SPECTRA OF CRITICAL EXPONENTS IN NONLINEAR HEAT EQUATIONS WITH ABSORPTION

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Abstract. It has been known from the beginning of the 1980's that the L^1-solutions of the classical porous medium equation with absorption $u_t = \Delta u^m - u^p$ in $\mathbb{R}^N \times \mathbb{R}_+$ change their large-time behaviour at the critical absorption exponent $p_0 = m + 2/N$. We show that, actually, there exists an infinite sequence $\{p_k,\, k \ge 0\}$ of critical exponents generating a countable subset of different non self-similar asymptotic patterns. The results are extended to the fully nonlinear dual porous medium equation with absorption.

1. Introduction

We describe new types of asymptotic behaviour of global compactly supported solutions of the porous medium equation (PME) with absorption

$$u_t = \Delta(|u|^{m-1}u) - |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad \text{with exponents } m > 1,\ p > 1, \tag{1.1}$$

which was extensively studied in the theory of nonlinear degenerate parabolic PDEs from the 1970's. The PME and related degenerate parabolic PDEs were the crucial models in the theory of free boundaries, [17]. Our aim is to introduce a sequence of the *critical* exponents $\{p = p_k,\, k \ge 0\}$ for (1.1) corresponding to special non-scaling invariant asymptotic behaviour of solutions as $t \to \infty$. Such critical asymptotic phenomena are not exceptional and are a common feature for other equations with power nonlinearities. As the second example, we extend the results to the fully nonlinear dual PME with absorption

$$u_t = |\Delta u|^{m-1}\Delta u - |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (m > 1,\ p > 1). \tag{1.2}$$

We consider the Cauchy problem with bounded, integrable, compactly supported initial data $u_0 \in X = L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. It has been known for a long period that such nonlinear heat equations admit unique global in time solutions vanishing as $t \to \infty$ with rates depending on the exponents m, p and the space dimension N. We refer to Kalashnikov's survey [25], DiBenedetto's book [14] and [26], [28], [6] for fully nonlinear equations.

Concerning the precise asymptotic behaviour of global solutions, a complete classification was achieved in the 1980-90's for nonnegative solutions of the PME with absorption (1.1). It was proved that $p_0 = m + 2/N$ is the critical exponent in the sense that (see key references in [10], [21], [27] and to Chapt. 2 in [34]):

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(I) In the subcritical range $p \in (1, p_0)$ the asymptotic behaviour of $u(x,t) \geq 0$ as $t \to \infty$ is governed by the unique *very singular* self-similar solution;

(II) In the supercritical range $p > p_0$ the solution converges as $t \to \infty$ to the self-similar Zel'dovich - Kompaneetz - Barenblatt (ZKB) solution of the PME $u_t = \Delta u^m$; and

(III) In the critical case $p = p_0$ the asymptotic behaviour is given by a unique ZKB solution with extra $\ln t$ scaling in u and x (see [21] and earlier references therein).

The main goal of the paper is to show that equations (1.1) and (1.2) admit a *monotone sequence of critical exponents* $\{p_k\}$, of which p_0 is the first. These critical exponents will be shown to generate special asymptotic patterns in the Cauchy problem with initial data u_0 that changes sign. Therefore, the critical behaviour at $p = p_k$ for any $k \geq 1$ cannot be observed in the class of nonnegative solutions. It is worth mentioning that critical phenomena are also of crucial importance for reaction-diffusion equations with source terms $+|u|^{p-1}u$, where p_0, known as the *Fujita* critical exponent, stays the same and affects various blow-up and stability properties; see references to Chapt. 4 in [34].

Concerning the free-boundaries, our patterns for (1.1) at $p = p_k$ exhibit the following spectrum of asymptotic behaviour as $t \to \infty$:

$$|x_k(t)| = Ct^{1/2}(t \ln t)^{-(m-1)\delta_k/2}(1 + o(1)) \quad \text{for } \; k = 0, 1, 2, ..., \tag{1.3}$$

with a positive sequence of exponents $\{\delta_k = \mu_0 - \lambda_k\}$, where $\{\lambda_k\}$ are the corresponding eigenvalues of a nonlinear elliptic operator.

Such asymptotic phenomena also occur for the semilinear heat equation $(m = 1)$

$$u_t = \Delta u - |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad p > 1 \tag{1.4}$$

(obviously, the free-boundary phenomena are not exhibited). In this case, the sequence of critical exponents is calculated explicitly

$$p_k = 1 + 2/(k + N), \quad k = 0, 1, 2, \ldots, \tag{1.5}$$

and is connected with the discrete spectrum of the linear differential operator (associated with the heat equation) in a weighted L^2-space

$$\mathbf{B} = \Delta + \frac{1}{2} y \cdot \nabla + \frac{N}{2} I. \tag{1.6}$$

Similar critical effects are observed for higher-order semilinear equations

$$u_t = -(-\Delta)^l u - |u|^{p-1}u \quad \text{with any integer } \; l \geq 2,$$

where $p_0 = 1 + 2l/N$, see the last example in [15]. For nonlinear second-order diffusion operators, such a connection with spectral theory of linear operators is not available, though we show how a discrete subset of asymptotic patterns ("nonlinear eigenfunctions") of purely diffusive equations generate the corresponding sequence of critical exponents in reaction-absorption equations (1.1) and (1.2). It is convenient to begin with the semilinear equation (1.4) and well established spectral properties of the linear operator (1.6). We next extend these ideas to quasilinear and fully nonlinear operators using different mathematical tools. We expect that similar critical phenomena occur for a class of quasilinear higher-order parabolic equations including the *thin film equation with absorption*

$$u_t = -(|u|^n u_{xxx})_x - |u|^{p-1}u$$

(the first critical exponent is $p_0 = n+5$ for $n \in (0,3)$), though the mathematical analysis becomes much more involved, as happens with several asymptotic results for the thin film equation, generating, as a rule, non-symmetric and non-potential rescaled operators.

2. Preliminaries: discrete spectrum of a linear self-adjoint operator

For future comparison with properties of nonlinear parabolic equations, we briefly describe well known facts concerning the linear diffusion operator with $m = 1$. Consider the heat equation

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x,0) = u_0(x) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N). \tag{2.1}$$

Denoting by

$$b(x,t) = t^{-N/2} f(y), \quad y = x/t^{1/2}, \quad f(y) = (4\pi)^{-N/2} e^{-|y|^2/4}, \tag{2.2}$$

the fundamental solution of the operator $\partial/\partial t - \Delta$, we have

$$u(x,t) = b(t) * u_0 = t^{-N/2} \int_{\mathbb{R}^N} f((x-z)t^{-1/2}) u_0(z)\, \mathrm{d}z. \tag{2.3}$$

Here $f(y)$ is the unique radial solution of the elliptic equation $\mathbf{B} f = 0$ satisfying $\int f = 1$, with the operator $\mathbf{B}$, as in (1.6), admitting the symmetric representation $\mathbf{B} \equiv \frac{1}{\rho} \nabla \cdot (\rho \nabla) + \frac{N}{2} I$ with weight $\rho = e^{|y|^2/4}$. It is well known [8] that $\mathbf{B} : H^2_\rho(\mathbb{R}^N) \to L^2_\rho(\mathbb{R}^N)$ is a bounded self-adjoint operator with compact resolvent and discrete spectrum $\sigma(\mathbf{B})$. We perform some auxiliary computations concerning this operator.

In order to classify the asymptotic behaviour of solutions as $t \to \infty$, as in the fundamental solution (2.2), we introduce the rescaled variables

$$u(x,t) = t^{-N/2} w(y,\tau), \quad y = x/t^{1/2}, \quad \tau = \ln t : \mathbb{R}_+ \to \mathbb{R}. \tag{2.4}$$

Then the rescaled solution w satisfies the evolution equation

$$w_\tau = \mathbf{B} w, \tag{2.5}$$

where $w(y,\tau)$ is a solution of the Cauchy problem for (2.5) in $\mathbb{R}^N \times \mathbb{R}_+$ with initial data given at $\tau = 0$ (hence, $t = 1$)

$$w_0(y) = u(y,1) \equiv b(1) * u_0 = f * u_0. \tag{2.6}$$

The linear operator $\partial/\partial\tau - \mathbf{B}$ is the rescaled version of the parabolic operator $\partial/\partial t - \Delta$ and hence the corresponding semigroup $e^{\mathbf{B}\tau}$ is obtained by rescaling from (2.3):

$$w(y,\tau) = \int_{\mathbb{R}^N} f(y - z e^{-\tau/2}) u_0(z)\, \mathrm{d}z \equiv e^{\mathbf{B}\tau} w_0, \quad \tau \geq 0. \tag{2.7}$$

Using the Taylor expansion of the rescaled analytic kernel f

$$f(y - z e^{-\tau/2}) = \textstyle\sum_{(\beta)} e^{-|\beta|\tau/2} \dfrac{(-1)^{|\beta|}}{\beta!} D^\beta f(y) z^\beta \equiv \textstyle\sum_{(\beta)} e^{-|\beta|\tau/2} \tilde{c}_\beta \psi_\beta(y) z^\beta \tag{2.8}$$

(here $z^\beta \equiv z_1^{\beta_1}...z_N^{\beta_N}$ and $\tilde c_\beta$ are normalisation constants, see below), which converges uniformly for z in compact subsets of $\mathbb{R}^N$, we obtain the following representation:

$$w(y,\tau) = \sum e^{-\lambda_\beta \tau} M_\beta(u_0)\psi_\beta(y), \tag{2.9}$$

where $\lambda_\beta = -|\beta|/2$ and $\psi_\beta(y)$ are the eigenvalues and eigenfunctions of $\mathbf{B}$,

$$\mathbf{B}\psi = \lambda\psi \quad \text{in } \mathbb{R}^N, \quad \psi \in H^2_\rho(\mathbb{R}^N), \;\; \psi \neq 0, \tag{2.10}$$

and $M_\beta(u_0) = \tilde c_\beta \int z^\beta u_0(z)\,\mathrm{d}z$ are the corresponding momenta of the initial datum w_0 (recall the relation (2.6) between w_0 and u_0). We next derive an equivalent representation of the semigroup by using another rescaling

$$u = (1+t)^{-N/2} w, \quad y = x/(1+t)^{1/2}, \quad \tau = \ln(1+t) : \mathbb{R}_+ \to \mathbb{R}_+.$$

Then $w(y,\tau)$ solves the Cauchy problem for equation (2.5) with initial data $w_0(y) \equiv u_0(y)$. Rescaling the convolution (2.3) yields

$$w(y,\tau) = (1-e^{-\tau})^{-N/2} \int_{\mathbb{R}^N} f((y - z e^{-\tau/2})(1-e^{-\tau})^{-1/2}) w_0(z)\,\mathrm{d}z \equiv e^{\mathbf{B}\tau} w_0. \tag{2.11}$$

The explicit representation of the resolvent of $\mathbf{B}$ is then constructed by the classical descent method. Let $\lambda \in \mathbb{C}$, and consider the auxiliary equation

$$w_\tau = \mathbf{B}w - e^{\lambda\tau} g \quad \text{for } \tau > 0, \quad w(0) = 0,$$

where $g \in L^2_\rho(\mathbb{R}^N)$. Setting $w = e^{\lambda\tau} v$, we obtain the equation $v_\tau = (\mathbf{B} - \lambda I)v - g$, and hence $v(\tau) = -\int_0^\tau e^{(\mathbf{B}-\lambda I)(\tau - s)} g\,\mathrm{d}s$. Setting $\tau - s = \eta$ and passing to the limit $\tau \to \infty$ yields that the limit $v(\infty) = -\int_0^\infty e^{(\mathbf{B}-\lambda I)\eta} g\,\mathrm{d}\eta \equiv (\mathbf{B} - \lambda I)^{-1} g$ exists provided that the integral converges. Using the semigroup representation (2.11) and performing the change of variable $e^{-\eta} = z \in (0,1)$ yields the integral operator

$$(\mathbf{B} - \lambda I)^{-1} g = \int_{\mathbb{R}^N} K(y,\zeta) g(\zeta)\,\mathrm{d}\zeta, \quad \text{with the kernel} \tag{2.12}$$

$$K(y,\zeta) = -\int_0^1 z^{\lambda - 1}(1-z)^{-N/2} f((y - \zeta z^{1/2})(1-z)^{-1/2})\,\mathrm{d}z. \tag{2.13}$$

This is the integral representation of the resolvent of $\mathbf{B}$ which is known to be a compact operator in $L^2_\rho(\mathbb{R}^N)$ for all $\lambda \in \mathbb{C} \setminus \sigma(\mathbf{B})$. Let us summarise the main spectral properties of $\mathbf{B}$, see [8].

