Product structure of heat phase space and Branching Brownian motion

Frederic P. Schuller

Department of Applied Mathematics and Theoretical Physics,

Centre for Mathematical Sciences,

University of Cambridge,

United Kingdom

E-mail: F.P.Schuller@damtp.cam.ac.uk

and

Pascal Vogt

Department of Mathematical Sciences,

University of Bath,

Bath BA2 7AY, United Kingdom

E-mail: P.Vogt@maths.bath.ac.uk

Version: May 30, 2003

A generical formalism for the discussion of Brownian processes with non-constant particle number is developed, based on the observation that the phase space of heat possesses a product structure that can be encoded in a commutative unit ring. A single Brownian particle is discussed in a Hilbert module theory, with the underlying ring structure seen to be intimately linked to the non-differentiability of Brownian paths. Multi-particle systems with interactions are explicitly constructed using a Fock space approach. The resulting ring-valued quantum field theory is applied to binary branching Brownian motion, whose Dyson-Schwinger equations can be exactly solved. The presented formalism permits the application of the full machinery of quantum field theory to Brownian processes.

Key Words: Brownian motion, branching process, Markov process, pseudo-complex ring

Contents

1. INTRODUCTION

The study of Brownian motion, and Markov processes based on it, has been a rapidly growing subject in both mathematics and physics since Einstein's pioneering paper [1], linking the macroscopic heat flow to microscopic Brownian motion. The rigorous probability theoretical formulation is due to Wiener [2], leading to a well-defined Feynman-Kac path integral representation for solutions of the heat equation. Remarkable non-trivial results, as the non-differentiability of Brownian paths [3], follow from Wiener's construction. Today, Brownian motion is a very active area of research in both pure mathematics and physics [4, 5, 6], as well as in applications to a wide variety of phenomena in nature and finance [7].

Multi-particle Brownian systems with dynamical creation and annihilation of particles, such as branching and dying processes, are conventionally studied by non-linear extensions of the heat equation. Such a treatment, however, requires tailor-made kinematics for each individual system at hand, in order to encode an intrinsically multi-particulate problem in the formalism of a single heat equation. This method becomes almost prohibitively difficult in application to the currently intensively studied catalytic processes [8]. The aim of this paper is to remedy this situation and to devise a generical kinematical framework for *any* Brownian process featuring a dynamically changing particle number.

Our investigation starts with the crucial observation that the phase space of the heat equation carries a product structure, in contrast to the complex structure of the phase space for the Schrödinger equation. The same way in which the latter gives rise to the ubiquitous occurrence of the complex numbers in quantum theory, the product structure can be absorbed into the commutative unit ring $\mathbb{P}$ of pseudo-complex numbers, governing the kinematics of Brownian particles. The study of one-particle Brownian motion then becomes a Hilbert module theory over $\mathbb{P}$, where the non-differentiability of Brownian paths emerges as a direct consequence of the pseudo-complex structure of heat phase space. This thorough geometrical understanding of the one-particle case can then be put to use in the construction of multi-particle systems, employing a Fock space approach. In contrast to the conventional treatment, the presented pseudo-complex Hilbert module description of the single particle is extended to a *model-independent* Fock space formalism, and the dynamics of an operator-valued second quantized field are derived. In this set-up, we discuss general interacting pseudo-complex quantum field theories as the appropriate kinematical framework for the discussion of Brownian processes with non-constant particle number.

In general, the advantage of this field theoretical point of view is that one can easily write down models for arbitrarily complex Brownian processes. Ideally, these models have exactly solvable Dyson-Schwinger equations. Otherwise, a truncated series expansion of interesting quantities in terms of Feynman diagrams may be just the appropriate tool to extract information on analytically inaccessible systems.

The abstract formalism is applied to the classically well-studied binary branching Brownian motion. The Dyson-Schwinger equations of this system can be exactly solved, and coincide with the equations of motion that are found classically for corresponding quantities. A point in case is the reproduction of the extinction probability for such a process, which appears as the dressed one-point function in the pseudo-complex quantum field theory. The formalism is versatile enough to embrace models with dynamical catalysts, which currently receive much attention [8].

The organization of the paper is as follows. Section II reviews the conventional treatment of multi-particle Brownian processes for the benefit of the non-specialist reader. Section III identifies the product structure of the heat phase space, and introduces the pseudo-complex ring. Section IV discusses a single Brownian particle in a pseudo-complex Hilbert module formalism, preparing the non-interacting multi-particle theory developed in Section V. Section VI studies theories with dynamical particle creation and annihilation. These abstract developments are applied to the binary branching model and a self-intoxicating particle in Section VII. We summarize and conclude in Section VIII.

2. A REVIEW OF BROWNIAN PROCESSES

The Brownian motion of a single particle is well-understood and can be given a rigorous probability theoretical description. More complex systems, such as spatial branching processes, are of current research interest [4, 5, 6, 9]. They are conventionally studied as extensions of the single-particle problem, but the kinematical framework has to be adapted from case to case. This section briefly reviews some of the well-studied processes derived from Brownian motion for the non-specialist reader, and closes with a critical discussion. In particular, we will argue that a general framework for the study of systems with non-constant particle number is desirable, such that a system is completely specified by its dynamical equations, without the need to adapt the underlying concepts. The construction of such a system-independent kinematical framework for Brownian processes is the aim of this paper.

2.1. Heat equation and free Brownian motion

Recall that the *heat equation* is given by

$$\partial_t \varphi(t, \mathbf{x}) = \frac{1}{2} \Delta \varphi(t, \mathbf{x}), \tag{1}$$

where φ is a real-valued field on non-relativistic space time $\mathbb{R}_+^0 \times \mathbb{R}^d$, and the diffusion constant is set to unity. Given the initial condition $\varphi(0, \mathbf{x}) = u(\mathbf{x})$, the linear partial differential equation (1) has a unique solution. Its physical significance is the following: suppose the function $\varphi(t, \mathbf{x})$ represents the temperature at time t and position x, then starting with the initial temperature distribution u, the heat equation describes the heat flow in time. In this sense, the heat equation governs the *macroscopic* behaviour of the temperature field.

In 1905, Einstein discovered the relation between the heat equation and a stochastic process [1] called *Brownian motion*. Intuitively, this process describes the *microscopic* behaviour of a *single particle* that contributes to the heat. Mathematically speaking, a Brownian motion with start in $x \in \mathbb{R}^d$ is a stochastic process, i.e., a random path $\{B_t : t \geqslant 0\}$ which is continuous and has independent, stationary and normally distributed increments. A Brownian motion can therefore be considered, in a rigorous probabilistic manner [2], as a random element of the canonical Wiener space $(C([0,\infty)), \mathcal{F}, \mathbb{P}_x)$, where $\mathbb{P}_x$ is a measure on the space $(C([0,\infty), \mathcal{F})$ of continuous paths in $\mathbb{R}^d$ equipped with the σ-field $\mathcal{F}$ which is generated by the family of projections $\pi_t : C[0,\infty) \to \mathbb{R}^d$, $\omega \mapsto \omega_t$. The Wiener measure $\mathbb{P}_x$ satisfies, for all times $t_1, t_2, \ldots, t_n$ and all Borel sets $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}^d$, the relation

$$\begin{aligned}&\mathbb{P}_x\{B_{t_1} \in A_1, B_{t_2} \in A_2, \ldots, B_{t_n} \in A_n\}\\ &\quad = \int_{A_1}\int_{A_2}\cdots\int_{A_n} d^d x_1 \ldots d^d x_n \, p_{t_1}(\mathbf{x}, \mathbf{x}_1) \ldots p_{t_n - t_{n-1}}(\mathbf{x}_{n-1}, \mathbf{x}_n), \quad (2)\end{aligned}$$

where the transition density of Brownian motion is given by the heat kernel

$$p_t(\mathbf{x}, \mathbf{y}) \equiv (2\pi t)^{-d/2} \exp\left(-\frac{\|\mathbf{x} - \mathbf{y}\|^2}{2t}\right). \quad (3)$$

The initial value problem for the classical heat equation,

$$\partial_t \varphi(t,\mathbf{x}) = \frac{1}{2}\Delta \varphi(t,\mathbf{x}), \qquad \varphi(0,\mathbf{x}) = u(\mathbf{x}), \tag{4}$$

is solved using the Brownian semigroup operators P_t,

$$\varphi(t,\mathbf{x}) = (P_t u)(\mathbf{x}) := \int d^d z\, u(\mathbf{z}) p_t(\mathbf{x},\mathbf{z}), \tag{5}$$

i.e., as a linear superposition of the fundamental solutions p_t.

The semigroup property of the operator P_t,

$$P_t \circ P_s = P_{t+s}, \tag{6}$$

implies the so-called Markov property of Brownian motion, i.e., any point on the Brownian path can be considered as the starting point of a new Brownian motion. In terms of the transition density p_t, the Markov property hence takes the form

$$p_{t+s}(\mathbf{x},\mathbf{y}) = \int d^d z\, p_t(\mathbf{x},\mathbf{z}) p_s(\mathbf{z},\mathbf{y}). \tag{7}$$

By definition of the expectation, we can rewrite (5) as an integral over the space of continuous functions with respect to the Wiener measure $\mathbb{P}_x$,

$$\varphi(t,\mathbf{x}) = \mathbb{E}_x\{u(B_t)\} = \int_{C_0[0,\infty)} \mathbb{P}_x(d\omega)\, u(\omega_t). \tag{8}$$

Hence, the solution of the initial value problem for the classical heat equation can be represented either using the Brownian semigroup or, equivalently, as a path integral. An important characteristic of Brownian paths was found by Paley, Wiener and Zygmund [3], who showed that with respect to the Wiener measure, almost all Brownian paths are nowhere differentiable. We will come back to this point in section 4.

