Real Options, Non-traded Assets and Utility Indifference Prices.

D.G. Hobson*

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Abstract

We show that the utility indifference (bid) price of a contingent claim is bounded above by the price under the minimal martingale measure. This bound is independent of both the utility function and initial wealth of the agent.

We prove this result in two ways, firstly by considering the HJB-equation for the value function of the agent, and secondly by representing the optimal target wealth as a function of the state-price density and considering the implications for option pricing. This second method yields that the marginal utility indifference (bid) price of a non-negative claim is the price under the minimal martingale measure. If the claim is bounded then the marginal ask price is also the price under the minimal martingale measure, but it is easy to construct counterexamples to this result if the claim is unbounded.

*dgh@maths.bath.ac.uk

1 Introduction

The key insights of the Black-Scholes option pricing methodology and formula are that in a complete market it is possible to replicate a contingent claim, and that the initial fortune which is required to finance the replicating strategy is therefore the fair price of the option. In particular the price of a contingent claim is determined unambiguously by the principles of no-arbitrage and is independent of the risk preferences of agents.

The conclusion that there are unique option prices in the Black-Scholes model is lost as soon incompleteness is introduced into the model. This can happen in many ways, for example following the introduction of transaction costs, or if the assumptions of the model do not allow the agent to follow the replicating strategy. In these cases any non-attainable contingent claim carries risk, and any pricing rule makes implicit or explicit assumptions about utilities and preferences.

The typical problem we have in mind (see [13], [2], [10], [11]) is as follows. There are two assets, one of which is traded, but the second is not. Although the price processes for the assets may be driven by correlated Brownian motions the coefficients of the dynamics for the traded asset do not depend on the untraded asset. An agent is due to receive a claim which is contingent on the non-traded asset. How much is that random claim worth? This is the situation in real options, see [3]. An illustration from Hubalek and Schachermayer [13] is when the two assets are different brands of crude oil, only one of which is liquidly traded.

This problem is an example of the problem of pricing a claim in an incomplete market and is similar to those considered in [7], [6], [20], [4] and many others. In common with [4] and [2] we model our agents as maximisers of expected utility. Another common approach is to select a martingale measure (for example the minimal martingale measure) and to use that for pricing.

For the problem detailed above Hubalek and Schachermayer [13] show that no-arbitrage arguments give only trivial bounds on the fair price of the claim, so that any price $p \in (0, \infty)$ is consistent with some martingale measure. Instead we consider agents who seek to maximise expected utility and their utility indifference price. For an agent with a given utility function the utility indifference bid price is the amount the agent is prepared to pay which leaves him indifferent between paying for and receiving the claim, and not paying for and not receiving the claim. The goal of this paper is to show that, even though the bid price offered by an agent must depend on her choice of utility function, we can find a simple upper bound on her bid price for the option which is independent of that choice. This bound is the price of the claim under the minimal martingale measure. Further this bound represents the unit price she would be prepared to pay for an infinitesimable quantity of the option.

The rest of this paper is structured as follows. In the next section we describe the model and the main concepts and state the results. In Section 3 we prove the first result, namely that if the agent prices by utility indifference then the bid-price for a contingent claim is bounded above by the discounted expectation under the minimal martingale measure. The key observation is that this bound is independent of the choice of utility function. The proof in Section 3 is based on an analysis of the HJB equation. In contrast, in the next section we use a dual approach to show that this bound also represents the marginal price the agent is prepared to pay for the claim. Section 5 describes the results for certain common parametric families of utility functions for which explicit calculations are sometimes possible. Section 6 concludes.

2 The model and main results

In this section we describe the model and the notion of utility indifference pricing. This allows us to state the main results of the paper. The first result is that the price under the minimal martingale measure (see Section 2.4) provides a bound on the utility indifference price of an option. This bound is independent of the choice of utility function, and of the initial wealth of the agent. The second result is stronger and says that the marginal price under utility indifference is the price under the minimal martingale measure.

2.1 The market

We consider a model for the pricing of contingent claims on non-traded assets. The market consists of a bond (paying a constant interest rate r[1]) and a traded asset with non-negative price process S which satisfies the dynamics[2]

(1) $$dS_t = S_t(\eta(S_t,t)dW + \mu(S_t,t)dt) \qquad S_0 = s_0.$$

[1] The results extend easily to deterministic interest rate processes.

[2] We assume that η and μ have sufficient regularity properties to ensure that the stochastic differential equation for S has a pathwise unique, non-explosive strong solution. See, for example, Rogers and Williams [18, Chapter V] for both terminology and sufficient conditions.

The reader is invited to think of the situation where $\eta(s,t) \equiv \sigma$ and $\mu(S_t, t) = \nu$ whence S is an exponential Brownian motion, but the results hold more generally. If an investor holds a portfolio which includes π units of the risky-asset then the fact that his portfolio is assumed to be self-financing means that his wealth process X is the solution to

$$dX_t = \pi_t dS_t + r(X_t - \pi_t S_t)dt.$$

We write X^π when we wish to emphasise the dependence of this trading wealth on the strategy π. Here π lies in some convex cone of admissible trading strategies $\mathcal{A}$. We make more comments on the space of admissible strategies below, but whilst admissibility is taken to exclude doubling strategies we do assume that the space of admissible strategies is suitably rich. Note that in general $\mathcal{A}$ may depend on the current values of x, s, y and the endowment of the claim $g \ldots$.

2.2 The contingent claim

Suppose there is a second asset with non-negative price process Y. This asset is not traded. Suppose that Y has dynamics[3]

$$dY_t = a(S_t, Y_t, t)dB + b(S_t, Y_t, t)dt \qquad\qquad Y_0 = y_0,$$

where the Brownian motion B is correlated to W with correlation ρ.[4] We let $dB = \rho dW + \overline{\rho} dW^\perp$ where $W^\perp$ is a Brownian motion independent of W and $\overline{\rho} = \sqrt{1-\rho^2}$. Again the canonical situation is when Y is an exponential Brownian motion.

The problem for the agent is to price a (European) contingent claim with payoff at time T given by a non-negative function $g(S_T, Y_T)$ with bounded derivatives. The introduction of the second non-traded asset makes the model incomplete (provided $|\rho| < 1$), so that the fair price of a contingent claim must depend on the risk preferences of the agent.