Lemma 2.1. *The spectrum of* $\mathbf{B}$ *in* $L^2_\rho(\mathbb{R}^N)$ *consists of real eigenvalues*

$$\sigma(\mathbf{B}) = \{\lambda_\beta = -|\beta|/2, \;\; |\beta| = 0,1,2,...\}. \tag{2.14}$$

The eigenvalues λ_β *have finite multiplicity with eigenfunctions*

$$\psi_\beta(y) = c_\beta D^\beta f(y) \equiv c_\beta H_\beta(y) f(y), \quad c_\beta = (2^{|\beta|}\beta!)^{-1/2}, \tag{2.15}$$

where $H_\beta(y)$ *are Hermite polynomials. The orthonormal subset of eigenfunctions* $\Phi = \{\psi_\beta\}$ *is complete in* $L^2_\rho(\mathbb{R}^N)$.

The eigenfunction expansion of $w(y,\tau)$, the solution of equation (2.5) with initial data $w_0 \in D(\mathbf{B}) = H^2_\rho(\mathbb{R}^N)$, takes the form

$$w(y,\tau) = \sum a_\beta e^{\lambda_\beta \tau} \psi_\beta(y), \quad a_\beta = \langle u_0, \psi_\beta \rangle_\rho, \tag{2.16}$$

where $\langle \cdot, \cdot \rangle_\rho$ denotes the inner product in $L^2_\rho(\mathbb{R}^N)$. Since the subset of eigenfunctions is complete and closed, we have that for any initial data $w_0 \in H^2_\rho(\mathbb{R}^N)$, $w_0 \neq 0$, there exists a finite integer $k = k(w_0) \geq 0$ such that as $\tau \to \infty$,

$$w(y,\tau) = e^{-k\tau/2}[\psi_k(y) + o(1)], \tag{2.17}$$

where ψ_k is an eigenfunction of $\mathbf{B}$ with eigenvalue $-k/2$, i.e., $\psi_k = \sum_{|\beta|=k} b_\beta \psi_\beta$, where $\sum b^2_\beta \neq 0$. Known spectral properties of $\mathbf{B}$ make it possible to give a complete description of asymptotic patterns which can occur in the linear evolution equation (2.5). For the original heat equation (2.1) with initial data $u_0 \in H^2_\rho(\mathbb{R}^N)$ this gives a discrete subset of asymptotic patterns

$$u(x,t) = t^{\lambda_k - N/2} \psi_k(y)(1 + o(1)), \;\; y = x/t^{1/2}, \quad \lambda_k = -k/2, \;\; k = 0, 1, 2, \ldots . \tag{2.18}$$

Since $\{\psi_\beta\}$ is complete in $L^2_\rho(\mathbb{R}^N)$, this subset of patterns is *evolutionary complete* in the sense that any nontrivial solution $u(\cdot,t) \in H^2_\rho(\mathbb{R}^N)$ has, for $t \gg 1$, the asymptotic behaviour (2.18) with a *finite* $k \geq 0$ that depends on initial data.

3. **Critical exponents in the semilinear heat equation with absorption**

3.1. **Rescaled equations and a sequence of critical exponents.** We show how the discrete spectrum of the linear operator $\mathbf{B}$ is associated with critical phenomena for the semilinear equation (1.4). Bearing in mind the typical asymptotic behaviour in the rescaled heat equation (see (2.17), where $k = 0, 1, ...$) we perform in (1.4) the rescaling

$$u(x,t) = t^{-(k+N)/2} v(y,\tau), \quad y = x/t^{1/2}, \quad \tau = \ln t : (1,\infty) \to \mathbb{R}_+. \tag{3.1}$$

The rescaled solution $v(y,\tau)$ satisfies the perturbed equation

$$v_\tau = (\mathbf{B} + \frac{k}{2} I)v - e^{-\gamma_k \tau} g(v), \quad g(v) \equiv |v|^{p-1}v, \;\; \gamma_k = \frac{1}{2}(p-1)(k+N) - 1. \tag{3.2}$$

It follows that

$$\gamma_k = 0 \quad \Longrightarrow \quad p = p_k = 1 + 2/(k+N). \tag{3.3}$$

In these critical cases we arrive at the autonomous parabolic equation

$$v_\tau = \mathbf{B}_k v - g(v), \;\; \text{where} \;\; \mathbf{B}_k = \mathbf{B} + \frac{k}{2} I \;\; \text{and} \;\; \sigma(\mathbf{B}_k) = \{\lambda^{(k)}_\beta = \frac{1}{2}(k - |\beta|)\}. \tag{3.4}$$

We consider sufficiently small initial data $v_0 \in H^2_\rho(\mathbb{R}^N)$ with exponential decay at infinity, $|v_0(y)| \leq ce^{-a|y|^2}$ in $\mathbb{R}^N$, with c, a positive. Let $O^+(v) = \{v(\tau), \, \tau > 0\}$ be the corresponding global forward orbit given by the rescaled equation (3.4). The linear operator $\mathbf{B}$ with the discrete spectrum and a single simple eigenvalue λ_0 with $\operatorname{Re} \lambda_0 = 0$, is sectorial in $L^2_\rho(\mathbb{R}^N)$ and generates a strong continuous analytic semigroup $\{e^{\mathbf{B}\tau}, \tau \geq 0\}$. The asymptotic behaviour with a finite-dimensional centre manifold is covered by the invariant manifold theory in [30], Chapt. 9 using interpolation spaces $E_i = D_{\mathbf{B}}(\theta+i, \infty)$ for $i = 0, 1$, $\theta \in (0, 1)$. The spectral set $\sigma_+(\mathbf{B}_k) = \{\lambda \in \sigma(\mathbf{B}_k) : \ \mathrm{Re}\, \lambda \geq 0\}$ consists of a finite number of eigenvalues. Introducing the second spectral set $\sigma_-(\mathbf{B}_k) = \{\lambda \in \sigma(\mathbf{B}_k) : \ \mathrm{Re}\, \lambda < 0\}$, we have a positive gap

$$\omega_- = -\sup\{\mathrm{Re}\, \lambda : \ \lambda \in \sigma_-(\mathbf{B}_k)\} = 1/2 > 0. \tag{3.5}$$

The projection P associated with the spectral set $\sigma_+(\mathbf{B}_k)$ satisfies $P(E_0) \subset E_1$ and is finite-dimensional, i.e., the equation for $X(\tau) = Pv(\tau)$ is a finite-dimensional dynamical system (DS)

$$\dot{X} = (\mathbf{B}_k)_+ X - Pg(X+Y), \quad \tau \geq 0, \tag{3.6}$$

where $(\mathbf{B}_k)_+ = \mathbf{B}_k|_{P(E_0)}$ and $Y(\tau) = (I-P)v(\tau)$. Existence of a finite-dimensional centre manifold is well established for several kinds of the lower-order operators g, see conditions in [30], Sect. 9.2. By Theorem 9.2.2, [30] there exists a finite-dimensional invariant local centre manifold $W^c(0)$ of the origin, which is the graph of a Lipschitz continuous function $\gamma : P(E_0) \to (I-P)(E_1)$. Moreover, it follows from (3.5) that it is exponentially attractive provided that $g(v)$ is twice continuously differentiable ($p \geq 2$), see Proposition 9.2.3 therein.

By the parabolic regularity theory, $v(y,\tau)$ for $\tau \geq 0$ is sufficiently smooth. In view of completeness and orthonormality of eigenfunctions, we use the uniformly converging eigenfunction expansion of the solution,

$$v(y,\tau) = \sum a_\beta(\tau)\psi_\beta(y), \tag{3.7}$$

where the expansion coefficients satisfy the DS

$$\dot{a}_\beta = \lambda_\beta^{(k)} a_\beta - \langle g(v), \psi_\beta\rangle_\rho \quad \text{for any} \ \ \beta. \tag{3.8}$$

3.2. **Centre manifold behaviour and generating algebraic system.** We now look for a solution $v(\cdot,\tau)$ with the behaviour for $\tau \gg 1$ on the centre manifold that is tangent to the centre subspace of the linearised operator $\mathbf{B}_k = \mathbf{B} + \frac{k}{2} I$. Such a centre subspace asymptotic dominance assumes that in the eigenfunction expansion of $v(y,\tau)$ of the form (2.16), the leading term as $\tau \to \infty$ is given by

$$v(\tau) = \sum_{|\beta|=k} a_\beta(\tau)\psi_\beta + \dots, \tag{3.9}$$

where we omit higher-order terms. Then the expansion coefficients satisfy a perturbed finite-order DS

$$\dot{a}_\beta = -\langle g(\textstyle\sum a_\beta\psi_\beta), \psi_\beta\rangle_\rho + \dots, \quad |\beta| = k. \tag{3.10}$$

We are interested in looking for the asymptotic solutions of (3.4) of the form

$$v(\tau) = a_k(\tau)\varphi_k + o(a_k(\tau)) \quad \text{with an eigenfunction} \quad \varphi_k = \textstyle\sum_{|\gamma|=k} b_\gamma \psi_\gamma, \tag{3.11}$$

where the leading term consists of functions of the same order of decay as $\tau \to \infty$. Here $a_k(\tau)$ is a single unknown function to be determined together with the coefficients $\{b_\beta\}$ which are not arbitrary (unlike in the linear expansion (2.17)). Substituting (3.11) into the equation (3.4) and multiplying by ψ_β in $L^2_\rho(\mathbb{R}^N)$, we have that the coefficients $\{b_\beta\}$ satisfy the following *generating algebraic system* (GAS)

$$\langle g(\sum b_\gamma \psi_\gamma), \psi_\beta \rangle_\rho = b_\beta \quad \text{for all } |\beta| = k, \tag{3.12}$$

characterising a subset of centre manifold patterns of the form (3.11). Then $a_k(\tau)$ is determined from the DS

$$\dot{a}_k = -g(a_k)[1 + o(1)] \quad \text{for } \tau \gg 1. \tag{3.13}$$

The asymptotic ODE (3.13) admits decaying solutions with the behaviour

$$a_k(\tau) = \pm[(p-1)\tau]^{-1/(p-1)}(1 + o(1)) \to 0 \quad \text{as } \tau \to \infty. \tag{3.14}$$

In terms of the original (x, t, u)-variables, such a behaviour takes a form of a logarithmically perturbed linearised pattern as $t \to \infty$

$$u(x,t) = \pm C_k (t \ln t)^{-(k+N)/2}[\varphi_k(x/t^{1/2}) + o(1)] \quad \text{with } C_k = [2/(k+N)]^{-(k+N)/2}. \tag{3.15}$$

Let us return to the solvability of the GAS (3.12). In general, for arbitrary $k \geq 1$ and in sufficiently large dimensions $N > 1$, it is a sophisticated nonlinear algebraic system of ν_k equations with ν_k unknowns $\{\beta, |\beta| = k\}$ (ν_k being number of distinct multiindices of the length k). A complete description of a (countable) subset of possible distinct solutions is unknown. We consider important particular examples.