We emphasize that the heat equation (1) describes a *single* Brownian particle, by the heat function $\varphi(t, \mathbf{x})$. As in the case of the Schrödinger equation, the formalism, as it stands, cannot be extended to accommodate a non-constant particle number, as this would require a non-constant number of functions φ_i in the dynamical equations. However, the desire to study killing and branching processes has led to the following two constructions.

2.2. Softly killed Brownian motion

It is interesting to study the heat equation with a potential term $v(\mathbf{x})$, which is assumed to be bounded from below,

$$\partial_t \varphi(t, \mathbf{x}) = \frac{1}{2} \Delta \varphi(t, \mathbf{x}) - v(\mathbf{x}) \varphi(t, \mathbf{x}). \tag{9}$$

Given the initial condition $\varphi(0, \mathbf{x}) = u(\mathbf{x})$, the solution to (9) is given by the Feynman-Kac formula [7]

$$\varphi(t, \mathbf{x}) = \mathbb{E}_{\mathbf{x}} \left\{ u(B_t) \exp\left(- \int_0^t ds \, v(B_s) \right) \right\}. \tag{10}$$

For a constant potential $v \equiv \gamma$, the solutions $\varphi(t, \mathbf{x})$ can be interpreted as the transition densities for a *softly killed* Brownian particle, surviving a time interval Δt only with probability $\exp(-\gamma \Delta t)$. The terminology refers to the fact that we are dealing with an inherent one-particle formalism, and killing is a matter of interpretation of the underlying random variables. This is seen most clearly when considering the continuous paths in Wiener space, which do not stop at a finite parameter value ('hard killing'), but must, in this context, rather be understood as random elements in $C([0,\infty)) \times \mathbb{R}^+$, i.e., as paths with a killing time assigned to them.

This extension of the Wiener space as the underlying probability space is necessary in order to encode a process with non-constant particle number, in a setting that *by construction* can only deal with a constant number of particles.

2.3. Branching Brownian motion

The standard approach to branching Brownian motion [9] is the study of trees composed of independent, softly killed one-particle Brownian motions, where a dying particle gives birth to k children, according to a probability distribution (p_k) on $\{0,1,2,\dots\}$. This is formalized by consideration of a measure-valued process $\{X_t : t \geqslant 0\}$, defined by the counting measure

$$X_t(C) := \#\{\text{particles in } C \text{ at time } t\}, \tag{11}$$

where C is a Borel set in $\mathbb{R}^d$. Hence, if we represent each individual particle at a point $\mathbf{x} \in \mathbb{R}^d$ by a unit point mass $\delta_{\mathbf{x}}$ and write Y_t^i for the position of the i-th particle alive at time t, we have

$$X_t = \sum_i \delta_{Y_t^i}. \tag{12}$$

As the exponential distribution has no memory, the measure-valued process X inherits the *Markov property* from the underlying Brownian motion. Due to Mc Kean (1975), the distribution of branching Brownian motion can be characterized in terms of a non-linear extension of the heat equation: Let ψ be a positive, bounded and continuous function on $\mathbb{R}^d$ such that $0 \leqslant \phi(\mathbf{x}) \leqslant 1$ for all $x \in \mathbb{R}^d$. Then the quantity

$$\varphi(t, \mathbf{x}) \equiv \mathbb{E}_{\delta_{\mathbf{x}}} \Big\{ \prod_i \phi(Y_t^i) \Big\} \tag{13}$$

satisfies the evolution equation

$$\begin{array}{c} \partial_t \varphi = \frac{1}{2}\Delta\varphi + \gamma(W(\varphi) - \varphi) \\ \varphi(0, \mathbf{x}) = \phi(\mathbf{x}) \end{array} , \tag{14}$$

where $W(\varphi) := \sum_{k=0}^{\infty} p_k \varphi^k$ is the probability generating function of the distribution (p_k), controlling the fertility of a dying particle. Now u is interpreted as describing an average mass flow of the measure-valued process X. This looks like a straightforward extension of the original heat equation by a non-linear term. However, the paths are now measure-valued, and hence we again deal with a somewhat different mathematical object, despite the formal similarity of equations (1) and (14). These adaptations were necessary to construct a spatially branching model within a formalism that originally described a one-particle system.

One can extract the extinction probability for such a branching and dying process via the following construction. Denote the total mass process of the branching Brownian motion X by $N_t := X_t(\mathbb{R}^d)$. It is well-known (see e.g. [16]) that $\{N_t : t \geqslant 0\}$ is a continuous time Galton-Watson branching process whose moment generating function is given by

$$\tilde{\varphi}_\theta(t) \equiv \mathbb{E}\{\theta^{N_t} \,|\, N_0 = 1\} = \sum_{k=0}^{\infty} \theta^k \, \mathbb{P}\{N_t = k \,|\, N_0 = 1\}, \tag{15}$$

for $\theta \geqslant 0$ (and $0^0 \equiv 1$). The function $\tilde{\varphi}_\theta$ solves the initial value problem

$$\begin{aligned} \partial_t \tilde{\varphi}_\theta(t) &= \gamma \left(W(\tilde{\varphi}_\theta(t)) - \tilde{\varphi}_\theta(t) \right), \\ \tilde{\varphi}_\theta(0) &= \theta. \end{aligned} \tag{16}$$

Compared to equation (14), this evolution equation is missing the Laplace operator that generates the spatial evolution of the process. Hence, equation (16) describes the pure branching process, without the spatial evolution. For the special case $\theta = 0$ we have by (15) that $\tilde{\varphi}_0(t) = \mathbb{P}\{N_t = 0 \,|\, N_0 = 1\}$. The solution of the initial value problem (16), with $\theta = 0$, hence is the extinction probability for the total mass process N_t in the limit $t \to \infty$. In terms of the process described by (14), this corresponds to the probability that the process dies out eventually. We will find equation (16, $\theta \equiv 0$) as the differential form of the recurrence relation for the so-called one-point function in our field-theoretical treatment in Section 7.

2.4. Discussion of the traditional approach

The one-particle Brownian motion is mathematically well-understood, and allows for a rigorous probability theoretical definition in terms of Wiener spaces. The desire to study more complex Markov processes, at the same level of mathematical rigor, led to the introduction of an exponential clock and the consideration of measure-valued paths. These generalizations of the one-particle case, however, come at the cost of an extension of the underlying probability spaces, generalizing paths $C([0,\infty))$ to $C([0,\infty))\times\mathbb{R}^+$, or even measure-valued paths, as explained in the preceding subsections. The necessity to perform these rather complicated and technical adaptations roots in the fact that these more complex Markov processes all feature a *non-constant particle number*, which must be encoded in a one-particle formalism, in a manner that varies from case to case.

The problem of dealing with systems of non-constant particle number is not a new one, but appears naturally in relativistic quantum physics. There, the relativistic mass-energy equivalence $E = mc^2$ allows for particle creation and annihilation, and hence is a feature of any quantum particle process at sufficiently high energy. It is clear that the Schrödinger equation,

$$i\partial_t\psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x), \tag{17}$$

describing the quantum behaviour of a one-particle quantum system in a potential V, cannot feature such a dynamical creation and annihilation of particles, due to the fixed number of functions ψ. An adequate solution was found in form of quantum field theories [10]. It was recognized [11] that quantum field theory can be formally applied to the discussion of Brownian motion, and vice versa, if one replaces the time t by imaginary time

it. This ad-hoc adaptation, however, bypasses the whole construction of a canonically quantized theory starting from its classical equations. There are *no* insights gained as to why one has to consider imaginary time, what the underlying geometrical structures are, or why there is no interference of heat in such a treatment. In fact, there exists no well-defined measure [12] for a Feynman path integral which formally solves the Schrödinger equation (17). Therefore, it seems that using quantum field theoretical methods to describe Brownian processes might imply a loss in mathematical rigour. In this paper, however, we show that careful consideration of the structure of the heat phase space leads to a canonical quantization of heat, whose path integral is well-defined.

In order to understand the occurrence of the imaginary unit in the quantum mechanical 'path integral', it is instructive to re-write the Lagrangian density for the free ($v = 0$) Schrödinger equation (17),

$$\mathcal{L} = \frac{i}{2}(\partial_t\psi\psi^* - \partial_t\psi^*\psi) - \frac{1}{2}\nabla\psi\nabla\psi^*, \tag{18}$$

in terms of the real and imaginary parts α and β of the complex-valued probability amplitude $\psi = \alpha + i\beta$,

$$\mathcal{L} = (\partial_t\alpha\beta - \alpha\partial_t\beta) - \frac{1}{2}(\nabla\alpha\nabla\alpha + \nabla\beta\nabla\beta). \tag{19}$$

The imaginary part β is recognized as the canonical momentum associated with the real part α,

$$\pi_\alpha \equiv \frac{\partial\mathcal{L}}{\partial\partial_t\alpha} = \beta, \tag{20}$$

and the time evolution of the field phase space point (α, β) thus reads

$$\frac{d}{dt}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{1}{2}\Delta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{21}$$

revealing the complex structure of the field phase space of the Schrödinger equation. This complex structure is the actual reason for the imaginary unit entering the exponential in the quantum mechanical analogue of the Feyman-Kac path integral (10). It is therefore worthwhile to investigate the phase space structure of the heat equation in the following section.

3. HEAT PHASE SPACE

This chapter studies the geometry of the heat phase space. The central result is the absorption of the occurring structure into the ring of pseudo-complex numbers. For the discussion of Brownian motion, these play as important a rôle as the complex numbers do in quantum mechanics.