2.3 Utility Indifference Prices

Suppose an agent has utility function U which is increasing and concave. In the absence of any contingent claim the problem faced by the agent is to maximise her expected utility

[3] Again we assume that the parameters a and b have sufficient regularity properties to guarantee that this equation has a unique strong solution. We also assume that a is bounded, although this may be weakened to a moment condition $\mathbb{E}\int_0^T a^2 ds < \infty$.

[4] More generally we can allow ρ to be a Markov function of S_t, Y_t and t.

of wealth at the horizon time T and her value function[5] is given by

$$V = V(x,s,t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X^\pi_T)|X_t = x, S_t = s].$$

Then V is increasing and concave in its first argument, and does not depend on Y since the dynamics of S do not depend on Y. If S is exponential Brownian motion then V does not depend on s either.

If the agent is to receive a random payout $g = g(S_T, Y_T)$ at time T then her value function becomes

$$\mathcal{V}_g = \mathcal{V}_g(x,s,y,t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X^\pi_T + g(S_T, Y_T))|X_t = x, S_t = s, Y_t = y].$$

(In this notation, $V = \mathcal{V}_0$.) The utility indifference (bid) price (Hodges and Neuberger [12]) is the amount of money the agent could pay now which would leave her indifferent between paying and receiving the claim at time T, and not paying, and not receiving the claim. In particular she has the same expected utilities under both scenarios so that, provided $V(x,s,t)$ is finite, the bid price $p \equiv p(g) \equiv p(x,s,y,t;g)$ is the solution[6] to

$$\mathcal{V}_g(x-p,s,y,t) \equiv V(x,s,t).$$

In order to emphasise the fact that p is a bid price we sometimes write $p \equiv p_b(g)$. We can also define the ask price of an agent. The utility-indifference ask price $p_a(g)$ is the smallest price at which the agent is prepared to sell a claim g, in the sense that selling the claim for $p_a(g)$ leaves the agent with at least as large an expected utility as if she had not sold the claim. We have p_a is the solution to

$$\mathcal{V}_{-g}(x+p_a,s,y,t) = V(x,s,t).$$

Finally we define the marginal (bid and ask) prices of the agent. For a fixed claim g we write $p^g_b(k) = p_b(kg)$ and $p^g_a(k) = p_a(kg)$ in order to emphasise the dependence of the

[5] Later we will make assumptions on V such as $V < \infty$. These assumptions are in effect joint assumptions on the asset dynamics and the utility function.

[6] Here we are assuming the existence of a unique solution which will follow if $\mathcal{V}_g$ is finite and continuous and strictly increasing in the first argument. More generally we can define the bid price to be the largest amount the agent is prepared to pay now without leaving herself with a lower expected utility

$$p = \sup\{q : \mathcal{V}_g(x-q,s,y,t) \geq V(x,s,t)\}.$$

price upon the number of units of claim involved. This is the total price for k units of the claim and the unit price, which will typically depend on the transaction size k, will be $p^g_{\cdot}(kg)/k$. Then the marginal bid and ask prices are defined via $\lim_{k\downarrow 0+}(p^g_b(k)/k)$ and $\lim_{k\downarrow 0+}(p^g_a(k)/k)$ respectively, assuming these limits exist.[7]

2.4 The minimal martingale measure

In an incomplete market there are many equivalent martingale measures each of which leads to a linear pricing rule. One such measure is the minimal martingale measure. The minimal martingale measure is characterised by the fact that as well as making the discounted prices of traded assets into martingales, it leaves the dynamics of all martingales orthogonal to the traded assets unchanged. In particular, if we define the Sharpe-ratio

$$\lambda(S,t) = \frac{\mu(S,t) - r}{\eta(S,t)}$$

and the Girsanov change-of-measure $\tilde{\mathbb{P}}$ by $(d\tilde{\mathbb{P}}/d\mathbb{P})|_{\mathcal{F}_t} = Z_t$ for $0 \le t \le T$ where

$$Z_t = \exp\left(-\int_0^t \lambda_u dW_u - \frac{1}{2}\int_0^t \lambda_u^2 du\right) \tag{2}$$

then $\tilde{\mathbb{P}}$ is the minimal martingale measure.[8] Under $\tilde{\mathbb{P}}$ both $\tilde{W}$ and $\tilde{W}^\perp$ are Brownian motions with

$$d\tilde{W}_t = dW_t + \lambda_t dt \qquad\qquad d\tilde{W}_t^\perp = dW_t^\perp.$$

In our simplified setting it turns out that the minimal martingale measure is also the minimal relative entropy measure and the variance-optimal measure. See Pham et al [17] for both a definition of these terms and a discussion of the 'almost complete' models for which this result is true.

2.5 The results

Let $h = h(s,y,t)$ be given by

$$h(s,y,t) = e^{-r(T-t)}\tilde{\mathbb{E}}[g(S_T,Y_T)|S_t = s, Y_t = y]$$

[7] Under the assumptions of Section 4, these limits do indeed exist.

[8] We assume that $\tilde{\mathbb{P}}$ defined via (2) is a probability measure, or equivalently that Z_t is a true martingale and not just a local martingale. By the Novikov condition a sufficient condition for this is that $\mathbb{E}[\exp(\int_0^T \lambda(S_t,t)^2 dt/2)] < \infty$. This is trivial if S is exponential Brownian motion under $\mathbb{P}$ and λ is constant.

Under Assumptions 3.1, 3.2 and 3.3, if the utility indifference price is defined then $p(g) \leq h$. In particular

Theorem 2.1. *h is an upper bound on the utility indifference (bid) price for the claim g, and a lower bound on the ask price.*

Note that h is independent of both the wealth of the agent and her particular utility function. This result should be compared with the conclusions of Hubalek and Schachermayer [13]. They show that if we consider the range of expected payoffs for the claim as the expectation measure ranges over all martingale measures, then there are only trivial bounds on this range. (For example the range of possible prices of a call option on the non-traded asset Y is from 0 to infinity.)

Under Assumptions 3.1 and 4.2 which essentially assume that the agent can follow any investment strategy in the traded asset subject to the budget constraint, we have the stronger result

Theorem 2.2. *The marginal bid price for the claim g is given by h. If g is bounded then h is also the marginal ask price.*

Again the key result is that the marginal bid price is independent of the initial wealth of the agent, and of her utility function.