The GAS is easily solved for the first critical exponent corresponding to $k = 0$ where the centre subspace $E^c = \text{Span}\{f\}$ (where $f = \psi_0$) of the operator $\mathbf{B}_0 = \mathbf{B}$ is one dimensional. Then $\varphi_0 = b_0 f$, where the constant $b_0 \neq 0$ is obtained from the equation (3.12) having the form

$$b_0 = \langle g(b_0 f), f \rangle_\rho \equiv |b_0|^{p-1} b_0 \langle |f|^{p+1}, 1 \rangle_\rho \implies b_0 = \pm \langle |f|^{p+1}, 1 \rangle_\rho^{-1/(p-1)}. \tag{3.16}$$

This gives a unique stable asymptotic pattern on the centre manifold (3.15). Such a generic asymptotic behaviour has been known about for a long time, see [19], [22] and Chapt. 2 in [34]. In [9], [10] such asymptotic behaviour was established by using the perturbation theory of linear self-adjoint operators.

Consider a more delicate case $k > 0$ corresponding to higher-order critical exponents $p = p_k$, where $\mathbf{B}_k$ has nontrivial unstable subspace $E^u = \text{Span}\{\psi_\beta, |\beta| < k\}$ and hence the asymptotic behaviour on the centre manifold is not stable (though exists).

One-dimensional geometry. If $N = 1$, then for any $k = 1, 2, ...$, the centre subspace $E^c = \text{Span}\{\psi_k\}$ is one-dimensional and the GAS (3.12) always gives a suitable solution

$$\varphi_k = b_k \psi_k, \quad \text{with } b_k = \pm \langle |\psi_k|^{p+1}, 1 \rangle_\rho^{-1/(p-1)}. \tag{3.17}$$

Multi-dimensional geometry. For $N > 1$, we first restrict our attention to the radially symmetric case and fix a unique radial eigenfunction in (3.11), $\varphi_k = b_k \psi_k(r)$ with $r = |y|$. It exists for any even $k > 0$ and is an eigenfunction of the ordinary differential operator

$$\frac{1}{r^{N-1}}(r^{N-1}\psi')' + \frac{1}{2}\psi' r + \frac{N}{2}\psi = -\frac{k}{2}\psi, \tag{3.18}$$

see [35]. In the radial setting $E^c = \text{Span}\{\psi_k\}$ is one-dimensional, the GAS (3.12) reduces to a single equation and, similar to (3.16), (3.17), we arrive at a unique asymptotic pattern with the constant

$$(3.19) \qquad \varphi_k = b_k \psi_k(r) \implies b_k = \pm \langle |\psi_k|^{p+1}, 1 \rangle_\rho^{-1/(p-1)}, \quad k = 2, 4, \dots .$$

Let us show that in $\mathbb{R}^N$ there exist non-symmetric patterns. Let $k = 1$. Setting for convenience $\varphi_1 = 2^{-1/2} \sum_{j=1}^N b_j f_j$, where $f_j \equiv \partial f / \partial y_j$, we arrive at the GAS

$$(3.20) \qquad 2^{-(p+1)/2} \langle g(\textstyle\sum b_j f_j), f_i \rangle_\rho = b_i, \quad i = 1, 2, \dots, N.$$

As a first solution, we choose equal coefficients, $b_j = b_0$ for all j. Then (3.20) reduces to a single equation for b_0

$$2^{-(p+1)/2} |b_0|^{p-1} \langle g(\textstyle\sum f_j), f_i \rangle_\rho = 1,$$

where i can be arbitrary by symmetry. On the other hand, there exists another solution $\{b_j\} = \{b_1, 0, \dots, 0\}$. Indeed, the system (3.20) reduces to the first equation for b_1, $2^{-(p+1)/2} |b_1|^{p-1} \int \rho |f_1|^{p+1} = 1$. A complete description of solutions of the GAS (3.12) remains an open problem, which becomes a difficult "nonlinear" part of the linearised centre manifold analysis. It is important that in all the cases the leading term of the asymptotic behaviour of the critical asymptotic patterns (3.15) does not depend on initial data.

Remark: on countable subset of exponentially decaying patterns on stable manifold. Equation (3.2) can admit orbits on the infinite-dimensional stable manifold of the origin, which can be seen by the eigenfunction expansion of solutions. The linear diagonal structure of the system (3.8) shows that if the nonlinear term g forms an exponentially decaying perturbation as $\tau \to \infty$ (unlike the centre manifold behaviour studied above), then there exist patterns with the following exponential decay as $\tau \to \infty$:

$$(3.21) \qquad v_\beta(y, \tau) = C e^{\lambda_\beta \tau} (\psi_\beta(y) + o(1)), \quad C = C(u_0) \neq 0,$$

where ψ_β is a suitable eigenfunction with $\lambda_\beta < 0$ for $|\beta| > 0$. Such results are well known in the linear perturbation theory, see [16], p. 226 and [13].

4. Discrete spectra and eigenfunctions for nonlinear operators

We now return to the nonlinear parabolic equations (1.1) and (1.2). Following the same lines as in the semilinear case, we first study the asymptotic behaviour for the purely diffusive equations: the PME

$$(4.1) \qquad u_t = \Delta(|u|^{m-1} u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad m > 1,$$

and the dual PME

$$(4.2) \qquad v_t = |\Delta v|^{m-1} \Delta v \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad m > 1.$$

We restrict our attention to a class of bounded compactly supported initial data.

4.1. **Discrete spectrum and similarity patterns for the PME.** The PME (4.1) is invariant under various groups of scaling transformations and admits self-similar solutions written in the following form:

$$u(x,t) = t^{\lambda-\mu_0}\psi(y), \quad y = x/t^{\beta}, \tag{4.3}$$

where $\mu_0 = N/[N(m-1)+2]$ and $\beta = [1+(m-1)(\lambda-\mu_0)]/2$. In the linear case $m=1$ we have $\beta = 1/2$, $\mu_0 = N/2$ and these similarity solutions reduce to (2.18) with $\lambda = \lambda_k$ being the spectrum (2.14). Substituting (4.3) into (4.1) yields that $\psi = \psi(y)$ is a weak continuous solution of the following *nonlinear eigenvalue problem* for the quasilinear elliptic equation:

$$\mathbf{B}(\psi) \equiv \Delta(|\psi|^{m-1}\psi) + \beta\nabla\psi \cdot y + \mu_0\psi = \lambda\psi \quad \text{in } \mathbb{R}^N, \quad \psi \in C_0(\mathbb{R}^N), \ \psi(y) \not\equiv 0, \tag{4.4}$$

which, in most cases, is a typical example of similarity solutions of the *second kind* (a term introduced by Ya.B. Zel'dovich [37]), where suitable values $\lambda \in \mathbb{R}$ are obtained from solvability of the elliptic equation in the prescribed functional class; see details in [3]. Dealing with the nonlinear operators, we continue to denote the real "point" spectrum (real "eigenvalues" $\lambda = \lambda_k$) of operator (4.4) by $\sigma(\mathbf{B})$. For $m = 1$, (4.4) becomes the Sturm-Liouville eigenvalue problem (2.10) for the linear bounded self-adjoint operator $\mathbf{B} : H^2_\rho(\mathbb{R}^N) \to L^2_\rho(\mathbb{R}^N)$ with real discrete spectrum (2.14).

One-dimensional case. The following lemma classifies all possible compactly supported similarity solutions (4.3) in the one-dimensional case.

Lemma 4.1. *Let $m > 1$ and $N = 1$. Then operator $\mathbf{B}$ in (4.4) has a strictly decreasing sequence of eigenvalues*

$$\sigma(\mathbf{B}) = \{\lambda_k\} \downarrow -2/(m^2-1), \quad \textit{where } \lambda_0 = 0 \textit{ and } \lambda_1 = -1/m(m+1), \tag{4.5}$$

in the sense that problem (4.4) has a compactly supported solution if, and only if, $\lambda = \lambda_k$ for some integer $k \geq 0$. Moreover, k equals exactly the number of sign changes of such $\psi_k(y)$, and $\psi_k(y)$ is symmetric (anti-symmetric) if k is even (odd).

Figure 1 shows the third eigenfunction of the PME plotted alongside its first two eigenfunctions, which are given explicitly below. This profile was produced using the Matlab boundary value problem solver bvp4c.

It follows from (4.5) that the exponents $\beta = \beta_k$ in (4.3) are strictly positive for all $k \geq 0$. For $N = 1$, equation (4.4) reduces to a first-order ODE with the phase plane already studied in detail in [2], where the first eigenfunction $\psi_0 \geq 0$ was proved to exist for more general equations including gradient-dependent diffusivity. The proof of Lemma 4.1 [23] is based on further delicate analysis of the phase-plane for different values of λ. In a special representation, the phase plane of (4.4) is known to admit limit cycles (see [11], [18], [36]) generating a non-compactly supported profile $\psi_\infty(y)$ with an infinite number of isolated zeros obtained as a result of an infinite number of rotations of the vector field. On this phase plane the eigenfunctions $\{\psi_k\}$ with $\lambda = \lambda_k$ correspond to exactly k rotations around the origin, see details in [11].

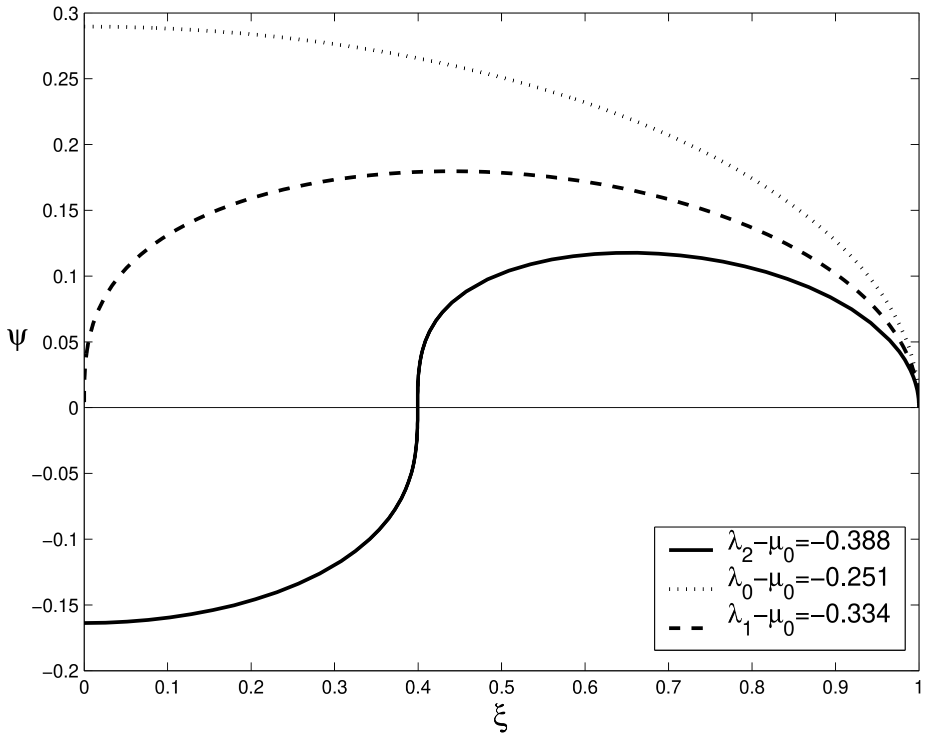

FIGURE 1. Three eigenvalues and their associated eigenvectors of the PME with $m = 3$.

In view of the scaling symmetry of equation (4.4), each ψ_k defines a one-parametric family of eigenfunctions

$$\psi_k(y; b) = b\psi_k(y/b^{(m-1)/2}) \quad \text{for any } \; b > 0. \tag{4.6}$$

Note that setting $m = 1$ in (4.5) yields precisely the spectrum (2.14) of the linear operator $\mathbf{B}$ in 1D, and then scaling (4.6) reduces to multiplication. It is also seen from (4.5) that in the linear case $m = 1$ the spectrum $\{\lambda_k\}$ is unbounded from below (a general property of spectra of self-adjoint operators in Hilbert spaces with compact resolvents, [8]).