3.1. The structure of heat phase space

In order to identify the canonical momentum associated with the heat ϕ, we need to devise a Lagrangian density for the heat equation. This, however, can only be done by introduction of a second dynamical field η. Without such, any attempt to generate the first order time derivative in (1) from a Lagrangian density inevitably results in a total time derivative, and hence does not contribute to the Euler-Lagrange equations at all. We therefore consider the Lagrangian density

$$\mathcal{L} = (\partial_t\varphi\eta - \varphi\partial_t\eta) + \nabla\varphi\nabla\eta, \tag{22}$$

yielding the Euler-Lagrange equations

$$\begin{aligned} \partial_t\varphi &= \frac{1}{2}\Delta\varphi & (23)\\ \partial_t\eta &= -\frac{1}{2}\Delta\eta. & (24) \end{aligned}$$

We observe that the additional dynamical field η, required by a Lagrangian treatment of heat, is the canonical momentum density associated with the heat field φ, as

$$\pi_\varphi \equiv \frac{\partial\mathcal{L}}{\partial\partial_t\varphi} = \eta. \tag{25}$$

The phase space picture of the heat evolution,

$$\frac{d}{dt}\begin{bmatrix} \varphi \\ \eta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{2}\Delta \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \tag{26}$$

reveals that the geometry of the heat phase space is governed by a *product structure* [13], i.e. the representing matrix squares to the identity. The above matrix representation of the product structure and the phase space element $\phi \equiv [\varphi, \eta]$ clearly depend on a particular choice of basis in phase space. In the following, we will show that there is a basis-independent formulation

$$I\partial_t\phi = \frac{1}{2}\Delta\phi \tag{27}$$

for equation (26). The obvious similarity of (27) with the Schrödinger equation (17) suggests to term ϕ the *heat amplitude*, and equation (27) the *heat amplitude equation*.

Consider a linear operator I on $\mathbb{R}^2$, with the properties $I^2 = \mathrm{id}_{\mathbb{R}^2}$ and $I \neq \pm\mathrm{id}_{\mathbb{R}^2}$. Then $\sigma_\pm \equiv \frac{1}{2}(\mathrm{id} \pm I)$ are orthogonal projection operators, i.e., $\sigma_\pm^2 = \sigma_\pm$ and $\sigma_+\sigma_- = 0$. As $\{\sigma_+, \sigma_-\}$ are linearly independent over $\mathbb{R}$, they provide a basis for $\mathbb{R}^2$. Thus the operator identity

$$I\sigma_\pm = \pm\sigma_\pm \tag{28}$$

can be read as an eigenvalue equation for the operator I, regarding $\sigma_\pm$ as *vectors* of $\mathbb{R}^2$. This shows that the matrix representation in (26) corresponds to the choice of the $\sigma_\pm$-basis for (27). In particular, the heat amplitude is given by the $\mathbb{R}^2$-valued function

$$\phi = \varphi\sigma_+ + \eta\sigma_-. \tag{29}$$

From this point of view, the abstract reasoning started after (26) simply reproduced the previous results. Now we note, however, that there is a *canonical multiplication* defined on the heat phase space $\mathbb{R}^2$, induced by the composition of operators $\sigma_\pm$. Hence, the heat phase space is not merely a vector space, but a commutative ring! We pause in our development of the physical aspects, in order to explore this structure a bit further.

3.2. The pseudo-complex ring

Let F be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. The pseudo-complex ring over F [14] is the set

$$\mathbb{P} \equiv \{p = a + Ib | a, b \in F\}, \tag{30}$$

equipped with addition and multiplication laws induced by those on F, with the additional rule that $I \notin F$ is a product structure, i.e. $I^2 = 1$. It is easily verified that $\mathbb{P}$ is a commutative unit ring with zero divisors $\mathbb{P}^0 \equiv \mathbb{P}^0_+ \cup \mathbb{P}^0_-$, where

$$\mathbb{P}^0_\pm \equiv \langle \sigma_\pm \rangle_F . \tag{31}$$

It is often convenient to employ the zero-divisor representation of a number $p \in \mathbb{P}$,

$$p = a + Ib = (a+b)\sigma_+ + (a-b)\sigma_- . \tag{32}$$

The zero-divisor branches $\mathbb{P}^0_+$ and $\mathbb{P}^0_-$ are the only multiplicative ideals in $\mathbb{P}$, thus they are both maximal ideals. Hence, the linear maps

$$\Gamma_\pm : \mathbb{P} \longrightarrow \mathbb{P}/\mathbb{P}^0_\pm \cong \mathbb{P}^0_\mp \cong F, \tag{33}$$

$$\Gamma_\pm(a + Ib) = a \pm b, \tag{34}$$

are the only non-trivial ring homomorphisms from $\mathbb{P}$ to F, i.e., for all $p, q \in \mathbb{P}$

$$\Gamma(p+q) = \Gamma(p) + \Gamma(q), \tag{35}$$

$$\Gamma(pq) = \Gamma(p)\Gamma(q). \tag{36}$$

Application of $\Gamma_\pm$ to the heat amplitude ϕ extracts the heat φ and antiheat η, respectively:

$$\Gamma_+(\phi) = \varphi, \tag{37}$$

$$\Gamma_-(\phi) = \eta. \tag{38}$$

Physically, this means that heat amplitudes do not interfere (linearity), and that one can compose heat amplitudes for complicated processes multiplicatively from elementary ones. In other words, the extraction of heat from heat amplitudes is compatible with the Markov property (6) of Brownian motion. Finally, we define the linear map $* : \mathbb{P} \longrightarrow \mathbb{P}$ by its action $*\sigma_\pm \equiv \sigma_\mp$. Clearly, $*$ is an involution, as $** = 1$. Acting on the heat amplitude, $*$ effects an exchange of heat and antiheat.

We have seen, from the analysis of the phase space structure of the heat equation, that the heat amplitude is a pseudo-complex valued field. For the discussion of Brownian motion, pseudo-complex numbers therefore play the same rôle as complex numbers do in quantum mechanics. The systematic exploitation of this insight will be seen to be intimately linked to the intrinsical properties of Brownian motion, like the non-differentiability of Brownian paths.

3.3. Stationary solutions and unitary time evolution

It is often sufficient to restrict the discussion of solutions $\phi(x)$ of the heat amplitude equation (27) to the solutions $\Phi(\mathbf{x})$ of the *stationary* heat amplitude equation

$$\frac{1}{2}\Delta\Phi(\mathbf{x}) = -E\Phi(\mathbf{x}), \qquad E \in \mathbb{R}^+, \tag{39}$$

which can be extended to solutions of the time-dependent dynamics by acting on them with the time evolution operators

$$U(t) \equiv \exp(-IEt), \tag{40}$$

such that $\phi(t,\mathbf{x}) = U(t)\Phi(\mathbf{x})$ solves (27). Note that the time evolution operators form a semigroup, i.e.,

$$U(t) \circ U(s) = U(t+s), \tag{41}$$

and are unitary with respect to pseudo-complex conjugation,

$$U^*(t) = U^{-1}(t) = U(-t). \tag{42}$$

In section 4 we will identify the stationary solutions Φ as elements of a Hilbert module $\mathcal{H}$, and the time evolution operators (40) as unitary operators on $\mathcal{H}$.

3.4. Lagrangian for the heat amplitude equation

Equipped with the constructions of the preceding sections, we are now in a position to write down the Lagrangian density (22) in terms of the pseudo-complex valued heat amplitude, i.e.,

$$\mathcal{L} = \frac{I}{2}(\phi\partial_t\phi^* - \partial_t\phi\phi^*) - \frac{1}{2}\nabla\phi\nabla\phi^*. \tag{43}$$

Note that the Lagrangian density is real-valued. For later developments, it will prove useful to consider a slightly more general form, namely including a quadratic term

$$\mathcal{L}_B = \frac{1}{2} I(\phi \partial_t \phi^* - \phi^* \partial_t \phi) - \frac{1}{2} \nabla\phi \nabla\phi^* - \gamma \phi\phi^*, \tag{44}$$

which will be seen to give rise to an exponential clock with clock rate γ. Observe that the Euler-Lagrange equations for (44) yield the equation of motion (9) for softly killed Brownian motion with a constant potential $v(x) \equiv \gamma$. Then the canonically conjugate fields to ϕ, ϕ^* are

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \partial_t \phi} = -\frac{I}{2}\phi^*, \tag{45}$$

$$\pi^* \equiv \frac{\partial \mathcal{L}}{\partial \partial_t \phi^*} = \frac{I}{2}\phi, \tag{46}$$

and we obtain the Hamiltonian density $\mathcal{H}$ as the Legendre transform of $\mathcal{L}$,

$$\mathcal{H} \equiv \pi \partial_t \phi + \pi^* \partial_t \phi^* - \mathcal{L} = \frac{1}{2} \nabla\phi \nabla\phi^* + \gamma\phi\phi^*. \tag{47}$$

We observe that the Hamiltonian density is real-valued, such that the Hamiltonian

$$H \equiv \int d^d x \mathcal{H} \tag{48}$$

can be consistently interpreted as the energy of the heat amplitude field.

4. SINGLE PARTICLE SYSTEM

For the intended development of the multi-particle theory in section 5, a thorough understanding of the one-particle system is inevitable. It is convenient to introduce a basis free Dirac notation for the heat amplitude equation (27), in terms of a Hilbert module formalism.