If the option payoff is contingent upon the traded asset alone then the bound on the bid price is $e^{-r(T-t)}\tilde{\mathbb{E}}_t(g(S_T))$ which is the Black-Scholes price. Further, under the assumptions of Section 4, we also find that the option can be replicated and both the marginal bid price and the marginal ask price are given by the Black-Scholes price. This is to be expected as in this case the market is complete. Hence utility indifference pricing is consistent with arbitrage-free pricing in a complete market.

In Section 3 we will consider the HJB equation for the problem. In that section we make explicit use of the Markov structure and the dynamics of the non-traded asset. In contrast in Section 4 we make no reference to the Markov structure or the form of the claim. Indeed the claim Y could be a random variable revealed at time T.

3 The Upper Bound

We first consider the problem with no random endowment or claim g. Recall that in this case

$$V(x,s,t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T^\pi)|X_t = x, S_t = s].$$

Assumption 3.1. (i) The utility function $U : \mathbb{R} \Rightarrow [-\infty, \infty)$ is increasing, and on the set $U(x) > -\infty$ it is strictly increasing and concave.
(ii) For each triple x, s, t we have $V(x,s,t) < U(\infty) \leq \infty$.

Assumption 3.2. The space $\mathcal{A}$ of admissible strategies is the set of strategies π such that $\tilde{\mathbb{E}} \int_0^T \pi_t^2 S_t^2 \eta(S_t,t)^2 < \infty$. In particular the discounted gains from trade are a martingale under the minimal martingale measure.

It follows from the first assumption above that V is strictly increasing and concave in its first argument.

Assumption 3.3. V has sufficient continuity properties that we can apply Itô's formula.

Using the fact that V is a martingale under the optimal strategy, and a supermartingale under any strategy we find from the dynamic programming equation that

$$\text{(3)} \quad \sup_{\pi \in \mathcal{A}} \left[\frac{1}{2}\pi^2 \eta^2 s^2 V_{xx} + \pi\{(\mu - r)sV_x + \eta^2 s^2 V_{xs}\} \right] + xrV_x + \mu s V_s + \frac{1}{2}\eta^2 s^2 V_{ss} + V_t = 0$$

Note that in general, and more particularly if 3.3 is not assumed then we can define the value function as the viscosity solution to (3), see for example Crandall et al [1], or for applications to finance, Duffie and Zariphopoulou [5].

Recall from above that $h(s,y,t) = e^{-r(T-t)}\tilde{\mathbb{E}}_t[g(S_T,Y_T)]$ and under $\tilde{\mathbb{P}}$ the price processes S and Y follow dynamics

$$\begin{aligned} dS_t &= S_t(\eta(S_t,t)d\tilde{W} + r\, dt) \\ dY_t &= a(S_t,Y_t,t)(\rho d\tilde{W} + \overline{\rho} d\tilde{W}^{\perp}) + (b(S_t,Y_t,t) - \rho\lambda(S_t,t)a(S_t,Y_t,t))dt. \end{aligned}$$

In particular, by the Feynman-Kac representation (Karatzas and Shreve [16, Section 5.7B], h solves

$$(b - \rho\lambda a)h_y + rsh_s + h_t + \frac{1}{2}\eta^2 s^2 h_{ss} + \rho\eta a s h_{sy} + \frac{1}{2}a^2 h_{yy} - rh = 0$$

subject to $h(s,y,T) = g(s,y)$. If we apply Itô's formula to h and use the fact that h satisfies the differential equation above then we find that

$$dh = h_s \eta S_t dW + h_y a(\rho dW + \overline{\rho} dW^{\perp}) + [(\mu - r)S_t h_s + \lambda a \rho h_y + rh]dt.$$

Now consider the problem facing the agent buying the claim. If she does not buy the claim then her value function is $V(x,s,t)$. If she buys the claim for a price p then her value function becomes $\mathcal{V}_g(x-p,s,y,t)$. If we can show that for each x

$$\mathcal{V}_g(x-h,s,y,t) \le V(x,s,t)$$

then it follows that $p(g) \le h$, and this is our main result. Equivalently, if we replace x with $x+h$ in the above it is sufficient to show that

$$\mathcal{V}_g(x,s,y,t) \le V(x+h,s,t) \tag{4}$$

Let π be any admissible strategy and consider $H_t^\pi = V(X_t^\pi + h(S_t, Y_t, t), S_t, t)$. Applying Itô's formula, and using the notation $\overset{m}{=}$ to denote equality up to a martingale term, we find

$$\begin{aligned} dH^\pi &= V_x(dX+dh) + \frac{1}{2}V_{xx}(dX+dh)^2 + V_s dS + \frac{1}{2}V_{ss}(dS)^2 + V_{xs}(dX+dh)dS + V_t dt \\ &\overset{m}{=} \Big[V_x\{\pi(\mu-r)S + rX + (\mu-r)Sh_s + \lambda a\rho h_y + rh\} \\ &\qquad + \frac{1}{2}V_{xx}\{(\pi\eta S + \eta S h_s + a\rho h_y)^2 + (a\overline{\rho}h_y)^2\} \\ &\qquad + V_s\mu S + \frac{1}{2}V_{ss}\eta^2 S^2 + V_{xs}\eta S(\pi\eta S + \eta S h_s + a\rho h_y) + V_t\Big]\, dt \\ &= \Big[V_x\{\psi(\mu-r)S + r(X+h)\} + \frac{1}{2}V_{xx}\{(\psi\eta S)^2 + (a\overline{\rho}h_y)^2\} \\ &\qquad + V_s\mu S + \frac{1}{2}V_{ss}\eta^2 S^2 + V_{xs}\eta^2 S^2\psi + V_t\Big]\, dt \end{aligned}$$

where we have used $\eta S\psi = \eta S\pi + S\eta h_s + a\rho h_y$. By Assumption 3.2, $\psi \in \mathcal{A}$ and using (3) we may conclude that up to a martingale term

$$dH^\pi \le\le \frac{1}{2}\{a\overline{\rho}h_y\}^2 V_{xx}(X_t + h, S_t, t)dt \le 0$$

where the last inequality comes from the fact that V is concave in its first argument.