In Lemma 4.1 the first two eigenvalues λ_0 and λ_1 and the corresponding eigenfunctions are obtained explicitly by using two known conservation laws for the PME. Namely, $\lambda_0 = 0$ ($\beta_0 = 1/(m+1)$) corresponds to the ZKB solution [38], [2] satisfying the mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u(x, t) \; \mathrm{d}x = 0. \tag{4.7}$$

The ODE (4.4) is integrated twice leading to the first eigenfunction

$$\psi_0(y) = \left[B_0(1 - y^2)_+\right]^{1/(m-1)}, \quad B_0 = (m - 1)/2m(m + 1). \tag{4.8}$$

For $\lambda_1 = -1/m(m+1)$, $\beta_1 = 1/2m$, the similarity solution is Barenblatt-Zel'dovich dipole solution [5] corresponding to the momentum conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int xu(x,t)\,\mathrm{d}x = 0. \tag{4.9}$$

Integrating the ODE (4.4) leads to the odd eigenfunction

$$\psi_1(y) = y^{1/m}\left[B_0(1 - y^{(m+1)/m})_+\right]^{1/(m-1)}, \quad y \geq 0. \tag{4.10}$$

Using the results of the phase-plane analysis of the ODE (4.4), we present two asymptotics of eigenfunctions at singular points. Let $y_0 = \sup \operatorname{supp} \psi_k$. Then

$$\psi_k(y) = \pm A_k (y_0 - y)^{1/(m-1)}(1 + o(1)) \quad \text{as } y \to y_0^-, \quad A_k = [(m-1)\beta_k/m]^{1/(m-1)}. \tag{4.11}$$

If $\psi_k(y)$ vanishes at an internal point y_1 of $\operatorname{supp} \psi_k$, then the asymptotics behaviour is

$$\psi_k(y) = B|y_1 - y|^{(1-m)/m}(y_1 - y)(1 + o(1)) \quad \text{as } y \to y_1, \tag{4.12}$$

where $B \neq 0$ is a constant. Actually, (4.11) is the limit case of (4.12) with $B = 0$.

We now describe similarity patterns in $\mathbb{R}^N$, where the results can be formulated for radially symmetric solutions [23].

Lemma 4.2. *Let $m > 1$ and $N > 1$. Then in the radial geometry, there exists a strictly monotone decreasing sequence of eigenvalues of operator* (4.4)

$$\sigma(\mathbf{B}) = \{\lambda_k,\ k = 0, 2, 4, ...\} \downarrow -2/(m-1)[N(m-1)+2], \quad \textit{where } \lambda_0 = 0, \tag{4.13}$$

so that problem (4.4) *has a radially symmetric, compactly supported solutions if, and only if, $\lambda = \lambda_k$ for some even integer $k \geq 0$, which is exactly the number of sign changes of $\psi_k(|y|)$ in $\mathbb{R}_+$.*

Note that $\beta_k > 0$ for $k \geq 0$. Each profile ψ_k generates a one-parametric family (4.6) of solutions. The first eigenvalue $\lambda_0 = 0$ corresponds to the ZKB solutions with the explicit eigenfunction

$$\psi_0(y) = \left[B_0(1 - |y|^2)_+\right]^{1/(m-1)}, \quad B_0 = (m-1)/2m[N(m-1)+2], \tag{4.14}$$

constructed via the mass conservation (4.7). A little is known for other non-radial eigenfunctions of the PME satisfying the elliptic equation (4.4). A multi-dimensional analogy of the dipole pattern $\psi_1(y)$ [24] seems to be the only known non-radial eigenfunction. Local asymptotic properties of radial eigenfunctions $\psi_k(\bar{y})$ with $\bar{y} = |y| \geq 0$ are the same as for $N = 1$ and in the new notation, (4.11) and (4.12) hold.

4.2. **Discrete spectrum and eigenfunctions for the dual PME.** We begin with the **one-dimensional case** $N = 1$, where the transformation of (4.1) into the dual PME (4.2) is straightforward. Performing first integration yields the p-Laplacian equation

$$w(x,t) = \int_{-\infty}^{x} u(s,t)\,\mathrm{d}s \implies w_t = (|w_x|^{m-1}w_x)_x,$$

and the second integration leads to the dual PME in 1D

$$v(x,t) = \int_{-\infty}^{x} w(s,t)\,\mathrm{d}s \implies v_t = |v_{xx}|^{m-1} v_{xx}. \tag{4.15}$$

We then translate the subset of similarity patterns for the PME from Lemma 4.1 to the dual PME (cf. [7]) and obtain a unique family of nonnegative compactly supported similarity solutions of (4.15) for any $k = 0, 1, 2, ...,$

$$v(x,t) = t^{\Lambda_k - \mu_0} \Psi_k(y), \;\; y = x/t^{\beta_k}, \tag{4.16}$$

$$\Lambda_k = \lambda_{k+2} + 2\beta_{k+2} \equiv -(m-1)\mu_0 + 1 + m\lambda_{k+2},$$

$$\Psi_k(y) = \int_{-\infty}^{y} \int_{-\infty}^{\xi} \psi_{k+2}(\zeta)\;\mathrm{d}\zeta\,\mathrm{d}\xi, \tag{4.17}$$

where eigenfunctions $\{\Psi_k\}$ satisfy the nonlinear eigenvalue problem

$$\mathbf{B}(\Psi) \equiv |\Psi''|^{m-1}\Psi'' + \beta\Psi' y + \mu_0 \Psi = \Lambda\Psi \quad \text{in } \mathbb{R}, \quad \Psi \in C_0^2(\mathbb{R}), \;\; \Psi(y) \not\equiv 0. \tag{4.18}$$

Transformation (4.17) on first two similarity PME profiles ψ_0 and ψ_1 yields unbounded and not compactly supported functions respectively not belonging to our functional class. We next classify the spectrum of similarity solutions (4.16). It follows that $\beta_k > 0$ for $k \geq 0$.

Lemma 4.3. *Let $m > 1$ and $N = 1$. Then the spectrum $\sigma(\mathbf{B})$ of operator* (4.18) *consists of a strictly decreasing sequence of negative eigenvalues*

$$\Lambda_k = 2/(m+1) + m\lambda_{k+2} \downarrow -2/(m^2-1), \quad k = 0, 1, 2, \dots, \tag{4.19}$$

so that problem (4.18) *has a compactly supported C^2 solution if, and only if, $\Lambda = \Lambda_k$ for some integer $k \geq 0$, which is the number of sign changes of $\Psi_k(y)$.*

It follows from (4.17) that $\Psi_k'' = \psi_{k+2}$ has precisely $k+2$ sign changes in $\mathbb{R}$. The standard Sturmian property of eigenfunctions is proved below.

Proposition 4.1. *Ψ_k has k sign changes.*

Proof. Recall that Ψ_k has $k+2$ isolated inflection points. We prove the result in three steps.

Step 1: For $y > 0$, $\Psi_k(y)$ has exactly one sign change between any two neighbouring inflection points. Suppose not. Then there exist two neighbouring inflection points, say $\Psi_k(a)$, $\Psi_k(b)$, such that either $\Psi_k(y) > 0$ or $\Psi_k(y) < 0$ for all $y \in (a,b)$. We note that an inflection point can only occur when $\Psi_k' < 0$ ($\Psi_k' > 0$) if $\Psi_k > 0$ ($\Psi_k < 0$). This follows from the ODE for Ψ_k given by $\mathbf{B}_k(\Psi_k) = 0$. When $\Psi_k'' = 0$ we have that

$$(\Lambda_k - \mu_0)\Psi_k(y) = \beta_k y \Psi_k'(y), \quad \text{where } \Lambda_k - \mu_0 < 0, \quad \beta_k > 0. \tag{4.20}$$

We assume that $\Psi_k, \Psi_k'' > 0$ on (a,b) and thus that Ψ_k' is an increasing function on this interval (all other combinations of signs of Ψ_k and Ψ_k'' require only slight variations to the proof below). For some $\varepsilon > 0$, there exists intervals $(a-\varepsilon, a)$, $(b, b+\varepsilon)$ such that $\Psi_k'' < 0$, and thus Ψ_k' is a decreasing function on these intervals. Since $\Psi_k' < 0$ at any inflection point we have that $\Psi_k'(y) < 0$ for $y \in (a-\varepsilon, b+\varepsilon)$. Thus at b, Ψ_k' achieves a local maximum whilst it is still negative which is a violation of the Maximum Principle. Thus, Ψ_k must have at least one sign change between two neighbouring inflection points. The proof that it has at most one sign change is a direct consequence of the fact that Ψ_k is a convex function on (a, b). A similar proof yields that Ψ_k must have exactly one sign change between any two neighbouring inflection points in the domain $y < 0$.

Step 2: If Ψ_k is a symmetric function then it has exactly k sign changes. Since Ψ_k is symmetric, k is even and we know by symmetry that Ψ_k has $k/2+1$ inflection points in the domain $y > 0$. By Step 1, Ψ_k has exactly $k/2$ sign changes in the domain $y > 0$, and thus k sign changes overall.

Step 3: If Ψ_k is an anti-symmetric function then it has exactly k sign changes. Since Ψ_k is anti-symmetric, k is odd and we know that Ψ_k has $\lfloor k/2+1 \rfloor$ inflection points in the domain $y > 0$ and one inflection point at $y = 0$. (Here $\lfloor \cdot \rfloor$ denotes the floor function for real numbers: $\lfloor k \rfloor$ is the largest integer j such that $j \le k$.) Thus it must have $\lfloor k/2 \rfloor$ sign changes in the domain $y > 0$ by step 1 and thus k sign changes overall. $\square$

The one-parametric families of eigenfunctions take the form

$$\Psi_k(y; b) = b\Psi_k(y/b^{(m-1)/2m}) \quad \text{for any } \ b > 0. \tag{4.21}$$

Instead of (4.11) and (4.12) we obtain the following asymptotics of eigenfunctions:

$$\Psi_k(y) = \pm A_k (y_0 - y)^{\delta}(1 + o(1)) \quad \text{as } \ y \to y_0^-, \quad \delta = (2m-1)/(m-1) > 2, \tag{4.22}$$

$$\Psi_k(y) = B|y_1 - y|^{(m+1)/m}(y_1 - y)(1 + o(1)) \quad \text{as } \ y \to y_1, \tag{4.23}$$

where $A_k = [\beta_k y_0 \delta^{1-m}(\delta-1)^{-m}]^{1/(m-1)}$ and $B \ne 0$ is an arbitrary constant. These expansions show that $\Psi_k \in C^{2+\alpha}$ are smooth classical solutions of the ODE (4.18).