4.1. Hilbert modules over $\mathbb{P}$

Let $(\mathcal{R}, \langle\cdot|\cdot\rangle_{\mathbb{R}})$ be a Hilbert vector space over $\mathbb{R}$. We define the associated *Hilbert* $\mathbb{P}$*-module* $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ by

$$\mathcal{H} = \mathcal{R}\sigma_+ \oplus \mathcal{R}\sigma_-, \tag{49}$$

with an *indefinite* inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{H}$ defined by

$$\langle f|g\rangle \equiv \langle f_-|g_+\rangle_{\mathbb{R}}\,\sigma_+ + \langle f_+|g_-\rangle_{\mathbb{R}}\,\sigma_-. \tag{50}$$

It follows immediately that $\langle\cdot|\cdot\rangle$ is pseudo-hermitian and semi-linear in the first argument,

$$\langle f|g\rangle = \langle g|f\rangle^*, \tag{51}$$

$$\langle \lambda f|\mu g\rangle = \lambda^* \mu \,\langle f|g\rangle, \qquad \lambda, \mu \in \mathbb{P}. \tag{52}$$

As any real Hilbert space $\mathcal{R}$ has a basis, so has $\mathcal{H}$, i.e., $\mathcal{H}$ is a free module. An operator $A = A_+\sigma_+ + A_-\sigma_-$ on $\mathcal{H}$ is seen to be self-adjoint with respect to $\langle\cdot|\cdot\rangle$ if, and only if, its zero-divisor components $A_\pm : \mathcal{R} \longrightarrow \mathcal{R}$ satisfy

$$A_+ = A_-^{\#}, \tag{53}$$

where # denotes the adjoint with respect to the inner product $\langle\cdot|\cdot\rangle_{\mathbb{R}}$ on $\mathcal{R}$. We further call an operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ *real* if $A = A^*$. It follows immediately that a self-adjoint real operator on $\mathcal{H}$ is a self-adjoint operator on $\mathcal{R}$. The spectral theorem for the Hilbert vector space $\mathcal{R}$ therefore guarantees that the Hilbert module $\mathcal{H}$ decomposes into a direct sum of pseudo-complex eigenspaces E_λ of any (i) selfadjoint and (ii) real operator A on $\mathcal{H}$,

$$\mathcal{H} = \bigoplus_\lambda E_\lambda(A). \tag{54}$$

Note that this slightly extended form of the spectral theorem for Hilbert modules over $\mathbb{P}$ does not recur to the underlying real Hilbert space $(\mathcal{R}, \langle\cdot|\cdot\rangle_{\mathbb{R}})$ any more, but is entirely formulated in terms of the Hilbert module $(\mathcal{H}, \langle\cdot|\cdot\rangle)$. An important example of a Hilbert module is the space of square-integrable pseudo-complex valued functions on $\mathbb{R}^d$, with the indefinite inner product $\langle f|g\rangle \equiv \int d^dx f^*(\mathbf{x})g(\mathbf{x})$. The operator $\hat{\mathbf{X}} \equiv \mathbf{x}$ is self-adjoint and real, and hence gives rise to a complete basis of $\mathcal{H}$. In contrast, the operator $\hat{\mathbf{P}} \equiv -I\nabla_{\mathbf{x}}$ is only self-adjoint, but not real. Hence the spectral theorem does not apply in the case of $\hat{\mathbf{P}}$. In the next section, we will see that the spectrum of $\hat{\mathbf{P}}$ is indeed empty.

Now let $\{|\mathbf{x}\rangle\}$ be a complete orthonormal basis of eigenvectors of a self-adjoint real operator $\mathbf{X} : \mathcal{H} \longrightarrow \mathcal{H}$, i.e.,

$$\hat{\mathbf{X}}\,|\mathbf{x}\rangle = \mathbf{x}\,|\mathbf{x}\rangle\,, \tag{55}$$

$$\langle\mathbf{x}|\mathbf{y}\rangle = \delta(\mathbf{x}-\mathbf{y}), \tag{56}$$

$$\mathrm{id}_{\mathcal{H}} = \int dx\,|\mathbf{x}\rangle\,\langle\mathbf{x}|\;. \tag{57}$$

We will refer to components $\langle\mathbf{x}|f\rangle$ of general states $|f\rangle \in \mathcal{H}$ with respect to the $|\mathbf{x}\rangle$-basis as the *position space representation* of $|f\rangle$. Stationary solutions (39) of the heat amplitude equation are elements $|\Phi\rangle \in \mathcal{H}$. Define states $|\mathbf{p}\rangle$ by their components in the position space representation,

$$\langle \mathbf{x}|\mathbf{p}\rangle \equiv \cos(\mathbf{px}) + I\sin(\mathbf{px}), \tag{58}$$

and observe that we can, in turn, express the $\{|\mathbf{x}\rangle\}$ in the $|\mathbf{p}\rangle$ basis,

$$|\mathbf{x}\rangle = \int dp\, |-\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{x}\rangle\,. \tag{59}$$

Hence $\{|\mathbf{p}\rangle\}$ constitutes another complete basis of $\mathcal{H}$, which one may call *twisted orthonormal*, because of the relation

$$\langle \mathbf{p}|-\mathbf{q}\rangle = \delta(\mathbf{p}-\mathbf{q}). \tag{60}$$

It is sometimes useful to insert the identity in the form of the twisted orthonormal basis,

$$\mathrm{id}_{\mathcal{H}} = \int dp\, |-\mathbf{p}\rangle \langle \mathbf{p}|\,. \tag{61}$$

4.2. Single particle Hilbert formalism

We now put the abstract developments of the previous subsection to use in the discussion of a single Brownian particle. Define an operator $\hat{\mathbf{P}}$ by its position space representation

$$\langle \mathbf{x}|\, \hat{\mathbf{P}}\, |\mathbf{y}\rangle \equiv -I\,\delta(\mathbf{x}-\mathbf{y})\,\nabla_{\mathbf{x}}, \tag{62}$$

and note its self-adjointness $\hat{\mathbf{P}}^{+} = \hat{\mathbf{P}}$. The unitary operator $\exp(I\mathbf{a}.\hat{\mathbf{P}})$, acting on the Hilbert module, translates states in position space by $\mathbf{a}$. We therefore identify its generator $\hat{\mathbf{P}}$ as the momentum operator for the one-particle states. We observe that the states $|\mathbf{p}\rangle$ satisfy the stationary heat amplitude equation

$$\hat{H}\;|\mathbf{p}\rangle = \frac{1}{2}\mathbf{p}^2\;|\mathbf{p}\rangle\,, \tag{63}$$

with $\hat{H} \equiv \frac{1}{2}\hat{P}^2$, such that the full time-dependent solution $|\mathbf{p}\rangle$ is given by

$$|\mathbf{p}(t)\rangle \equiv \exp(-\frac{I}{2}\mathbf{p}^2 t)\;|\mathbf{p}\rangle \tag{64}$$

Hence, any linear combination $|\phi(t)\rangle = \int dp\,\tilde{\phi}(\mathbf{p})\;|\mathbf{p}(t)\rangle$ satisfies the time-dependent heat amplitude equation

$$I\partial_t\;|\phi(t)\rangle = \hat{H}\;|\phi(t)\rangle\,. \tag{65}$$

We emphasize that the states $|\mathbf{p}\rangle$ are *not* eigenstates of the momentum operator $\hat{P}$, because we have

$$\hat{P}\;|\mathbf{p}\rangle = -\mathbf{p}\;|-\mathbf{p}\rangle\,. \tag{66}$$

This shows that the fundamental solutions of the heat amplitude equation have no well-defined momentum, but a well-defined energy. Because the spectrum of $-I\nabla_{\mathbf{x}}$ lies in $iI\mathbb{R}$, there are also no linear combinations over $\mathbb{P}_{\mathbb{R}}$ with a well-defined momentum. This corresponds to the classical result that Brownian paths are almost surely nowhere differentiable. The probability density (3) for a Brownian particle to propagate from $\mathbf{x}$ to $\mathbf{y}$ within time t, is given in Dirac notation as the overlap of the initial and final states:

$$p_t(\mathbf{x},\mathbf{y}) = \Gamma_+\;\langle\mathbf{y}(t)|\mathbf{x}(0)\rangle\,, \tag{67}$$

after projection Γ_+ to the heat part, according to our discussion of the heat projection operator Γ_+ in section 3.2. Using that $\Gamma_+ : \mathbb{P} \longrightarrow \mathbb{R}$ is a ring homomorphism, it is easy to show that the transition probabilities defined by (67) satisfy the Markov property (7). This concludes our discussion of one-particle Brownian motion in terms of the heat amplitude equation.

5. MULTI-PARTICLE SYSTEMS

We now want to give the idea of a multi-particle system a rigorous meaning. The basis for the understanding of a multi-particle system is the understanding of the one-particle system given by a Hilbert module $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ and an equation of motion $I\partial_t\,|\phi(t)\rangle = \hat{H}\,|\phi(t)\rangle$, as discussed in the previous section. In the one-particle theory, the state of a system is given as an element of the space $\mathcal{H}$.

5.1. Fock space

The one-particle space $\mathcal{H}$ is now extended, in order to accommodate the case of zero, one, two, ... particles. the appropriate structure is the so-called *Fock space* generated by $\mathcal{H}$

$$F(\mathcal{H}) \equiv \bigoplus_{n=0}^{\infty} \bigotimes_{i=1}^{n} \mathcal{H} \equiv \mathbb{P} \oplus \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H} \oplus \ldots, \tag{68}$$

where $\mathcal{H}^0 \equiv \mathbb{P}$. We will consider indistinguishable particles obeying Bose statistics, i.e., an exchange of two particles goes unnoticed. This choice is reflected in considering the boson Fock space

$$\odot\mathcal{H} \equiv \bigoplus_{n=0}^{\infty} \bigodot_{i=1}^{n} \mathcal{H}, \tag{69}$$

where $\odot$ denotes a symmetrized tensor product. The symmetrization removes any order of the factors in any of the spaces $\mathcal{H}^{\odot n}$, which therefore encodes the indistinguishability of the particles represented by these factors. Antisymmetrization would also lead to indistinguishable particles, but an exchange of an even number of them would not go completely unnoticed, but rather effect a sign change.