Now $\mathcal{V}_g(x,s,y,t) = \sup_{\pi\in A} \mathbb{E}_t[U(X_T^\pi + g(S_T,Y_T))]$. Conversely $V(x+h,s,T) = U(x+g(s,y))$ and $V(X_t^\pi + h(S_t,Y_t,t),S_t,t)$ is a supermartingale for any π so

$$V(X_t^\pi + h(S_t,Y_t,t),S_t,t) \geq \mathbb{E}_t U(X_T^\pi + g(S_T,Y_T)).$$

In particular

$$V(x + h(s,y,t),s,t) \geq \sup_\pi \mathbb{E}_t U(X_T^\pi + g(S_T,Y_T)) = \mathcal{V}_g(x,s,y,t)$$

and (4) is proved.

The result on the ask price follows if we replace g with $-g$.

The results of this section depend crucially on the assumption that the dynamics of S as given in (1) are independent of Y. This is perfectly reasonable if we are modelling (S,Y) as a pair of correlated assets such that we are only allowed to trade in the former. However, this is not the case if we think of S as a traded asset and Y as its stochastic volatility. As a simple example suppose that (1) is replaced by

$$dS_t = S_t(\eta(Y_t)dW + \nu dt) \qquad\qquad S_0 = s_0.$$

For this model the agent's value function V is no longer independent of y. As a result the expression for the Itô derivative dH^π includes an extra term $a^2\overline{\rho}^2 h_y V_{xy}$. This term is of indeterminate sign and it is not possible to conclude that $dH^\pi \leq 0$. It seems unlikely therefore that there is a non-trivial, utility independent, bound on the bid price. However, in the case where attention is restricted to the family of exponential utilities (see Example 5.1) Sircar and Zariphopoulou [21] have some interesting results which give bounds on the option prices which are independent of the risk aversion parameter.

4 Marginal Pricing

The result of the previous section is that the price under the minimal martingale measure provides a bound on the utility indifference option price. The aim of this section is firstly to rederive that result using a dual approach, and secondly to deduce a stronger conclusion, namely that this bound is also the marginal price for the option. To this end we make stronger assumptions on the utility function, and use a slightly different definition of admissibility. One advantage of the approach in this section is that we do not need to make assumptions about the dynamics of the non-traded asset Y, nor to make a Markov assumption. Instead the claim may be any random variable Y which is measurable at time T. The only proviso is that the value of Y does not affect the dynamics of the traded asset.

We begin by deducing some general properties of the unit prices for contingent claims. Recall that we defined

$$\begin{aligned} V(x,s,t) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^\pi)|X_t=x,S_t=s], \\ \mathcal{V}_g(x,s,y,t) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^\pi+g(S_T,Y_T))|X_t=x,S_t=s,Y_t=y]. \end{aligned}$$

Throughout this section we will consider the case $t=0$, $S_0=s$ and $Y_0=y$ so that we will abbreviate these functions to $V(x)$ and $\mathcal{V}_g(x)$ respectively.

In this section we want to consider a fixed non-negative claim g, but to allow the agent to purchase or sell this claim in arbitrary amounts k. Hence we write

$$\mathcal{V}(x,k)=\mathcal{V}_{kg}(x)=\mathcal{V}_{kg}(x,s,y,0)=\sup_{\pi\in\mathcal{A}}\mathbb{E}[U(X_T^\pi+kg(S_T,Y_T))|X_0=x,S_0=s,Y_0=y],$$

where the convex space $\mathcal{A}$ of strategies is as defined below. It can be shown (see below) that provided $V(x)<U(\infty)$ and g has finite expectation under $\tilde{\mathbb{E}}$ then $\mathcal{V}(x,k)<\infty$ for each positive k and $\mathcal{V}$ is strictly increasing in its first component. Recall that $p_b^g(k)=p_b^g(x,s,y,0;kg)$, the bid price for k units of the claim g, is given as the solution[9] to

$$\mathcal{V}(x-p_b^g(k),k)\equiv\mathcal{V}_{kg}(x-p_b^g(k),s,y,0)=V(x,s,0),$$

and similarly the ask price for k units of the claim is defined via

$$\mathcal{V}(x+p_a^g(k),-k)\equiv\mathcal{V}_{-kg}(x+p_a^g(k),s,y,0)=V(x,s,0).$$

Lemma 4.1. *$\mathcal{V}$ is concave in the (x,k) plane. Further $p_b^g(k)\le p_a^g(k)$. For $k>0$, the unit price $p_b^g(k)/k$ is non-increasing in k, conversely if defined then $p_a^g(k)/k$ is non-decreasing in k.*

Proof

The concavity of $\mathcal{V}$ follows easily from the fact that U is concave and $\mathcal{A}$ is convex.

[9] Since $\mathcal{V}$ is strictly increasing this solution is unique, but existence will only follow if $\mathcal{V}$ is continuous in x. When this fails we can use the extended definition given in the footnote to Section 2.3. It can be shown that Lemma 4.1 remains true in this case.

Further, if $p_a^g(k) < \infty$, else there is nothing to prove,

$$\mathcal{V}(x,0) = \frac{1}{2}\mathcal{V}(x - p_b^g(k), k) + \frac{1}{2}\mathcal{V}(x + p_a^g(k), -k) \le \mathcal{V}(x + (p_a^g(k) - p_b^g(k))/2, 0).$$

Since by assumption $V(x) = \mathcal{V}(x,0)$ is strictly increasing we deduce that $p_a^g(k) \ge p_b^g(k)$. Similarly, for $0 < k < k'$,

$$\mathcal{V}(x - p_b^g(k), k) = \mathcal{V}(x,0) = \frac{k}{k'}\mathcal{V}(x - p_b^g(k'), k') + \frac{(k'-k)}{k'}\mathcal{V}(x,0) \le \mathcal{V}(x - kp_b^g(k')/k', k).$$

□

Note that it follows from the Lemma that the marginal bid and marginal ask prices both exist, and it follows from the previous section that the marginal bid price is bounded above by h, and that the marginal ask price is bounded below, again by h. To prove that these marginal prices are equal to h we need to make some additional assumptions.