Radial eigenfunctions in $\mathbb{R}^{\mathbf{N}}$. In the multi-dimensional case $N > 1$ the dual PME (4.2) is related to (4.1) by the change $u = -\Delta v$. Denoting by Γ the fundamental solution of Laplace's operator in $\mathbb{R}^N$:

$$\Gamma(x) = \frac{1}{N(2-N)\omega_N}|x|^{2-N} \text{ for } N \ge 3, \quad \Gamma(x) = \frac{1}{2\pi}\ln|x| \text{ for } N = 2,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$, we have $v = -\Gamma * u$, and Lemma 4.2 gives a unique family of similarity solutions (4.16) of the dual PME corresponding to radially symmetric profiles

$$\Psi_k(y) = -\int_{\mathbb{R}^N} \Gamma(y - \xi)\psi_{k+2}(\xi)\,\mathrm{d}\xi, \quad k = 0, 2, 4, \ldots, \tag{4.24}$$

where each function Ψ_k is a classical radial C^2 solution of the elliptic eigenvalue problem

$$\mathbf{B}(\Psi) \equiv |\Delta\Psi|^{m-1}\Delta\Psi + \beta\nabla\Psi \cdot y + \mu_0\Psi = \Lambda\Psi \quad \text{in } \mathbb{R}^N, \ \ \Psi \in C_0^2(\mathbb{R}^N), \ \ \Psi(y) \not\equiv 0. \tag{4.25}$$

Similar to the case $N = 1$, for $N \ge 3$ this transformation of the first radial profile ψ_0 yields a bounded but not compactly supported pattern. For $N = 2$ the transformation of similarity solutions (4.3) takes the form

$$v(x,t) = -t^{\lambda_k+2\beta_k-\mu_0}\left[\frac{\beta_k \ln t}{2\pi}\int_{\mathbb{R}^2}\psi_k(\xi)\,{\mathrm d}\xi + \int_{\mathbb{R}^2}\Gamma(y-\xi)\psi_k(\xi)\,{\mathrm d}\xi\right], \tag{4.26}$$

and therefore for $k=0$, where $\int \psi_0 > 0$, it does not belong to our functional class. From Lemma 4.2 we obtain the following subset of similarity solutions and again discover that $\beta_k > 0$ for $k \ge 0$.

Lemma 4.4. *Let $m>1$ and $N>1$. Then in radial geometry, operator* (4.25) *has a strictly decreasing sequence of negative eigenvalues*

$$\Lambda_k = \frac{2}{N(m-1)+2} + m\lambda_{k+2} \downarrow -\frac{2}{(m-1)[N(m-1)+2]}, \quad k=0,2,4,..., \tag{4.27}$$

so that problem (4.25) *has a compactly supported, radially symmetric solution if, and only if, $\Lambda = \Lambda_k$, and then Ψ_k has exactly k sign changes in $\mathbb{R}_+$.*

Sturmian zero-property of eigenfunctions is proved similarly, see Proposition 4.1. Scalings (4.21) determine one-parametric families of nonlinear radial eigenfunctions. Asymptotics (4.22) and (4.23) remain valid in terms of the radial variable $\bar{y} = |y|$.

5. Critical asymptotic behaviour for the PME with absorption

Using the above "nonlinear spectral analysis" of operator (4.4), similar to the semilinear case in Section 3, we study critical asymptotic behaviour for the PME with absorption (1.1). According to (4.3), we introduce the rescaled variables (cf. (3.1))

$$u(x,t) = t^{\lambda_k-\mu_0}v(y,\tau), \quad y = x/t^{\beta_k}, \quad \tau = \ln t, \tag{5.1}$$

to get the following rescaled equation:

$$v_\tau = \mathbf{B}_k(v) - e^{-\gamma_k\tau}g(v), \quad \gamma_k = p(\mu_0-\lambda_k) - (1+\mu_0-\lambda_k), \tag{5.2}$$

where $g(v) = |v|^{p-1}v$, $\mathbf{B}_k = \mathbf{B} - \lambda_k I$ and $\mathbf{B}$ is as in (4.4). Similar to (3.3) we obtain a sequence of critical exponents generated by the nonlinear spectrum $\sigma(\mathbf{B})$

$$\gamma_k = 0 \quad \Longrightarrow \quad p = p_k = 1 + 1/(\mu_0-\lambda_k), \tag{5.3}$$

where $k=0,1,2,...$ for $N=1$ (Lemma 4.1) and $k=0,2,4,...$ for $N>1$ (Lemma 4.2). It follows from (4.13) that

$$p_k \downarrow m/(m-1) \;\text{ as }\; k \to \infty. \tag{5.4}$$

Hence, $p_\infty > 1$ if $m>1$ unlike the linear case (1.5), where $p_\infty = 1$ due to the unboundedness from below of the spectrum (2.14) of the linear self-adjoint operator.

In the critical case $p=p_k$ we arrive at the autonomous rescaled equation (cf. (3.4))

$$v_\tau = \mathbf{B}_k(v) - g(v), \quad \tau \gg 1. \tag{5.5}$$

We will describe a special asymptotic behaviour admitted by equation (5.5), where

$$v(\cdot,\tau) \to 0, \;\text{ not exponentially fast, }\; \text{ as }\; \tau \to \infty. \tag{5.6}$$

By construction of the nonlinear eigenfunctions, $\mathbf{B}_k(\psi_k) = 0$, and therefore, similar to the linear expansion case (3.9), using the scaling invariance (4.6) we will study the asymptotic behaviour close to the one-dimensional manifold $W^{(k)} = \{\pm\psi_k(y;b),\, b > 0\}$. In order to describe such "slow motion" of these orbits close to $W^{(k)}$, given a solution $v(y,\tau)$ of equation (5.5) and an eigenfunction $\psi_k(y)$, we perform an extra rescaling (as suggested by (4.6))

$$v(y,\tau) = b(\tau)w(\xi,\tau), \quad \xi = y/b^{(m-1)/2}(\tau), \quad \text{with a positive function } b(\tau), \tag{5.7}$$

$$b(\tau) \to 0, \text{ not exponentially fast, as } \tau \to \infty, \tag{5.8}$$

to be determined in such a way that the new rescaled orbit $\{w(\cdot,\tau)\}$ stabilises as $\tau \to \infty$ to a non-trivial limit equilibrium. In view of the scaling invariance (4.6), w satisfies an equation with a non-autonomous perturbation

$$w_\tau = \mathbf{B}_k(w) + q(\tau)\mathbf{C}w - b^{p-1}(\tau)g(w), \quad \mathbf{C} = \nu_*\xi\cdot\nabla - I, \; q(\tau) = \dot b(\tau)/b(\tau)\,, \tag{5.9}$$

where $\nu_* = (m-1)/2$. If (5.8) holds and $q(\tau) \to 0$ as $\tau \to \infty$, then (5.9) is an asymptotically small perturbation of the autonomous equation

$$w_\tau = \mathbf{B}_k(w) \tag{5.10}$$

admitting a one-parametric family (4.6) of equilibria. We are going to show that for a special choice of the scaling function $b(\tau)$ there exists a unique eigenfunction ψ_k such that

$$w(\cdot,\tau) \to \psi_k \text{ as } \tau \to \infty. \tag{5.11}$$

In 1D or in radial geometry in $\mathbb{R}^N$, the asymptotic analysis of the perturbed equation (5.9) is based on the classical theory of singular ordinary differential operators [31].

5.1. **One-dimensional case.** We consider a class of symmetric (anti-symmetric) solutions $u(x,t)$ for even (odd) k and choose

$$b^{(m-1)/2}(\tau) = \sup \operatorname{supp} v(\cdot,\tau), \quad \tau \gg 1. \tag{5.12}$$

Regularity of interfaces for the PME with reaction-absorption terms are well-known, see [17], [25] and [4], [33] for solutions changing sign. Hence, we may assume that $b(\tau)$ is sufficiently smooth for $\tau \gg 1$. Note that equation (5.9) is understood in the weak sense so that actually at this stage we do not need C^1-regularity of the interfaces. Then (5.12) implies that

$$w(\pm 1,\tau) = 0 \quad \text{and} \quad \operatorname{supp} w(\cdot,\tau) \subseteq [-1,1] \text{ for } \tau \gg 1. \tag{5.13}$$

Using (4.6), we choose the eigenfunction $\psi_k(\xi)$ such that

$$\operatorname{supp} \psi_k = [-1,1]. \tag{5.14}$$

We first treat two special cases.

Example 1: mass conservation for $\mathbf{k = 0}$. The analysis becomes easier for $k = 0$ when the operator $\mathbf{B}_0 = \mathbf{B}$ admits the mass conservation, $\int \mathbf{B}_0(w)d\xi \equiv 0$. Following

[19] and [21], integrating equation (5.9) over $\mathbb{R}^N$, we obtain that the mass $M_0(\tau) = \int w$ satisfies

$$M' = q(\tau) \int \mathbf{C} w - b^{p-1} \int g(w). \tag{5.15}$$

Assuming that (5.11) holds for the unique ψ_0 satisfying (5.14), we have

$$M' = -\varepsilon_1 \frac{\dot b}{b}(1+o(1)) - \varepsilon_2 b^{p-1}(1+o(1)) \quad \text{for } \tau \gg 1, \tag{5.16}$$

where $\varepsilon_1 = -\int \mathbf{C}\psi_0 > 0$ and $\varepsilon_2 = \int g(\psi_0) > 0$. Since the mass $M(\tau)$ of a compact orbit is uniformly bounded and non-zero (see details in [21]) and the first perturbation $\dot b/b$ is not integrable on $(1,\infty)$ by (5.8), we have to have that $b(\tau)$ satisfies the "ODE"

$$\dot b = -b^p(\varepsilon_2/\varepsilon_1)(1+o(1)) \implies b(\tau) = C\tau^{-1/(p-1)}(1+o(1)) \text{ for } \tau \gg 1, \tag{5.17}$$

where $C = [(p-1)\varepsilon_2/\varepsilon_1]^{-1/(p-1)}$.

Example 2: momentum conservation for k = 1. Then there exists the momentum conservation, $\int \xi \mathbf{B}_1(w)\mathrm{d}\xi \equiv 0$. Therefore, for the momentum $M(\tau) = \int \xi w$ we obtain equation (5.16), where $\varepsilon_1 = \int \xi \mathbf{C}\psi_1$ and $\varepsilon_2 = \int \xi g(\psi_1)$ and finally the asymptotic behaviour (5.17), which is generic for such anti-symmetric solutions of zero mass, see [20].

Arbitrary k ≥ 2. Then no explicit conservation laws for the rescaled operators $\mathbf{B}_k$ are available and it seems that an independent asymptotic ODE for the scaling function $b(\tau)$ in (5.7) cannot be derived. We then have to study the behaviour of $b(\tau)$ as $\tau \to \infty$ together with the rate of convergence in (5.11).

We begin with the standard linearisation by setting

$$w(\xi,\tau) = \psi_k(\xi) + Y(\xi,\tau) \quad \text{for } \tau \gg 1, \quad \operatorname{supp} Y(\cdot,\tau) \subseteq [-1,1], \tag{5.18}$$

leading to the perturbed equation

$$Y_\tau = \mathbf{A}_k Y + q(\tau)P_1 - b^{p-1}(\tau)P_2 + q(\tau)\mathbf{C}Y + \mathbf{D}(Y) - b^{p-1}(\tau)\mathbf{E}(Y), \tag{5.19}$$

where $\mathbf{A}_k = \mathbf{B}_k'(\psi_k)$, $P_1(\xi) = \mathbf{C}\psi_k(\xi)$, $P_2(\xi) = g(\psi_k(\xi))$ and nonlinear perturbations

$$\mathbf{D}(Y) = \mathbf{B}_k(\psi_k + Y) - \mathbf{B}_k'(\psi_k)Y, \quad \mathbf{E}(Y) = g(\psi_k + Y) - g(\psi_k) \tag{5.20}$$

are quadratic in $Y \to 0$ in the corresponding metrics. We need to describe special asymptotic behaviour of global uniformly bounded solutions of (5.19) satisfying

$$Y(\cdot,\tau) \to 0 \quad \text{as } \tau \to \infty \text{ not exponentially fast.} \tag{5.21}$$

Let us write down the Frechet derivative $\mathbf{A}_k$ in the Sturm-Liouville form

$$\mathbf{A}_k Y = m(|\psi_k|^{m-1}Y)'' + \beta_k Y'\xi + (\mu_0 - \lambda_k)Y \equiv \frac{1}{\rho}[(pY')' - qY], \tag{5.22}$$

$$\text{with } p(\xi) = |\psi_k(\xi)|^{2(m-1)} \exp\left\{\frac{\beta_k}{m}\int |\psi_k(\xi)|^{1-m}\xi\,\mathrm{d}\xi\right\}, \ \rho(\xi) = \frac{p(\xi)}{m|\psi_k(\xi)|^{m-1}} \tag{5.23}$$

and $q(\xi) = \rho(\xi)[\lambda_k - \mu_0 - m(|\psi_k(\xi)|^{m-1})'']$. We next need spectral properties of the singular ordinary differential operator $\mathbf{A}_k$ on the bounded interval $(-1,1)$. It is symmetric relative to the inner product $\langle\cdot,\cdot\rangle_\rho$ in the weighted space $L^2_\rho(I)$ with the induced norm denoted by