The one-dimensional space $\mathcal{H}^0$ accommodates a system of zero particles, and we call its normalized basis vector

$$|\Omega\rangle \equiv 1 \oplus 0 \oplus 0 \odot 0 \oplus \cdots \in \odot\mathcal{H} \tag{70}$$

the *vacuum* of the multi-particle theory. The vacuum is a new concept that does not occur in the one-particle theory. The boson Fock space $\odot\mathcal{H}$ inherits the inner product on $\mathcal{H}$ in a natural way,

$$\langle u_1 \odot \cdots \odot u_n | v_1 \odot \cdots \odot v_n \rangle \equiv \langle u_1 | v_1 \rangle \dots \langle u_n | v_n \rangle \,. \tag{71}$$

Define a set of linear operators $a^+_{|\mathbf{p}\rangle}$ acting on the Fock space $\odot\mathcal{H}$, labelled by elements of the one-particle Hilbert space $\mathcal{H}$. Their action on the n-particle subspace $\mathcal{H}^{\odot n} \subset \odot\mathcal{H}$ is defined by

$$a^+_{|\mathbf{p}\rangle} \, |u_1\rangle \odot \cdots \odot |u_n\rangle \equiv |\mathbf{p}\rangle \odot |u_1\rangle \odot \cdots \odot |u_n\rangle \,. \tag{72}$$

In particular, we can generate the entire one-particle space $\mathcal{H}$ from the vacuum,

$$a^{+}_{|\mathbf{p}\rangle}\,|\Omega\rangle = |\mathbf{p}\rangle\,. \tag{73}$$

The operators a^{+} are called *creation operators*. Similarly, we define linear *annihilation operators* $a^{-}_{\langle\mathbf{p}|}: \odot\mathcal{H} \longrightarrow \odot\mathcal{H}$, labelled by elements of the dual space $\mathcal{H}^{*}$. They act on an element of the $(n+1)$-particle subspace $\mathcal{H}^{\odot(n+1)}$ by

$$a^{-}_{\langle\mathbf{p}|}\,|u_0\rangle \odot \cdots \odot |u_n\rangle \equiv \sum_{i=0}^{n} \langle\mathbf{p}|u_i\rangle\,|u_0\rangle \odot \ldots |\hat{u}_i\rangle \odot \cdots \odot |u_n\rangle\,, \tag{74}$$

where the hat on $|\hat{u}_i\rangle$ denotes the omission of this vector in the tensor product. In particular, all a^{-} annihilate the vacuum,

$$a^{-}_{\langle\mathbf{p}|}\,|\Omega\rangle = 0 \oplus 0 \oplus 0 \odot 0 \oplus \ldots\,. \tag{75}$$

Linearity allows to extend the definitions of a^{+}, a^{-} to the entire Fock space [15]. One easily checks that

$$\left[a^{-}_{\langle\mathbf{p}|}\,, a^{+}_{|\mathbf{q}\rangle}\right] = \langle\mathbf{p}|\mathbf{q}\rangle \tag{76}$$

from definitions (72) and (74). The choice of Bose statistics for the multi-particle spaces immediately requires

$$\left[a^{+}_{|\mathbf{p}\rangle}, a^{+}_{|\mathbf{q}\rangle}\right] = 0, \tag{77}$$

$$\left[a^{-}_{\langle\mathbf{p}|}\,, a^{-}_{\langle\mathbf{q}|}\right] = 0, \tag{78}$$

such that $a^{+}_{|\mathbf{p}\rangle}a^{+}_{|\mathbf{q}\rangle}\,|\Omega\rangle \equiv |\mathbf{p}\rangle \odot |\mathbf{q}\rangle$ is well-defined. An important property for calculations is that creation and annihilation operators are adjoint with respect to the inner product (71),

$$\langle a^{+}_{|\mathbf{p}\rangle} x | y \rangle = \langle x | a^{-}_{\langle\mathbf{p}|}\, y \rangle, \tag{79}$$

for $x \in \mathcal{H}^{\odot n}$, $y \in \mathcal{H}^{\odot(n+1)}$. Finally, we define an operator-valued field

$$\hat{\Phi}(\mathbf{x}) \equiv \int \frac{d^d p}{(2\pi)^{d/2}}\; a^{-}_{\langle\mathbf{p}|}\; \langle \mathbf{x}|\mathbf{p}\rangle\,, \tag{80}$$

from which we can extract the annihilation operators by

$$a^{-}_{\langle\mathbf{p}|} = (2\pi)^{d/2} \int dx\; \hat{\Phi}(\mathbf{x})\, \langle -\mathbf{p}|\mathbf{x}\rangle\,. \tag{81}$$

Note that up to this point, the creation and annihilation operator gymnastics have no physical meaning. They just describe the 'kinematics' of a multi-particle system, independent of any particular dynamics.

5.2. Second quantization of heat

We now want to derive the stationary dynamics of the free multi-particle theory from the stationary dynamics (63) of the one-particle sector $\mathcal{H}$. To this end we require that the one-particle states (73) of the multi-particle theory still satisfy the equation (63),

$$\hat{H} a^{+}_{|\mathbf{p}\rangle}\; |\Omega\rangle = \frac{1}{2}\mathbf{p}^2 a^{+}_{|\mathbf{p}\rangle}\; |\Omega\rangle\,. \tag{82}$$

The same way in which we understand the one-particle states in terms of creation and annihilation operators, we can construct the appropriate 'Hamiltonian' $\hat{H}$ in terms of creation and annihilation operators,

$$\hat{H} \equiv \int d^d p \frac{\mathbf{p}^2}{2} a^{+}_{|-\mathbf{p}\rangle} a^{-}_{\langle\mathbf{p}|}\,, \tag{83}$$

such that (82) holds, as is easily checked using the commutation relations for the a^+, a^-. Note the sign of the label of the creation operator in (83), due to the pseudo-complex structure. It is now possible to express the Hamiltonian in terms of the operator-valued field $\hat{\Phi}$,

$$\hat{H} = \int d^d x \frac{1}{2} \nabla \hat{\Phi}(\mathbf{x}) \nabla \hat{\Phi}^+(\mathbf{x}), \tag{84}$$

where Φ^+ denotes the adjoint of Φ with respect to the inner product on $\mathcal{H}$.

We know that the time-dependent one-particle states evolve according to (64). In the multi-particle theory, it is convenient to transfer this time dependence of the states to the operators a^+, a^- or, equivalently, $\hat{\Phi}$. Clearly, this change to the so-called *Heisenberg picture* must preserve the inner product between states. If we denote the time evolution operator $U(t) \equiv \exp(I\hat{H}t)$, and note its unitarity $UU^+ = 1$ with respect to the inner product on $\mathcal{H}$, then the Heisenberg picture $\phi(t, \mathbf{x})$ of the stationary field $\hat{\Phi}(\mathbf{x})$ must be given by

$$\hat{\phi}(x) \equiv \hat{\phi}(t, \mathbf{x}) \equiv U(t)\hat{\Phi}(\mathbf{x})U^+(t), \tag{85}$$

in order to preserve the inner product on $\mathcal{H}$. This definition can be rephrased as the Heisenberg equation of motion

$$I\partial_t \hat{\phi}(x) = \left[\hat{H}, \hat{\phi}(x)\right], \tag{86}$$

using that H is self-adjoint. Using the identity $[A, BC] = [A, B]C + B[A, C]$ and (84), one can verify that the operator-valued field $\hat{\phi}(x)$ satisfies

$$I\partial_t \hat{\phi}(x) = \frac{1}{2} \Delta \hat{\phi}(x). \tag{87}$$

This is the most important result of this section: We have shown that the dynamics of the one-particle sector fully determine the dynamics of the operator-valued field $\hat{\phi}(x)$, i.e. the dynamics of the creation and annihilation operators! Indeed, the Lagrangian density for (87),

$$\mathcal{L} = \frac{I}{2}(\phi^+\partial_t\phi - \phi\partial_t\phi^+) - \frac{1}{2}\nabla\phi^+\nabla\phi, \tag{88}$$

gives rise to the Hamiltonian (84). Note that now the ordering of the fields in (88) is relevant, and must be chosen as above to yield (84). An explicit integral representation for the Heisenberg picture field $\hat{\phi}(x)$, whose equations of motion follow from the more general Lagrangian density (44), including a quadratic term $\gamma\phi\phi^+$, can be obtained by observing that

$$\hat{H}^n a_{\mathbf{p}} = a_{\mathbf{p}}(\hat{H} - \frac{1}{2}\mathbf{p}^2 - \gamma)^n, \tag{89}$$

which is immediate by an induction argument. It follows that for these slightly more general dynamics, the Heisenberg picture field $\hat{\phi}$ is given by the explicit expression

$$\hat{\phi}(x) = \int \frac{d^dp}{(2\pi)^{d/2}} a_{\mathbf{p}} \exp(-I(\frac{1}{2}\mathbf{p}^2 + \gamma)x^0)\left\{\cos(\mathbf{px}) + I\sin(\mathbf{px})\right\}. \tag{90}$$

For completeness, we remark that the field $\hat{\phi}$ and its canonically conjugate field $\hat{\pi} \equiv \frac{\partial\mathcal{L}}{\partial\partial_t\hat{\phi}} = -\frac{I}{2}\hat{\phi}^+$ satisfy the *equal time* commutation relations

$$[\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = -\frac{I}{2}\langle\mathbf{x}|\mathbf{y}\rangle, \tag{91}$$

$$[\hat{\phi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{y})] = 0, \tag{92}$$

$$[\hat{\pi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = 0, \tag{93}$$

as is easily checked from the commutation relations of the time-independent a^+ and a^- operators. These equal time commutation relations contain no dynamical information, but simply encode the multi-particle kinematics. The next subsection deals with the explicit calculation of the commutator of fields at different times, yielding dynamical information on the system.