We assume that the assumptions of Assumption 3.1 still hold. Let the inverse to the derivative of U be denoted by I. The assumptions on U ensure that I is a well-defined continuous, strictly decreasing function on $\mathbb{R}^+$.

Assumption 4.2. (i) The utility function $U : (-\infty, \infty) \to [-\infty, \infty)$ is increasing and has the property that on the set $U(x) > -\infty$ it is strictly increasing, strictly concave and has continuous first derivative which tends to zero as the wealth increases to infinity.
(ii) For each $w > 0$ we have $\mathbb{E}[|Z_T I(wZ_T)|] < \infty$.
(iii) The market consisting of the traded asset alone is complete.
(iv) Suppose that the claim g is bounded below with $g \ge ze^{rT}$. Then any strategy available to an agent with initial wealth $x + z$ is also available to an agent with initial wealth x who is entitled to the claim g at time T.

Since the market is complete we can identify the space of admissible portfolios $\mathcal{A}$ with the set of attainable terminal wealths $\mathcal{A}^*(x) = \{X_T : X_T \in m\mathcal{F}_T^W, e^{-rT}\mathbb{E}(Z_T X_T) \le x, \mathbb{E}(U(\xi)^-) < \infty\}$. Here $\mathcal{F}_t^W \equiv \mathcal{F}_t^S$ is the filtration generated by the Brownian motion which drives the price process of the traded asset. The condition $\mathbb{E}(Z_T X_T) \le x$ is a budget constraint, and the condition $\mathbb{E}(U(\xi)^-) < \infty$ helps rule out doubling strategies. Stated another way, in saying that the market is complete we mean that any candidate terminal wealth which is measurable with respect to the filtration generated by W alone can be represented as the sum of an initial wealth plus a gains from trade from an *admissible* strategy π.

The next result, adapted from Karatzas et al [15], describes the form of the optimal target wealth random variable $\hat{X}_T$.

Lemma 4.3. *Suppose $U(xe^{rT}-) > -\infty$. Then, for initial wealth x, the optimal target portfolio $\hat{X}_T \equiv \hat{X}_T^x$ is given by $\hat{X}_T^x \equiv I(\chi Z_T)$ where χ is a Lagrange multiplier chosen to satisfy $x \equiv e^{-rT}\mathbb{E}[Z_T I(\chi Z_T)]$.*

Proof
For $w > 0$ define

$$\gamma(w) = e^{-rT}\mathbb{E}[Z_T I(wZ_T)].$$

Then $\gamma(w)$ is a continuous strictly decreasing function and we can define an inverse Γ. It is simple to show that $\gamma(\infty) = e^{-rT}I(\infty)$ so that if $U(xe^{rT}-) > -\infty$ then $\Gamma(x) < \infty$.

$\hat{X}_T(x) \equiv I(\Gamma(x)Z_T)$ satisfies $e^{-rT}\tilde{\mathbb{E}}(\hat{X}_T) = \gamma(\Gamma(x)) = x$ and

$$U(\hat{X}_T) \geq U(xe^{rT}) + (\hat{X}_T - xe^{rT})U'(\hat{X}_T)$$

so that

$$U(\hat{X}_T)^- \leq U(xe^{rT})^- + x^+e^{rT}\Gamma(x)Z_T + \hat{X}_T^-\Gamma(x)Z_T$$

This last term is finite in expectation by assumption so we may conclude that $\hat{X}_T(x) \in \mathcal{A}^*(x)$. It remains to show that $\hat{X}_T$ is optimal: for any $X_T^\pi \in \mathcal{A}^*(x)$

$$\begin{aligned} U(\hat{X}_T) &\geq U(X_T^\pi) + (\hat{X}_T - X_T^\pi)U'(\hat{X}_T) \\ &= U(X_T^\pi) + (\hat{X}_T - X_T^\pi)\Gamma(x)Z_T \end{aligned} \tag{5}$$

and this last term has non-negative expectation under $\mathbb{P}$. Hence $\mathbb{E}[U(\hat{X})] \geq \mathbb{E}[U(X_T^\pi)]$. $\square$

Corollary 4.4. *For $z > 0$, $\hat{X}_T^{x+z} \geq \hat{X}_T^x$, almost surely.*

Remark 4.5. Assumption 4.2(ii) is used to show that the functions γ and Γ are well defined and hence that $\hat{X}_T$ is optimal. This assumption can be weakened, see for example Schachermayer [19], where the notion of 'reasonable asymptotic utility' is introduced and used to show that $\hat{X}_T = I(wZ_T)$ is still optimal.

If $U(xe^{rT}) = -\infty$ then for any strategy π, $\mathbb{P}(X_T^\pi \leq xe^{rT}) > 0$ so that $V(x) = -\infty$. In this case $p_b(g)$ is not defined.

Lemma 4.6. *Suppose $U(xe^{rT}) > -\infty$. For any non-negative claim $g = g(S_T, Y_T)$ the bid price $p_b(g)$ for a unit of the claim is bounded above by $h = e^{-rT}\tilde{\mathbb{E}}[g(S_T, Y_T)]$.*

Proof.
Consider an agent with initial fortune x such that $U(xe^{rT}) > -\infty$. In the absence of any claim, denote the optimal target wealth for the agent by $\hat{X}^x_T$.

Now suppose the agent has paid h to receive the claim g. Denote his terminal trading wealth generated by an admissible strategy π by $X_T^{\pi,x-h}$. The superscript $x-h$ represents the fact that her after she has paid h the budget constraint that her wealth must satisfy is $e^{-rT}\tilde{\mathbb{E}}[X_T] \leq x-h$. Then, using $U(a+b) \leq U(a) + bU'(a)$,

$$\begin{aligned}
\mathbb{E}[U(X_T^{\pi,x-h} + g)] - \mathbb{E}[U(\hat{X}^x_T)] &\leq \mathbb{E}[(X_T^{\pi,x-h} + g - \hat{X}^x_T)U'(\hat{X}^x_T)] \\
&= \mathbb{E}[(X_T^{\pi,x-h} + g - \hat{X}^x_T)\Gamma(x)Z_T] \\
&\leq \Gamma(x)(x - h + \mathbb{E}[Z_T g] - x) \;\; = \;\; 0
\end{aligned}$$

Optimising over π we find

$$\mathcal{V}_g(x-h) \leq V(x,0) = \mathcal{V}_g(x - p_b(g))$$

and the result follows.