$\|\cdot\|_\rho$. It follows from (4.12) that $1/p(\xi)$ is locally integrable, $1/p \in L^1_\rho$, in a neighbourhood of any internal point $\xi_1 \in (-1,1)$, where $p(\xi)$ vanishes. Hence, $\mathbf{A}_k$ has only two singular points: $\xi = \pm 1$, see [31], Chapter V. It suffices to consider $\xi = 1$. Denoting $1-\xi = s > 0$, we now use the asymptotics (4.11) to get that

$$p(s) \sim s^\kappa \ \text{ and } \ \rho(s),\, q(s) \sim s^{\kappa-1}, \tag{5.24}$$

where $\kappa = (2m-3)/(m-1)$. In order to get the deficiency indices of $\mathbf{A}_k$, we fix $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and, using (5.24), solve equation the $\mathbf{A}_k Y = \lambda Y$ to get two linearly independent solutions as $s \to 0$

$$Y_1 = 1 + O(s) \ (Y,\, Y' \text{ are bounded}) \quad \text{and } \ Y_2 \sim s^{1-\kappa},\ m \neq 2;\ \ Y_2 \sim \ln s,\ m = 2. \tag{5.25}$$

Since $Y_2 \in L^2_\rho$ and $Y_1 \in L^2_\rho$ if, and only if, $\kappa > 0$, i.e., $m > 3/2$, we have that the deficiency indices are $(2,2)$ if $m > 3/2$ ($\xi = 1$ is in the limit-circle case of singular end-point) and are $(1,1)$ if $m \in (1, 3/2]$ (the limit-point case). Therefore, for $m > 3/2$ any real self-adjoint extension of $\mathbf{A}_k$ has a discrete spectrum, [31], p. 84. For $m \in (1, 3/2]$ a similar result follows from L^2_ρ integrability of the kernel of the inverse operator, see examples in [31], Section 23. For the second-order operator (5.22), the non-oscillatory asymptotics (5.25) as $s \to 0$ imply that the spectrum of $\mathbf{B}$ is discrete; see Lemma 3.1.1 in [29], p. 74. On the other hand, it is easy to see by a direct construction via variation of constants formula that the inverse operator $(\mathbf{A}_k + \varepsilon I)^{-1} : L^2_\rho \to L^2_\rho$ for some $\varepsilon < 0$ has a Hilbert-Schmidt kernel, see [31], p. 88. We summarise these results as follows.

Proposition 5.1. *(i) If $m \in (1, 3/2]$, then operator (5.22) in $L^2_\rho(I)$ has a discrete spectrum $\sigma(\mathbf{A}_k) = \{\lambda_j^{(k)},\ j = 0,1,2,...\}$ which is a strictly monotone decreasing sequence of eigenvalues. The corresponding eigenfunctions $\Phi^{(k)} = \{\psi_j^{(k)}\}$ form an orthonormal basis in $L^2_\rho(I)$. (ii) If $m \in (3/2, 2)$, then the same is true in the class of functions satisfying*

$$Y(\pm 1) = 0. \tag{5.26}$$

(iii) If $m = 2$, then the same is true with the condition

$$Y(\xi) \quad \textit{is bounded as } \ \xi \to \pm 1. \tag{5.27}$$

(iv) For $m \in (1, 2]$ the centre subspace E^c of such self-adjoint extensions $\mathbf{A}_k$ is one-dimensional and is spanned by the eigenfunction

$$\psi_k^{(k)}(\xi) = c_k \frac{\mathrm{d}}{\mathrm{d}b} \psi_k(\xi; b)\Big|_{b=1} \equiv -c_k \mathbf{C} \psi_k(\xi) \quad \textit{with } \lambda_k^{(k)} = 0, \tag{5.28}$$

where $c_k > 0$ is a normalisation constant.

In the cases (i) - (iii) of Proposition 5.1 suitable choices of real self-adjoint extensions are given by choosing the corresponding symmetric unitary 2×2 matrix $[u]$ describing the singularity behaviour of admitted solutions as $\xi \to \pm 1$, [31], pp. 74-81.

Note that for $m > 2$ there exist real self-adjoint extensions of $\mathbf{A}_k$ with the condition at singular end-points $\xi = \pm 1$ admitting singular behaviour $Y(\xi) = O((1 \mp \xi)^{1-\kappa})$ as $\xi \to \pm 1$, where $\kappa > 1$. In case (ii) there exist infinitely many suitable self-adjoint extensions for which (5.28) is an eigenfunction, cf. Theorem 4.1 in Chapter 9, [12]. Nevertheless, being applied to the PDE (5.9) by using of uniformly converging eigenfunction expansions, these extensions assume a special behaviour of solutions $w(\xi,\tau)$ at singular end-points $\xi = \pm 1$, which we cannot guarantee in general. The eigenfunction (5.28) is obtained by differentiating equation (4.4) with respect to b and using (4.6). By (5.22) this yields $\mathbf{A}_k\psi_k^{(k)} = 0$ where $\psi_k^{(k)}$ satisfies necessary growth conditions at singular endpoints, e.g. (5.26) or (5.27).

We now prove that $\psi_k^{(k)}$ has precisely k zeros on I so that it is k-th eigenfunction by Sturm's Theorem.

Proposition 5.2. *$\psi_k^{(k)}$ has exactly k zeros.*

Proof. We know that ψ_k has exactly k zeros (Lemma 4.1). Take an interval $[c,d]$ such that $\psi_k' > 0$ on (c,d), $\psi_k'(c) = \psi_k'(d) = 0$ and $\psi_k(\xi) = 0$ for exactly one $\xi \in (c,d)$. We prove that $\psi_k^{(k)}$ must also have exactly one zero in (c,d). (On an interval in which $\psi_k' < 0$ replace ψ_k by $-\psi_k$ in the proof below to gain a similar contradiction.)

When $\psi_k' = 0$ we have, by (5.28) and (5.9) that $\operatorname{sign}(\psi_k^{(k)}) = \operatorname{sign}(\psi_k)$. This shows that $\psi_k^{(k)}$ has at least one zero in $[c,d]$ since $\psi_k(c) < 0$ and $\psi_k(d) > 0$ by the Maximum Principle. Hence, it remains to prove that $\psi_k^{(k)}$ can have at most one zero in (c,d). We do this by examining the sign of $\left(\psi_k^{(k)}\right)'$ at a zero of $\psi_k^{(k)}$. Now, for $\xi \in (c,d)$, substituting for ψ_k'' using $\mathbf{B}_k(\psi_k) = 0$ yields

$$\left(\psi_k^{(k)}(\xi)\right)' = -c_k\left(\frac{(m-3)}{2}\psi_k'(\xi) + \frac{m-1}{2}\xi\psi_k''(\xi)\right)$$

$$= -c_k\left\{\frac{(m-3)}{2}\psi_k' + \frac{m-1}{2}\xi\left[\frac{(\lambda_k-\mu_0)\psi - \beta_k\xi\psi'}{m|\psi(\xi)|^{m-1}} - (m-1)\frac{\psi}{|\psi|^2}(\psi')^2\right]\right\}. \tag{5.29}$$

At any zero ξ_0 of $\psi_k^{(k)}$ we have that

$$(m-1)\psi_k'(\xi_0)\xi_0 = 2\psi_k(\xi_0). \tag{5.30}$$

We now have two distinct cases to consider.

Case (i): $\xi_0 \neq 0$. Substituting for ψ_k' in (5.29) using (5.30) yields

$$\left(\psi_k^{(k)}\right)'(\xi_0) = -c_k\left[\frac{(m-3)a}{(m-1)\xi_0} + \frac{(m-1)\xi_0(\lambda_k-\mu_0)a}{2m|a|^{m-1}} - \frac{a\beta_k\xi_0}{m|a|^{m-1}} - \frac{2a}{\xi_0}\right], \tag{5.31}$$

where $a = \psi(\xi_0)$. Noting that $\lambda_k < 0$, $\beta_k > 0$ by Lemma 4.1 and that at a zero of $\psi_k^{(k)}$, $\operatorname{sign}(a) = \operatorname{sign}(\xi_0)$ by equation (5.30) we have that $(m-3)a/(m-1)\xi_0 < 2a/\xi_0$, $(m-1)\xi_0(\lambda_k-\mu_0)a/2m|a|^{m-1} < 0$ and $a\beta_k\xi_0/m|a|^{m-1} < 0$ if $m > 1$. Thus, at any zero $\xi_0 \in (c,d)$ of $\psi_k^{(k)}$ we know that $(\psi_k^{(k)})'$ is positive. Hence, there can be only one.

Case (ii): $\xi_0 = 0$. We have shown that at any other zero ξ_1 of $\psi_k^{(k)}$ the gradient must be positive so a second zero clearly cannot exist. $\square$

It follows from completeness in $L^2_\rho(I)$ of the subset $\Phi^{(k)} = \{\psi^{(k)}_k\}$ of eigenfunctions that

$$L^2_\rho(I) = E^u \oplus E^c \oplus E^s, \quad \text{where} \tag{5.32}$$

$$E^u = \mathrm{Span}\{\psi^{(k)}_1, ..., \psi^{(k)}_{k-1}\}, \;\; E^c = \mathrm{Span}\{\psi^{(k)}_k\}, \;\; E^s = \mathrm{Span}\{\psi^{(k)}_{k+1}, \psi^{(k)}_{k+2}, ...\}. \tag{5.33}$$

Using Proposition 5.1 and (5.13), we get the uniformly converging eigenfunction expansion of the bounded continuous weak solution

$$Y(\xi,\tau) = \sum a_j(\tau)\psi^{(k)}_j(\xi). \tag{5.34}$$

The expansion coefficients $\{a_j(\tau)\}$ satisfy an infinite-dimensional DS obtained by substituting (5.34) into (5.19) and multiplying by $\psi^{(k)}_j$ in $L^2_\rho(I)$,

$$\begin{aligned} \dot a_j = \lambda^{(k)}_j a_j &+ q(\tau)\langle P_1, \psi^{(k)}_j\rangle_\rho - b^{p-1}(\tau)\langle P_2, \psi^{(k)}_j\rangle_\rho + q(\tau)\langle \mathbf{C}Y, \psi^{(k)}_j\rangle_\rho \\ &+\langle \mathbf{D}(Y), \psi^{(k)}_j\rangle_\rho - b^{p-1}(\tau)\langle \mathbf{E}(Y), \psi^{(k)}_j\rangle_\rho, \;\; j = 0,1,2,... \,. \end{aligned} \tag{5.35}$$

By the PME regularity theory [25], [14], we may assume that (5.21) holds in $L^2_\rho(I)$ and in $H^2_\rho(I)$ and the expansion coefficients are uniformly small

$$\|Y(\cdot,\tau)\|^2_\rho = \sum a^2_j(\tau) \to 0 \quad \text{as } \tau \to \infty. \tag{5.36}$$

We are interested in the critical asymptotic behaviour corresponding to the evolution close to the centre subspace of $\mathbf{A}_k$, so that we exclude both the unstable and stable exponentially decaying patterns. Unlike the semilinear equation in Section 3, in the case of the quasilinear parabolic equation (5.19) with the degenerate singular linear operator $\mathbf{A}_k$, we do not know that a centre manifold exists and we cannot use a standard invariant manifold theory, cf. [30] and [32]. Moreover, to our knowledge, for the case of singular operators like (5.22) with degenerate non-constant coefficients, the only known rigorous result is Angenent's analysis [1] of the rate of convergence to the ZKB-profiles for the PME (i.e., for equation like (5.9) without non-autonomous perturbations), where $\ln t$-perturbations were shown to exist (but actual convergence of the asymptotic series was not achieved).