5.3. Propagator for free Brownian motion

We are now in a position to determine the one-particle propagator for Brownian particles within the formalism of the multi-particle theory. The result will present a check on our construction, as it must coincide with the transition amplitude (67) obtained in the one-particle theory. However, the calculation demonstrates the abstract concepts developed in the previous section at work, and we therefore present it in some detail. In order to slim down the notation, we will omit the hat on the operator-valued field $\hat{\phi}$ from now on. We are no longer dealing with the classical heat amplitude ϕ, so that there is no danger of confusion. Consider the amplitude for a particle to be created at spacetime point x and to be destroyed at y,

$$B(x,y) \equiv \langle\Omega|\ \phi(y)\phi^+(x)\ |\Omega\rangle\,. \tag{94}$$

It is convenient to rewrite this in terms of a commutator

$$B(x,y) \equiv \langle\Omega|\ [\phi(x),\phi^+(y)]\ |\Omega\rangle \tag{95}$$

for the Heisenberg field ϕ. From the expansion (90) we obtain

$$B(x,y) = \int \frac{d^d p}{(2\pi)^d} \exp(-I(\frac{1}{2}\mathbf{p}^2 + \gamma)\Delta t) \cos(\mathbf{p}.\Delta\mathbf{x}), \tag{96}$$

where $\Delta t \equiv y^0 - x^0$ and $\Delta\mathbf{x} \equiv \mathbf{y} - \mathbf{x}$. We are interested in the transition amplitude for a Brownian particle, and hence must project $B(x,y)$ by Γ_+, yielding

$$\Gamma_+ B(x,y) = \int \frac{d^d p}{(2\pi)^d} \exp\left(-(\frac{1}{2}\mathbf{p}^2 + \gamma)\Delta t\right) \exp(i\mathbf{p}.\Delta\mathbf{x}), \tag{97}$$

where we have used the symmetry of the integral in $\mathbf{p}$ to extend the $\cos(\mathbf{p}.\Delta\mathbf{x})$ to $\exp(\mathbf{p}.\Delta\mathbf{x})$. Completing the square and shifting of the integration variable $\mathbf{p} \to \mathbf{p} + i\frac{\Delta\mathbf{x}}{\Delta t}$ gives

$$\Gamma_+ B(x-y) = e^{-\gamma\Delta t} e^{-\frac{(\Delta\mathbf{x})^2}{2\Delta t}} \int_{\mathbb{R}^d} \frac{d^d p}{(2\pi)^d} \exp(-\frac{1}{2}\Delta t \mathbf{p}^2). \tag{98}$$

We define the *retarded propagator*

$$B_R(x,y) \equiv \theta(y^0 - x^0) B(x,y), \tag{99}$$

with $\theta(0) = 0$, and extract the transition probability from the retarded propagator via the projection (37),

$$\begin{aligned} p(x,y) &\equiv \Gamma_+(B_R(x,y)) \\ &= \exp(-\gamma\Delta t)(2\pi\Delta t)^{-\frac{d}{2}} \exp\left(-\frac{(\Delta\mathbf{x})^2}{2\Delta t}\right). \end{aligned} \tag{100}$$

This is exactly the transition density for Brownian motion in d spatial dimensions for $\gamma = 0$, already obtained in the one-particle formalism. For $\gamma > 0$, the Brownian particle is seen to carry an independent exponential clock with clock rate γ. Later, we will put this observation to use in the construction of branching Brownian motion, which possesses the Markov property if, and only if, the lifetime of each generic path is exponentially distributed (see section 7).

Also observe that the multiplicativity of Γ_+ corresponds to the semigroup structure of Brownian transition operators. The *Feynman propagator* is defined as the *time-ordered* product of field operators,

$$B_F(x,y) \equiv \langle\Omega|\, T\left\{\phi(x)\phi^+(y)\right\}\, |\Omega\rangle\,. \tag{101}$$

Using the integral representation

$$\theta(x^0) = \int \frac{dp^0}{2\pi}\frac{\sin(p^0x^0)}{p^0}, \tag{102}$$

one finds

$$B_F(x,y) = \int_{\mathbb{R}^d} \frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{1}{ip^0 - \frac{\mathbf{p}^2}{2} - \gamma}\exp\left\{-Iik(y-x)\right\}, \tag{103}$$

where $k \equiv (k^0, \mathbf{k})$ and the inner product is euclidean.

6. INTERACTING MULTI-PARTICLE SYSTEMS

Our criticism in section 2.4, of the standard treatment of Brownian processes, was focused on the fact that such models are not fully specified by their dynamics, but also require an undesirable case-to-case adaptation of the underlying mathematical structures. The pseudo-complex Fock space formalism, developed in section 5, provides the solution to this problem.

One can now study almost arbitrary extensions of the operator-valued field equation (87), in order to describe multi-particle systems, where the particle number changes dynamically. In the field theoretical framework, such dynamical creation and annihilation of particles are already fully encoded in so-called interaction terms added to the free Lagrangian. In this section, we first develop the theory a little for general interaction terms $\mathcal{L}_{\text{int}}$. As an example, we will study the binary branching and dying model in the next section.

6.1. Unitary time evolution

Consider Brownian motion with a general interaction term $\mathcal{L}_{\text{int}}$, giving rise to a total Lagrangian density

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_{\text{int}}[\phi, \phi^*]. \tag{104}$$

In terms of the Hamiltonian formalism, this gives rise to an additional interaction term

$$H_{\text{int}} = -\int d^d x \mathcal{L}_{\text{int}} \tag{105}$$

in the complete Hamiltonian $H = H_0 + H_{\text{int}}$. It is convenient to define the so-called *interaction picture* field ϕ_I as the Heisenberg picture field of the free theory,

$$\phi_I(x) = \exp(IH_0 t)\Phi(\mathbf{x}) \exp(-IH_0 t), \tag{106}$$

where $x \equiv (T, \mathbf{x})$ Hence, the full Heisenberg field of the interacting theory can be expressed in terms of ϕ_I as

$$\phi(x) = V(t)^{+} \phi_I(x) V(t) \tag{107}$$

with

$$V(t) = \exp(IH_0 t) \exp(-IHt). \tag{108}$$

The unitary evolution operator $V(t)$ satisfies the differential equation

$$-I\partial_t V(t) = H_I(t) V(t), \tag{109}$$

where

$$H_I(t) \equiv \exp(IH_0 t) H_{\text{int}} \exp(-IH_0 t) \tag{110}$$

is the interaction Hamiltonian in the interaction picture. Equation (109), with the initial condition $V(0) = 1$ has the iterative solution

$$V(t) = 1 - I \int_0^t dt_1 H_I(t_1) V(t). \tag{111}$$

By iteration one gets, with $t_0 \equiv t$,

$$V(t) = \sum_{n=0}^{\infty} (-I)^n \prod_{i=1}^{n} \left(\int_0^{t_{i-1}} dt_i H_I(t_i) \right), \tag{112}$$

such that the introduction of a time ordering operator and appropriate combinatorial factors [10] yields

$$\begin{aligned} V(t) &= \sum_{n=0}^{\infty} \frac{(-I)^n}{n!} T \left(\int_0^{t_0} dt' H_I(t') \right)^n \\ &\equiv T \exp\left(-I \int_0^{t_0} dt' H_I(t') \right). \end{aligned} \tag{113}$$

6.2. Correlation functions

The field ϕ encodes the full time-dependence of the interacting theory, whereas ϕ_I simply describes the free theory, according to its definition (106). The (n, m)-point correlation function of the free theory is defined by

$$\langle\Omega|\, T\{\phi_I(x_1)\ldots\phi_I(x_n)\phi_I^+(x_{n+1})\ldots\phi_I^+(x_{n+m})\}\,|\Omega\rangle\,, \tag{114}$$

such that the Feynman propagator (101) is recognized as the free two-point correlation function. The vacuum $|\Omega\rangle$ is the ground state of the free theory, i.e., it is the eigenstate of H_0 with minimal eigenvalue. Correspondingly, we define the vacuum of the interacting theory, denoted by $|\Sigma\rangle$, as the lowest eigenvalue eigenstate of the full Hamiltonian H. Using (113), it is a standard exercise [10] to show that the (n, m)-point correlation function for the interacting theory,

$$\langle\Sigma|\, T\{\phi(x_1)\ldots\phi(x_n)\phi^+(x_{n+1})\ldots\phi^+(x_{n+m})\}\,|\Sigma\rangle\,, \tag{115}$$

where ϕ is the Heisenberg picture field (107) of the interacting theory, can be calculated from the interaction picture field ϕ_I (106) and the free vacuum $|\Omega\rangle$ as

$$\frac{\langle\Omega|\, T\left\{\phi_I(x_1)\ldots\phi_I(x_n)\phi_I^+(x_{n+1})\ldots\phi_I^+(x_{n+m})\exp(-I\int d^{d+1}z\mathcal{H}_I(z))\right\}\,|\Omega\rangle}{\langle\Omega|\, T\exp(-I\int d^{d+1}z\,\mathcal{H}_I(z))\,|\Omega\rangle}, \tag{116}$$

where $\mathcal{H}_I$ is the Hamiltonian density for the interaction Hamiltonian H_I. It is desirable to simplify this expression by replacing the calculationally awkward time ordering operator T by so-called normal-ordered expressions. The following subsection introduces Feynman diagrams as the appropriate technology.