Theorem 4.7. *Suppose $U(xe^{rT}-) > -\infty$. For any non-negative claim $g = g(S_T, Y_T)$, if we define the bid price for k units $p^g_b(k) = p_b(x,s,y,t;kg)$ then*

$$\lim_{k\downarrow 0+} \frac{p^g_b(k)}{k} = \left.\frac{\partial p^g_b(k)}{\partial k}\right|_{k=0+} = e^{-rT}\tilde{\mathbb{E}}(g(S_T,Y_T)).$$

Conversely, if g is bounded, then the ask price satisfies

$$\lim_{k\downarrow 0+} \frac{p^g_a(k)}{k} = \left.\frac{\partial p^g_a(k)}{\partial k}\right|_{k=0+} = e^{-rT}\tilde{\mathbb{E}}(g(S_T,Y_T)).$$

Proof.
By assumption U' is continuous, which implies in turn that both I and γ are strictly decreasing, and hence Γ is continuous.

Consider an agent with initial fortune x, such that $U(xe^{rT}) > -\infty$, and no units of the claim. Let the optimal portfolio and the optimal terminal wealth for this agent be given by $\hat{\pi}^x_t$ and $\hat{X}^x_T$ respectively. It follows that we have $\hat{X}^x_T = xe^{rT} + \int_0^T \hat{\pi}^x_t d(S_t e^{r(T-t)})$ and by the optimality of the terminal wealth $\hat{X}^x_T = I(\Gamma(x)Z_T)$ so that $U'(\hat{X}^x_T) = \Gamma(x)Z_T$. Now

$$\mathcal{V}(x+z,0) - \mathcal{V}(x,0) = \mathbb{E}[U(\hat{X}^{x+z}_T)] - \mathbb{E}[U(\hat{X}^x_T)].$$

By concavity of U, $U(a+b) \le U(a) + bU'(a)$ and

$$\begin{aligned} \mathcal{V}(x+z,0) - \mathcal{V}(x,0) &\le \mathbb{E}\left[\left(ze^{rT} + \int_0^T \{\hat{\pi}_t^{x+z} - \hat{\pi}_t^x\} d(S_t e^{r(T-t)})\right) U'(\hat{X}_T^x)\right] \\ &= \Gamma(x)\mathbb{E}[ze^{rT} Z_T] + \Gamma(x)\mathbb{E}\left[\int_0^T \{\hat{\pi}_t^{x+z} - \hat{\pi}_t^x\} d(S_t e^{r(T-t)}) Z_T\right] \\ &= \Gamma(x) z e^{rT} \end{aligned}$$

since each of $\int_0^T \hat{\pi}_t^{x+z} d(S_t e^{r(T-t)})$ and $\int_0^T \hat{\pi}_t^x d(S_t e^{r(T-t)})$ is a martingale under $\tilde{\mathbb{P}}$.

Conversely, $U(a+b) \ge U(a) + bU'(a+b)$ and a similar argument yields

$$\mathcal{V}(x+z,0) - \mathcal{V}(x,0) \ge \Gamma(x+z) z e^{rT}.$$

By the continuity of Γ

$$\left.\frac{\partial \mathcal{V}}{\partial x}\right|_{x,0} = \Gamma(x) e^{rT}.$$

Now consider $\partial\mathcal{V}/\partial k$. On the one hand, for any admissible target wealth X_T^π which can be financed from initial fortune x,

$$\begin{aligned} \mathbb{E}[U(X_T^\pi + kg)] - \mathbb{E}[U(\hat{X}_T^x)] &\le \mathbb{E}[\{(X_T^\pi - \hat{X}_T^x) + kg\} U'(\hat{X}_T^x)] \\ &\le \mathbb{E}[kg\Gamma(x) Z_T] \\ &= k\Gamma(x)\tilde{\mathbb{E}}[g(S_T, Y_T)], \end{aligned}$$

so that

$$\mathcal{V}(x,k) - \mathcal{V}(x,0) = \sup_{\pi, X_0^\pi = x} \mathbb{E}[U(X_T^\pi + kg)] - \mathbb{E}[U(\hat{X}_T^x)] \le k\Gamma(x)\tilde{\mathbb{E}}[g(S_T, Y_T)]. \tag{6}$$

In particular if $\tilde{\mathbb{E}}(g) < \infty$ then $\mathcal{V}(x,k) < \infty$ for all k.

For $k > 0$ we derive a reverse inequality. Suppose that $\tilde{\mathbb{E}}(g) < \infty$, then $\mathcal{V}(x,k)$ is well defined and

$$\begin{aligned} \mathcal{V}(x,k) - \mathcal{V}(x,0) &\ge \mathbb{E}[U(\hat{X}_T^x + kg)] - \mathbb{E}[U(\hat{X}_T^x)] \\ &\ge k\mathbb{E}[gU'(\hat{X}_T^x + kg)]. \end{aligned}$$

By the positivity of g we have that $U'(\hat{X}_T^x + kg)$ is non-decreasing as $k \downarrow 0$ and converges almost surely to $U'(\hat{X}_T^x) = \Gamma(x) Z_T$. Hence, by the Monotone Convergence Theorem,

$$\mathbb{E}[gU'(\hat{X}_T^x + kg)] \uparrow \mathbb{E}[gU'(\hat{X}_T^x)] = \Gamma(x)\tilde{\mathbb{E}}(g).$$

We have that

$$\lim_{k\downarrow 0+}\frac{\mathcal{V}(x,k)-\mathcal{V}(x,0)}{k}\geq \Gamma(x)\tilde{\mathbb{E}}[g(S_T,Y_T)]. \tag{7}$$

Similarly, if $\tilde{\mathbb{E}}(g)=\infty$ then by considering $g_n=g\wedge n$ and letting $\mathcal{V}_n$ denote the solution to the maximisation problem involving the claim g_n, we get that

$$\mathcal{V}(x,k)-\mathcal{V}(x,0)\geq \mathcal{V}_n(x,k)-\mathcal{V}(x,0)\geq k\Gamma(x)\tilde{\mathbb{E}}(g_n)\uparrow\infty.$$

In either case we can conclude that $\partial\mathcal{V}/\partial k|_{(x,0+)}=\Gamma(x)\tilde{\mathbb{E}}[g]$ and hence that

$$\left.\frac{\partial\mathcal{V}/\partial k}{\partial\mathcal{V}/\partial x}\right|_{k=0+}=e^{-rT}\tilde{\mathbb{E}}[g(S_T,Y_T)]. \tag{8}$$

Now, by definition, $V(x)=\mathcal{V}(x-p_b^g(k),k)$ so that

$$-\frac{\partial p_b^g(k)}{\partial k}\mathcal{V}_x+\mathcal{V}_k=0$$

which translates to the first result in Theorem 4.7.