Therefore, we perform further asymptotic analysis under the assumption of centre subspace dominance, i.e., assuming that the behaviour for $\tau \gg 1$ of the k-th coefficient is dominant in the sense that

$$Y(\tau) = a_k(\tau)\psi^{(k)}_k + o(a_k(\tau)) \quad \text{as } \tau \to \infty \tag{5.37}$$

uniformly on compact subsets and in H^2_ρ. Under this assumption, performing necessary expansions on the right-hand side of (5.35), we obtain the following system:

$$\dot a_j = \lambda^{(k)}_j a_j - \varepsilon_{1,j}\frac{\dot b}{b} - \varepsilon_{2,j} b^{p-1} + A_j a_k^2 + B_j a_k^3 + C_j a_k \frac{\dot b}{b} + D_j a_k b^{p-1} \tag{5.38}$$

$$+O(a_k^2(b^{p-1}+a_k^2)),\ \varepsilon_{1,j}=-\langle \mathbf{C}\psi_k,\psi_j^{(k)}\rangle_\rho,\ \ \varepsilon_{2,j}=\langle g(\psi_k),\psi_j^{(k)}\rangle_\rho,$$
$$A_j=\frac{1}{2}m(m-1)\langle[|\psi_k|^{m-3}\psi_k(\psi_k^{(k)})^2]'',\psi_j^{(k)}\rangle_\rho,\ \ C_j=\langle \mathbf{C}\psi_k^{(k)},\psi_j^{(k)}\rangle_\rho.$$
$$B_j=\frac{1}{6}m(m-1)(m-2)\langle[|\psi_k|^{m-3}(\psi_k^{(k)})^3]'',\psi_j^{(k)}\rangle_\rho,\ \ D_j=\langle g'(\psi_k),\psi_j^{(k)}\rangle_\rho>0,$$

Using asymptotics (4.11), one can see that all the expansion coefficients above are finite if $m\in(1,3/2)$. Roughly speaking, this means that the expansion techniques apply if the PME operator is not "very nonlinear". Such a restriction is natural when dealing with weak solutions, where, in general, pointwise expansion methods are hard to apply.

Setting $j=k$ in (5.38) with $\lambda_k^{(k)}=0$ yields the ODE for a_k describing the behaviour close to the centre subspace

$$\dot a_k=-\varepsilon_{1,k}\frac{\dot b}{b}-\varepsilon_{2,k}b^{p-1}+A_ka_k^2+B_ka_k^3+C_ka_k\frac{\dot b}{b}+D_ka_kb^{p-1}+O(a_k^2(b^{p-1}+a_k^2)), \tag{5.39}$$

$$\varepsilon_{1,k}=-\langle P_1,\psi_k^{(k)}\rangle_\rho=c_k\|\mathbf{C}\psi_k\|_\rho^2>0, \tag{5.40}$$

$$\varepsilon_{2,k}=\langle P_2,\psi_k^{(k)}\rangle_\rho=c_k\langle g(\psi_k),\psi_k\rangle_\rho-\frac{1}{2}(m-1)c_k\langle g(\psi_k),\psi_k'\xi\rangle_\rho. \tag{5.41}$$

Observe that $A_k=B_k=0$ for $m=1$ so that these quadratic and cubic terms on the right-hand side of (5.39) do not occur in the linear case (cf. (3.10)), which makes the centre manifold analysis in Section 3 essentially easier.

Consider the DS (5.38), (5.39). It is of crucial importance that under assumption (5.8), $\dot b/b\notin L^1(\mathbb{R}_+)$. Therefore, in all the equations this term cannot be balanced by the derivatives $\dot a_j(\tau)$ which by (5.36) are integrable. We then observe a typical "centre manifold" case where all non-integrable terms on the right-hand sides must cancel each other.

Proposition 5.3. *Let $m\in(1,3/2)$ and (5.8) hold, $\varepsilon_{2,k}>0$, and let the rescaled orbit $\{Y(\tau)\}$ approach the centre subspace in the sense of (5.37) with the rate of convergence satisfying*

$$a_k^2(\tau)=o(b^{p-1}(\tau))\quad \textit{as }\tau\to\infty. \tag{5.42}$$

Then the scaling function $b(\tau)$ given in (5.7) satisfies

$$b(\tau)=C_k\tau^{-1/(p-1)}(1+o(1))\quad \textit{as }\tau\to\infty,\quad C_k=[(p-1)\varepsilon_{2,k}/\varepsilon_{1,k}]^{-1/(p-1)}. \tag{5.43}$$

Proof. Under the assumption (5.42) the last five terms depending on the rate of convergence $a_k(\tau)$ on the right-hand side of (5.39) are negligible in comparison with the first two terms which thus form non-integrable perturbations. Therefore, $b(\tau)$ can be determined from the asymptotic ODE

$$-\varepsilon_{1,k}[\dot b(\tau)/b(\tau)]-\varepsilon_{2,k}b^{p-1}(\tau)+\nu(\tau)=0\quad\text{for }\tau\gg 1, \tag{5.44}$$

where $\nu(\tau)\in L^1((1,\infty))$ is an integrable function. Writing down (5.44) in the form

$$(\varphi(\tau)b)'=-\frac{\varepsilon_{2,k}}{\varepsilon_{1,k}}b^p\varphi(\tau),\quad \text{where }\ \varphi(\tau)=\exp\left\{-\frac{1}{\varepsilon_{1,k}}\int_0^\tau\nu(s)\mathrm{d}s,\right\}$$

and integrating it in terms of the new function $B(\tau) = b(\tau)\varphi(\tau)$ using that by the assumption on $\nu(\tau)$ there exists the limit $\varphi(\infty)$, we obtain (5.43). Actually, this means that $b(\tau)$ can be obtained from the following equivalent "autonomous" ODE:

$$\dot{b} = -[\varepsilon_{2,k}/\varepsilon_{1,k}]\, b^p(1 + o(1)), \quad \tau \gg 1, \tag{5.45}$$

which was observed earlier in particular examples. □

It is important that (5.43) establishes the same rate of decay of such asymptotic patterns as those in (5.17) already known for two particular cases. The assumption on the centre subspace dominance and non-exponential decay rate as $\tau \to \infty$ are essential. By Proposition 5.1, equation (5.22) is expected to admit a countable subset of other exponentially decaying patterns corresponding to the behaviour close to the infinite-dimensional stable subspaces corresponding to negative eigenvalues $\lambda_j^{(k)} < 0$ for any $j > k$.

Finally, bearing in mind the critical exponents (5.3) and going back to the original variables, by using the two scalings (5.1) and (5.7) and equality (5.43), we obtain the following subset of critical asymptotic patterns as $t \to \infty$:

$$u(x,t) = C_k(t\ln t)^{-\delta_k}[\psi_k(\eta) + o(1)], \quad \eta = \frac{x}{t^{1/2}}(t\ln t)^{(m-1)\delta_k/2}, \;\; \delta_k = \mu_0 - \lambda_k > 0. \tag{5.46}$$

It follows from the rescaled variable η in (5.46) that the interface (free boundary) has the behaviour given in (1.3).

Remark: on a possibility of other asymptotic patterns. We expect that the centre subspace behaviour described in Proposition 5.3 is a stable generic one. Nevertheless, in general it is difficult to prove this even if the existence of the centre manifold can be justified. In the DS for the expansion coefficients, several higher-order terms can affect the asymptotic behaviour. Namely, one can see that the DS (5.38), (5.39) admits the following formal solution as $\tau \to \infty$ satisfying (5.37):

$$a_j(\tau) = O(\tau^{-1}) \;\text{ for } j \ne k, \quad a_k(\tau) = z_1\tau^{-1/2}(1 + o(\tau^{-1/2})), \tag{5.47}$$

$$b(\tau) = z_2\tau^{-1/(p-1)}(1 + o(\tau^{-1/2})), \tag{5.48}$$

where constants $z_1,\, z_2 > 0$ solve the algebraic system

$$\varepsilon_{1,k}/(p-1) - \varepsilon_{2,k}z_2^{p-1} + A_k z_1^2 = 0, \;\; -z_1/2 = B_k z_1^3 - C_k z_1/(p-1) + D_k z_1 z_2^{p-1}. \tag{5.49}$$

Indeed, substituting $a_k(\tau)$, as given in (5.47), into (5.39) yields (5.49), and then $a_j(\tau) = O(\tau^{-1})$, $j \ne k$, follows from equations (5.38), $\lambda_j^{(k)} \ne 0$, for "quasi-stationary" solutions satisfying $|\dot{a}_j| \ll |a_j|$ as $\tau \to \infty$.

Of course, (5.49) admits a solution with $z_1 = 0$ which is equivalent to (5.43). On the other hand, the algebraic system can admit other solutions with $z_1 \ne 0$, where (5.49) reduces to a linear system for positive unknowns $\{z_1^2, z_2^{p-1}\}$

$$B_k z_1^2 + D_k z_2^{p-1} = C_k/(p-1) - 1/2, \quad A_k z_1^2 - \varepsilon_{2,k} z_2^{p-1} = -\varepsilon_{1,k}/(p-1), \tag{5.50}$$

and an extra analysis of solvability is necessary. The algebraic system (5.49) shows that the coefficient z_2 in (5.48) can be calculated independently of z_1 (i.e., of the rate of convergence) if, and only if, $A_k = 0$. The second equality in (5.49) shows that the positivity of the ratio $\varepsilon_{1,k}/\varepsilon_{2,k}$ is necessary (even for the case $A_k = 0$) or $\varepsilon_{2,k} > 0$ since $\varepsilon_{1,k}$ is positive by (5.40). It follows from (5.41) that $\varepsilon_{2,k} > 0$ for any odd k's since the last term on the right-hand side of (5.41) vanishes. On the other hand, for $k = 0$, where $\psi_k \geq 0$, the last term is positive.

5.2. **Radial multi-dimensional case.** In the radial geometry with the single spatial variables $\bar{y} = |y| \geq 0$ and $\bar{\xi} = |\xi|$, $k \geq 0$ is always even and we keep the same scalings and transformations as for $N = 1$. Example 1 is true [21]. The Frechet derivative $\mathbf{A}_k$ in (5.19) is a singular ordinary differential operator on $(0, 1)$

$$\mathbf{A}_k Y = m\bar{\xi}^{1-N}(\bar{\xi}^{N-1}(|\psi_k|^{m-1}Y)')' + \beta_k Y'\bar{\xi} + (\mu_0 - \lambda_k)Y, \tag{5.51}$$

which admits a symmetric Sturm-Liouville representation similar to (5.22), where the coefficients include the Jacobian $\bar{\xi}^{N-1}$. Then (5.23) for p reads

$$p(\bar{\xi}) = \bar{\xi}^{N-1}|\psi_k(\bar{\xi})|^{2(m-1)} \exp\left\{\frac{\beta_k}{m}\int |\psi_k(\bar{\xi})|^{1-m}\bar{\xi}\,\mathrm{d}\bar{\xi}\right\}$$

and the formula for the weight ρ stays the same. The singular end-point $\bar{\xi} = 1$ has the same properties and deficiency indices as for $N = 1$. In view of the radial Laplacian in (5.51), the origin $\bar{\xi} = 0$ is a singular point (in the limit circle or point case). The symmetry Neumann condition

$$Y'(0) = 0 \quad (\text{or } Y(\bar{\xi}) \text{ is bounded as } \bar{\xi} \to 0) \tag{5.52}$$

does not change the spectral properties of self-adjoint extensions formulated in Proposition 5.1. The rest of the critical asymptotic construction is the same and finally we arrive at asymptotics (5.46) for even $k \geq 0$ provided that $\varepsilon_{2,k} > 0$.