6.3. Wick's Theorem and Feynman diagrams

We introduce the normal ordering operator N, which acts on a product of creation and annihilation operators by regrouping the factors such that all annihilation operators are sent to the right of all creation operators, e.g,

$$N(a^{+}_{|\mathbf{p}\rangle} a^{-}_{\langle\mathbf{q}|} a^{+}_{|\mathbf{r}\rangle}) \equiv a^{+}_{|\mathbf{p}\rangle} a^{+}_{|\mathbf{r}\rangle} a^{-}_{\langle\mathbf{q}|} . \tag{117}$$

Wick's theorem states how a time-ordered product of heat amplitude fields ϕ_I, as occurring in (116), can be expressed by normal ordered expressions and contractions,

$$\begin{aligned} & T\left\{\phi_I(x_1)\phi_I(x_2)\phi_I^{+}(x_3)\dots\phi_I(x_f)\right\} \\ = \; & N\left[\phi_I(x_1)\phi_I(x_2)\phi_I^{+}(x_3)\dots\phi_I(x_f)\right] && (118) \\ & +N\left[\text{all (possibly incomplete) contractions}\right], && (119) \end{aligned}$$

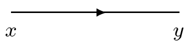
where a contraction is defined as

$$C[\phi_I(x)\phi_I^{+}(y)] \equiv B_F(x-y). \tag{120}$$

Note that for Brownian processes, only *future directed* contractions contribute in the above expression, because we find from (99) and (101) that $B_F = B_R$. As an example, assume the following string of fields is already time-ordered, then

$$\begin{aligned} T\left\{\phi_I(a)\phi_I^+(b)\phi_I(c)\phi_I^+(d)\right\} &= N\left[\phi_I^+(b)\phi_I^+(d)\phi_I(a)\phi_I(c)\right] \\ &\quad +B_F(a,b)N\left[\phi_I(c)\phi_I^+(d)\right] \\ &\quad +B_F(a,d)N\left[\phi_I(b)^+\phi_I(c)\right] \\ &\quad +B_F(c,d)N\left[\phi_I(a)\phi_I^+(b)\right] \\ &\quad +B_F(a,b)B_F(c,d). \end{aligned} \tag{121}$$

A proof of Wick's theorem can be found in [10]. Representing the propagator $B_F(x,y)$ by a directed line joining the points x and y,



$= B_F(x,y)$

we can write correlation functions in diagrammatic form. Note that the normal ordered terms will not contribute to the correlation function, as the sandwiching between vacuum states annihilates them. The integral over $\mathcal{H}_I$ in (116) will lead to diagrams with vertices, where all occurring vertices are integrated over.

Before we undertake the enterprise of calculating correlation functions for a particular theory, however, we can use Wick's theorem to simplify expression (116) significantly. As only future-directed contractions survive, we immediately have

$$\langle\Omega|\, T\exp(-I\int d^{d+1}z\mathcal{H}_I(z))\,|\Omega\rangle = 1, \tag{122}$$

and hence formula (116) for the computation of correlation functions of the interacting theory simplifies to

$$\langle\Omega|\, T\left\{\phi_I(x_1)\ldots\phi_I(x_n)\exp(-I\int d^{d+1}z\mathcal{H}_I(z))\right\}|\Omega\rangle\,. \tag{123}$$

This is consistent with a general result for any interacting quantum field theory, namely that the denominator in (116) cancels all disconnected vacuum diagrams in the numerator (see e.g. [10]). In the present case of Brownian motion, however, we saw above that the Feynman propagator is retarded, and so there simply are no non-vanishing vacuum bubbles.

For general interactions $\mathcal{H}_I$, it is non-trivial to calculate the (n, m)-point correlation functions, and very often it is impossible to do so exactly. In those cases, which present the rule rather than the exception in realistic quantum field theories, one must resort to a truncated series expansion of (123). The diagrammatic representation of a summand in such a perturbation series is called a Feynman diagram.

In the next section, we will show that the binary branching and dying model is exactly solvable. Nevertheless, we will first expand the corresponding correlation functions, and then exactly sum the resulting series by integration of a set of recurrence relations.

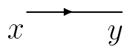
7. EXAMPLE: BINARY BRANCHING BROWNIAN MOTION

7.1. Feynman Rules

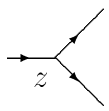
Binary branching and dying processes of Brownian particles are easily formulated as interaction terms in the field theoretical treatment of Brownian motion. This model is completely specified by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_B[\phi] + \gamma\alpha\phi^* + \gamma\beta\phi^*\phi\phi, \tag{124}$$

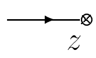
where $\gamma \in \mathbb{R}_0^+$, $\alpha, \beta \in [0,1]$, and $\alpha + \beta = 1$. The fertility distribution $\{p_k : k = 0,1,2\}$, encoding the probability that a dying particle will give birth to k children, will turn out to be $p = (\alpha, 0, \beta)$. The parameter γ will be identified below as the clock rate of the exponential clock controlling the process. Note, that the equation of motion corresponding to $\mathcal{L}$ is the reaction diffusion equation (14) with $p = (\alpha, 0, \beta)$ for classical binary branching Brownian motion. The free theory $\mathcal{L}_B$ is as defined in (44). Applying the analysis of the previous section to the model (124), one obtains the position space Feynman rules [10]



$$= B_F(x,y)$$



$$= I\gamma\beta \int d^{d+1}z$$



$$= I\gamma\alpha \int d^{d+1}z$$

as the fundamental building blocks of the trees generated by the branching process. Thus, at last, we recover the trees of branching Brownian motion from the field theoretical formulation! It is an interesting question of whether and how one can extract information on the genealogy of the process from the Feynman series, or indeed the exact summability shown below. The development of such ideas, however, is beyond the scope of the present paper. Before studying more complicated questions within the binary branching model, we calculate the probability that a single particle will either branch or die within a specified time interval.

7.2. Exponential Clock

The amplitude for a particle starting at spacetime point x to die within a time interval $\Delta\tau$ is given by

$$I\gamma\alpha \int_{x^0}^{x^0+\Delta\tau} dy^0 \int d^d y B_F(x,y) = I\alpha \left(1 - e^{-\gamma\Delta\tau}\right), \qquad (125)$$

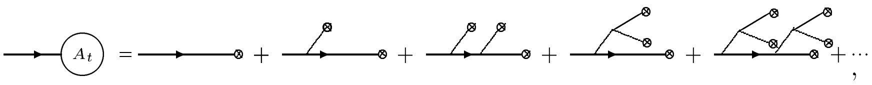
using the Feynman rules above. Similarly, one obtains the probability for a branching event within the time interval $\Delta\tau$. Hence the total probability for the occurrence of branching or dying within $\Delta\tau$ is found by the Γ_+ projection as

$$1 - e^{-\gamma\Delta\tau}. \qquad (126)$$

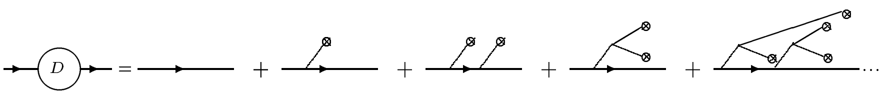
This exponential clock behaviour is crucial for the process being of Markov type, as explained in section 2. In the field theoretical treatment, it need not be imposed, but is *enforced*, unless $\gamma = 0$ in (124), but then all interaction is switched off and one is left with free Brownian motion. Hence, the description of binary branching by the Lagrangian density (124), within the field theoretical framework, implicitly contains the branching tree, the exponential clock, and the fertility distribution, which must all be separately specified in the conventional approach.

7.3. Exact summation of the binary branching Brownian motion

The fully interacting model (124) provides a non-trivial test bed for our formalism. Here, the one-point function already has contributions from any order in α and β,

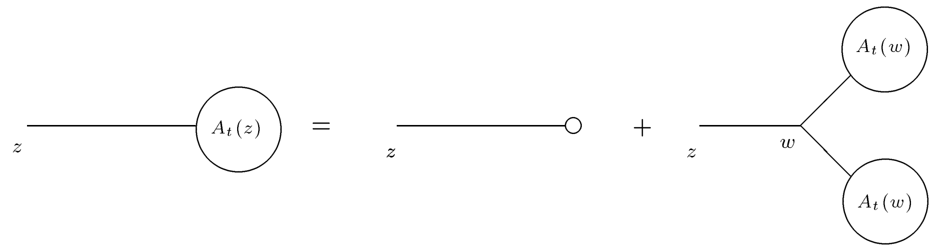


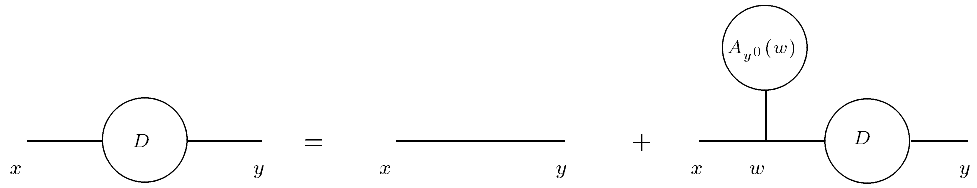
where the subscript t denotes the earliest time at which all particles occurring in the series have died. Similarly, the two-point correlation function is



with contributions of order $\mathcal{O}(\alpha) = \mathcal{O}(\beta)$.

