Asymptotic Behaviour of Solutions of Ordinary Differential Equations and Inclusions: Generalizations of Barbălat's Lemma and LaSalle's Invariance Principle

Hartmut Logemann[*]
Department of Mathematical Sciences,
University of Bath,
Bath BA2 7AY, United Kingdom
Email: `hl@maths.bath.ac.uk`

Eugene P Ryan
Department of Mathematical Sciences,
University of Bath,
Bath BA2 7AY, United Kingdom
Email: `epr@maths.bath.ac.uk`

Abstract: This paper, which has a tutorial flavour, develops a self-contained, elementary and unified approach to a variety of results (including LaSalle's Invariance Principle and generalizations thereof) pertaining to asymptotic behaviour of solutions of (non-autonomous) ordinary differential equations and (autonomous) differential inclusions. Concepts of meagreness and weak meagreness of functions form the basis of the approach. These concepts, in conjunction with hypotheses of uniform continuity on particular subsets of $[0,\infty)$, capture certain asymptotic properties of functions defined on $[0,\infty)$ and provide generalizations of Barbălat's Lemma. Two illustrative examples are detailed.
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1 Introduction

This paper attempts to provide a self-contained, elementary and unified approach to the analysis of certain aspects of the asymptotic behaviour of solutions of ordinary differential equations and differential inclusions; some ramifications for semiflows on locally compact metric spaces are also highlighted. As a starting point, we make the simple observation – usually referred to as Barbălat's Lemma – that if a function $y : [0,\infty) \to \mathbb{R}$ is uniformly continuous and integrable, then $y(t)$ necessarily approaches 0 as $t \to \infty$. In the context of an autonomous ordinary differential equation $\dot{x} = f(x)$ with locally Lipschitz $f : \mathbb{R}^N \to \mathbb{R}^N$, this observation leads - in an entirely elementary manner - to the invariance principle of LaSalle, as the following sketch proof shows. Let $V : \mathbb{R}^N \to \mathbb{R}$ be continuously differentiable and assume that $V_f(\xi) := \langle \nabla V(\xi), f(\xi)\rangle \le 0$ for all $\xi \in \mathbb{R}^N$. Moreover, assume that x is a bounded solution of the differential equation. Then $V \circ x$ is bounded

[*] Corresponding author.

with derivative $(\mathrm{d}/\mathrm{d}t)V(x(t)) = (V_f \circ x)(t) \le 0$ for all t. Therefore, $(V \circ x)(t)$ converges to a finite limit as $t \to \infty$, whence integrability of $V_f \circ x$. Since x is bounded, the derivative $\dot{x} = f(x)$ is also bounded, and thus x is uniformly continuous. Therefore, we may conclude that the integrable function $V_f \circ x$ is uniformly continuous. By Barbălat's Lemma, $V_f(x(t)) \to 0$ as $t \to \infty$ implying that $x(t)$ approaches the zero-level set $V_f^{-1}(0)$ of V_f as $t \to \infty$. Consequently, the ω-limit set $\Omega(x)$ must be contained in $V_f^{-1}(0)$. Combining this with the fact that $\Omega(x)$ is invariant with respect to the flow generated by the differential equation, we see that $x(t)$ approaches the largest invariant subset contained in $V_f^{-1}(0)$: this is LaSalle's Invariance Principle. We mention that the "standard" proof of the invariance principle, ubiquitous in the literature (see, for example, [1, 10, 13, 14, 15, 20]), is somewhat different insofar as Barbălat's Lemma is not usually invoked. Instead, the convergence of $V(x(t))$ as $t \to \infty$ is used to conclude that V is constant on $\Omega(x)$, which in turn implies (via a straightforward argument based on the invariance of $\Omega(x)$) that $V_f(\xi) = 0$ for all $\xi \in \Omega(x)$. In this paper, we show that suitable generalizations of the first argument (involving Barbălat's Lemma) lead to diverse results on asymptotic dynamic behaviour in the more general setting of non-autonomous ordinary differential equations and (autonomous) differential inclusions. Our goal is first to develop a compendium of results pertaining to asymptotic behaviour of functions and constituting generalizations of Barbălat's Lemma. We achieve this by elementary arguments based on concepts of meagreness and weak meagreness of functions, which, in conjunction with uniform continuity on particular subsets of $[0,\infty)$, capture certain asymptotic properties of functions $t \mapsto y(t)$ as $t \to \infty$. This compendium then forms the basis for a unified approach to various results (including generalizations of LaSalle's Invariance Principle) on asymptotic behaviour of solutions of (non-autonomous) ordinary differential equations and (autonomous) differential inclusions. The paper has a tutorial flavour and, for purposes of illustration, we have included detailed descriptions of two examples.

2 Preliminaries

In order to render the paper essentially self-contained, we first assemble some familiar facts, notation and terminology. Throughout, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}_+ := [0,\infty)$, and μ denotes Lebesgue measure on $\mathbb{R}_+$. The Euclidean inner product and induced norm on $\mathbb{R}^N$ are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively. Let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be a Lebesgue measurable function: if $0 < p < \infty$ and the function $t \mapsto \|x(t)\|^p$ is Lebesgue integrable (respectively, locally Lebesgue integrable), then we write $x \in L^p$ (respectively, $x \in L^p_{\mathrm{loc}}$); if the function $t \mapsto \|x(t)\|$ is essentially bounded (respectively, locally essentially bounded), then we write $x \in L^\infty$ (respectively, $x \in L^\infty_{\mathrm{loc}}$).

Let $A \subset \mathbb{R}^N$ be non-empty and $h : A \to \mathbb{R}^P$. For $U \subset \mathbb{R}^P$, $h^{-1}(U)$ denotes the preimage of U under h, that is, $h^{-1}(U) := \{\xi \in A : h(\xi) \in U\}$; for notational simplicity, if $u \in \mathbb{R}^P$, then we write $h^{-1}(u)$ in place of the more cumbersome $h^{-1}(\{u\})$. We recall that h is continuous at $\xi_0 \in A$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|h(\xi_0) - h(\xi)\| \le \varepsilon$ for all $\xi \in A$ with $\|\xi_0 - \xi\| \le \delta$. If h is continuous at ξ for all $\xi \in B \subset A$, then h is said to be continuous on B; if $B = A$, then we simply say that h is continuous. The function h is uniformly continuous on $B \subset A$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|h(\xi_1) - h(\xi_2)\| \le \varepsilon$ for all $\xi_1, \xi_2 \in B$ with $\|\xi_1 - \xi_2\| \le \delta$; if $B = A$, then we simply say that h is uniformly continuous. It is convenient to adopt the convention that h is deemed to be uniformly continuous on the empty set $\emptyset$. If h is scalar-valued (that is, $P = 1$), then h is said to be lower semicontinuous if $\liminf_{\xi' \to \xi} h(\xi') \ge h(\xi)$ for all $\xi \in A$. We say that

h is upper semicontinuous if $-h$ is lower semicontinuous.

The Euclidean distance function for a non-empty set $A \subset \mathbb{R}^N$ is the function $d_A : \mathbb{R}^N \to \mathbb{R}_+$ given by $d_A(v) := \inf\{\|v - a\| : a \in A\}$. The function d_A is globally Lipschitz with Lipschitz constant 1, that is, $\|d_A(v) - d_A(w)\| \leq \|v - w\|$ for all $v, w \in \mathbb{R}^N$. A function $x : \mathbb{R}_+ \to \mathbb{R}^N$ is said to approach A if $d_A(x(t)) \to 0$ as $t \to \infty$. For $\varepsilon > 0$, $\mathbb{B}_\varepsilon(A) := \{\xi \in \mathbb{R}^N : d_A(\xi) < \varepsilon\}$ (the ε-neighbourhood of A); for $a \in \mathbb{R}^N$, we write $\mathbb{B}_\varepsilon(a)$ in place of the more cumbersome $\mathbb{B}_\varepsilon(\{a\})$. It is convenient to set $\mathbb{B}_\varepsilon(\emptyset) := \emptyset$. The closure of A is denoted by $\mathrm{cl}(A)$.

The notion of an ω-limit point is usually introduced in the context of trajectories of flows or semiflows. However, for the purposes of this paper, it is useful to define the concept of an ω-limit point for arbitrary $\mathbb{R}^N$-valued functions defined on $\mathbb{R}_+$ (see also [11], pp. 112). Let $x : \mathbb{R}_+ \to \mathbb{R}^N$. A point $\xi \in \mathbb{R}^N$ is said to be an ω-limit point of x if there exists an unbounded sequence (t_n) in $\mathbb{R}_+$ such that $x(t_n) \to \xi$ as $n \to \infty$; the (possibly empty) ω-limit set of x, denoted by $\Omega(x)$, is the set of all ω-limit points of x. The following well-known properties of ω-limit sets (see, for example, [1] and [11]) are summarized here for later reference.

Lemma 2.1. *The following hold for any function $x : \mathbb{R}_+ \to \mathbb{R}^N$.*

(a) *$\Omega(x)$ is closed.*

(b) *$\Omega(x) = \emptyset$ if, and only if, $\|x(t)\| \to \infty$ as $t \to \infty$.*

(c) *If x is continuous and bounded, then $\Omega(x)$ is non-empty, compact, connected, is approached by x and is the smallest closed set approached by x.*

(d) *If x is continuous and $\Omega(x)$ is non-empty and bounded, then x is bounded and x approaches $\Omega(x)$.*

3 Motivation: Barbălat's Lemma and LaSalle's Invariance Principle

A function $y : \mathbb{R}_+ \to \mathbb{R}$ is said to be Riemann integrable (on $\mathbb{R}_+$) if the improper Riemann integral $\int_0^\infty y(s)\mathrm{d}s$ exists, that is, for all $t \geq 0$, y is Riemann integrable on $[0, t]$ and the limit $\lim_{t\to\infty} \int_0^t y(s)\mathrm{d}s$ exists and is finite. If $y \in L^1$ and, for all $t \geq 0$, y is Riemann integrable on $[0, t]$, then y is Riemann integrable on $\mathbb{R}_+$. Furthermore, if y is ultimately non-negative (respectively, non-positive) in the sense that, for some $\tau \in \mathbb{R}_+$, $y(t) \geq 0$ (respectively, $y(t) \leq 0$) for all $t \geq \tau$, then Riemann integrability of y implies that $y \in L^1$. However, if y is neither ultimately non-negative nor ultimately non-positive, the improper Riemann integral y may exist, but y may fail to be Lebesgue integrable.

As our starting point, we highlight the following simple observation, due to Barbălat [3].

Lemma 3.1. (Barbălat's Lemma) *If $y : \mathbb{R}_+ \to \mathbb{R}$ is uniformly continuous and Riemann integrable, then $y(t) \to 0$ as $t \to \infty$.*

Proof. For contradiction, suppose $y(t) \not\to 0$ as $t \to \infty$. Then there exist $\varepsilon > 0$ and a sequence (t_n) in $\mathbb{R}_+$ such that $t_{n+1} - t_n > 1$ and $|y(t_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. By uniform continuity of y, there exists $\delta \in (0, 1)$ such that, for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}_+$,

$$|t_n - t| \leq \delta \quad \Longrightarrow \quad |y(t_n) - y(t)| \leq \varepsilon/2\,.$$

Therefore,

$$|y(t)| \geq |y(t_n)| - |y(t_n) - y(t)| \geq \varepsilon/2 \quad \forall\, t \in [t_n, t_n + \delta], \quad \forall\, n \in \mathbb{N},$$

whence

$$\left| \int_{t_n}^{t_n+\delta} y(t)\mathrm{d}t \right| = \int_{t_n}^{t_n+\delta} |y(t)|\mathrm{d}t \geq \frac{\varepsilon\delta}{2} \quad \forall\, n \in \mathbb{N},$$

which contradicts the existence of the improper Riemann integral $\int_0^\infty y(t)\mathrm{d}t$. ■

The following corollary is an immediate consequence of statement (c) of Lemma 2.1 and Lemma 3.1.