6. Critical asymptotic behaviour for the dual PME with absorption

Similar to Section 5 we study the critical asymptotic behaviour of the dual PME with absorption (1.2). According to (4.16) we introduce the rescaled variables (cf. (5.1))

$$u(x,t) = t^{\Lambda_k - \mu_0}v(y,\tau), \quad y = x/t^{\beta_k}, \quad \tau = \ln t, \tag{6.1}$$

to get the following rescaled equation:

$$v_\tau = \mathbf{B}_k(v) - e^{-\Gamma_k\tau}g(v), \quad \Gamma_k = p(\mu_0 - \Lambda_k) - (1 + \mu_0 - \Lambda_k), \tag{6.2}$$

where $g(v) = |v|^{p-1}v$ and $\mathbf{B}_k(v) = |\Delta v|^{m-1}\Delta v + \beta\nabla v \cdot y + (\mu_0 - \Lambda_k)v$. Similar to (5.3) we obtain a sequence of critical exponents

$$\Gamma_k = 0 \quad \Longrightarrow \quad p = p_k = 1 + 1/(\mu_0 - \Lambda_k), \tag{6.3}$$

where $k = 0, 1, 2, \ldots$ for $N = 1$ (Lemma 4.3) and $k = 0, 2, 4, \ldots$ for $N > 1$ (Lemma 4.4). It follows from (4.27) that (5.4) holds. In the critical case $p = p_k$, the rescaled equation is exactly (5.5) and is autonomous.

We study exponentially decaying asymptotic patterns of equation (5.5) satisfying (5.6). By construction, $\mathbf{B}_k(\Psi_k) = 0$, and by scaling invariance (4.21) we describe the asymptotic behaviour close to the one-dimensional manifold $W^{(k)} = \{\pm\Psi_k(y;b), b>0\}$. To this end we perform an extra rescaling (cf. (5.7))

$$v(y,\tau) = b(\tau)w(\xi,\tau), \quad \xi = y/b^{(m-1)/2m}(\tau), \tag{6.4}$$

with a positive function $b(\tau)$ satisfying (5.8) such that the rescaled orbit $\{w(\cdot,\tau)\}$ stabilises as $\tau \to \infty$ to a non-trivial equilibrium. In view of the scaling invariance (4.21), w satisfies the perturbed equation (5.9) with $\nu_* = (m-1)/2m$, which is an asymptotically small perturbation of the autonomous equation (5.10) admitting equilibria (4.21). We will show that for a special choice of the scaling function $b(\tau)$ there exists a unique eigenfunction Ψ_k such that $w(\cdot,\tau) \to \Psi_k$ as $\tau \to \infty$. The operator $\mathbf{B}_k$ for any $k \geq 0$ does not admit any conservation laws (unlike the corresponding operator for the PME) and thus the only relative simplification occurs in the one-dimensional case.

6.1. **One-dimensional case.** In the class of symmetric (anti-symmetric) solutions $u(x,t)$ for even (odd) k we set (cf. (5.12))

$$b^{(m-1)/2m}(\tau) = \sup \operatorname{supp} v(\cdot,\tau), \quad \tau \gg 1. \tag{6.5}$$

In view of transformation (4.15), we may assume that $b(\tau)$ is sufficiently smooth for $\tau \gg 1$ as follows from the PME theory. Then (5.13) holds and by (4.21), $\operatorname{supp} \Psi_k = [-1,1]$. Since we do not have any explicit conservation laws we study the behaviour of $b(\tau)$ as $\tau \to \infty$ together with the rate of convergence. The linearisation (5.18) gives the perturbed equation (5.19) with the coefficients

$$\mathbf{A}_k = \mathbf{B}_k'(\Psi_k), \quad P_1(\xi) = \mathbf{C}\Psi_k(\xi), \quad P_2(\xi) = g(\Psi_k(\xi)), \tag{6.6}$$

where $\mathbf{C}$ is as in equation (5.9) with $\nu_* = (m-1)/2m$, and the quadratic perturbation terms given by (5.20) with ψ_k replaced by Ψ_k. We study global uniformly bounded solutions of (5.19) satisfying (5.21). Writing down $\mathbf{A}_k$ in the Sturm-Liouville form

$$\mathbf{A}_k Y = m|\Psi_k''|^{(m-1)} Y'' + \beta_k \xi Y' + (\mu_0 - \Lambda_k)Y \equiv \frac{1}{\rho}[(pY')' - qY], \tag{6.7}$$

$$\text{with} \quad p(\xi) = \exp\left\{\frac{\beta_k}{m}\int |\Psi_k''|^{1-m}\xi \, \mathrm{d}\xi\right\}, \quad \rho(\xi) = \frac{p(\xi)}{m|\Psi_k''|^{m-1}}, \tag{6.8}$$

$q(\xi) = \rho(\xi)(\Lambda_k - \mu_0)$, we next need spectral properties of this singular ordinary differential operator in $L^2_\rho(I)$ having two singular points $\xi = \pm 1$. Consider $\xi = 1$. Setting $1 - s = \xi$, we use the asymptotics (4.22) to get that (5.24) holds with the exponent $\kappa = -1/(m-1)$. Calculating the deficiency indices of $\mathbf{A}_k$, for any fixed $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we obtain solutions (5.25) without the special case when $m = 2$, and hence it follows that the deficiency indices are always $(1,1)$. It is not difficult to see that the inverse operator $(\mathbf{A}_k + \varepsilon I)^{-1}$ with a large constant $\varepsilon < 0$, has L^2_ρ-kernel, so we have a discrete spectrum.

Proposition 6.1. *(i) The operator* (6.7) *in* $L^2_\rho(I)$ *has a discrete spectrum* $\sigma(\mathbf{A}_k) = \{\Lambda_j^{(k)}, j = 0,1,2,...\}$ *which is a strictly monotone decreasing sequence of eigenvalues. The corresponding eigenfunctions* $\Phi^{(k)} = \{\Psi_j^{(k)}\}$ *form an orthonormal basis in* $L^2_\rho(I)$.

(ii) The centre subspace E^c of such self-adjoint extensions $\mathbf{A}_k$ is one-dimensional and is spanned by the eigenfunction

$$\Psi_k^{(k)}(\xi) = c_k \frac{\mathrm{d}}{\mathrm{d}b} \Psi_k(\xi; b)\Big|_{b=1} \equiv -c_k \mathbf{C}\Psi_k(\xi), \quad i.e., \ \Lambda_k^{(k)} = 0, \tag{6.9}$$

where $c_k > 0$ is a normalisation constant and $\mathbf{C}$ is as in (5.9) with $\nu_* = (m-1)/2m$.

The eigenfunction (6.9) is obtained by differentiating equation (4.25) with respect to b and using (4.21). By (6.7) this yields $\mathbf{A}_k \Psi_k^{(k)} = 0$. Using a proof similar to that of Proposition 4.1, one can establish that $\Psi_k^{(k)}$ has precisely k zeros.

By completeness in $L^2_\rho(I)$ of the subset $\Phi^{(k)}$ of eigenfunctions, using Proposition 6.1, equation (5.19) with operators (6.6) and (5.20) with ψ_k replaced by Ψ_k, we get the uniformly converging eigenfunction expansion of the bounded smooth solution $Y(\xi,\tau) = \sum a_j(\tau)\Psi_j^{(k)}(\xi)$. The expansion coefficients $\{a_j(\tau)\}$ satisfy the DS (5.35) with $\lambda_j^{(k)}$ replaced by $\Lambda_j^{(k)}$, subject to (6.6). We may assume that the expansion coefficients are uniformly small. Being interested in the critical asymptotic behaviour corresponding to the evolution close to the centre subspace of $\mathbf{A}_k$, we exclude both the stable and unstable exponentially decaying patterns. We perform further formal asymptotic analysis under the assumption of centre subspace dominance and assume that the behaviour for $\tau \gg 1$ of the k-th coefficient is dominant in the sense that (cf. (5.37))

$$Y(\tau) = a_k(\tau)\Psi_k^{(k)} + o(a_k(\tau)) \quad \text{as} \ \ \tau \to \infty \tag{6.10}$$

uniformly on compact subsets and in H^2_ρ. Under this assumption, by expanding the right-hand side of (5.35), (6.6) and using (5.20) with ψ_k replaced by Ψ_k, we obtain the DS (5.38) with $\lambda_j^{(k)}$ replaced by $\Lambda_j^{(k)}$ and coefficients

$$\begin{aligned}
&\varepsilon_{1,j} = -\langle \mathbf{C}\Psi_k, \Psi_j^{(k)}\rangle_\rho, \ \ \varepsilon_{2,j} = \langle g(\Psi_k), \Psi_j^{(k)}\rangle_\rho,\\
&A_j = \frac{1}{2} m(m-1)\langle |\Psi_k''|^{m-3}\Psi_k[(\Psi_k^{(k)})'']^2, \Psi_j^{(k)}\rangle_\rho, \quad D_j = \langle g'(\Psi_k), \Psi_j^{(k)}\rangle_\rho > 0,\\
&B_j = \frac{1}{6} m(m-1)(m-2)\langle |\Psi_k''|^{m-3}[(\Psi_k^{(k)})'']^3, \Psi_j^{(k)}\rangle_\rho, \ \ C_j = \langle \mathbf{C}\Psi_k^{(k)}, \Psi_j^{(k)}\rangle_\rho.
\end{aligned}$$

By asymptotics (4.22) all the expansion coefficients above are finite if $m \in [1,2)$. For $j = k$ with $\Lambda_k^{(k)} = 0$, we obtain the key ODE (5.39) for $a_k(\tau)$ with the coefficients $\varepsilon_{1,k} = -\langle P_1, \Psi_k^{(k)}\rangle_\rho = c_k \|\mathbf{C}\Psi_k\|^2_\rho > 0$ and $\varepsilon_{2,k} = \langle P_2, \psi_k^{(k)}\rangle_\rho = c_k\langle g(\Psi_k), \psi_k\rangle_\rho - \frac{c_k}{2m}(m-1)\langle g(\Psi_k), \Psi_k'\xi\rangle_\rho$. Obviously, $\mathbf{A}_k = \mathbf{B}_k = 0$ for $m = 1$.

The asymptotic analysis of this DS is the same as in Proposition 5.3 (see also the remark afterwards). Finally, in the original variables we obtain the critical asymptotic patterns:

$$u(x,t) = C_k (t \ln t)^{-\delta_k}[\Psi_k(\eta) + o(1)], \quad \eta = \frac{x}{t^{1/2m}}(t \ln t)^{(m-1)\delta_k/2m}, \tag{6.11}$$

where $\delta_k = \mu_0 - \Lambda_k > 0$ and C_k is as given in (5.43).

6.2. **Radial geometry.** The analysis is quite similar, see the end of Section 5. $\mathbf{A}_k$ is a singular ordinary differential operator on $(0,1)$

$$\mathbf{A}_k Y = m|\Delta_{\bar\xi}\Psi_k|^{m-1}\Delta_{\bar\xi}Y + \beta_k\bar\xi Y' + (\mu_0 - \Lambda_k)Y \tag{6.12}$$

with the symmetric representation as in (6.7), where

$$p(\bar\xi) = \bar\xi^{N-1}\exp\left\{\frac{\beta_k}{m}\int \bar\xi|\Delta_{\bar\xi}\Psi_k|^{1-m}\mathrm{d}\bar\xi\right\}, \quad \rho(\bar\xi) = \frac{p(\bar\xi)}{m|\Delta_{\bar\xi}\Psi_k|^{m-1}}.$$

One can see that the end-point $\bar\xi = 1$ has the same deficiency indices as for $N = 1$. At the origin $\bar\xi = 0$ a symmetry condition (5.52) is imposed. Finally, we obtain critical asymptotics (6.11) for any $k = 0, 2, 4, \dots$.

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