We will put to use the special properties of the model at hand to exactly sum these Feyman series. The one-point function $A(z)$ and the two-point function $D(x, y)$ of the interacting theory satisfy the coupled recurrence relations





Algebraically, these read

$$
\begin{aligned}
A_t(z) \;&=\; I\gamma\alpha \int_{z_0}^{t} dw^0 \int d^dw B_F(z,w) + I\gamma\beta \int_{z^0}^{t} dw^0 \int d^dw\, B_F(z,w) A_t^2(w), \\
D(x,y) \;&=\; B_F(x,y) + I\,\gamma\beta \int_{x^0}^{y^0} dw^0 \int d^dw\, B_F(x,w) A_{y_0}(w) D(w,y).
\end{aligned}
$$

The translational invariance of the theory implies that the one-point function $A_t(z)$ only depends on the difference $\tau \equiv t - z^0$, and hence we denote $A_t(z)$ as $A(t - z^0)$. The recurrence relation for A_t can then be written

$$
A(\tau) = I\gamma\alpha \int_0^{\tau} dw^0 e^{-\gamma w^0} + I\gamma\beta \int_0^{\tau} dw^0 e^{-\gamma w^0} A^2(\tau - w^0), \qquad (127)
$$

using that the spatial integral over B_F is normalized to unity. Multiplication of (127) by $e^{\gamma\tau}$ renders the second integrand independent of τ. By differentiation with respect to τ we obtain the differential form of the recurrence relation (127). Writing $\tilde{A}$ for the projection $\Gamma_+ A$, it reads

$$
\frac{d\tilde{A}(\tau)}{d\tau} = \gamma\alpha - \gamma\tilde{A}(\tau) + \gamma\beta\tilde{A}^2(\tau), \qquad \tilde{A}(0) = 0. \qquad (128)
$$

Indeed, equation (128) is equal to (16) for $\theta = 0$, because there we have $W(\varphi) = \alpha + \beta\varphi^2$ for a binary branching mechanism. The unique solution of (128) subject to the constraint $\alpha + \beta = 1$ is

$$\tilde{A}(\tau) = \begin{cases} 1 - \frac{2}{\gamma\tau+2}, & \text{iff} \quad \alpha = \beta = \frac{1}{2} \\ \\ \frac{1}{2\beta}\left(1 - \sqrt{1-4\alpha\beta}\tanh\left[\frac{1}{2}\sqrt{1-4\alpha\beta}\gamma\tau + C\right]\right), \end{cases} \tag{129}$$

where $\tanh(C) = (1-4\alpha\beta)^{-\frac{1}{2}}$.

From this result we can easily extract the *extinction probability* of the process, i.e., the probability for the process to have died completely at temporal infinity $\tau \to \infty$:

$$\lim_{\tau\to\infty} \tilde{A}(\tau) = \frac{1 - |1-2\alpha|}{2-2\alpha} = \begin{cases} \frac{\alpha}{1-\alpha} & \text{for } \alpha < \frac{1}{2} \\ \\ 1 & \text{for } \alpha \geqslant \frac{1}{2} \end{cases} \tag{130}$$

with almost sure extinction for $\alpha \geqslant \frac{1}{2} \geqslant \beta$, as one would intuitively expect. Indeed, (130) is a classical result for branching Brownian motion [16]. As the function $D(x,y)$ only depends on the difference $y-x$ of the space-time points x and y, we may write the recurrence relation for $D(x,y)$ in the form

$$D(z) = B(z) + I\gamma\beta\int_0^{z^0} dw^0 \int d^dwB(w)A(z^0-w^0)D(z-w). \tag{131}$$

Requiring D to be integrable, standard arguments using Banach's fixpoint theorem, guarantee existence and uniqueness of a solution to the integral equation (131).

7.4. Extension to Brownian catalysts

The extension of the model to Brownian catalysts is straightforward in the field theoretical treatment. One simply provides terms for both the free Brownian particle ϕ and the free Brownian catalyst ψ, say. The type of interaction between them is then modeled in the interaction terms. As an example, self-intoxicating particles ϕ are modeled by

$$\mathcal{L} = \mathcal{L}_B[\phi] + \mathcal{L}_B[\psi] + \gamma\alpha\psi^*\phi^* + \gamma\beta\phi^*\phi\psi, \tag{132}$$

where ψ presents the toxic substance. The one-point function for this model is exactly solvable, now due to its simple *two-point* function, which can be exactly summed and immediately gives the one-point function. The analysis of catalytic models will, in general, be more involved than the exactly solvable binary branching or self-intoxicating model. However, expressing catalytic Brownian motion in the form of a pseudo-complex field theory, most of the apparatus of quantum field theories is applicable to these questions. This is expected to allow insights into such processes that are hidden in the standard formalism.

8. CONCLUSION

The product structure of heat phase space emerges as the key to a geometrical understanding of characteristic properties of Brownian motion, most notably the non-differentiability of Brownian paths. This structure is concisely encoded in the commutative ring of pseudo-complex numbers, giving rise to dynamics for a pseudo-complex valued heat amplitude. The real-valued heat can be extracted from the heat amplitude by a unique additive and multiplicative projection, reflecting the absence of interference and Markov property for heat propagation. The discussion of an abstract Hilbert module over the pseudo-complex ring provides a convenient formulation of one-particle Brownian motion in terms of heat amplitudes. The appropriate kinematical framework for the discussion of Brownian processes with non-constant particle number is the pseudo-complex Fock space generated by the one-particle system. The operator-valued second quantized field equations are rigorously constructed and extended by interactions. A standard derivation of the correlation functions for interacting theories in terms of the free fields yields their diagrammatical representations.

The central result of the paper is the realization that multi-particle Brownian processes can be conveniently studied as pseudo-complex quantum field theories, with Brownian particles emerging as the quanta of the heat amplitude field. Arbitrarily complex models are easily written down in the formalism. Indeed, the mere specification of the model dynamics, by a Lagrangian, generates *all* elements of the conventional constructions: The Feynman diagrams of the pseudo-complex quantum field theory naturally generate the trees of the standard approach; interactions cannot be switched on without also starting an exponential clock, which guarantees the Markov property of the process; the fertility and death rates are concisely encoded in the dynamics of the model. No adaptations of the formalism must be made if the dynamics are extended by additional fields and arbitrary local interactions. This is in contrast to the conventional approach discussed in section 2, and indeed solves the discussed shortcomings of the latter.

The well-known binary branching Brownian motion provides an ideal test bed for the field theoretical description, and its one-point function is explicitly calculated, illustrating the extraction of information in the presented formalism. The extinction probability of binary branching Brownian motion is a point in case and yields, of course, the classical result.

A field theoretical treatment easily allows the discussion of dynamical catalysts, which attract much interest in the present probabilistic literature on the subject [8]. The application of the theory to quantum systems immersed in a heat bath, a question that currently achieves a lot of attention in the context of quantum information theory, is under investigation.

An important restriction of the systems that can be rigorously discussed, in the context of mathematical questions, is the requirement of exact solvability of the Dyson-Schwinger equations. However, for phenomenological applications of branching processes to practical problems, a diagrammatic expansion may just be the appropriate tool for extracting information on a complex system.

Exerting some care in the application of standard field theoretical results to the pseudo-complex case, much of the vast apparatus of modern quantum field theory is now applicable to problems on Brownian motion, and vice versa.

REFERENCES

[1] A. Einstein (1905), Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Annalen der Physik und Chemie*, **17**, IV, 549-560

[2] N. Wiener (1923), Differential space, *J. Math. Phys.* **2**, 131-174

[3] R. Paley, N. Wiener, and A. Zygmund (1933), A note on random functions, *Math. Z.* **37**, 647–668

[4] N. Ikeda, M. Nagasawa and S. Watanabe (1968), Branching Markov processes I, II, *J. Math. Kyoto Univ.* **8**, 233-278 and 365-410

[5] E. B. Dynkin (1991), Branching particle systems and superprocesses, *Ann. Prob.* **19**, 1157-1194

[6] D. A. Dawson (1993), Measure-valued Markov processes, *Ecole d'été de probabilités de Saint Flour*, vol. 1541, Springer

[7] L. Karatzas, S. E. Shreve (1991), Brownian motion and stochastic calculus, *Springer Verlag*

[8] A. Klenke (1999), A Review on Spatial catalytic branching. Stochastic Models, A Conference in Honor of Don Dawson (L. Gorostiza and G. Ivanoff, eds.), Conference Proceedings, vol. **26**, *Canadian Mathematical Society*, 245-264.

[9] A. M. Etheridge (2000), An introduction to superprocesses, *AMS Lecture Series*, Rhode Island

[10] M. E. Peskin, D. V. Schroeder (1996), An Introduction to Quantum Field Theory, *Addison-Wesley*

[11] M. Nagasawa (1993), Schrödinger equations and diffusion theory, Monographs in Mathematics, vol. 86, *Birkhäuser Verlag*, Basel

[12] J. B. Keller and D. W. MacLaughlin (1975), The Feynman Integral, *Am. Math. Monthly* **82** 5

[13] K. Yano (1965), Differential geometry on complex and almost complex spaces, Int. Series of monographs in pure and appl. math., vol. 49, *Macmillan*, New York

[14] F. P. Schuller (2002), Born-Infeld Kinematics, *Annals of Physics* **299**, 174 – 207

[15] M. Masujima, Path integral quantization and stochastic quantization. (English), *Springer Tracts in Modern Physics* **165**, Berlin, Springer

[16] K. B. Athreya and P. E. Ney (1972), Branching Processes, *Springer*