If we try to take limits for $k\uparrow 0$ then extra conditions are needed on the claim g in order to derive the analogue of (7). A sufficient condition is that g is bounded by K say and that for some ϵ, $V(x-\epsilon)>-\infty$. In that case, for small enough k we have $\Gamma(x+kKe^{-rT})<\infty$, $\hat{X}_T^{x+kKe^{-rT}}$ is well defined and $\hat{X}_T^{x+kKe^{-rT}}-kK$ is an attainable wealth from initial fortune x. Then

$$\begin{aligned}\mathcal{V}(x,k)-\mathcal{V}(x,0) &\geq \mathbb{E}[U(\hat{X}_T^{x+kKe^{-rT}}+k(g-K))]-\mathbb{E}[U(\hat{X}_T^{x})]\\ &\geq \mathbb{E}[\{k(g-K)+\hat{X}_T^{x+kKe^{-rT}}-\hat{X}_T^{x}\}U'(\hat{X}_T^{x+kKe^{-rT}}+k(g-K))]\\ &\geq k\mathbb{E}[(g-K)U'(\hat{X}_T^{x+kKe^{-rT}}+k(g-K))]\\ &\qquad +\mathbb{E}[\{\hat{X}_T^{x+kKe^{-rT}}-\hat{X}_T^{x}\}U'(\hat{X}^{x+kKe^{-rT}})]\end{aligned}$$

where we have used in the last term the fact that $\hat{X}_T^{x+kKe^{-rT}}\geq\hat{X}_T^{x}$, and U' is decreasing. The first term converges by the dominated convergence theorem to

$$k\mathbb{E}[(g-K)U'(\hat{X}_T^{x+kKe^{-rT}})]=k\Gamma(x+kKe^{-rT})\tilde{\mathbb{E}}[g-K],$$

whereas the final term is equal to $\Gamma(x+kKe^{-rT})kK$.

Thus

$$\begin{aligned}\mathcal{V}(x,k)-\mathcal{V}(x,0) &\geq \Gamma(x+kKe^{-rT})\{\tilde{\mathbb{E}}[k(g-K)]+kK\}\\ &= \Gamma(x+kKe^{-rT})k\tilde{\mathbb{E}}[g]\end{aligned}$$

Combining this result with (6) it follows by the continuity of Γ that

$$\lim_{k\uparrow 0} \frac{\mathcal{V}(x,k)-\mathcal{V}(x,0)}{k} = \Gamma(x)\tilde{\mathbb{E}}[g(S_T, Y_T)]. \tag{9}$$

□

Remark 4.8. In this section we can replace the discount factor e^{-rT} and the minimal martingale change of measure Z_T by a state-price density ξ_T. Provided ξ_T does not depend on the non-traded asset Y the above proofs still hold. Hence the results remain valid in models with non-deterministic interest rates provided the dynamics of the interest rate process are driven by the same Brownian motion as the traded price process.

Remark 4.9. The key result that makes the theorem work is the fact that Γ is a continuous function. In the proof this was achieved by assuming that U had a continuous first derivative. However this is not a necessary assumption and if the change of measure random variable Z is a continuous random variable then Γ may still be continuous even if the derivative of U has jumps.

Remark 4.10. In Davis [2], the author gives an expression for what he calls the 'fair price' of a claim g which corresponds to our utility indifference marginal price (when both bid and ask marginal prices are defined). However, he gives the price as

$$\hat{p} = \frac{\mathbb{E}[U'(\hat{X}_T^x)g]}{V_x(x)}.$$

It is not at all apparent from this formulation that this price is independent of the choice of utility function, which is the key result of Theorem 4.7.

5 Examples

For this section we assume that the traded asset follows exponential Brownian motion with constant parameters so that

$$dS_t = S_t(\sigma dW + \nu dt) \qquad\qquad S_0 = s.$$

Set $\lambda = (\nu - r)/\sigma$. We will also assume that the claim is contingent upon a second traded asset with price process Y given by

$$dY_t = Y_t(adB + bdt) \qquad\qquad Y_0 = y.$$

Here the Brownian motions W and B are assumed to have constant correlation coefficient ρ. The minimal martingale measure has density of the form

$$Z_T = \exp\left(-\lambda W_T - \frac{1}{2}\lambda^2 T\right). \tag{10}$$

In this section we will only price claims at time $t = 0$.

5.1 Exponential Utility

Consider the family of utility functions with constant absolute risk aversion. These utilities are defined for negative wealth and take the form

$$U(x) = -\frac{1}{\theta}e^{-\theta x} \qquad\qquad x \in \mathbb{R},$$

with θ a positive parameter. For the dynamics given above we find that the value function of the agent is given by

$$V(x, s, 0) = -\frac{1}{\theta}\exp\left(-\theta x e^{rT} - \frac{\lambda^2 T}{2}\right) \qquad\qquad x \in \mathbb{R}.$$

Observe that this function is independent of S and twice differentiable in x so that Itô's formula can be applied to $V(X_t, S_t, t)$ and Assumptions 3.1 and 3.3 are both satisfied.

Furthermore $U'(x) = e^{-\theta x}$ is continuous and tends to zero for large x. The inverse function I is given by $I(y) = -(\ln y)/\theta$. It follows that Assumptions 4.2(i) and (ii) are satisfied and $\gamma(w) = (-1/\theta)\mathbb{E}[Z_T(\ln w + \ln Z_T)] = -(\ln w + \lambda^2 T/2)/\theta$. We find $\Gamma(x) = \exp(-\theta x - \lambda^2 T/2)$.