Corollary 3.2. *Let $G \subset \mathbb{R}^N$ be non-empty and closed and let $g : G \to \mathbb{R}$ be continuous. Assume that $x : \mathbb{R}_+ \to \mathbb{R}^N$ is bounded and uniformly continuous with $x(\mathbb{R}_+) \subset G$. If $g \circ x$ is Riemann integrable, then $\Omega(x) \subset g^{-1}(0)$ and x approaches $g^{-1}(0)$.*

Elaborating on the arguments sketched in Section 1, we will use Corollary 3.2 to derive LaSalle's Invariance Principle. Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz and consider the initial-value problem

$$\dot{x} = f(x), \quad x(0) = x^0 \in \mathbb{R}^N. \tag{1}$$

Let φ denote the corresponding local flow, that is $t \mapsto \varphi(t, x^0)$ is the unique solution of (1) defined on $I(x^0)$, its maximal interval of existence. It is well known that, if for given $x^0 \in \mathbb{R}^N$, $\mathbb{R}_+ \subset I(x^0)$ and $\varphi(\cdot\,, x^0)$ is bounded on $\mathbb{R}_+$, then $\Omega(\varphi(\cdot\,, x^0))$ is invariant with respect to the local flow φ (see, for example, [1]).

The following "integral-invariance principle" is an easy consequence of Corollary 3.2.

Proposition 3.3. *Let $G \subset \mathbb{R}^N$ be non-empty and closed and let $g : G \to \mathbb{R}$ be continuous. Let $x^0 \in G$. Assume that $\mathbb{R}_+ \subset I(x^0)$, $\varphi(\cdot\,, x^0)$ is bounded on $\mathbb{R}_+$ and $\varphi(\mathbb{R}_+, x^0) \subset G$. If the function $t \mapsto g(\varphi(t, x^0))$ is Riemann integrable on $\mathbb{R}_+$, then $\varphi(\cdot\,, x^0)$ approaches the largest invariant subset contained in $g^{-1}(0)$.*

Proof. Since $\varphi(\cdot\,, x^0)$ is bounded on $\mathbb{R}_+$ and satisfies the differential equation, it follows that the derivative of $\varphi(\cdot\,, x^0)$ is bounded on $\mathbb{R}_+$ and consequently, $\varphi(\cdot\,, x^0)$ is uniformly continuous on $\mathbb{R}_+$. An application of Corollary 3.2 together with the invariance property of the ω-limit set of $\varphi(\cdot\,, x^0)$ yields the claim. ■

Proposition 3.3 is essentially contained in [4, Theorem 1.2]: the proof given therein is not based on Barbălat's Lemma. The above proof of Proposition 3.3 is from [6].

LaSalle's Invariance Principle (announced in [12], with proof in [13]) is a straightforward consequence of Proposition 3.3.

Corollary 3.4. (LaSalle's Invariance Principle) *Let $D \subset \mathbb{R}^N$ be non-empty and open, let $V : D \to \mathbb{R}$ be continuously differentiable and set $V_f(\xi) := \langle \nabla V(\xi), f(\xi) \rangle$ for all $\xi \in D$. Let $x^0 \in D$. Assume that $\mathbb{R}_+ \subset I(x^0)$ and there exists a compact set $G \subset \mathbb{R}^N$ such that $\varphi(\mathbb{R}_+, x^0) \subset G \subset D$. If $V_f(\xi) \leq 0$ for all $\xi \in G$, then $\varphi(\cdot\,, x^0)$ approaches the largest invariant subset contained in $V_f^{-1}(0) \cap G$.*

Proof. By compactness of G and continuity of V on G, the function V is bounded on G. Combining this with

$$\int_0^t V_f(\varphi(s,x^0))\mathrm{d}s = \int_0^t (\mathrm{d}/\mathrm{d}s)V(\varphi(s,x^0))\mathrm{d}s = V(\varphi(t,x^0)) - V(x^0)\,,$$

we may conclude that the function $t \mapsto \int_0^t V_f(\varphi(s,x^0))\mathrm{d}s$ is bounded from below. But this function is also non-increasing (because $V_f \le 0$ on G) and so the limit of $\int_0^t V_f(\varphi(s,x^0))\mathrm{d}s$ exists and is finite as $t \to \infty$, showing that the function $t \mapsto V_f(\varphi(t,x^0))$ is Riemann integrable on $\mathbb{R}_+$. An application of Proposition 3.3 (with $g = V_f|_G$) completes the proof. ■

We mention that there are many situations of interest in control theory where the integrability condition in Proposition 3.3 is automatically satisfied, for example, in optimal control (finiteness of an integral performance criterion). Proposition 3.3 is particularly useful in the context of observed systems. To illustrate the latter comment, consider the observed system given by (1) and the observation

$$y = c(x)\,, \tag{2}$$

where $c : \mathbb{R}^N \to \mathbb{R}^P$ is continuous with $c(0) = 0$. The system given by (1) and (2) is said to be *zero-state observable* if, for each $x^0 \in \mathbb{R}^N$, the following holds

$$y(\cdot) = c(\varphi(\cdot\,,x^0)) = 0 \quad \Rightarrow \quad \varphi(\cdot\,,x^0) = 0\,.$$

The next corollary of Proposition 3.3 is contained in [4, Theorem 1.3] and essentially states that, for a zero-state observable system, every bounded trajectory with observation in L^p necessarily converges to zero.

Corollary 3.5. *Assume that the observed system given by* (1) *and* (2) *is zero-state observable. Let $x^0 \in \mathbb{R}^N$. Assume that $\mathbb{R}_+ \subset I(x^0)$ and $\varphi(\cdot\,,x^0)$ is bounded on $\mathbb{R}_+$. If, for some $p \in (0,\infty)$, $\int_0^\infty \|c(\varphi(t,x^0))\|^p \mathrm{d}t < 0$, then $\lim_{t\to\infty} \varphi(t,x^0) = 0$.*

Proof. By continuity and boundedness of $\varphi(\cdot\,,x^0)$, it follows from Lemma 2.1 that $\varphi(\cdot\,,x^0)$ approaches its ω-limit set $\Omega := \Omega(\varphi(\cdot\,,x^0))$ and Ω is the smallest closed set approached by $\varphi(\cdot\,,x^0)$. An application of Proposition 3.3 with $G = \mathbb{R}^N$ and $g(\cdot) = \|c(\cdot)\|^p$ shows that $\Omega \subset g^{-1}(0) = c^{-1}(0)$. Let $\xi \in \Omega$. By the invariance property of Ω, $\varphi(t,\xi) \in \Omega$ for all $t \in \mathbb{R}$. Consequently, $c(\varphi(\cdot\,,\xi)) = 0$, and so, by zero-state observability, $\varphi(\cdot\,,\xi) = 0$, showing that $\xi = 0$. Hence, $\Omega = \{0\}$, and thus, $\lim_{t\to\infty} \varphi(t,x^0) = 0$. ■

4 Generalizations of Barbălat's Lemma

In Theorems 4.4 and 4.5 below, we present generalizations of Barbălat's Lemma and of Corollary 3.2 which allow interesting applications to differential equations. To this end we introduce the concept of (weak) meagreness which will replace the assumption of Riemann integrability in Barbălat's Lemma. Recall that μ denotes the Lebesgue measure on $\mathbb{R}_+$.

Definition 4.1. (a) *A function $y : \mathbb{R}_+ \to \mathbb{R}$ is said to be* meagre *if y is Lebesgue measurable and $\mu(\{t \in \mathbb{R}_+ : |y(t)| \ge \lambda\}) < \infty$ for all $\lambda > 0$.*

(b) *A function $y : \mathbb{R}_+ \to \mathbb{R}$ is said to be* weakly meagre *if $\lim_{n\to\infty}(\inf_{t\in I_n} |y(t)|) = 0$ for every family $\{I_n : n \in \mathbb{N}\}$ of non-empty mutually disjoint closed intervals $I_n \subset \mathbb{R}_+$ with $\inf_{n\in\mathbb{N}} \mu(I_n) > 0$.*

We remark that, in the theory of rearrangements of functions, the property of meagreness is sometimes referred to as "vanishing at infinity" (see [16], p. 72). From the above definitions, it immediately follows that a meagre function is weakly meagre. The converse is not true: consider the function

$$\chi : \mathbb{R}_+ \to \mathbb{R}, \quad t \mapsto \begin{cases} 0, & \text{if } t \in \mathbb{R}_+ \cap \mathbb{Q}, \\ 1, & \text{if } t \in \mathbb{R}_+ \setminus \mathbb{Q}, \end{cases}$$

which is weakly meagre, but evidently not meagre; moreover, in the restricted context of continuous functions, the converse again fails to hold (in the Appendix, Example 8.1, we provide an example of a continuous function which is weakly meagre but not meagre). It is clear that if a function $y : \mathbb{R}_+ \to \mathbb{R}$ is weakly meagre, then $0 \in \Omega(y)$.

The following result gives sufficient conditions for, respectively, meagreness and weak meagreness.

Proposition 4.2. *Let $y : \mathbb{R}_+ \to \mathbb{R}$ be measurable. Then the following statements hold.*

(a) *If there exists a Borel function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ such that $\alpha^{-1}(0) = \{0\}$, $\inf_{s \geq \sigma} \alpha(s) > 0$ for all $\sigma > 0$ and $\alpha(|y(\cdot)|) \in L^1$, then y is meagre.*

(b) *If there exists $\tau > 0$ such that $\lim_{t\to\infty} \int_t^{t+\tau} |y(s)|\mathrm{d}s = 0$, then y is weakly meagre.*

(c) *If y is continuous and for every $\delta > 0$ there exists $\tau \in (0, \delta)$ such that $\int_t^{t+\tau} y(s)\mathrm{d}s$ converges to 0 as $t \to \infty$, then y is weakly meagre.*

Proof. We only prove part (c) (the proofs of parts (a) and (b) are even more straightforward). Let $y : \mathbb{R}_+ \to \mathbb{R}$ be continuous. We show that if y is not weakly meagre, then there exists $\delta > 0$ such that, for every $\tau \in (0, \delta)$, the integral $\int_t^{t+\tau} y(s)\mathrm{d}s$ does not converge to 0 as $t \to \infty$. The claim follows then from contraposition. So assume that y is not weakly meagre. Then there exists a family $\{I_n : n \in \mathbb{N}\}$ of non-empty disjoint closed intervals with $\inf_{n\in\mathbb{N}} \mu(I_n) =: \delta > 0$ and a number $\varepsilon > 0$ such that

$$\inf_{t\in I_n} |y(t)| \geq \varepsilon \quad \forall\, n \in \mathbb{N}.$$

Since y is continuous, the function y has the same sign on I_n for each n. Without loss of generality, we may assume that there infinitely many intervals I_n on which y is positive. Then there exists a sequence (n_k) in $\mathbb{N}$ such that y has positive sign on I_{n_k} for all $k \in \mathbb{N}$. Denoting the left-end points of I_{n_k} by t_k, we obtain that

$$\int_{t_k}^{t_k+\tau} y(s)\mathrm{d}s \geq \varepsilon\tau > 0 \quad \forall\, k \in \mathbb{N}, \;\; \forall\, \tau \in (0, \delta),$$

showing that the integral $\int_t^{t+\tau} y(s)\mathrm{d}s$ does not converge to 0 as $t \to \infty$. ∎

It follows immediately from part (a) of Proposition 4.2 that, if $y \in L^p$ for some $p \in (0, \infty)$, then y is meagre. Part (c) shows, in particular, that, if $y : \mathbb{R}_+ \to \mathbb{R}$ is continuous and Riemann integrable on $\mathbb{R}_+$, then y is weakly meagre. However, we mention that continuity and Riemann integrability of a function $y : \mathbb{R}_+ \to \mathbb{R}$ do not guarantee that y is meagre (an example of a non-meagre function which is both continuous and Riemann integrable is provided in the Appendix, Example 8.1). The sufficient conditions for weak meagreness provided by parts (b) and (c) of Proposition 4.2 are not necessary. To illustrate this, in the Appendix (Example 8.2) we construct a continuous function y which is meagre (and so *a fortiori* is weakly meagre), but is such that, for all $\tau > 0$, $\int_t^{t+\tau} y(s)\mathrm{d}s$ does not converge to 0 as $t \to \infty$. By contrast, the sufficient condition for meagreness given in part (a) of Proposition 4.2 is also necessary: for a proof of this assertion, see [17]).

The following result will play a rôle in the subsequent derivation of generalized versions of Barbălat's Lemma.

