta), \text{ for } \theta \in \mathbb{R} \ .$$

As $g := (hU)^{-1}\,(f_{\mathrm{D}} + 2i\lambda H^{-1}(\partial_\lambda(hU))(hU)^{-1})\,(hU)$ is a rigid motion of f_{D}, we conclude that $f_{\mathrm{D}}|_{\lambda_0=1}$ is a Delaunay surface if $g|_{\lambda_0=1}$ is. Defining

$$A = \begin{pmatrix} 0 & a\lambda^{-1}+b \\ b+a\lambda & 0 \end{pmatrix},$$

we have

$$\begin{aligned}\mathcal{T}_\theta^*\, g|_{\lambda_0} &= \mathcal{T}_\theta^*\,\left(-2i\lambda H^{-1}(\partial_\lambda(F_{\mathrm{D}}h_\infty^{-1}))(F_{\mathrm{D}}h_\infty^{-1})^{-1}\right)\big|_{\lambda_0} \\ &= \left[\exp(i\theta A)\,g\,\exp(-i\theta A) - 2i\lambda H^{-1}(\partial_\lambda \exp(i\theta A))\exp(-i\theta A)\right]_{\lambda_0},\end{aligned} \tag{12.11}$$

since h_∞ is independent of the variable y. Equation (12.11) represents a rotation of $g|_{\lambda_0}$ by the angle θ about a fixed axis (independent of θ). Hence $g|_{\lambda_0}$ is a surface of revolution and so f_{D} is a Delaunay surface. This concludes the proof. □

Although the condition $\alpha \in \mathfrak{A}$ in Theorem 12.1 appears restrictive, it is in fact satisfied by the dressed Delaunay surfaces as well as by the dressed n-noids in [20].

We have made extensive use of the classical theory of regular singular points to prove both existence of new classes of CMC surfaces as well as exhibiting their asymptotic behaviour. The unitarisation problem for three regular singularities is solved here, and these ideas can be extended to n singularities if special symmetries are imposed. The general case of $n \geq 4$ remains unsolved.

It might also be interesting to see similar results in a setting where not all singularities are regular. In Section 10 we constructed cylinders with one Delaunay and one 'Smyth like' end by utilising the theory of regular singular points, even though the singularity corresponding to the Smyth like end was not regular singular. Is it possible to construct trinoids whose ends are not all asymptotically Delaunay and prove asymptotic results in such cases?

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