Suppose the claim takes the form $g = g(Y_T)$. Then the utility indifference bid price for k units of the claim is given in Henderson and Hobson [10, 11] as

$$p_b^g(k) = -\frac{e^{-rT}}{\theta(1-\rho^2)}\ln\tilde{\mathbb{E}}\left[\exp\left(-k\theta(1-\rho^2)g(Y_T)\right)\right]. \tag{11}$$

For g a non-negative claim, and $k > 0$, it follows form Jensen's inequality that $p_b^g(k) \geq e^{-rT}k\tilde{\mathbb{E}}[g(Y_T)]$. Further, on differentiation we find

$$\lim_{k\downarrow 0}\frac{p_b^g(k)}{k} = e^{-rT}\tilde{\mathbb{E}}[g(Y_T)]$$

Hence $e^{-rT}\tilde{\mathbb{E}}[g(Y_T)]$, which is independent of the risk aversion parameter θ is both an upper bound on the bid price for the claim and the marginal bid price.

Now suppose that we are interested in the ask price. A formula for the ask price is obtained by replacing k with $-k$ in (11) and multiplying by -1. However if $g(Y_T) = Y_T$ then the exponential moment is infinite, and the utility indifference ask price is also infinite. Hence, trivially, $e^{-rT}\tilde{\mathbb{E}}[g(Y_T)]$ is a lower bound on the ask price for the claim, but it does not represent the marginal ask price.

5.2 Power Law Utility

For a positive risk aversion parameter R consider the utility function

$$U(x) = \frac{x^{1-R}}{1-R} \qquad x \geq 0 \tag{12}$$

with $U(x) = -\infty$ for negative x. The value function for the agent with no endowment is given by

$$V(x,s,t) = \frac{x^{1-R}e^{(1-R)r(T-t)}}{1-R} \exp\left\{ \frac{(1-R)\lambda^2}{2R}(T-t) \right\},$$

with $V(x) = -\infty$ for $x < 0$. For positive x this function is twice differentiable, and Assumption 3.3 is satisfied.

The inverse to U' is given by $I(y) = y^{-1/R}$ and we have

$$\gamma(w) = w^{-1/R}\mathbb{E}[Z_T^{1-(1/R)}] = w^{-1/R}\exp\{(1-R)\lambda^2/(2R^2)\}.$$

For each R, Assumption 3.3(i) and (ii) hold.

In general there are no closed form expressions for options prices. However Henderson [9, Theorem 4.2] (see also Henderson and Hobson [11, Theorem 5.4]) gives an expansion for the price in powers of the number of options bought. Suppose the wealth x is positive. If the claim is units of the non-traded asset then for $k \geq 0$ the price $p_b^Y(k) = p_b(x,s,y,0;kY)$ is given as

$$p_b^Y(k) = ke^{-rT}\tilde{\mathbb{E}}[Y_T] - k^2 e^{-rT}\frac{y^2}{x}\frac{a^2R(1-\rho^2)}{2\Lambda}e^{2(b-(\nu\rho b)/\sigma)T}[e^{\Lambda T}-1] + O(k^3)$$

where $\Lambda = a^2 - \frac{2\nu\rho a}{\sigma R} + \frac{\nu^2}{\sigma^2 R^2}$. For a more general claim $g = g(Y_T)$ the price to leading order is

$$p_b^g(k) = p_b(x,s,y,0;kg) = ke^{-rT}\tilde{\mathbb{E}}[g(Y_T)].$$

In either case the marginal bid price is $e^{-rT}\tilde{\mathbb{E}}[g(Y_T)]$ which is independent of the risk aversion parameter R. As for exponential utility the marginal ask price can be infinite for unbounded claims.

Note that if we take $R = 1$ then we recover the formulae for logarithmic utility.

5.3 A counterexample for which the bounds are not attained.

Consider an agent with power-law utility function given by (12) with $R < 1$. Suppose that this agent has zero initial wealth and $r = 0$. What is the bid price of this agent for k units of the claim Y_T where $Y_T = \exp(aB_T + (b - a^2/2)T)$, where for simplicity B is independent of the Brownian motion driving the traded asset?

In this case $V(0, s, 0) = 0$. If the agent bids any positive amount p for the claim then there is a positive probability that the agent has negative final wealth, and therefore her utility is $-\infty$. However, even if the agent follows the strategy of investing zero in the traded asset her value function is

$$\mathbb{E}[U(k \exp(aB_T + (b - a^2/2)T))] = \frac{k^{1-R}}{1-R} \exp\left(b(1-R)T - a^2 \frac{R(1-R)}{2} T\right) > 0.$$

Hence there is no solution to the equation

$$\mathcal{V}(-p, k) = 0.$$

If the agent pays any positive amount for the claim then her expected utility is reduced.

6 Concluding Remarks

In an incomplete market there are no unique, preference independent option prices. Instead it is necessary either to choose a pricing measure from the family of equivalent martingale measures, or to model the the preferences directly. Utility indifference option pricing is a consistent pricing scheme which reduces to Black-Scholes option pricing in a complete market. Prices are non-linear, so that in general agents will pay a lower unit price for larger quantities of a financial asset which are associated with higher risk.

The perceived disadvantage of the utility based option pricing approach is that since the price depends on the choice of utility function, it seems unlikely that there are any general pricing principles which hold uniformly across all utility functions, and for all initial wealths. In our non-traded asset model we have shown that this is not the case and that there is a bound for the utility indifference bid and ask price for the option which is valid for all sufficiently regular utility choices. Furthermore, if the set of trading opportunities is sufficiently broad then this bound also acts as the marginal price.

If the bid price for every agent is below the ask price for every agent then under no circumstances would any trading occur. Hence if all agents have zero initial endowment of

Y, and all agents have the same physical measure $\mathbb{P}$, then no trading in a claim with payoff $g(Y)$ would occur, even if the agents have different initial wealths or utility functions. Trading only occurs if the agents have initial endowments of non-traded asset, or if they have different expectations of Y.

The results of this paper can clearly be extended to a multidimensional model where there is more than one traded asset or more than one untraded asset, all driven by correlated Brownian motions. However the assumption in Section 2.1 that the dynamics of the traded asset are not affected by the non-traded asset is crucial for the results, so that although the results hold for models for the pricing of non-traded assets they cannot be extended directly to stochastic volatility models.

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