Lemma 4.3. *Let $A, B \subset \mathbb{R}^N$ be non-empty such that $\mathrm{cl}(\mathbb{B}_\lambda(B)) \subset A$ for some $\lambda > 0$. Let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be such that x is uniformly continuous on $x^{-1}(A)$. Then there exists $\tau > 0$ such that*

$$t \in \mathbb{R}_+,\ x(t) \in B \quad \Longrightarrow \quad x(s) \in \mathbb{B}_\lambda(B) \ \ \forall\, s \in [t-\tau\,,\, t+\tau] \cap \mathbb{R}_+\,. \tag{3}$$

Proof. Seeking a contradiction, suppose that property (3) does not hold. Then there exist sequences $(s_n), (t_n) \subset \mathbb{R}_+$ such that

$$x(t_n) \in B,\ \ x(s_n) \notin \mathbb{B}_\lambda(B) \quad \text{and} \quad s_n - t_n \to 0 \ \text{ as } n \to \infty\,.$$

Evidently, $s_n \neq t_n$ for all $n \in \mathbb{N}$. Define

$$I_n := \begin{cases} [s_n, t_n], & s_n < t_n \\ [t_n, s_n], & s_n > t_n \end{cases}, \qquad T_n := \{s \in I_n : s \notin x^{-1}(\mathbb{B}_\lambda(B))\} \quad \forall\, n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $\tau_n \in T_n$ (a compact set) be such that

$$|\tau_n - t_n| = \min_{s \in T_n} |s - t_n|.$$

Clearly, $d_B(x(\tau_n)) = \lambda$ and $d_B(x(t_n)) = 0$ for all $n \in \mathbb{N}$. Combining this with the facts that $\tau_n \in I_n$ and $\lim_{n\to\infty}(s_n - t_n) = 0$, we may conclude that

(i) $\|x(t_n) - x(\tau_n)\| \geq |d_B(x(t_n)) - d_B(x(\tau_n))| = \lambda > 0 \quad \forall\, n \in \mathbb{N}$,

(ii) $t_n, \tau_n \in x^{-1}(A) \ \ \forall\, n \in \mathbb{N}$,

(iii) $|t_n - \tau_n| \to 0 \ \text{ as } n \to \infty$,

contradicting the hypothesis of uniform continuity of x on $x^{-1}(A)$. Therefore, property (3) holds. ∎

The following two theorems, the main results of this section, provide our generalizations of Barbălat's Lemma.

Theorem 4.4. *Let $G \subset \mathbb{R}^N$ be non-empty and closed. Let $g : G \to \mathbb{R}$ have the property that each $\xi \in G$, with $g(\xi) \neq 0$, has a neighbourhood U such that*

$$\inf\{|g(w)| : w \in G \cap U\} > 0. \tag{4}$$

Furthermore, let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be continuous with $x(\mathbb{R}_+) \subset G$. If (i) *for every $\xi \in G$ there exists a neighbourhood U' of ξ such that x is uniformly continuous on $x^{-1}(U')$ and* (ii) *$g \circ x$ is weakly meagre, then the following statements hold.*

(a) $\Omega(x) \subset g^{-1}(0)$.

(b) *If $g^{-1}(0)$ is bounded and $\Omega(x) \neq \emptyset$, then x is bounded and x approaches $g^{-1}(0)$.*

(c) *If x is bounded, then $g^{-1}(0) \neq \emptyset$ and x approaches $g^{-1}(0)$.*

(d) *If x is bounded and $g^{-1}(0)$ is totally disconnected, then $\Omega(x) = \{x^\infty\} \subset g^{-1}(0)$ (and so, $\lim_{t\to\infty} x(t) = x^\infty \in g^{-1}(0)$).*

Proof. (a) If $\Omega(x) = \emptyset$, then statement (a) holds trivially. Now assume that $\Omega(x) \neq \emptyset$. Let $\xi \in \Omega(x)$. Since G is closed and $x(\mathbb{R}_+) \subset G$, $\Omega(x) \subset G$ and thus $\xi \in G$. We shall show that $g(\xi) = 0$. Seeking a contradiction, suppose that $g(\xi) \neq 0$. By the hypotheses, there exists a neighbourhood U of ξ such that (4) holds and x is uniformly continuous on $x^{-1}(U)$. Choose $\delta > 0$ such that $\mathbb{B}_\delta(\xi) \subset U$. Then, we have

$$\inf\{|g(w)| : w \in G \cap \mathbb{B}_\delta(\xi)\} =: \varepsilon > 0\,. \tag{5}$$

Choose $\delta_1, \delta_2 \in (0, \delta)$ with $\delta_1 < \delta_2$. Since $\xi \in \Omega(x)$, there exists a sequence (t_n) in $\mathbb{R}_+$ with $t_{n+1} - t_n > 1$ and $x(t_n) \in \mathbb{B}_{\delta_1}(\xi)$ for all $n \in \mathbb{N}$. An application of Lemma 4.3 (with $A = U$, $B = \mathbb{B}_{\delta_1}(\xi)$ and $\lambda = \delta_2 - \delta_1$) shows that there exists $\tau \in (0, 1)$ such that

$$x(t) \in \mathbb{B}_{\delta_2}(\xi) \quad \forall\, t \in [t_n, t_n + \tau]\,, \ \ \forall\, n \in \mathbb{N}\,.$$

Therefore, by (5),

$$|(g \circ x)(t)| \geq \varepsilon \quad \forall\, t \in [t_n, t_n + \tau]\,, \ \ \forall\, n \in \mathbb{N}\,. \tag{6}$$

Finally, since $t_{n+1} - t_n > 1$ for all $n \in \mathbb{N}$ and $\tau \in (0, 1)$, the intervals $[t_n, t_n + \tau]$ are mutually disjoint. Combined with (6) this contradicts the weak meagreness of $g \circ x$.

(b)-(d) A combination of statement (a) and Lemma 2.1 yields statements (b)-(d). ∎

We remark that lower semicontinuity of the function $\xi \mapsto |g(\xi)|$ is sufficient to ensure that (4) holds for some neighbourhood U of every $\xi \in G$ with $g(\xi) \neq 0$.

Barbălat's Lemma follows immediately from an application of statement (b) of Theorem 4.4 to the situation wherein $N = 1$, $G = \mathbb{R}$, $g = \mathrm{id}_\mathbb{R}$ and $x = y$, in conjunction with the observation that a uniformly continuous and Riemann integrable function $y : \mathbb{R}_+ \to \mathbb{R}$ is weakly meagre and thus $0 \in \Omega(y)$, implying that $\Omega(y) \neq \emptyset$. Corollary 3.2 is a simple consequence of statements (a) and (c) of Theorem 4.4.

When compared with Theorem 4.4, the next result (Theorem 4.5) posits that x be uniformly continuous on $x^{-1}(\mathbb{B}_\varepsilon(g^{-1}(0)))$ for some $\varepsilon > 0$: we remark that, in certain situations (for example, if $g^{-1}(0)$ is finite), this assumption is weaker than the uniform continuity assumption imposed on x in Theorem 4.4. On the other hand, the assumption imposed on g in Theorem 4.5 is stronger than that in its counterpart in Theorem 4.4. However, under these modified hypotheses, Theorem 4.5 guarantees that x approaches $g^{-1}(0) \neq \emptyset$ without positing non-emptiness of $\Omega(x)$ or boundedness of x.

Theorem 4.5. *Let $G \subset \mathbb{R}^N$ be non-empty and closed. Let $g : G \to \mathbb{R}$ be such that $g^{-1}(0)$ is closed and*

$$\inf_{\xi \in K} |g(\xi)| > 0 \quad \textit{for every closed set } K \subset G \textit{ with } K \cap g^{-1}(0) = \emptyset. \tag{7}$$

Furthermore, let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be continuous with $x(\mathbb{R}_+) \subset G$. If (i) *$x$ is uniformly continuous on $x^{-1}(\mathbb{B}_\varepsilon(g^{-1}(0)))$ for some $\varepsilon > 0$ and* (ii) *$g \circ x$ is weakly meagre, then the following statements hold.*

(a) *$g^{-1}(0) \neq \emptyset$, x approaches $g^{-1}(0)$ and $\Omega(x) \subset g^{-1}(0)$.*

(b) *If $g^{-1}(0)$ is bounded, then x is bounded, x approaches $g^{-1}(0)$ and $\emptyset \neq \Omega(x) \subset g^{-1}(0)$.*

(c) *If $g^{-1}(0)$ is bounded and totally disconnected, then $\Omega(x) = \{x^\infty\} \subset g^{-1}(0)$ and hence, $\lim_{t\to\infty} x(t) = x^\infty$.*

Proof. For convenience, we set $Z := g^{-1}(0)$. It is clear that $Z \neq \emptyset$ (otherwise, by (7) and the closedness of G, $\inf_{\xi\in G} |g(\xi)| =: \gamma > 0$ and so $|g(x(t))| \geq \gamma$ for all $t \in \mathbb{R}_+$ which contradicts weak meagreness of $g \circ x$). To prove statements (a) and (b), it now suffices to show that x approaches Z (since, by closedness of Z, it then follows immediately that $\Omega(x) \subset Z$; moreover, if Z is bounded, then we may conclude that x is bounded and so $\Omega(x) \neq \emptyset$). Since, by assumption, the trajectory of x is contained in G, it is immediate that, if $G = Z$, then x approaches Z. Consider the remaining case wherein $Z \subsetneq G$. By closedness of Z, there exists $\delta \in (0, \varepsilon/3)$ such that $G \setminus \mathbb{B}_\delta(Z) \neq \emptyset$. For $\theta \in (0, \delta)$, define

$$\iota(\theta) := \inf\{|g(\xi)| : \xi \in G \setminus \mathbb{B}_\theta(Z)\} > 0\,,$$

wherein positivity is a consequence of (7) and the closedness of $G \setminus \mathbb{B}_\theta(Z)$. Seeking a contradiction, suppose that $\lim_{t\to\infty} d_Z(x(t)) \neq 0$. Then there exist $\lambda \in (0, \delta)$ and a sequence (t_n) in $\mathbb{R}_+$ with $t_n \to \infty$ as $n \to \infty$ and $d_Z(x(t_n)) \geq 3\lambda$ for all $n \in \mathbb{N}$. By weak meagreness of $g \circ x$, there exists a sequence (s_n) in $\mathbb{R}_+$ with $s_n \to \infty$ as $n \to \infty$ and $|g(x(s_n))| < \iota(\lambda)$ for all $n \in \mathbb{N}$ and so $d_Z(x(s_n)) \leq \lambda$ for all $n \in \mathbb{N}$. Extracting subsequences of (t_n) and (s_n) (which we do not relabel), we may assume that $s_n \in (t_n, t_{n+1})$ for all $n \in \mathbb{N}$. We now have

$$d_Z(x(t_n)) \geq 3\lambda, \quad d_Z(x(s_n)) \leq \lambda, \quad s_n \in (t_n, t_{n+1}) \quad \forall\, n \in \mathbb{N},$$

and so, by continuity of $d_Z \circ x$, for each $n \in \mathbb{N}$ there exists $\sigma_n \in (t_n, s_n)$ such that $x(\sigma_n) \in B := \{\xi \in G : d_Z(\xi) = 2\lambda\}$. Extracting a subsequence (which, again, we do not relabel), we may assume that $\sigma_{n+1} - \sigma_n > 1$ for all $n \in \mathbb{N}$. Noting that $\mathrm{cl}\big(\mathbb{B}_\lambda(B)\big) \subset A := \mathbb{B}_\varepsilon(Z)$ and invoking Lemma 4.3, we may conclude the existence of $\tau \in (0, 1)$ such that, for all $n \in \mathbb{N}$,

$$d_Z(x(t)) \geq \lambda \quad \forall\, t \in [\sigma_n, \sigma_n + \tau]\,.$$

Therefore, $\{t \in \mathbb{R}_+ : |g(x(t))| \geq \iota(\lambda)\} \supset \cup_{n\in\mathbb{N}}[\sigma_n, \sigma_n + \tau]$, which (on noting that the intervals $[\sigma_n, \sigma_n + \tau]$, $n \in \mathbb{N}$, are each of length $\tau > 0$ and form a mutually disjoint family) contradicts weak meagreness of $g \circ x$. Therefore, x approaches Z, implying that statements (a) and (b) hold. Finally, invoking the fact that the ω-limit set of a bounded continuous function is connected, statement (c) follows from statement (b). ■

We mention that Barbălat's Lemma follows immediately from statement (a) of Theorem 4.5. Furthermore, as a particular refinement of Barbălat's lemma, wherein the hypothesis of uniform continuity of y on $\mathbb{R}_+$ is weakened to that of uniform continuity of y on $y^{-1}(-\varepsilon, \varepsilon) := \{t \in \mathbb{R}_+ : |y(t)| < \varepsilon\}$ for some $\varepsilon > 0$, we state the following simple corollary of Theorem 4.5.

Corollary 4.6. *Let $y : \mathbb{R}_+ \to \mathbb{R}$ be continuous and Riemann integrable. If there exists $\varepsilon > 0$ such that y is uniformly continuous on $y^{-1}(-\varepsilon, \varepsilon)$, then $y(t) \to 0$ as $t \to \infty$.*

5 Applications to ordinary non-autonomous differential equations

Consider the initial-value problem for a non-autonomous ordinary differential equation

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x^0 \in \mathbb{R}^N. \tag{8}$$

Throughout, $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. Precisely: for all $\xi \in \mathbb{R}^N$, $f(\cdot, \xi)$ is Lebesgue measurable; for all $t \in \mathbb{R}_+$, $f(t, \cdot)$ is continuous; f is locally integrably bounded on compact sets, specifically, for each compact set $K \subset \mathbb{R}^N$, there exists $m \in L^1_{\text{loc}}$ such that $\|f(t,\xi)\| \le m(t)$ for all $(t,\xi) \in \mathbb{R}_+ \times K$. For each $x^0 \in \mathbb{R}^N$, there exists $\omega > 0$ such that (8) has a solution on $[0,\omega)$, that is, a locally absolutely continuous function $x : [0,\omega) \to \mathbb{R}^N$, with $x(0) = x^0$, and satisfying the differential equation in (8) for almost all $t \in [0,\omega)$. Recall that a function is locally absolutely continuous if, and only if, it is a primitive of a locally integrable function. A solution of (8) on $\mathbb{R}_+$ is said to be a global solution. A solution x on $I = [0,\omega)$ is said to be maximal (with maximal interval of existence I) if, for every $\tilde{\omega} > \omega$ and every solution $\tilde{x}$ of (8) on $[0,\tilde{\omega})$, there exists $\tilde{t} \in (0,\omega)$ such that $\tilde{x}(\tilde{t}) \neq x(\tilde{t})$ (equivalently, the solution x is maximal if it has no proper right extension which is also a solution). Every solution x of (8) can be extended to a maximal solution on its maximal interval, henceforth denoted by $[0,\omega_x)$; moreover, if x is maximal and $\omega_x < \infty$, then x is unbounded (see Theorem 1.3 on p. 47 in [5]). It follows that if a maximal solution is bounded, then it is global. .

Definition 5.1. *A function $m : \mathbb{R}_+ \to \mathbb{R}$ is said to be uniformly locally integrable if $m \in L^1_{loc}$ and, for all $\varepsilon > 0$, there exists $\tau > 0$ such that*

$$\int_t^{t+\tau} |m(s)| \mathrm{d}s \le \varepsilon \quad \forall\, t \in \mathbb{R}_+ .$$

Clearly, a locally integrable function $m : \mathbb{R}_+ \to \mathbb{R}$ is uniformly locally integrable if, and only if, the function $t \mapsto \int_0^t |m(s)| \mathrm{d}s$ is uniformly continuous. It is readily verified that, if $m \in L^p$ for some p $(1 \le p \le \infty)$, then m is uniformly locally integrable.

Definition 5.2. *For non-empty $A \subset \mathbb{R}^N$, let $\mathcal{F}(A)$ be the class of Carathéodory functions $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ with the property that there exists a uniformly locally integrable function m such that $\|f(t,\xi)\| \le m(t)$ for all $(t,\xi) \in \mathbb{R}_+ \times A$.*

The following proposition shows that under suitable uniform local integrability assumptions relating to f, solutions of (8) satisfy the uniform continuity assumptions required for an application of Theorems 4.4 and 4.5.

Proposition 5.3. *Let $A, B \subset \mathbb{R}^N$ be non-empty. Assume that, for some $\varepsilon > 0$, $\mathbb{B}_\varepsilon(A) \cap B \neq \emptyset$ and $f \in \mathcal{F}(\mathbb{B}_\varepsilon(A) \cap B)$. If $x : \mathbb{R}_+ \to \mathbb{R}^N$ is a global solution of (8) such that $x(\mathbb{R}_+) \subset B$, then x is uniformly continuous on $x^{-1}(A)$.*

Proof. If $x^{-1}(A) = \emptyset$, then the claim holds trivially. Assume $x^{-1}(A) \neq \emptyset$. Since $f \in \mathcal{F}(\mathbb{B}_\varepsilon(A) \cap B)$, there exists a uniformly locally integrable function m such that $\|f(t,w)\| \le m(t)$ for all $(t,w) \in \mathbb{R}_+ \times (\mathbb{B}_\varepsilon(A) \cap B)$. Let $\delta \in (0,\varepsilon)$ be arbitrary. Choose $\tau > 0$ such that $\int_t^{t+\tau} m \le \delta$ for all $t \in \mathbb{R}_+$. Let $t_1, t_2 \in x^{-1}(A)$ with $0 \le t_2 - t_1 \le \tau$. We will complete the proof by showing that $\|x(t_2) - x(t_1)\| \le \delta$. Defining

$$J := \{t > t_1 : x(s) \in \mathbb{B}_\varepsilon(A) \ \ \forall\, s \in [t_1, t]\},$$

it follows that $\|x(t) - x(t_1)\| \le \int_{t_1}^t m(s)\mathrm{d}s \le \int_{t_1}^{t_1+\tau} m(s)\mathrm{d}s \le \delta$ for all $t \in J$ with $t \le t_1 + \tau$. Since $\delta < \varepsilon$, , $t_1 + \tau \in J$, and so, $\|x(t_2) - x(t_1)\| \le \delta$. ■

In the following, we combine Proposition 5.3 with Theorems 4.4 and 4.5 to derive results on the asymptotic behaviour of solutions of (8).

Theorem 5.4. *Let $G \subset \mathbb{R}^N$ be non-empty and closed. Let $g : G \to \mathbb{R}$ have the property that each $\xi \in G$, with $g(\xi) \neq 0$, has a neighbourhood U such that (4) holds. Assume that, for every $\xi \in G$, there exists exists a neighbourhood U' of ξ such that $f \in \mathcal{F}(U' \cap G)$. If $x : \mathbb{R}_+ \to \mathbb{R}^N$ is a global solution of (8) with $x(\mathbb{R}_+) \subset G$ and $g \circ x$ is weakly meagre, then statements* (a)-(d) *of* Theorem 4.4 *hold.*

Proof. Let $\xi \in G$ be arbitrary. By the hypotheses, there exists a neighbourhood U of ξ such that (4) holds and $f \in \mathcal{F}(U \cap G)$. Let $\varepsilon > 0$ be sufficiently small so that $\mathbb{B}_{2\varepsilon}(\xi) \subset U$. Then, setting $A := \mathbb{B}_\varepsilon(\xi)$, we have that $f \in \mathcal{F}(\mathbb{B}_\varepsilon(A) \cap G)$. By Proposition 5.3, it follows that x is uniformly continuous on $x^{-1}(A)$. An application of Theorem 4.4 completes the proof. ∎

We remark that Theorem 5.4 contains a recent result by Teel [22, Theorem 1] as a special case.

In the next theorem, it is assumed that $f \in \mathcal{F}(\mathbb{B}_\varepsilon(g^{-1}(0)) \cap G)$ for some $\varepsilon > 0$. Under the additional assumption that g satisfies (7), it is then guaranteed that x approaches $g^{-1}(0)$ (without positing boundedness of x).

Theorem 5.5. *Let $G \subset \mathbb{R}^N$ be non-empty and closed. Let $g : G \to \mathbb{R}$ be such that $g^{-1}(0)$ is closed and (7) holds. Assume that $f \in \mathcal{F}(\mathbb{B}_\varepsilon(g^{-1}(0)) \cap G))$ for some $\varepsilon > 0$. If $x : \mathbb{R}_+ \to \mathbb{R}^N$ is a global solution of (8) with $x(\mathbb{R}_+) \subset G$ and $g \circ x$ is weakly meagre, then statements* (a)-(c) *of* Theorem 4.5 *hold.*

Proof. Let $\delta \in (0, \varepsilon)$. By Proposition 5.3, x is uniformly continuous on $x^{-1}(\mathbb{B}_\delta(g^{-1}(0)))$. An application of Theorem 4.5 completes the proof. ∎

In the following we shall use Theorem 5.4 to obtain a version of a well-known result on ω-limit sets of solutions of non-autonomous ordinary differential equations. For a non-empty, open set $D \subset \mathbb{R}^N$ and a continuously differentiable function $V : \mathbb{R}_+ \times D \to \mathbb{R}$, we define $V_f : \mathbb{R}_+ \times D \to \mathbb{R}$ (the derivative of V with respect to (8) in the sense that $(\mathrm{d}/\mathrm{d}t)V(t, x(t)) = V_f(t, x(t))$ along a solution x of (8)) by

$$V_f(t,\xi) := \frac{\partial V}{\partial t}(t,\xi) + \sum_{i=1}^{N} \frac{\partial V}{\partial \xi_i}(t,\xi) f_i(t,\xi) \quad \forall\, (t,\xi) \in \mathbb{R}_+ \times D\,,$$

where $f_1, \ldots, f_N$ denote the components of f.

Corollary 5.6. *Let $D \subset \mathbb{R}^N$ be non-empty and open and let $V : \mathbb{R}_+ \times D \to \mathbb{R}$ be continuously differentiable. Assume that V satisfies the following two conditions:*

(a) *for every $\xi \in \mathrm{cl}(D)$ there exists a neighbourhood U of ξ such that V is bounded from below on the set $\mathbb{R}_+ \times (U \cap D)$;*

(b) *there exists a lower semicontinuous continuous non-negative-valued function $W : \mathrm{cl}(D) \to \mathbb{R}_+$ such that $V_f(t,\xi) \leq -W(\xi)$ for all $(t,\xi) \in \mathbb{R}_+ \times D$.*

Furthermore, assume that, for every $\xi \in \mathrm{cl}(D)$, there exists a neighbourhood U' of ξ such that $f \in \mathcal{F}(U' \cap D)$. Under these assumptions, if $x : \mathbb{R}_+ \to \mathbb{R}^N$ is a global solution of (8) with $x(\mathbb{R}_+) \subset D$, then $\Omega(x) \subset W^{-1}(0)$.

Proof. If $\Omega(x) = \emptyset$ there is nothing to show. So let us assume that $\Omega(x) \neq \emptyset$. Since $(\mathrm{d}/\mathrm{d}t)V(t, x(t)) = V_f(t, x(t))$ for all $t \in \mathbb{R}_+$, it follows from assumption (b) that the function $t \mapsto V(t, x(t))$ is non-increasing, showing that the limit l of $V(t, x(t))$ as $t \to \infty$ exists, where $l \in \mathbb{R} \cup \{-\infty\}$. Let $\xi \in \Omega(x) \subset \mathrm{cl}(D)$. Then there exists a non-decreasing unbounded sequence (t_n) in $\mathbb{R}_+$ such that $\lim_{n\to\infty} x(t_n) = \xi$. By assumption (a) there exists a neighbourhood U of ξ such that V is bounded from below on $\mathbb{R}_+ \times (U \cap D)$. Now $x(\mathbb{R}_+) \subset D$, and so there exists $n_0 \in \mathbb{N}$ such that $x(t_n) \in U \cap D$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Consequently, the non-increasing sequence $(V(t_n, x(t_n)))$ is bounded from below, showing that $l > -\infty$. Therefore

$$0 \leq \int_0^\infty (W \circ x)(t)\mathrm{d}t \leq -\int_0^\infty V_f(t, x(t))\mathrm{d}t = -\int_0^\infty (\mathrm{d}/\mathrm{d}t)V(t, x(t))\mathrm{d}t = V(0, x^0) - l < \infty\,,$$

showing that $W \circ x \in L^1$ and hence is weakly meagre. By assumption, for every $\xi \in \mathrm{cl}(D)$, there exists an open neighbourhood U' of ξ such that $f \in \mathcal{F}(U' \cap D)$, implying that $f \in \mathcal{F}(U' \cap \mathrm{cl}(D))$. Therefore, an application of Theorem 5.4 with $G = \mathrm{cl}(D)$ and $g = W$ yields the claim. ∎

Corollary 5.6 is essentially due to LaSalle [14] (see also [11], Satz 6.2 on p. 140). However, we point out that the assumption imposed on f in Corollary 5.6 is weaker then that in [11] and [14], wherein it is required that, for every $\xi \in \mathrm{cl}(D)$, there exists a neighbourhood U' of ξ such that f is bounded on the set $\mathbb{R}_+ \times (U' \cap D)$. Furthermore, we impose only lower semicontinuity on the function W (in contrast to [11] and [14], wherein continuity of W is assumed).

The next result is a consequence of Theorem 5.5 and, in particular, shows that, under a mild assumption on f, every global L^p solution of (8) converges to zero.

Corollary 5.7. *Assume that there exists $\varepsilon > 0$ such that $f \in \mathcal{F}(\mathbb{B}_\varepsilon(0))$ and let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be a global solution of (8). Then the following hold.*

(a) *If $\|x(\cdot)\|$ is weakly meagre, then $\lim_{t\to\infty} x(t) = 0$.*

(b) *If $x \in L^p$ for some $p \in (0, \infty)$, then $\lim_{t\to\infty} x(t) = 0$.*

Proof. (a) If $\|x(\cdot)\|$ is weakly meagre, then an application of Theorem 5.5 with $G = \mathbb{R}^N$ and $g = \|\cdot\|$ shows that $\lim_{t\to\infty} x(t) = 0$.

(b) If $x \in L^p$ for some $p \in (0, \infty)$, then by statement (a) of Proposition 4.2, the function $\|x(\cdot)\|$ is meagre and hence is weakly meagre. By part (a) above, $\lim_{t\to\infty} x(t) = 0$. ∎

In view of part (a) of Proposition 4.2, we may infer that the second assertion of the above corollary remains valid if the L^p assumption on x is replaced by the weaker hypothesis that $\alpha(\|\cdot\|) \in L^1$ for some Borel function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ with $\alpha^{-1}(0) = \{0\}$ and $\inf_{s\geq\sigma} \alpha(s) > 0$ for all $\sigma > 0$.

Obviously, if (8) is autonomous (that is, there exists a continuous function $f^* : \mathbb{R}^N \to \mathbb{R}^N$ such that $f(t, \xi) = f^*(\xi)$ for all $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^N$), then the assumption that $f \in \mathcal{F}(\mathbb{B}_\varepsilon(0))$ for some $\varepsilon > 0$ is trivially satisfied. Thus we may conclude that every weakly meagre global solution $t \mapsto x(t)$ of an autonomous ordinary differential equation converges to 0 as $t \to \infty$.

Example 5.8. In this example we describe a typical application of Theorem 5.5. In part (a) of the example we analyze a general class of second-order systems with non-autonomous "damping"; in part (b) we discuss a special case. We will return to this example in Section 6 to further refine the result.

(a) Consider the second-order system

$$\ddot{y}(t) + d(t, \dot{y}(t)) + k(y(t)) = 0, \quad (y(0), \dot{y}(0)) = (p^0, v^0) \in \mathbb{R}^2, \tag{9}$$

where $k : \mathbb{R} \to \mathbb{R}$ is a continuous function with the property that

$$\lim_{|p|\to\infty} \int_0^p k(s)\mathrm{d}s = \infty\,. \tag{10}$$

We assume that $d : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(i) $d(t, v)v \geq 0$ for all $(t, v) \in \mathbb{R}_+ \times \mathbb{R}$;

(ii) for every bounded set $K \subset \mathbb{R}$ there exists a constant $b \geq 0$ such that $|d(t, v)| \leq b$ for all $(t, v) \in \mathbb{R}_+ \times K$;

(iii) the function $d_* : \mathbb{R} \to \mathbb{R}_+,\ v \mapsto \inf_{t\in\mathbb{R}_+} |d(t, v)|$ is a Borel function such that $\inf_{v\in K} d_*(v) > 0$ for every compact set $K \subset \mathbb{R}$ with $K \cap \mathrm{cl}(d_*^{-1}(0)) = \emptyset$.

Since d is a Carathéodory function, we have that, for each fixed $t \in \mathbb{R}_+$, the function $v \mapsto d(t, v)$ is continuous. Therefore it follows from assumption (i) that $d(t, 0) = 0$ for all $t \in \mathbb{R}_+$, showing that $0 \in d_*^{-1}(0)$.

By standard existence theory, (9) has a solution and every solution has a maximal extension. A solution y of (9) is said to be global if y is defined on $\mathbb{R}_+$.

Claim. For each $(p^0, v^0) \in \mathbb{R}^2$, every maximal solution y of (9) is global and bounded, with bounded derivative $\dot{y}$. Moreover, $\dot{y}$ approaches $d_*^{-1}(0)$ and, if 0 is an isolated point of $d_*^{-1}(0)$, then $\lim_{t\to\infty} \dot{y}(t) = 0$.

We proceed to prove this claim. Setting $x(t) := (y(t), \dot{y}(t))$, the second-order system (9) can be expressed in the equivalent form (8), where $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ and $x^0 \in \mathbb{R}^2$ are given by

$$f(t, (p, v)) = (v, -k(p) - d(t, v)) \quad \text{and} \quad x^0 = (p^0, v^0)\,, \tag{11}$$

respectively. Define

$$V : \mathbb{R}^2 \to \mathbb{R},\ \ (p, v) \mapsto \int_0^p k(s)\mathrm{d}s + v^2/2\,.$$

Observe that, by (10), V is bounded from below and is such that, for every sequence (ξ_n) in $\mathbb{R}^2$,

$$\|\xi_n\| \to \infty \ \text{ as } n \to \infty \quad \Longrightarrow \quad V(\xi_n) \to \infty \ \text{ as } n \to \infty\,. \tag{12}$$

Let $x = (y, \dot{y}) : [0, \omega_x) \to \mathbb{R}^2$ be a maximal solution of (8) (with f and x^0 given by (11)). Then, for almost all $t \in [0, \omega_x)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leq -d(t, \dot{y}(t))\dot{y}(t) = -|d(t, \dot{y}(t))||\dot{y}(t)| \leq -d_*(\dot{y}(t))|\dot{y}(t)| \leq 0\,,$$

wherein we have invoked assumption (i) on d. Consequently, $c \le V(x(t)) \le V(x^0)$ for all $t \in [0, \omega_x)$ and for some constant c. Combining this with (12), it follows that x is bounded and so $\omega_x = \infty$; moreover, $\lim_{t\to\infty} V(x(t)) =: l \in \mathbb{R}$ exists and hence

$$0 \le \int_0^\infty d_*(\dot y(t))|\dot y(t)|\mathrm{d}t \le V(x^0) - l < \infty\,. \tag{13}$$

Since x is bounded, there exists a compact interval $I \subset \mathbb{R}$ such that $x(\mathbb{R}_+) \subset I \times I$. Define

$$g : I \times I \to \mathbb{R}_+, \ \ (p, v) \mapsto \begin{cases} 0, & \text{if } (p, v) \in I \times (\mathrm{cl}(d_*^{-1}(0)) \cap I), \\ d_*(v)|v|, & \text{otherwise.} \end{cases}$$

Then $g^{-1}(0) = I \times (\mathrm{cl}(d_*^{-1}(0)) \cap I)$ and so, in particular, $g^{-1}(0)$ is closed. It follows from assumption (iii) on d that g satisfies (7). Moreover, assumption (ii) on d implies that $f \in \mathcal{F}(B)$ for every bounded set $B \subset \mathbb{R}^2$ and so $f \in \mathcal{F}(\mathbb{B}_\varepsilon(g^{-1}(0)) \cap (I \times I))$ for $\varepsilon > 0$. By (13), $g \circ x$ is weakly meagre, and thus we may apply Theorem 5.5, statement (b), to conclude that $x = (y, \dot y)$ approaches $g^{-1}(0)$. Consequently, $\dot y$ approaches $d_*^{-1}(0)$. Finally, assume that 0 is an isolated point of $d_*^{-1}(0)$ and hence of $\mathrm{cl}(d_*^{-1}(0))$. Now $\dot y$ is continuous and bounded, and so, by part (c) of Lemma 2.1, the ω-limit set $\Omega(\dot y)$ of $\dot y$ is non-empty, compact, connected, is approached by $\dot y$ and is the smallest closed set approached by $\dot y$. Consequently, there exists $\varepsilon > 0$ such that either $\Omega(\dot y) \subset (-\infty, -\varepsilon]$ or $\Omega(\dot y) \subset [\varepsilon, \infty)$ or $\Omega(\dot y) = \{0\}$. Since y is bounded, it follows that the first two alternatives are impossible. Therefore, we may conclude that $\lim_{t\to\infty} \dot y(t) = 0$, completing the proof of the claim.

(b) As a particular case, consider $d : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ given by

$$d(t, v) = \begin{cases} v, & \text{if } (t, v) \in \mathbb{R}_+ \times [-1, 1], \\ \mathrm{sgn}(v) \max\{0, 1 - (t+1)(|v| - 1)\}, & \text{if } (t, v) \in \mathbb{R}_+ \times \{v \in \mathbb{R} : |v| > 1\}, \end{cases} \tag{14}$$

where $\mathrm{sgn}(v)$ denotes the sign of v. For generic $t \in \mathbb{R}_+$, the graph of $v \mapsto d(t, v)$ is depicted in Figure 1 below.

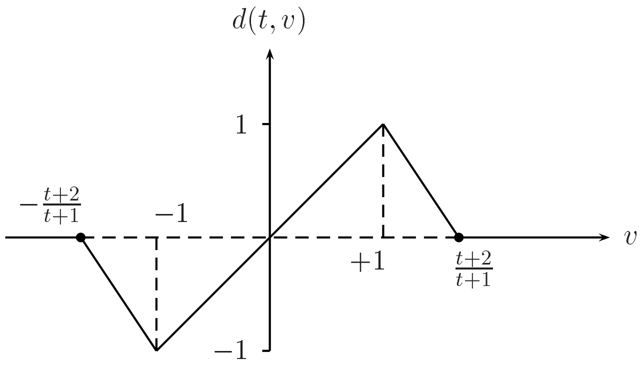

Figure 1

Observe that d satisfies assumptions (i) and (ii). Furthermore, it is readily verified that the function $d_* : \mathbb{R} \to \mathbb{R}_+$, $v \mapsto \inf_{t\in\mathbb{R}_+} |d(t, v)|$ is given by

$$d_*(v) = \begin{cases} |v|, & \text{if } v \in [-1, 1], \\ 0, & \text{if } |v| > 1. \end{cases}$$

Clearly, d_* is piecewise continuous (with jump discontinuities at $v = \pm 1$) and therefore is a Borel function. Moreover, $\mathrm{cl}(d_*^{-1}(0)) = \{v \in \mathbb{R} : |v| \geq 1\} \cup \{0\}$ and it is clear that $\inf_{v \in K} d_*(v) > 0$ for every compact set $K \subset \mathbb{R}$ with $K \cap \mathrm{cl}(d_*^{-1}(0)) = \emptyset$, showing that assumption (iii) is satisfied. Since, trivially, 0 is an isolated point of $d_*^{-1}(0)$, it follows from part (a) that $\lim_{t \to \infty} \dot{y}(t) = 0$.

In Section 6, we will return to part (b) of the above example and further refine the result to conclude that $(y, \dot{y})$ approaches the set $k^{-1}(0) \times \{0\}$.

6 Applications to autonomous differential inclusions

Let $\mathcal{U}$ denote the class of set-valued maps $\xi \mapsto F(\xi) \subset \mathbb{R}^N$, defined on $\mathbb{R}^N$, that are upper semicontinuous at each $\xi \in \mathbb{R}^N$ and take non-empty convex compact values. We recall that a set-valued map F is said to be upper semicontinuous at $\xi \in \mathbb{R}^N$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $F(\xi') \subset \mathbb{B}_\varepsilon(F(\xi))$ for all $\xi' \in \mathbb{B}_\delta(\xi)$. Consider the initial-value problem for an autonomous differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x^0 \in \mathbb{R}^N. \tag{15}$$

We will study asymptotic properties of solutions of (15), where, by a solution on $[0, \omega)$, we mean a locally absolutely continuous function $x : [0, \omega) \to \mathbb{R}^N$ satisfying (15) almost everywhere on $[0, \omega)$. A solution on $\mathbb{R}_+$ is said to be a global solution. The concept of a maximal solution of (15) is the natural analogue of that for differential equations (see Section 5). We record some well-known facts in Lemmas 6.1 and 6.2 which form a distillation of results in, for example, [2], [9] and [18].

Lemma 6.1. *Let $F \in \mathcal{U}$. For each $x^0 \in \mathbb{R}^N$,* (15) *has a solution with $x(0) = x^0$ and every such solution can be extended to a maximal solution with maximal interval of existence $[0, \omega_x)$. If $\omega_x < \infty$, then x is unbounded.*

It follows from Lemma 6.1 that bounded maximal solutions of (15) are global. With respect to (15), a non-empty set $A \subset \mathbb{R}^N$ is said to be *weakly invariant* if, for each $x^0 \in A$, (15) has at least one maximal solution $x : [0, \omega_x) \to \mathbb{R}^N$, with $x(0) = x^0$ and $x(t) \in A$ for all $t \in [0, \omega_x)$.

Lemma 6.2. *Let $F \in \mathcal{U}$. If $x : \mathbb{R}_+ \to \mathbb{R}^N$ is a bounded global solution of* (15)*, then $\Omega(x)$ is non-empty, compact, connected, is approached by x* (*and is the smallest closed set so approached*)*, and is weakly invariant with respect to* (15)*.*

Example 6.3. Let us revisit the special case (b) of Example 5.8. Let f and x^0 be given by (11) and let $x = (y, \dot{y}) : [0, \omega_x) \to \mathbb{R}^N$ be a maximal solution of (8). We already know that $\omega_x = \infty$, x is bounded and $\emptyset \neq \Omega(x) \subset \mathbb{R} \times \{0\}$. Defining a set-valued map Δ on $\mathbb{R}$ by

$$\Delta(v) := \begin{cases} \{v\}, & \text{if } |v| \leq 1 \\ [-1, 1], & \text{if } |v| > 1\,, \end{cases}$$

we have $d(t, v) \in \Delta(v)$ for all $(t, v) \in \mathbb{R}_+ \times \mathbb{R}$, where d is defined by (14). On $\mathbb{R} \times \mathbb{R}$, define the set-valued map $(p, v) \mapsto F(p, v) := \{v\} \times \{-k(p) - w : w \in \Delta(v)\}$. Evidently, $F \in \mathcal{U}$ and $f(t, (p, v)) \in F(p, v)$ for all $(t, (p, v)) \in \mathbb{R}_+ \times \mathbb{R}^2$. Therefore, the solution x of (8) is, *a fortiori*, a solution of the differential inclusion $\dot{x}(t) = F(x(t))$ and so, by Lemma 6.2, $\Omega(x)$ is weakly invariant with respect to that differential inclusion. Since x approaches $\Omega(x) \subset \mathbb{R} \times \{0\}$, it follows that x must approach the largest subset $E \subset \mathbb{R} \times \{0\}$ that is weakly invariant with respect to the differential inclusion. Letting $(p,v)\in E$, there exists, by weak invariance of E, a maximal solution $(z,\dot z)$ of the differential inclusion such that $(z(0),\dot z(0))=(p,v)=(p,0)$ and $(z(t),\dot z(t))\in E\subset\mathbb{R}\times\{0\}$ for all $t\in[0,\omega_{(z,\dot z)})$. Therefore, $z(t)=p$ for all $t\in\mathbb{R}_+$ and, noting that $\Delta(0)=\{0\}$, we have $(0,0)=(\dot z(t),\ddot z(t))\in F(z(t),\dot z(t))=F(p,0)=\{(0,-k(p))\}$ for almost all $t\in[0,\omega_{(z,\dot z)})$ and so $k(p)=0$. Thus, $E\subset k^{-1}(0)\times\{0\}$ and so x approaches the set $k^{-1}(0)\times\{0\}$. Finally, note that, if $k^{-1}(0)$ is totally disconnected, then x approaches an equilibrium of the non-autonomous differential equation (8) (with f and x^0 given by (11)).

The following proposition shows that, under suitable local boundedness assumptions on F, the solutions of (15) satisfy the uniform continuity assumptions required for an application of Theorems 4.4 and 4.5. For a subset $A\subset\mathbb{R}^N$, we denote (with slight abuse of notation) the set $\cup_{a\in A}F(a)$ by $F(A)$.

Proposition 6.4. *Let $A,B\subset\mathbb{R}^N$. Assume that $F(\mathbb{B}_\varepsilon(A)\cap B)$ is bounded for some $\varepsilon>0$ and that $x:\mathbb{R}_+\to\mathbb{R}^N$ is a global solution of (8) with $x(\mathbb{R}_+)\subset B$. Then x is uniformly continuous on $x^{-1}(A)$.*

Proof. If $x^{-1}(A)=\emptyset$, then the claim holds trivially. Assume $x^{-1}(A)\neq\emptyset$ and let $\delta\in(0,\varepsilon)$ be arbitrary. Define $\theta:=\sup\{\|v\|\,:\,v\in F(\mathbb{B}_\varepsilon(A)\cap B)\}$ and let $\tau>0$ be sufficiently small so that $\tau\theta\le\delta$. Adopting an argument similar to that used in the proof of Proposition 5.3, it can be shown that $\|x(t_2)-x(t_1)\|\le\delta$ for all $t_1,t_2\in x^{-1}(A)$ with $0\le t_2-t_1\le\tau$, proving that x is uniform continuous on $x^{-1}(A)$. ∎

We now use Theorems 4.4 and 4.5 to derive counterparts of Theorems 5.4 and 5.5 for differential inclusions.

Theorem 6.5. *Let $G\subset\mathbb{R}^N$ be non-empty and closed. Let $g:G\to\mathbb{R}$ have the property that each $\xi\in G$, with $g(\xi)\neq0$, has a neighbourhood U such that* (4) *holds. If $x:\mathbb{R}_+\to\mathbb{R}^N$ is a global solution of* (15) *with $x(\mathbb{R}_+)\subset G$ and $g\circ x$ is weakly meagre, then statements* (a) *and* (d) *of Theorem 4.4 hold. Moreover,*

- (b′) *If $g^{-1}(0)$ is bounded and $\Omega(x)\neq\emptyset$, then x is bounded and x approaches the largest subset of $g^{-1}(0)$ that is weakly invariant with respect to* (15).
- (c′) *If x is bounded, then $g^{-1}(0)\neq\emptyset$ and x approaches the largest subset of $g^{-1}(0)$ that is weakly invariant with respect to* (15).

Proof. Let $\xi\in G$. By hypothesis, there exists $\varepsilon>0$ such that (4) holds with $U:=\mathbb{B}_\varepsilon(\xi)$. By upper semicontinuity of F, together with compactness of its values, $F(\mathbb{B}_\varepsilon(U)\cap G)$ is bounded (see [2], Proposition 3, p. 42) and so, by Proposition 6.4, x is uniformly continuous on $x^{-1}(U)$. Therefore, the hypotheses of Theorem 4.4 are satisfied and so statements (a)-(d) thereof hold. Combining statements (b) and (c) of Theorem 4.4 with the weak invariance of $\Omega(x)$ yields statements (b′) and (c′). ∎

Theorem 6.6. *Let $G\subset\mathbb{R}^N$ be non-empty and closed. Let $g:G\to\mathbb{R}$ be such that $g^{-1}(0)$ is closed and* (7) *holds. Assume that $F(\mathbb{B}_\varepsilon(g^{-1}(0))\cap G)$ is bounded for some $\varepsilon>0$. If $x:\mathbb{R}_+\to\mathbb{R}^N$ is a global solution of* (8) *with $x(\mathbb{R}_+)\subset G$ and $g\circ x$ is weakly meagre, then statements* (a) *and* (c) *of* Theorem 4.5 *hold. Moreover,*

- (b′) *If $g^{-1}(0)$ is bounded, then x is bounded and x approaches the largest subset of $g^{-1}(0)$ that is weakly invariant with respect to* (15).

Proof. Let $\delta \in (0, \varepsilon)$. By Proposition 6.4, x is uniformly continuous on $x^{-1}(\mathbb{B}_\delta(g^{-1}(0)))$. It immediately follows from Theorem 4.5 that statements (a)-(c) thereof hold. Assuming that $g^{-1}(0)$ is bounded, a combination of statements (b) of Theorem 4.5 with the weak invariance of $\Omega(x)$ yields statement (b$'$). ■

If there exists a local Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $F(x) = \{f(x)\}$ (that is, the differential inclusion (15) "collapses" to an autonomous differential equation which, for every $x^0 \in \mathbb{R}^N$, has a unique solution satisfying $x(0) = x^0$), then the conclusions of Theorems 6.5 and 6.6 remain true when every occurence of "weakly invariant" is replaced by "invariant". We mention that precursors of Theorems 6.5 and 6.6 have appeared in [6] and [19].

We now use Theorem 6.6 to generalize LaSalle's Invariance Principle (see Corollary 3.4) to differential inclusions.

Corollary 6.7. *Let $D \subset \mathbb{R}^N$ be non-empty and open, let $V : D \to \mathbb{R}$ be continuously differentiable and set $V_F(\xi) := \max_{y \in F(\xi)} \langle \nabla V(\xi), y \rangle$ for all $\xi \in D$. Let $x : \mathbb{R}_+ \to \mathbb{R}^N$ be a (global) solution of (15) and assume that there exists a compact set $G \subset \mathbb{R}^N$ such that $x(\mathbb{R}_+) \subset G \subset D$. If $V_F(\xi) \le 0$ for all $\xi \in G$, then x approaches the largest subset of $V_F^{-1}(0) \cap G$ that is weakly invariant with respect to (15).*

Proof. For later convenience, we first show that the function $V_F : D \to \mathbb{R}$ is upper semicontinuous. Let (ξ_n) be a convergent sequence in D, with limit $\xi \in D$. Define $l := \limsup_{n\to\infty} V_F(\xi_n) \in (-\infty, 0]$. From $(V_F(\xi_n))$, extract a subsequence $\big(V_F(\xi_{n_k})\big)$ with $V_F(\xi_{n_k}) \to l$ as $k \to \infty$. For each k, let y_k be a maximizer of the continuous function $y \mapsto \langle \nabla V(\xi_{n_k}), y\rangle$ over the compact set $F(\xi_{n_k})$, and so $V_F(\xi_{n_k}) = \langle \nabla V(\xi_{n_k}), y_k\rangle$. Let $\varepsilon > 0$ be arbitrary. By upper semicontinuity of F, $F(\xi_{n_k}) \subset \mathbb{B}_\varepsilon(F(\xi))$ for all k sufficiently large. Since $y_k \in F(\xi_{n_k})$ and $F(\xi)$ is compact and $\varepsilon > 0$ is arbitrary, we may infer that (y_k) has a subsequence (which we do not relabel) converging to $y^* \in F(\xi)$. Therefore,

$$\limsup_{n\to\infty} V_F(\xi_n) = l = \lim_{k\to\infty} V_F(\xi_{n_k}) = \lim_{k\to\infty} \langle \nabla V(\xi_{n_k}), y_k\rangle = \langle \nabla V(\xi), y^*\rangle \le V_F(\xi)$$

and so V_F is upper semicontinuous.

Evidently,

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \langle \nabla V(x(t)), \dot{x}(t)\rangle \le V_F(x(t)) \le 0 \quad \text{for a.a. } t \in \mathbb{R}_+$$

and so

$$V(x(t)) - V(x(0)) \le \int_0^t V_F(x(s))\mathrm{d}s \le 0 \quad \forall\, t \in \mathbb{R}_+\,. \tag{16}$$

Since x is bounded, we may conclude that the function $t \mapsto \int_0^t V_F(x(s))\mathrm{d}s$ is bounded from below. But this function is also non-increasing (because $V_F \le 0$ on G) and so $\lim_{t\to\infty} \int_0^t V_F(x(s))\mathrm{d}s$ exists and is finite. Consequently, $V_F \circ x \in L^1$, showing that $V_F \circ x$ is weakly meagre. Since V_F is upper semicontinuous and $V_F \le 0$ on G, the function $G \to \mathbb{R},\ \xi \mapsto |V_F(\xi)|$ is lower semicontinuous. Therefore, each $\xi \in G$, with $V_F(\xi) \ne 0$, has a neighbourhood U such that $\inf\{|V_F(w)| : w \in G \cap U\} > 0$. By statement (c$'$) of Theorem 6.5 (with $g = V_F|_G$) it follows that x approaches the largest subset of $V_F^{-1}(0) \cap G$ that is weakly invariant with respect to (15). ■

In Corollary 6.7, it is assumed that the solution x is global and has trajectory in some compact subset $G \subset D$. These conditions may be removed at the expense of strengthening the conditions on V by assuming that its sublevel sets are bounded and that $V_F(\xi) \le 0$ for all $\xi \in D$.

Corollary 6.8. *Let D, V and V_F be as in* Corollary 6.7. *Assume that the sublevel sets of V are bounded and that $V_F(\xi) \le 0$ for all $\xi \in D$. If $x : [0, \omega_x) \to \mathbb{R}^N$ is a maximal solution of* (15) *such that* $\mathrm{cl}(x([0,\omega_x))) \subset D$, *then x is bounded, $\omega_x = \infty$ and x approaches the largest subset of $V_F^{-1}(0)$ that is weakly invariant with respect to* (15).

Proof. Since $(\mathrm{d}/\mathrm{d}t)V(x(t)) = V_F(x(t)) \le 0$ for almost all $t \in [0, \omega_x)$, we have the counterpart of (16): $V(x(t)) \le V(x(0))$ for all $t \in \omega_x$. Since the sublevel sets of V are bounded, it follows that x is bounded and so, by Lemma 6.1, $\omega_x = \infty$. An application of Corollary 6.7, with $G = \mathrm{cl}(x(\mathbb{R}_+))$, completes the proof. ∎

Example 6.9. In this example we describe a typical application of Corollary 6.8. In part (a) of the example we analyze a general class of second-order differential inclusions; in part (b) we discuss a special case, a mechanical system subject to friction of Coulomb type.

(a) Let $k : \mathbb{R} \to \mathbb{R}$ be as in Example 5.8, that is, k is continuous with property (10). Let $(p, v) \mapsto C(p, v) \subset \mathbb{R}$ be upper semicontinuous with non-empty, convex, compact values and such that

$$C_*(p, v) := \max\{vw : w \in C(p, v)\} \le 0 \quad \forall\ (p, v) \in \mathbb{R}^2 . \tag{17}$$

Consider the system

$$\ddot{y}(t) + k(y(t)) \in C(y(t), \dot{y}(t)), \quad (y(0), \dot{y}(0)) = (p^0, v^0) \in \mathbb{R}^2 . \tag{18}$$

Setting $x(t) := (y(t), \dot{y}(t))$, the second-order initial-value problem (18) can be expressed in the equivalent form

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x^0 = (p^0, v^0) \in \mathbb{R}^2 , \tag{19}$$

where the set-valued map $F \in \mathcal{U}$ is given by

$$F(p, v) = \{v\} \times \{-k(p) + w : w \in C(p, v)\}. \tag{20}$$

By Lemma 6.1, (19) has a solution and every solution can be extended to a maximal solution; moreover, every bounded maximal solution is global.

Claim A. For each $x^0 = (p^0, v^0) \in \mathbb{R}^2$, every maximal solution $x = (y, \dot{y})$ of (19) is bounded (and hence global) and approaches the largest subset $E \subset C_*^{-1}(0)$ that is weakly invariant with respect to (19).

To prove this claim, we define (as in Example 5.8),

$$V : \mathbb{R}^2 \to \mathbb{R}, \ \ (p, v) \mapsto \int_0^p k(s)\mathrm{d}s + v^2/2 .$$

Observe that, by (10), V is such that, for every sequence (ξ_n) in $\mathbb{R}^2$, (12) holds and so every sublevel set of V is bounded. Moreover,

$$V_F(p, v) = \max_{\theta \in F(p,v)} \langle \nabla V(p, v), \theta \rangle = C_*(p, v) \le 0 \quad \forall\ (p, v) \in \mathbb{R}^2 .$$

Let $x^0 = (p^0, v^0) \in \mathbb{R}^2$ and let $x = (y, \dot{y})$ be a maximal solution of (19). An application of Corollary 6.8, with $D = \mathbb{R}^N$, completes the proof of Claim A.

(b) As a particular example, the mechanical system depicted in Figure 2 below, wherein a mass is subject to a friction force of Coulomb type on a rough surface of length $2L$ (where $L > 0$) and is friction free off the surface, may be represented by a second-order autonomous differential inclusion of the form (18).

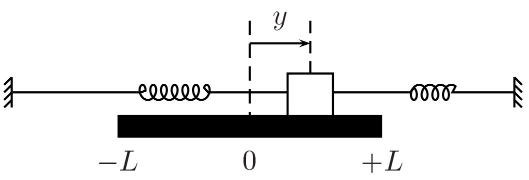

Figure 2

In this specific example, the function k (continuous with property (10)) corresponds to the spring force and is assumed to be such that $k^{-1}(0) = \{0\}$. The (upper semicontinuous) set-valued map C, which models the Coulomb friction effects, is given by

$$C(p, v) := \begin{cases} \{-\mathrm{sgn}(v)\}, & \text{if } |p| < L, \quad v \neq 0, \\ [-1, 1], & \text{if } |p| \leq L, \quad v = 0, \\ [-1, 0], & \text{if } |p| = L, \quad v > 0, \\ [0, 1], & \text{if } |p| = L, \quad v < 0, \\ \{0\}, & \text{if } |p| > L, \quad v \in \mathbb{R}. \end{cases} \tag{21}$$

Claim B. For each $x^0 = (p^0, v^0) \in \mathbb{R}^2$, every maximal solution $x = (y, \dot{y})$ of (19) (with F and C given by (20) and (21) is bounded, global and approaches the set $\big([-L, L] \cap k^{-1}([-1, 1])\big) \times \{0\}$.

To prove the claim, we first note that in this case the function C_* (defined in (17)) is given by

$$C_*(p, v) = \begin{cases} -|v|, & \text{if } |p| < L, \\ 0 & \text{if } |p| \geq L. \end{cases}$$

Therefore, $C_*^{-1}(0) = S_1 \cup S_2 \cup S_3$, where

$$S_1 := [-L, L] \times \{0\}, \quad S_2 := \{(p, v) \in \mathbb{R}^2 : |p| = L, \quad v \neq 0\}, \quad S_3 := \{(p, v) \in \mathbb{R}^2 : |p| > L\}.$$

By Claim A, for each $x^0 = (p^0, v^0) \in \mathbb{R}^2$, every maximal solution $x = (y, \dot{y})$ of (19) is bounded, global and approaches the largest subset $E \subset S_1 \cup S_2 \cup S_3$ that is weakly invariant with respect to (19). To conclude Claim B, it suffices to show that $E \subset \big([-L, L] \cap k^{-1}([-1, 1])\big) \times \{0\}$. To this end, we first show that

$$E \cap (S_2 \cup S_3) = \emptyset\,. \tag{22}$$

By (18), the following observations are immediate:

(i) if $(p^0, v^0) \in S_3$, then $(y(t), \dot{y}(t)) \in S_3$ for all $t > 0$ sufficiently small;

(ii) if $(p^0, v^0) \in S_2$ and $p^0 v^0 > 0$, then $(y(t), \dot{y}(t)) \in S_3$ for all $t > 0$ sufficiently small;

(iii) if $(p^0, v^0) \in S_2$ and $p^0 v^0 < 0$, then $(y(t), \dot{y}(t)) \notin S_1 \cup S_2 \cup S_3$ for all $t > 0$ sufficiently small.

Seeking a contradiction, suppose that (22) does not hold. Then there exists $(p,v) \in E \cap \big(S_2 \cup S_3\big)$ and a global bounded solution $(z,\dot z)$ such that $(z(0),\dot z(0)) = (p,v)$ and $(z(t),\dot z(t)) \in E \subset S_1 \cup S_2 \cup S_3$ for all $t \in \mathbb{R}_+$. If $(p,v) \in S_2$, then $pv > 0$ as, otherwise, observation (iii) leads to the contradiction: $(z(t),\dot z(t)) \notin S_1 \cup S_2 \cup S_3$ for all $t > 0$ sufficiently small. Therefore, by observations (i) and (ii), $(z(t),\dot z(t)) \in S_3$ for all $t > 0$ sufficiently small. We consider the following two alternatives

$$(\alpha) \quad (z(t),\dot z(t)) \in S_3 \;\; \forall\, t > 0\,, \qquad\qquad (\beta) \quad (z(t_0),\dot z(t_0)) \notin S_3 \;\text{ for some } t_0 > 0\,,$$

and show that they both lead to contradictions. Firstly, suppose that alternative (α) holds. Then $|z(t)| > L > 0$ for all $t > 0$. Moreover, z is bounded and so, since k and z are continuous and $k^{-1}(0) = \{0\}$, we may conlude that there exists $\varepsilon > 0$ such that $k(z(t)) \geq \varepsilon$ for all $t \in (0,\infty)$ or $k(z(t)) \leq -\varepsilon$ for all $t \in (0,\infty)$. On noting that $\ddot z(t) = -k(z(t))$ for all $t \in (0,\infty)$, we obtain a contradiction to the boundedness of $\dot z$. Secondly, suppose that alternative (β) holds. Define $\tau := \inf\{t > 0 : (z(t),\dot z(t)) \notin S_3\} > 0$. Then $|z(\tau)| = L$ and

$$z(\tau)p > 0\,. \tag{23}$$

Since $\ddot z(t) = -k(z(t))$ for all $t \in (0,\tau)$, a straightforward calculation shows that $V(z(t),\dot z(t))$ is constant on the interval $t \in [0,\tau]$ and thus

$$\int_0^{z(t)} k(s)\mathrm{d}s + \big(\dot z(t)\big)^2/2 = \int_0^p k(s)\mathrm{d}s + v^2/2 \quad \forall\, t \in [0,\tau]. \tag{24}$$

If $(p,v) \in S_2$, then $z(\tau) = p$ and $v \neq 0$, and so, by (24), $\dot z(\tau) \neq 0$. If $(p,v) \in S_3$, then $|p| > |z(\tau)|$. Combining this with (23) and the inequality $k(\xi)\xi > 0$ for all real $\xi \neq 0$ (the latter being a consequence of (10), the continuity of k and the fact that $k^{-1}(0) = \{0\}$), it follows again from (24) that $\dot z(\tau) \neq 0$. Consequently, $(z(\tau),\dot z(\tau)) \in S_2$. Furthermore, it is clear that $z(\tau)\dot z(\tau) < 0$, and therefore, by observation (iii), there exists $\delta > 0$ such that $(z(t),\dot z(t)) \notin S_1 \cup S_2 \cup S_3$ for all $t \in (\tau,\tau+\delta)$. This contradicts the fact that $(z(t),\dot z(t)) \in E \subset S_1 \cup S_2 \cup S_3$ for all $t \in \mathbb{R}_+$. We may now conclude that (22) holds.

By (22), we have $E \subset S_1$ and so, if $(p,v) \in E$, then $|p| \leq L$ and $v = 0$. By weak invariance of E, there exists a maximal solution $(z,\dot z)$ of (19) such that $(z(0),\dot z(0)) = (p,0)$ and which never leaves E. Consequently, $(z(t),\dot z(t)) \equiv (p,0)$ and so, by (18), $k(p) \in C(p,0) = [-1,1]$, whence $p \in k^{-1}([-1,1])$. Therefore, $E \subset \big([-L,L] \cap k^{-1}([-1,1])\big) \times \{0\}$. Now x approaches E, and so, *a fortiori*, x approaches the set $\big([-L,L] \cap k^{-1}([-1,1])\big) \times \{0\}$.

7 Semiflows on locally compact metric spaces

In this section we highlight some ramifications of the ideas from Sections 4 and 5 in a more general setting wherein $\mathbb{R}^N$ is replaced by a locally compact metric space. Henceforth, X is a metric space with metric ϱ. We say that a subset $A \subset X$ is relatively compact if $\mathrm{cl}(A)$ is compact. Recall that X is said to be locally compact if, for every $\xi \in X$, there exists a compact neighbourhood of ξ. The definition of an ω-limit set for $\mathbb{R}^N$-valued functions given in Section 1 extends, *mutatis mutandis*, to X-valued functions.

Lemma 7.1. *Let $x : \mathbb{R}_+ \to X$ be a function and let $\Omega(x)$ denote the ω-limit set of x. Then the following statements hold.*

(a) *$\Omega(x)$ is closed.*

(b) *If x is continuous and $x(\mathbb{R}_+)$ is relatively compact, then $\Omega(x)$ is non-empty, compact, connected, is approached by x and is the smallest closed set approached by x.*

(c) *Assume that X is locally compact. If x is continuous and $\Omega(x)$ is non-empty and relatively compact, then $x(\mathbb{R}_+)$ is relatively compact and x approaches $\Omega(x)$.*

Proof. The proofs of statements (a) and (b) are the same as in the $\mathbb{R}^N$-valued case. This comment also applies to statement (c), once we realize that, by local compactness of X and relative compactness of $\Omega(x)$, there exists $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon(\Omega(x))$ is relatively compact (see, for example, [7], p. 62). ■

Suitably modified (for example, replacing boundedness assumptions by relative compactness assumptions), Theorems 4.4 and 4.5 remain true in the more general context of a metric space X, wherein X replaces $\mathbb{R}^N$. We restrict ourselves here to a detailed formulation of the generalization of Theorem 4.5.

Theorem 7.2. *Let $G \subset X$ be non-empty and closed. Let $g : G \to \mathbb{R}$ be such that $g^{-1}(0)$ is closed and*

$$\inf_{\xi \in K} |g(\xi)| > 0 \quad \text{for every closed set } K \subset G \text{ with } K \cap g^{-1}(0) = \emptyset. \tag{25}$$

Furthermore, let $x : \mathbb{R}_+ \to X$ be continuous with $x(\mathbb{R}_+) \subset G$. If (i) *$x$ is uniformly continuous on $x^{-1}(\mathbb{B}_\varepsilon(g^{-1}(0)))$ for some $\varepsilon > 0$ and* (ii) *$g \circ x$ is weakly meagre, then*

(a) *$g^{-1}(0) \neq \emptyset$, x approaches $g^{-1}(0)$ and $\Omega(x) \subset g^{-1}(0)$.*

(b) *If $g^{-1}(0)$ is relatively compact, then $x(\mathbb{R}_+)$ is relatively compact, x approaches $g^{-1}(0)$ and $\emptyset \neq \Omega(x) \subset g^{-1}(0)$.*

(c) *If $g^{-1}(0)$ is relatively compact and totally disconnected, then $\Omega(x) = \{x^\infty\} \subset g^{-1}(0)$ and hence, $\lim_{t\to\infty} x(t) = x^\infty$.*

Proof. An inspection of its proof shows that Lemma 4.3 remains true when $\mathbb{R}^N$ is replaced by X. Statement (a) can be proved by the same arguments as used in establishing statement (a) of Theorem 4.5. To prove statement (b), we note that, by statement (a), x approaches $g^{-1}(0)$. Using continuity of x and relative compactness of $g^{-1}(0)$, it is readily shown that $x(\mathbb{R}_+)$ is relatively compact. It now follows from Lemma 7.1 that $\Omega(x) \neq \emptyset$. To prove statement (c), we observe that, by statement (b), $x(\mathbb{R}_+)$ is relatively compact. Combining this with the continuity of x, Lemma 7.1 shows that $\Omega(x)$ is connected, yielding statement (c). ■

A map $\varphi : \mathbb{R}_+ \times X \to X$ is said to be a semiflow on X if φ is continuous, $\varphi(0, x^0) = x^0$ for all $x^0 \in X$ and $\varphi(s + t, x^0) = \varphi(s, \varphi(t, x^0))$ for all $s, t \in \mathbb{R}_+$ and all $x^0 \in X$.

Lemma 7.3. *Let $\varphi : \mathbb{R}_+ \times X \to X$ be a semiflow on X and let $K \subset X$ be relatively compact. Then, for every $x^0 \in X$, the function $\varphi(\cdot, x^0)$ is uniformly continuous on $\{t \in \mathbb{R}_+ : \varphi(t, x^0) \in K\}$.*

Proof. Let $K \subset X$ be relatively compact and let $x^0 \in X$. Define $T := \{t \in \mathbb{R}_+ : \varphi(t, x^0) \in K\}$. Seeking a contradiction, suppose that $\varphi(\cdot, x^0)$ is not uniformly continuous on T. Then there exist $\varepsilon > 0$ and sequences (s_n) and (t_n) in T such that $t_n > s_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty}(t_n - s_n) = 0$ and

$$\varrho(\varphi(t_n, x^0), \varphi(s_n, x^0)) \geq \varepsilon \quad \forall\, n \in \mathbb{N}. \tag{26}$$

Since (s_n) is a sequence in T, $\varphi(s_n, x^0) \in K$ for all $n \in \mathbb{N}$, and therefore, by relative compactness of K, there exists a subsequence (s_{n_k}) of (s_n) and $z \in \mathrm{cl}(K)$ such that $\lim_{k\to\infty} \varphi(s_{n_k}, x^0) = z$. Consequently, using the semiflow properties and continuity of φ

$$\begin{aligned}\lim_{k\to\infty} \varrho(\varphi(t_{n_k}, x^0), \varphi(s_{n_k}, x^0)) &= \lim_{k\to\infty} \varrho(\varphi(t_{n_k} - s_{n_k}, \varphi(s_{n_k}, x^0)), \varphi(s_{n_k}, x^0)) \\ &= \varrho(\varphi(0, z), z) = \varrho(z, z) = 0\,,\end{aligned}$$

contradicting (26). ■

Let $A \subset X$ be non-empty and let $\varphi : \mathbb{R}_+ \times X \to X$ be a semiflow. Recall that A is said to be invariant with respect to φ if, for every $\xi \in A$, there exists a continuous function $\psi : \mathbb{R} \to X$ such that (i) $\psi(0) = \xi$, (ii) $\varphi(t, \psi(\tau)) = \psi(t+\tau)$ for all $(t, \tau) \in \mathbb{R}_+ \times \mathbb{R}$ and (iii) $\psi(\mathbb{R}) \subset A$. In the literature, a continuous function $\psi : \mathbb{R} \to X$ satisfying conditions (i) and (ii) is sometimes referred to as a globally defined solution (or motion) of φ through ξ. It is not difficult to show that A is invariant with respect to φ if, and only if, $\varphi(t, A) = A$ for all $t \in \mathbb{R}_+$ (see, for example, [21], p. 16). The following result provides a counterpart of Theorem 6.6 in the context of semiflows on locally compact metric spaces.

Theorem 7.4. *Assume that X is locally compact and let $\varphi : \mathbb{R}_+ \times X \to X$ be a semiflow on X. Furthermore, let $G \subset X$ be non-empty and closed, and let $g : G \to X$ be such that $g^{-1}(0)$ is compact and (25) holds. If $x^0 \in X$ is such that $\varphi(\mathbb{R}_+, x^0) \subset G$ and $g(\varphi(\cdot, x^0))$ is weakly meagre, then the following statements hold.*

(a) *$\varphi(\mathbb{R}_+, x^0)$ is relatively compact and $\varphi(\cdot, x^0)$ approaches the largest subset of $g^{-1}(0)$ which is invariant with respect to φ.*

(b) *If $g^{-1}(0)$ is totally disconnected, then there exists $x^\infty \in g^{-1}(0)$ such that $\varphi(t, x^0) \to x^\infty$ as $t \to \infty$.*

Proof. By local compactness of X and compactness of $g^{-1}(0)$ we have that $\mathbb{B}_\varepsilon(g^{-1}(0))$ is relatively compact for all sufficiently small $\varepsilon > 0$ (see, for example, [7], p. 62). Therefore, by Lemma 7.3, $\varphi(\cdot\, x^0)$ is uniformly continuous on the set $\{t \in \mathbb{R}_+ : \varphi(t, x^0) \in \mathbb{B}_\varepsilon(g^{-1}(0))\}$, provided that $\varepsilon > 0$ is sufficiently small. The claim now follows from an application of Theorem 7.2 combined with the fact that the ω-limit set of $\varphi(\cdot, x^0)$ is invariant with respect to φ (see, for example, [1], p. 226 or [21], p. 19). ■

Let $A \subset X$ be non-empty. Extending the definition of the distance function in $\mathbb{R}^N$ to X, we define $d_A : X \to \mathbb{R}_+$ by $d_A(v) = \inf\{\varrho(v, a) : a \in A\}$. The following corollary is an immediate consequence of Theorem 7.4.

Corollary 7.5. *Assume that X is locally compact, let $\varphi : \mathbb{R}_+ \times X \to X$ be a semiflow on X, let $x^0 \in X$ and let $K \subset X$ be compact. If $d_K(\varphi(\cdot, x^0))$ is weakly meagre, then $\varphi(\cdot, x^0)$ approaches K. In particular, if $d_K(\varphi(\cdot, x^0)) \in L^p$ for some $p \in (0, \infty)$, then $\varphi(\cdot, x^0)$ approaches K.*

By standard results in topology, a finite-dimensional (topological) manifold (that is, a Hausdorff space which has a countable basis and which is locally homeomorphic to $\mathbb{R}^N$ for some $N \in \mathbb{N}$) is metrizable and hence is a locally compact metric space (see, for example, [8], Corollary 9.2 on p. 195 and Theorem 6.4 on p. 238). Consequently, Theorem 7.4 and Corollary 7.5 apply to semiflows on finite-dimensional manifolds.

8 Appendix

Example 8.1. *A continuous non-meagre function that is both weakly meagre and Riemann integrable.*

Consider the continuous function $y : \mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \sum_{n\in\mathbb{N}} y_n(t)$, where, for each $n \in \mathbb{N}$, $y_n : \mathbb{R}_+ \to [-1, 1]$ is the piecewise linear continuous function, compactly supported on $\mathrm{supp}(y_n) = [n, n+1/n]$, with graph as shown in Figure 3 below (wherein the corners occur at $t = n + k/(5n)$, $k = 0, ..., 5$).

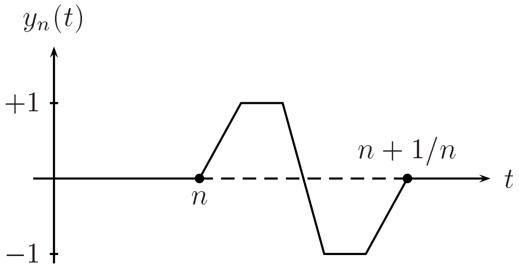

Figure 3

To show Riemann integrability of y, note that, for all $t \in [1, \infty)$, we have that

$$\left| \int_0^t y(s)\mathrm{d}s \right| \le \int_{\lfloor t \rfloor}^t y_{\lfloor t \rfloor}(s)\mathrm{d}s < \frac{1}{2\lfloor t \rfloor},$$

where $\lfloor t \rfloor := \max\{n \in \mathbb{N} : \; n \le t\}$ is the integer part of $t \ge 1$, and so $\lim_{t\to\infty} \int_0^t y(s)\mathrm{d}s = 0$. Therefore, y is Riemann integrable. Invoking part (c) of Proposition 4.2, we may conclude tha y is weakly meagre. Alternatively, weak meagreness of y can be established by a straightforward verification of the defining property of weak meagreness. However, y is not meagre, as the following argument shows. Noting that $|y_n(t)| = 1$ for all $t \in [n + 1/(5n), n + 2/(5n)] \cup [n + 3/(5n), n + 4/(5n)]$, it follows that

$$\mu(\{t \in \mathbb{R}_+ : |y(t)| \ge 1\}) = \sum_{n\in\mathbb{N}} \frac{2}{5n} = \infty$$

and so y is not meagre.

Example 8.2. *A continuous meagre function y such that, for all $\tau > 0$, $\int_t^{t+\tau} y(s)\mathrm{d}s$ does not converge to 0 as $t \to \infty$.*

Consider the continuous non-negative function $y : \mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \sum_{n\in\mathbb{N}} y_n(t)$, where, for each $n \in \mathbb{N}$, $y_n : \mathbb{R}_+ \to [0, n^2]$ is the piecewise linear continuous function, compactly supported on $\mathrm{supp}(y_n) = [n, n + 1/n^2]$, with graph as shown in Figure 4 below (wherein the corners occur at $t = n + k/(2n^2)$, $k = 0, 1, 2$).

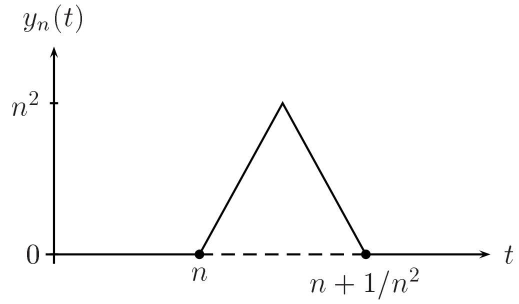

Figure 4

For any $\lambda > 0$ we have that

$$\mu(\{t \in \mathbb{R}_+ : |y(t)| \geq \lambda\}) \leq \sum_{n=1}^{\infty} 1/n^2 < \infty\,,$$

showing that y is meagre (and so, *a fortiori*, y is weakly meagre). However, for any $\tau > 0$,

$$\int_n^{n+\tau} y(t)\mathrm{d}t \geq \int_n^{n+1/n^2} y_n(t)\mathrm{d}t = \frac{1}{2} \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq 1/\sqrt{\tau}\,,$$

and so $\int_t^{t+\tau} y(s)\mathrm{d}s$ does not converge to 0 as $t \to \infty$.

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